Obstructions to existence in fast-diffusion equations

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Abstract

The study of nonlinear diffusion equations produces a number of peculiar phenomena not present in the standard linear theory. Thus, in the sub-field of very fast diffusion it is known that the Cauchy problem can be ill-posed, either because of non-uniqueness, or because of non-existence of solutions with small data. The equations we consider take the general form $u_t = (D(u, u_x)u_x)_x$ or its several-dimension analogue. Fast diffusion means that $D \to \infty$ at some values of the arguments, typically as $u \to 0$ or $u_x \to 0$.

Here we describe two different types of non-existence phenomena. Some fast diffusion equations with very singular D do not allow for solutions with sign changes, while other equations admit only monotone solutions, no oscillations are allowed. The examples we give of both types of anomaly are closely related. The most typical examples are $v_t = (v_x/|v|)_x$ and $u_t = u_{xx}/|u_x|$. For these equations we investigate what happens to the Cauchy problems when we take incompatible initial data and perform a standard regularization. It is shown that the limit gives rise to an initial layer where the data become admissible (positive or monotone, respectively), followed by a standard evolution for all $t > 0$, once the obstruction has been removed.

Keywords and Phrases. Non-existence, fast diffusion equations, solutions with changing sign, monotone solutions, initial layer.

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1 Introduction and main results

In this paper we investigate the effect of very fast diffusion rates on the existence theory of nonlinear parabolic equations. Nonlinear diffusion equations produce a number of peculiar phenomena not present in the standard linear theory. Typical examples are finite propagation, limited regularity, and the occurrence of singularities which appear after a finite time. In particular, when we consider equations of the *very fast diffusion type*, $u_t = (D(u, u_x)u_x)_x$, with D very divergent as u or u_x tend to zero, the Cauchy problem can be ill-posed, either because of non-uniqueness, or because of non-existence of solutions with small data, cf. e.g. [16], [35]. This contrasts with the known fact that for the standard range of exponents $1 < p < \infty$, equation $u_t = (|u_x|^{p-2}u_x)_x$ generates a semigroup of contractions in all spaces $L^p(\mathbb{R})$, $1 \le p$, while $u_t = (|u|^{p-2}u)_{xx}$ generates a similar semigroup in $L^1(\mathbb{R})$.

1.1. The basic equations. We perform here the study of a limit case of fast diffusion in two versions and derive unusual properties. Our first example is the following equation

$$u_t = \frac{u_{xx}}{|u_x|}. \tag{1.1}$$

We can also write equation (1.1) in the divergence form $u_t = (\log|u_x|\,\mathrm{sign}(u_x))_x$. If we differentiate these forms with respect to x and put $v = u_x$, we arrive at the second basic example,

$$v_t = \left(|v|^{-1}v_x\right)_x, \tag{1.2}$$

which can be written as $v_t = (\log|v|\ \mathrm{sign}(v))_{xx}$. Due to this relationship, the mathematical study of both equations is basically equivalent. They are border-line cases of the standard theory of nonlinear parabolic equations. Thus, we may view equation (1.1) as the limit of the equations

$$u_t = \frac{u_{xx}}{|u_x|^n} = \frac{1}{p-1}\left(|u_x|^{p-2}u_x\right)_x \tag{1.3}$$

with $n < 1$ and $p = 2 - n > 1$, limit taken as $n \to 1$, i.e., $p \to 1$. Upon differentiation equation (1.3) gives

$$v_t = \left(|v|^{-n}\,v_x\right)_x = \frac{1}{p-1}(|v|^{p-2}v)_{xx} \tag{1.4}$$

for $v = u_x$, which in the limit becomes, at least formally, (1.2). We will refer to (1.2) and (1.4) as the *differentiated equations*, and (1.1) and (1.3) as the *integrated equations*. The latter are examples of the type known as nonlinear diffusion equations with gradient-dependent diffusivity. More specifically, we call (1.1) the *integrated limit equation* and (1.2) the *differentiated limit equation*, where the word *limit* aims at specifying the difference with respect to the approximate models (1.3), (1.4), and will be dropped when no confusion arises. Let us note for the reader's convenience that for equation (1.4) one writes in the usual "porous-medium notation" $m = p - 1 = 1 - n$, $v_t = (|v|^{m-1}v)_{xx}$, and then $m \to 0$.

1.2. EXISTENCE AND NON-EXISTENCE. Equations (1.3) and (1.4) are well-known in the nonlinear diffusion literature. Thus, (1.4) is a case of the generalized filtration equation $v_t = \Phi(v)_{xx}$, where Φ is a monotone function in $\mathbb{R}$, while (1.2) is not, since the function $\Phi(v) = \log|v|\,\mathrm{sign}(v)$ has an asymptote with change of sign at $v = 0$, and is therefore not monotone when both positive and negative signs are considered. In the same spirit (1.3) can be written as $u_t = \Phi(u_x)_x$ with the same monotone Φ, while (1.1) cannot. The asymptote of Φ will have serious consequences for the existence theory of equations (1.1) and (1.2) that we will investigate below. The theory for these pairs of equations is closely related and will be pursued in parallel in this paper.

Thus, in the case of the approximate equations (1.3) and (1.4), existence and uniqueness of solutions in a suitable framework, i.e. in $L^1(\mathbb{R})$ as a base space, is a consequence of results which apply to a large class of nonlinear parabolic equations, see Section 6 below. We stress that existence and uniqueness of weak solutions is true for equation (1.4) with $p > 1$ without the restriction of positivity. The authors have investigated in [15] the theory of nonnegative solutions of equation (1.2) and found existence of solutions of the Cauchy problem with arbitrary locally integrable data (or even locally finite measures), but also non-uniqueness, a phenomenon which is studied in [25], [27]. We have

Theorem 1.1 *For every nonnegative function $v(x,0) \in L^1(\mathbb{R})$ there exist infinitely many functions $v(x,t) \in C([0,T) : L^1(\mathbb{R})) \cap C^\infty(Q_T)$ for some $T > 0$ with $Q_T = \mathbb{R} \times (0,T)$, which are positive and classical solutions of (1.2) and take the initial data $v_0(x)$. There exists a unique maximal function in the class of such solutions, which exists globally in time and is characterized by the equality (conservation law)*

$$\int v(x,t)\,dx = \int v(x,0)\,dx \quad \text{for all } t > 0. \tag{1.5}$$

More details are given in Section 2. We are interested here in the existence of solutions with changing sign for equation (1.2). We introduce a natural class of weak solutions for this equation and prove the following result.

Theorem 1.2 *There exist no continuous weak solutions of equation (1.2) with changing sign. Actually, all such solutions are positive everywhere.*

In other words, the extension of Theorem 1.1 to solutions with changing sign is false for equation (1.2), though such solutions exist for equation (1.4) with $p > 1$. We want to remark that the assumption of continuity, or some alternative assumption, is needed. Indeed, we construct in Subsection 5.5 an example of discontinuous weak solution with changing sign. This example is closely tied to the special behaviour of the function $\Phi(v) = \log|v|\mathrm{sign}(v)$.

As for equation (1.1), we can produce a unique classical solution of this equation out of a classical and integrable solution of (1.2), by means of the formula

$$u(x,t)=\int_a^x v(x,t)\,dx+\int_0^t |v(a,t)|^{-1}v_x(a,t)\,dt+c, \tag{1.6}$$

with arbitrary constants $a, c \in I\!R$. This construction, which will be extended to continuous weak solutions, is unique but for a constant. In this way we can translate Theorem 1.1 into the existence of classical solutions $u(x,t)$ of equation (1.1) which are monotone functions of x, given monotone and bounded initial data. See a detailed statement in Section 2. On the other hand, we can translate Theorem 1.2 into a negative result for non-monotone solutions for (1.1).

Theorem 1.3 *All regular weak solutions of (1.1) are monotone.*

Regular means that u and u_x are continuous. A more precise statement will be given in Section 4 below. The result can be re-phrased in a stronger form as follows: if we take non-monotone data, say, with the extra conditions of being smooth, positive and bounded, solve some approximate problems, say, equations (1.3) for $p > 1$, and pass to the limit, this limit cannot be oscillating like the data, hence the initial-value problem cannot be solved. Such a result can be seen as an extreme form of incompatibility with oscillations, i.e., no *lap numbers* larger than 1 are allowed.

1.3. OTHER EQUATIONS. The non-existence results extend to a large class of very-fast diffusion equations which include the previous models past the critical exponent, i.e., to

$$u_t=\frac{1}{p-1}\left(|u_x|^{p-2}u_x\right)_x=\frac{u_{xx}}{|u_x|^n}\,,\quad p<1\quad (\text{i.e.,}\ \ n>1), \tag{1.7}$$

and its differentiated version

$$v_t=\frac{1}{p-1}(|v|^{p-2}v)_{xx}=\left(\frac{v_x}{|v|^n}\right)_x,\quad p<1\quad (\text{i.e.,}\ \ n>1), \tag{1.8}$$

or in standard porous-medium notation, $m = p-1 < 0$. Again, when $p < 1$ the graph $\Phi_p(s) = (p-1)^{-1}|s|^{p-2}s$ is not a monotone function in the whole of $I\!R$ and has an asymptote at $s = 0$. Indeed, the non-oscillation results are formulated and proved in Section 3, 4 for equations of the forms $u_t = \Phi(u_x)_x$ and $v_t = (\Phi(v))_{xx}$ under the condition of divergence of the function $\Phi(s)$ at $s = 0$.

We recall in this respect that the case $p \le 0$ in (1.8) is even more radical with respect to non-existence. The differentiated equation does not even admit positive solutions if they have finite integral at $x = \infty$ or $x = -\infty$, cf. [22] and [16] for the limit $p \to 0$. For non-existence of small positive solutions in several dimensions cf. [35].

1.4. INITIAL LAYER. MESA PROBLEMS. The second part of this paper discusses the important question of the sense that can be given to the evolution process when we impose incompatible initial data, i.e., non-monotone for (1.1) or non-positive for equation (1.2). In order to illustrate this for the latter equation we assume changing-sign data $v_0 = f$, consider the standard regularization by means of equation (1.4) and take the limit as $p \to 1$. We show that this problem is very similar mathematically to the famous *mesa problem* for the porous media equation, cf. [4], [9], [17], [31], which happens in the limit $p \to \infty$ of equation (1.4). Indeed, the limit gives rise to an initial discontinuity (*initial layer*), which allows to pass in the limit from the incompatible data f to a compatible function g at $t = 0+$.

There is however a noticeable difference with the standard mesa problem: in the present problem *three choices* of limit are possible: $g \geq 0$, and then $v > 0$; $g \leq 0$ with $v < 0$; and finally, $g = v = 0$. Besides, when $g \neq 0$ the situation here is more complex than in the standard mesa problem in the following respect: the limit of the approximate solutions for $t > 0$ is an evolutionary solution of the limit equation (1.2) with initial data g, while in the porous medium case is constant in time after the initial layer. By integration the results are translated into equation (1.1) and the corresponding initial layer is characterized as a projection.

In a final section we explain the simple extension of some of the above results to analogous equations of the form $v_t = \Delta(\Phi(v))$ posed several space dimensions, and we comment on the relationship with recent developments in the literature of related topics of singular diffusion and mesa problems.

2 Existence theory for $p = 1$ and $0 < p < 1$

2.I. For nonnegative data the differentiated equations (1.2), (1.8) has been studied in [16, 26, 28]. The following result completes the information of Theorem 1.1.

Theorem 2.1 *For every nontrivial $v_0 \in L^1(I\!R)$, $v_0 \geq 0$, there exist infinitely many solutions $v(x,t)$ of the Cauchy problem for (1.2). One of them is the maximal one and can be characterized by the conservation of mass as stated in Theorem 1.1, and also by the property of zero flux at infinity. Moreover, for every pair of integrable, nonnegative functions $f(t), g(t) \geq 0$ (with some regularity) there exists exactly a solution satisfying the flux conditions*

$$\lim_{x\to-\infty}(\log(v(x,t))_x \to f(t), \qquad \lim_{x\to\infty}(\log(v(x,t))_x \to -g(t). \tag{2.1}$$

and this solution exists for all times $t > 0$ such that

$$\int_{I\!R} v_0(x)\,dx > \int_0^t (f(s) + g(s))\,ds. \tag{2.2}$$

If equality is reached at a time $T > 0$ (extinction time), then

$$v(x,t) \to 0 \quad \forall x \text{ as } t \nearrow T. \tag{2.3}$$

All these solutions are classical, $v \in C^\infty(Q_T)$, $Q_T = I\!R \times (0,T)$ and take the initial data as a limit in $L^1(I\!R)$, or continuously when v_0 is continuous. The flux data f and g are taken as suitable limits, depending on their regularity. Standard properties apply, like Maximum Principle and L^1-contraction, see [26]. The results extend literally to the differentiated equation (1.8) in the range $1 > p > 0$ (in the more standard notation $0 > m > -1$), only replacing (2.1) by

$$\lim_{x\to-\infty} v^{p-2}v_x \to f(t), \qquad \lim_{x\to\infty} v^{p-2}v_x \to -g(t). \tag{2.4}$$

When $f = g = 0$ the solutions also satisfy the homogeneity property, which comes for instance from the semigroup property, [6]:

$$v_t \le \frac{v}{(2-p)t}. \tag{2.5}$$

As we have mentioned, for $p \le 0$ no solutions with finite mass exist.

On the other hand, similar results hold for negative solutions with nonpositive data (change v into $-v$). Only the solutions with changing sign will present a problem.

2.II. These results can be translated to the integrated equation using formula (1.6) with $a = -\infty$, i.e.,

$$u(x,t) = \int_0^t f(t)\,dt + \int_{-\infty}^x v(x,t)\,dx + c, \tag{2.6}$$

and we have an existence theorem for monotone solutions.

Theorem 2.2 *For every monotone increasing $u_0 \in L^\infty(I\!R)$, $u_0 \ge 0$, there exist infinitely many classical solutions $u(x,t)$ of the Cauchy problem for equation (1.1) which are strictly monotone in x. One of them is characterized by the conservation of values at infinity:*

$$u(\infty,t) = u_0(\infty), \qquad u(-\infty,t) = u_0(-\infty), \tag{2.7}$$

and exists for all times $t > 0$. Moreover, for every pair of integrable, nonnegative functions $f(t), g(t) \ge 0$ (with some regularity) there exists exactly a solution satisfying the conditions at infinity

$$u(-\infty,t) = u_0(-\infty) + \int_0^t f(s)\,ds, \quad u(\infty,t) = u_0(\infty) - \int_0^t g(s)\,ds. \tag{2.8}$$

existing in a maximal time interval $(0,T)$ with $0 < T \le \infty$ as before.

These solutions are C^∞ smooth in Q_T. They satisfy the Maximum Principle in terms of u_0, f and g. Observe that $u(-\infty,t)$ is nondecreasing and $u(\infty,t)$ is nonincreasing. Corresponding to the maximal solution of the differentiated equation we have solutions of the integrated equation with constant data at $x = \pm\infty$. It is easy to see that equation

(1.1) admits no maximal or minimal solutions, since the conditions at infinity counteract each other. We also have $u(\infty,t)-u(-\infty,t)\to 0$ as $t\to T$ if T is finite.

Similar results hold for monotone decreasing solutions (change x into $-x$). As above, the results can be extended to the very fast-diffusion equation (1.7) in the range $0<p<1$.

2.III. For non-integrable data we have in the whole range $0<p\le 1$:

Theorem 2.3 *For every nontrivial $v_0\in L^1_{loc}(I\!R)$, $v_0\ge 0$, there exists a maximal solution $v(x,t)$ of the Cauchy problem for the differentiated equations. Even when v_0 is not integrable at $x=\infty$ and $x=-\infty$, uniqueness can only be imposed by stating suitable conditions at infinity, but then the solutions with nontrivial fluxes must have finite integrals on the corresponding end, $x=-\infty$ or $x=\infty$.*

Translated into the integrated equations, (1.1) and (1.7), monotone solutions with infinite values at $x=\pm\infty$ for $t=0$ can be finite for $t>0$, since f and g can be chosen to be singular at $t=0$. The solution that keeps both values infinite is unique (for fixed u_0).

In contrast to these results, there is a theory of existence and uniqueness for the good equations, (1.3) and (1.4), with $p>1$, which will be reviewed in Section 6 before using it in a regularization process. On the other hand, the theory of existence and uniqueness for $p\le 1$ posed on bounded intervals offers novelties that fall into the next section.

3 Non-existence of solutions I

We have seen existence of a multiplicity of classical solutions for the differentiated equation (1.2), and for (1.8) when $1>p>0$, whenever the initial data are nonnegative. By contrast, there is an intrinsic impossibility for these equations to admit solutions which change sign, or even nonnegative solutions which touch the level $v=0$. This obstruction is local in form and is valid for a large class of equations, so we will adopt a more general framework than up to now and consider a general equation of the form

$$v_t=(D(v)v_x)_x \tag{3.1}$$

where we assume that

(H) $D(s)$ is a positive C^1 real function away from $s=0$, which has a singularity at $s=0$ and is not integrable in $(0,1)$. Let Φ be a primitive for $s>0$ (defined up to a constant), so that Φ is monotone and $\Phi(0+)=-\infty$. When solutions with negative values are considered we have to introduce the primitive of D for $s<0$ and then the two branches of Φ cannot combine into a globally monotone function. We also assume in that case that $\Phi(0-)=+\infty$. For easy reference we refer to these properties as *singular* Φ.

All of this clearly generalizes equations (1.2) and (1.8). The consideration of bounded of semibounded space intervals leads to non-existence when the boundary data vanish.

For a bounded interval with zero data on both sides and for equation (1.2) and (1.8) it is a consequence of the results of the previous section, cf. [26]. We will need here an extension of such result that shows that positive solutions cannot take zero boundary values. By the Maximum Principle we only need to prove it in a particular case.

Lemma 3.1 *Let $\Phi(s)$ be a smooth function defined on $(0,\infty)$ with $\Phi'(s) = D(s) > 0$ $\forall s > 0$ and let us consider in the rectangle $\mathcal{R} = (a,b)\times(0,T)$ the problem*

$$\left\{\begin{array}{lll} v_t = (\Phi(v))_{xx} & & \text{in } \mathcal{R}, \\ v(x,0) = M & & \text{for } a \le x \le b, \\ v(a,t) = \varepsilon, & v(b,t) = M & 0 \le t \le T, \end{array}\right. \tag{3.2}$$

with $0 < \varepsilon < M$. This problem has a unique classical solution v_ε, $\varepsilon < v_\varepsilon < M$ which is $C^\infty([a,b]\times(0,T))$. If $\Phi(0+) = -\infty$ then as $\varepsilon \to 0$ we have

$$\lim_{\varepsilon\to 0} v_\varepsilon(x,t) = 0 \tag{3.3}$$

uniformly inside $\mathcal{R}$. If $\Phi(0)$ is finite the limit is nontrivial.

Proof. The existence and uniqueness part is standard quasilinear parabolic theory, [25]. The fact that the limit is a nontrivial weak solution when Φ is not singular at $s = 0$, like the cases $p > 1$, is a part of known results that will be reviewed (for power nonlinearities) in Section 6. In any case, it is easy to derive that the solution, that we call v_ε satisfies $\varepsilon < v_\varepsilon < M$, $v_{\varepsilon,x} \ge 0$. As a consequence of the fact that $v\varepsilon(x,t) \le v_\varepsilon(x,0) = M$ we have $v_{\epsilon,x} \ge 0$ and $v_{\varepsilon,t} \le 0$, hence $\Phi(v_\varepsilon)_{xx} \le 0$. Moreover

$$f_\varepsilon(t) = \lim_{x\to a}(\Phi(v_\varepsilon)(x,t))_x, \quad g_\varepsilon(t) = \lim_{x\to b}(\Phi(v_\varepsilon)(x,t))_x$$

exist and are finite. Both functions are positive. By the concavity $f_\varepsilon(t) \ge g_\varepsilon(t)$. The monotonicity of v_ε in t implies that $f_\varepsilon(t)$ is decreasing with t and $g_\varepsilon(t)$ increasing. We get

$$0 \le \int v_\varepsilon(x,t)\,dx = M(b-a) - \int_0^t f_\varepsilon(t)\,dt + \int_0^t g_\varepsilon(t)\,dt \le M(b-a). \tag{3.4}$$

We now pass to the limit $\varepsilon \to 0$, $v_\varepsilon(x,t) \to v_*(x,t)$. Since $v(a,t) \to 0$ and $\Phi(0) = -\infty$ (here the singularity of Φ comes into play), we get necessarily

$$\lim_{\varepsilon\to 0} f_\varepsilon(t) = \infty$$

uniformly in t. If $\lim g_\varepsilon(t)$ is finite for some t_1, then the limit v_* cannot be trivial for $0 < t < t_1$, but this contradicts estimate (3.4). Therefore, $\lim_\varepsilon g_\varepsilon(t) = \infty$ and the concavity implies that $v_\varepsilon(x,t) \to 0$ uniformly in sets of the form $[a < x < b - \delta) \times (0 < t < T)$, $\delta > 0$. ∎

To proceed, we need a suitable concept of weak solution.

Definition 3.1 *A weak solution of equation (3.1) defined in a sub-domain $\Omega \subset Q_T$ is a function $v \in L^1_{loc}(\Omega)$ with $\Phi(v) \in L^1_{loc}(\Omega)$ and the equation is satisfied in the sense of distributions. When $\Omega = Q_T$ a weak solution of the Cauchy problem with initial data $v_0 \in L^1_{loc}(I\!R)$ is a weak solution such that*

$$v(\cdot, t) \to v_0 \quad \text{in } \ L^1_{loc}(I\!R). \tag{3.5}$$

This definition implies for singular Φ that $\{v = 0\}$ is a zero-measure set. In practice we will work with *continuous weak solutions*. For convenience, the trivial function $v \equiv 0$ will be admitted as a solution. The non-existence result will be stated in local form, which implies a stronger result. A first result follows easily from Lemma 3.1.

Theorem 3.2 *For the equations (3.1) with singular Φ, if a continuous weak solution defined in a finite rectangle $\mathcal{R} = (A, B) \times (0, T)$ vanishes at one point $(x_0, t_0) \in \mathcal{R}$, then it vanishes identically at the time t_0.*

Proof. By restricting the rectangle is necessary we assume that v is continuous up to the boundary, hence it is bounded above in $\mathcal{R}$. Let M be larger than this upper bound. If the solution is nor strictly positive, then for some $t = t_0 \geq 0$ the solution vanishes at a point $x = a \in I = (A, B)$. Without loss of generality we can take $a < B$ and $t_0 = 0$. Then we compare v with the solution v_ε constructed in the previous lemma with $b = B$, taking large enough M. We will show that comparison applies as long as the parabolic boundary data are ordered. Since $v(a, 0) = 0$ we conclude that there exists $\tau_\varepsilon > 0$ such that $v(a, t) \leq \varepsilon = v_\varepsilon(a, t)$ for $0 < t < \tau_\varepsilon$.

Let us compare in a rectangle of the form $\mathcal{R}_1 = (a, b) \times (0, \tau_\varepsilon)$ the solution v with the solution v_ε produced by Lemma 3.1 with left-hand lateral data ε. On the open set where $v > v_\varepsilon$ we have $\varepsilon < v < M$, hence equation (3.1) is uniformly parabolic and the v is a C^∞ solution by standard local theory. Now we subtract both equations and multiply in a rather usual way by $p(\Phi_\varepsilon(v) - \Phi_\varepsilon(v_\varepsilon))$, where p is the typical smooth approximation of Heaviside function, $0 \leq p \leq 1$, $p(s) = 0$ for $s < 0$, $p'(s) > 0$ for $s > 0$. The graph Φ_ε is introduced to avoid the lack of monotonicity of Φ in crossing $s = 0$. It is defined as $\Phi_\varepsilon(s) = \Phi(s)$ for $s > \varepsilon$, and in any smooth way that makes it monotone for $s < \varepsilon$. Now we have the typical calculation

$$\int_a^b (v - v_\varepsilon)_t p(w)\, dx = \int_a^b (\Phi(v) - \Phi(v_\varepsilon))_{xx} p(w)\, dx, \tag{3.6}$$

where $w = \Phi_\varepsilon(v) - \Phi_\varepsilon(v_\varepsilon)$. Since $p(w) = 0$ for $w \leq 0$ the integrations are performed only on the open set $D = \{v > v_\varepsilon\}$, hence $\Phi(v) = \Phi_\varepsilon(v)$ and $\Phi(v_\varepsilon) = \Phi_\varepsilon(v_\varepsilon)$ in D. By integration by parts in the last term of the displayed formula we get

$$\int\int_{D \cap \mathcal{R}_1} (v - v_\varepsilon)_t p(w)\, dx dt,$$

and letting p tend to the Heaviside function $\text{sign}^+(s)$, we finally arrive $p(w) \to \text{sign}^+(v - v_\varepsilon)$, therefore

$$\int\int_{D\cap\mathcal{R}_1} (v - v_\varepsilon)_t \text{sign}^+(v - v_\varepsilon)\, dxdt \le 0$$

which implies that $v \le v_\varepsilon$ in $\mathcal{R}_1$.

Letting $\varepsilon \to 0$ we prove that the integral $I_v(t) = \int_a^b v(x,t)\, dx$ is as small as we like for $t > 0$, $t \sim 0$. By continuity $v(x,0) = 0$ in the whole interval $a \le x \le B$. Since the same happens on the right we have $v(x,t_0) \le 0$ for every x. On the assumption that Φ is also singular on the negative side,$\Phi(0-) = \infty$, we just change signs in the equation and the solution becomes trivial at $t = 0$. ∎

We can now formulate a simultaneous space extinction result.

Corollary 3.3 *A continuous weak solution of the singular equation (3.1) defined in Q_T that vanishes at a point (x_0,t_0) is identically zero for all times $t \ge t_0$. The same happens in a bounded space domain with homogeneous Neumann conditions.*

The result is a simple consequence of comparison in the whole space with solutions of the form $v_\varepsilon = \pm\varepsilon$. Next, we get a non existence result that contains Theorem 1.2.

Corollary 3.4 *A continuous weak solution of the singular equation (3.1) cannot change sign as a function of x for any fixed time. Posed in the whole space, it must be wither positive or negative everywhere and for every time $t > 0$.*

Proof. If we assume that it changes sign at some time t_0 then it must have a zero value and we have proved that in that case it is trivial at this time. The only option to change sing is being say positive in a strip $(a,b) \times (t_1,t_2)$ and then passing to negative after t_2 through the value $v \equiv 0$ at $t = t_2$. Note that this may happen on bounded intervals thorough the influence of the boundary conditions. ∎

REMARK. The continuity assumption is essential as we will see below with examples of discontinuous weak solutions. We have used continuous solutions to justify the comparison arguments. Other types of solution which admit comparison with nonnegative weak solutions could have been used.

4 Non-existence of solutions II

We now translate these results to equations (1.1), (1.7) and their generalization

$$u_t = (\Phi(u_x))_x \tag{4.1}$$

under the conditions on Φ stated in the previous section using the relation between both equations.

Definition 4.1 *We define a weak solution of equation (4.1) is a function* $u \in L^1_{loc}(\Omega)$ *with* $u_x, \Phi(u_x) \in L^1_{loc}(\Omega)$*, and the equation is satisfied in the sense of distributions. A weak solution of the Cauchy problem posed in* Q_T *with* $u_0 \in L^1_{loc}(\mathbb{R})$ *is a weak solution such that* $u(\cdot,t) \to u_0$ *in* $L^1_{loc}(\mathbb{R})$*. We define a regular weak solution as a continuous weak solution such that* u_x *is also continuous.*

Theorem 4.1 *Under the same assumptions on the singular* Φ *a regular weak solution of equations (4.1) defined in a rectangle is necessarily strictly monotone in the space variable for every fixed time.*

Proof. The poof relies on observing that $v = u_x$ is then a continuous weak solution of the differentiated equation. $\square$

5 Some solutions for $p = 1$

As a test of the theory it is interesting to consider some classes of explicit solutions and investigate the way in which monotonicity or positivity are imposed in the limit $p \to 1$.

5.1. TRAVELLING WAVES. There are well-known globally defined travelling wave solutions of (1.2), given by the explicit formula

$$v(x,t) = \frac{A}{c + e^{A(ct-x+x_0)}}. \tag{5.1}$$

Given a speed $c > 0$ there exists for every $A > 0, x_0 \in \mathbb{R}$ a positive solution with nonzero flux at minus infinity, $(\log v)_x = A$. It is bounded and goes monotonically from the value 0 at $x = -\infty$ to the value $h = A/c$ at $x = \infty$. For $c < 0$ we get the same solutions after the symmetry $x \to -x$, but now they travel to the left. For $c = 0$ we get stationary solutions $v = Be^{Ax}$. If $A < 0$ we get negative solutions. For equation (1.1) this translates into

$$u(x,t) = At + \frac{1}{c}\log(1 + ce^{A(x-ct-x_0)}) \tag{5.2}$$

(mind the extra term At). All these solutions are monotone. Note that there is another class of travelling waves of the form $v(x,t) = A/(c - e^{A(ct \pm x + x_0)})$, but they are not properly defined for all $x \in \mathbb{R}$ since they exhibit an asymptote at a finite x. In a very rough way, they can be seen as changing sign solutions across the singularity.

It is interesting to examine types of travelling waves with changing sign for equation (1.4) with $p > 1$, and see how they pass to the limit. Indeed, introducing the form $v_p(x,t) = f_p(\xi)$, $\xi = x - x_0 - ct$ we get for the profile $f = f_p(\xi)$ the equation $(|f|^{p-2}f)'' + (p-1)cf' = 0$, hence by integration

$$(|f|^{p-2}f)' = (p-1)(A - cf). \tag{5.3}$$

If we want solutions with a sign change we take $A \neq 0$, say $A > 0$. Take also $c > 0$. By normalization we may assume that $f(0) = 0$, and then A is the flux at the sign change, $A = (p-1)^{-1}(|f|^{p-2}f)'(0)$. The profile f is monotone and goes from $-\infty$ at a certain asymptote $\xi = \xi_p < 0$ up to the height $h = A/c$ as $\xi \to \infty$. Near the transition point $\xi = 0$ we have $f \sim ((p-1)A\xi)^{1/(p-1)}\mathrm{sign}(\xi)$, which becomes increasingly flat as $p \to 1$. The idea we derive is that in the limit the change of sign will take infinite space.

Let us now pass to the limit $p \to 1$. We find in this context the three limit options already mentioned in the introduction that will be discussed below in a general setting.

(i) Keeping the normalization $f_p(0) = 0$ we get a trivial limit, $f = 0$ everywhere.

(ii) On the other hand, if we shift the ξ-axis and fix $f_p(0) = h_1 \in (0, A/c)$ we will get in the limit $p \to 1$ the positive travelling wave (5.1), while the negative part is pushed to minus infinity.

(iii) Finally, when we normalize $f_p(0) = -h_1 < 0$ the limit will be a negative solution with an asymptote on its left-hand. Now the positive part disappears through the right-hand part of the real line.

Other signs of A and c give reflected variants of the above solutions, and the case $A = 0$ is already one-signed for $p > 1$, and gives in the limit one-signed solutions with an asymptote.

5.2 Source-type solutions. For $1 < p < 2$ there are solutions of equation (1.3) given by the formulas

$$u(x,t;C) = t^{-\frac{1}{2(p-1)}}\left(C + \frac{2-p}{p}\left(\frac{1}{2}\right)^{\frac{1}{p-1}}\left(\frac{|x|}{t^{\frac{1}{2(p-1)}}}\right)^{\frac{p}{p-1}}\right)^{\frac{p-1}{p-2}}, \tag{5.4}$$

where $C > 0$ is an arbitrary positive constant. These *source-type solutions* take as initial data a Dirac mass

$$u(x,0;C) = M\,\delta(x), \tag{5.5}$$

where the mass $M = \int u(x,t;C)dx$ is related to C by the formula

$$C = c(p)\,M^{-\sigma}, \qquad \sigma = \frac{p(2-p)}{2(p-1)^2}. \tag{5.6}$$

These solutions can be differentiated in x to produce so-called *dipole solutions* of the fast diffusion equation (1.4), which are anti-symmetric, $v(-x,t) = -v(x,t)$. Passing to the limit $p \to 1$ in this example, (5.4), we get

$$\lim_{p\to 1} u(x,t;C) = 0 \tag{5.7}$$

uniformly in sets of the form $\{(x,t) : t \geq \tau > 0\}$ for every $\tau > 0$. This means that there will be no source-type solution in the limit and the solution existing for $p > 1$ will disappear in zero time even if we now conserve the initial data; there arises a discontinuity of the type called *initial layer*, a phenomenon that will discussed at length below. By a symmetrization argument, [33], the same applies to equation (1.1) for any finite distributed initial mass, $u_0 \in L^1(I\!R)$.

5.3. A VERY SINGULAR SOLUTION. When we let the mass tend to infinity in (5.4) we get for $1<p<2$ the explicit solution

$$u(x,t;\infty)=K(p)\,\frac{t^{\frac{1}{2-p}}}{|x|^{\frac{p}{2-p}}},\qquad K=\left(\frac{2^{\frac{1}{p-1}}\,p}{2-p}\right)^{\frac{p-1}{2-p}}, \tag{5.8}$$

In the limit $p\to 1$ this solution becomes

$$u(x,t)=\frac{2t}{|x|}, \tag{5.9}$$

which is a strange *non-monotone solution* of (1.1). It has a permanent non-integrable singularity at $x=0$. Therefore, u is not a global solution of the equation in any reasonable sense. By differentiation we get the function

$$v(x,t)=\frac{2t}{x^2}\,\mathrm{sign}(x), \tag{5.10}$$

a solution of equation (1.2) away from the line $x=0$, $t>0$. It has changing sign but the singularity at $x=0$ is essential. We will see below less singular cases of change of sign.

5.4. SELF-SIMILAR SOLUTIONS. Next, we will perform a general self-similarity analysis. We consider the differentiated equation (1.4) and look for solutions of the form

$$v(x,t)=t^{\alpha}F(\xi),\quad \xi=x\,t^{-\beta} \tag{5.11}$$

Substituting into (1.1) we get in the usual way the exponent relation

$$\alpha(2-p)+2\beta=1, \tag{5.12}$$

and the profile equation

$$(|F|^{p-2}F')'+\beta\xi F'=\alpha F, \tag{5.13}$$

where primes denote differentiation with respect to ξ. In order to do a phase plane analysis we may assume (at least locally) that $F>0$, $\xi>0$ without loss of generality, since the equation is invariant under the changes $F\to -F$, $\xi\to-\xi$. Under these conditions we perform the change of variables

$$\eta=\log(\xi),\quad x=\frac{\xi\,F'}{F},\quad y=\xi^2F^{2-p}. \tag{5.14}$$

We then obtain the quadratic system

$$\left\{\begin{array}{l}\dot x=x-(p-1)\,x^2+\alpha y-\beta\,xy\\ \dot y=y\,(2+(2-p)x),\end{array}\right. \tag{5.15}$$

where dot represents derivative with respect to η: $\dot{x} = dx/\eta$, $\dot{y} = dy/d\eta$. In the limit we get the system for equation (1.2) as

$$\left\{ \begin{array}{l} \dot{x} = x + (1 - 2\beta)y - \beta\, xy \\ \dot{y} = y\,(2 + x), \end{array} \right. \tag{5.16}$$

where now $\alpha = 1 - 2\beta$ and $y = \xi^2 F$, the other variables being defined as before. We obtain a complete family of solutions with positive sign for the different values of α. By change of sign, $v \mapsto -v$, we obtain a family of negative solutions.

5.5. A DISCONTINUOUS SOLUTION WITH CHANGING SIGN. Let us now consider the self-similar solution corresponding to $\alpha = 0$, $\beta = 1/2$, which takes the form

$$v_1(x, t) = F(x\, t^{-1/2}), \tag{5.17}$$

were F is positive, symmetric and decreasing for $\xi > 0$ with $F(\infty) = 0$. We may normalize so that $F(0) = 1$. We now take the function $\overline{v}$ defined in an antisymmetric way as

$$\overline{v}(x, t) = \left\{ \begin{array}{ll} v_1(x, t) & \text{for } x > 0, \\ -v_1(x, t) & \text{for } x < 0. \end{array} \right. \tag{5.18}$$

It is easy to see that $\overline{v}$ is in fact a weak solution of equation (1.2) in the sense of Definition 3.1. However, $\overline{v}$ does not satisfy the assumptions of Corollary. 3 because it is not continuous: it has a jump discontinuity at $x = 0$, $\overline{v}(0+, t) - \overline{v}(0-, t) = 2$. Observe that since $\log(\overline{v}) = 0$ at the points of discontinuity ($x = 0$, $t > 0$), the function $\log(\overline{v})$ is continuous in Q.

The existence of antisymmetric weak solutions is still valid for other similarity exponents in the profile equation (5.13), written in the form

$$(\log |F| \operatorname{sign}(F))'' + \beta\xi F' = \alpha F,$$

on the condition that the original profile is normalized to $F(0) = 1$. However, for $\alpha \neq 0$ they do not produce weak solutions of equation (1.2) because $\log(\overline{v})$ has a jump discontinuity at $x = 0$ for $t \neq 1$.

6 Theory of the equations with $p > 1$

Let us review the known properties of the good equations, (1.3) and (1.4), which will be used next in the approximation process as regularizations of the limit equations with $p = 1$. The Cauchy problem for equation (1.3) with initial data

$$u(x, 0) = u_0(x), \quad x \in I\!R, \tag{6.1}$$

has been extensively studied in the literature for all $p > 1$. The analysis has been also carried out for the corresponding problem in several space dimensions where the problem offers some extra difficulties. In dimension $n = 1$ we have the following results.

Theorem 6.1 *Let $1 < p < 2$ and let $u_0 \in L^1_{loc}(I\!R)$, $u_0 \geq 0$. Then the Cauchy problem admits a unique weak solution in the following sense. There exists a unique function $u \in C([0,T] : L^1_{loc}(I\!R)) \cap C^\alpha_{loc}(Q_T)$ for some $\alpha \in (0,1)$, such that $|u_x| \in L^p_{loc}(Q)$, $u_t \in L^1_{loc}(Q)$, the equation is satisfied in the sense of distributions and the initial data are taken in the sense of L^1_{loc} as $t \to 0$. Besides, we have for all solutions the estimates*

$$-\frac{u}{2(p-1)t} \leq u_t \leq \frac{u}{(2-p)t}. \tag{6.2}$$

The upper estimate comes from Bénilan-Crandall's homogeneity argument, cf. [6], while the lower bound is due to Esteban and Vazquez [15].

Theorem 6.2 *The accretivity of the differential operator implies that whenever $u_0 \in L^r(I\!R)$ for some $r \in [1,\infty]$ then $u(\cdot,t) \in L^r(I\!R)$ for all $t > 0$ and*

$$\|u(\cdot,t)\|_r \leq \|u_0\|_r. \tag{6.3}$$

Moreover, the map $u_0 \mapsto u(\cdot,t)$ is an ordered L^r-contraction. Finally, we have

$$\|u(\cdot,t)\|_\infty \leq C(r)\frac{\|u_0\|_r^\gamma}{t^\beta} \tag{6.4}$$

with

$$\gamma = \frac{p(r+1)-2}{2r(p-1)}, \quad \beta = \frac{r-1}{2r(p-1)}. \tag{6.5}$$

In this case there is no sign restriction on the data or the solutions, $u \in C([0,T] : L^r(I\!R)) \cap L^\infty(I\!R \times [\tau,T])$.

We refer to [14] for the results in several space dimensions with locally integrable data. For integrable data cf. [3], [6]. The L^∞ estimates can be obtained by symmetrization cf. [32, 33]. In case the data are monotone we can differentiate and apply to $v = u_x$ the existing theory for the fast diffusion equation (1.4), thus obtaining a classical theory without any sign restriction.

Theorem 6.3 *If u_0 is a locally absolutely continuous function with is monotone nonincreasing (or monotone nondecreasing), then there exists a unique weak solution of the Cauchy problem for equation (1.1) which is classical in Q_T and strictly monotone in x for every $t > 0$.*

We remind the reader that equations (1.1) and (1.3) are invariant under translations of the form $u \mapsto u + c$, hence the theory done for nonnegative solutions applies immediately to solutions which are merely *bounded below* by a constant, $u \geq c$. Or bounded above, $u \leq c$ after the reflection $u \mapsto -u$.

On the other hand, we may consider the problem with *anti-symmetric* initial data, possibly unbounded below, by reducing it to the Cauchy-Dirichlet problem in the domain $Q = \{(x,t) : x > 0, t > 0\}$ with boundary data $u(0,t) = 0$ for $0 \le t \le \infty$. All the above assertions and estimates except inequality (6.2)-left hold when we consider locally integrable data

$$u(x,0) = u_0(x) \ge 0 \quad \text{for} \quad x > 0.$$

Extending this solution to the whole line in the anti-symmetric way $u(-x,t) = -u(x,t)$, we obtain a solution of the original problem with anti-symmetric data. Unless constant, these solutions are nor monotone.

Let us turn our attention to the differentiated equation (1.4) with $p > 1$. Existence of a solution is obtained in the framework of locally integrable functions, cf. [23]. Thus, for every $v_0 \in L^1_{loc}(\mathbb{R})$ there exists a unique function $v \in C([0,\infty) : L^1_{loc}(\mathbb{R}))$ such that (i) v satisfies (1.4) in distribution sense in $Q = \mathbb{R} \times (0,\infty)$, and (ii) $v(\cdot,t) \to v_0$ in $L^1_{loc}(\mathbb{R})$. Moreover, the solution is locally bounded and continuous in Q. For integrable data $v_0 \in L^1(\mathbb{R})$ we have a unique mild solution $v \in C([0,\infty] : L^1(\mathbb{R})) \cap L^\infty(\mathbb{R} \times (\tau,\infty))$ for every $\tau > 0$ and mass is conserved. By the results of [7] this solution is strong: $v_t \in L^1_{loc}(Q_T)$. In this case we may integrate

$$u(x,t) = \int_{-\infty}^{x} v(s,t)\, ds$$

to obtain a solution of the Cauchy problem for (1.3) in the class described above. Conversely, differentiation of a solution of Theorem (6.1) produces a weak solution $v = u_x$ of (1.4) with initial data taken in L^1_{loc}-weak.

7 Approximation. Monotonicity in the limit as $p \to 1$

In agreement with the results of the Sections 3 and 4, the construction of solutions for non-monotone data finds obstructions that we describe in the subsequent sections.

A natural method to construct solutions of equations (1.1), (1.2) consists in passing to the limit of the corresponding problems for $p > 1$. We obtain the following monotonicity result.

Theorem 7.1 *Any bounded function obtained as limit of bounded solutions of equations (1.3) as $p \to 1$ is necessarily monotone in x.*

Proof. First of all, by displacing the graph up by a constant we may assume that $u \ge c > 0$ uniformly in x. Multiplying equation (1.3) by $u\,\zeta^p$, with $\zeta = \zeta(x)$ a cutoff function, and integrating by parts in $\mathbb{R} \times (t_1,t_2)$ with $0 < t_1 < t_2$, we obtain an energy estimate for all solutions $u = u^{(p)}(x,t)$ of the equation

$$\int\!\!\int |u_x|^p \zeta^p \, dx dt \le C \int\!\!\int u^p\, |\zeta_x|^p \, dx dt + O(p-1). \tag{7.1}$$

This estimate is uniform as $p \to 1$ as long as functions are bounded in $L^p_{loc}(I\!R)$. Another estimate is based on homogeneity, cf. [6]

$$u_t \leq \frac{u}{(2-p)t}. \tag{7.2}$$

This means that

$$(|u_x|^{p-2}u_x)_x \leq \frac{(p-1)}{(2-p)t}u.$$

We have omitted the superscripts for convenience. Putting $V^{(p)} = |u_x^{(p)}|^{p-2}u_x^{(p)}/u$, we write $|u_x^{(p)}|^{p-2}u_x^{(p)} = u^{(p)}V^{(p)}$, hence, omitting again superscripts,

$$u_x\, V + u\, V_x \leq \frac{(p-1)u}{(2-p)t}.$$

Since $u_x\, V = |u_x|^p/u \geq 0$ we get

$$V_x^{(p)} \leq \frac{(p-1)}{(2-p)t}. \tag{7.3}$$

Let us now pass to the limit. By compactness (using estimates (7.1), (7.2)) we have $u^{(p)} \to u$ locally in L^1 and a.e. Since

$$-\frac{u}{2t} \leq (|u_x|^{p-2}u_x)_x \leq \frac{(p-1)}{(2-p)t}u,$$

we also know that $|u_x^{(p)}|^{p-2}u_x^{(p)}$ converges to a function w, and w is a selection of the graph $\text{sign}(u_x)$. Finally, since u is bounded above and below away from zero, $V^{(p)}$ converges as $p \to 1$,

$$V^{(p)} \to V = \frac{w}{u},$$

for which in the limit of (7.3) we get

$$V_x \leq 0.$$

This is the monotonicity result that we wanted to prove, since it means that w is monotone nonincreasing. As w is the sign of u_x there are three options for w: it can only be either positive everywhere or negative everywhere or jump from positive to negative. The first two cases agree with our result. The third case is eliminated when u is bounded above by taking as new function $\tilde{u} = M - u$, where M is an upper bound for u. $\square$

Corollary 7.2 *For fixed bounded initial data $u_0(x)$ the limit as $p \to 1$ of the solutions of the Cauchy problem for equation (1.3) is a bounded solution of equation (1.1), which is monotone in x for all $t > 0$.*

The fact that solutions are monotone poses no problem when the initial data are also monotone, and in fact we recover for such data one of the solutions described in Section 2. A problem arises for non-monotone data.

We may now translate these results to the differentiated equation (1.2). We know that when the data are nonnegative then the initial data and the limit solution is just the maximal solution constructed in Section 2. For general data we have in view of the previous result and the relation between equations (1.3) and (1.4).

Corollary 7.3 *Let $v_0(x) \in L^1(\mathbb{R})$. Then the limit of the solutions of the approximate problems (1.4) has constant sign, either nonnegative or nonpositive. If the data are antisymmetric, $v_0(-x) = -v_0(x)$, the solution vanishes.*

8 Initial layer and initial projection

We have concluded that the solutions of the limit equations, both in the integrated and differentiated case, have special geometric properties, monotonicity or positivity respectively. Let us consider now the following question: we consider the solutions of the Cauchy problems for $p > 1$ with fixed initial data and pass to the limit as $p \to 1$. Does the limit satisfy the same initial conditions? The answer is positive when the initial data have the above-mentioned properties, since then known theory implies that the solutions for $p > 1$ converge to the solution for $p = 1$, see Section 6. However, when the initial data do not satisfy such restrictions, then it must necessarily happen that the limit solution does not satisfy the initial condition. Then a discontinuity takes place at $t = 0+$, which we call the *initial layer*. We want to prove that the discontinuity of the initial layer takes the form of the "projection" into a different initial function

$$P : u_0 \mapsto \widetilde{u}_0, \tag{8.1}$$

such that $\widetilde{u}_0$ is monotone, and the limit solution $u = \lim_{p\to 1} u_p$ is precisely the unique maximal solution of (1.1) with initial data $\widetilde{u}_0$. Hence, for monotone data $P(u_0) = u_0$. A similar description holds for the differentiated equation, where we want to find an operator

$$Q : v_0 \mapsto \widetilde{v}_0. \tag{8.2}$$

such that the limit solution $v = \lim_{p\to 1} v_p$ is precisely the maximal solution of (1.2) with data $\widetilde{v}_0$. Hence, $Q(v_0) = v_0$ for nonnegative v_0.

The simplest instance of this phenomenon happens when the data are also symmetric.

Proposition 8.1 *If we solve the equations (1.3) for $p > 1$ with fixed initial function u_0 and moreover this function is bounded and symmetric, $u_0(x) = u_0(-x)$, then the limit as $p \to 1$ is constant.*

Proof. The fact that the limit exists is easy in view of the estimates. The symmetry property holds for the solutions of the approximate problems, $u_p(x,t)$, hence for the limit $u(x,t)$. In view of the monotonicity, this limit must be constant. $\square$

Proposition 8.2 *Whenever the data u_0 are L^r-integrable, $1 \le r < \infty$, the limit of the approximate problems (1.3) is zero. This is also true if $u_0(x) \to 0$ as $x \to \infty$ (even in a weak sense).*

Proof. We consider first the case where the data are rearranged, i.e., $u_0(-x) = u_0(x)$ for every $x < 0$ and u_0 is nonincreasing for $x \ge 0$ so that $u_0(x) \to 0$ as $x \to \infty$. We also assume that u_0 is bounded. The previous result implies that the limit is zero (uniformly on compacts). For general bounded data (we may start first with nonnegative for convenience and then use Maximum Principle) we use symmetrization [33] to compare the spatial rearrangement of the solutions of the problem with the solutions $\overline{u}_p$ of the rearranged data. We conclude that for every set $\Omega \subset I\!R$ with finite measure $2l$ we have

$$\int_\Omega |u_p(x,t)|\,dx \le \int_{-l}^{l} \overline{u}_p(x,t)\,dx. \tag{8.3}$$

We then conclude that $u_p(x,t) \to 0$ locally in L^1 as $p \to 1$. The convergence holds also in L^∞. Finally, for unbounded data we use the L^r contraction property to approximate by good data as before. For the last statement we use comparison (with functions cut at the levels ε and $-\varepsilon$). $\square$

Comparison allows to derive the following result

Corollary 8.3 *If we have initial data with a limit $u_0 \to a$ at $x = -\infty$ and another limit $u_0 \to b$ at $x = \infty$ then the solution $u(x,t)$ of (1.1) has values in $[a,b]$ and is monotone in x. Both asymptotic limits are preserved in time.*

The last assertion follows from comparison with solutions with monotone step-wise initial data. In case $a \le u_0(x) \le b$ for a.e. x we fall into the standard theory explained in Section 2. For initial functions which jut out of the interval $I = (a,b)$, the corollary shows that the *limit solution* is contained in I for all times. Note that the asymptotic limits imply that the solutions will not be trivial if $a \ne b$.

Translation of Proposition 8.2 for the differentiated equation gives a statement for the existence of a trivial limit.

Theorem 8.4 *Whenever the data v_0 are integrable and have zero integral, $\int v_0(x)\,dx = 0$, the limit of the solutions of the approximate problems (1.4) as $p \to 1$ is identically zero in $Q = I\!R \times (0,\infty)$.*

9 Nontrivial initial layer. Differentiated equation with positive finite mass

We now address the question of finding classes of initial data for which there is an initial layer (the data are not taken in the limit) but the limit solution is not nontrivial. This gives rise to a more complex initial boundary layer. We start with the differentiated equation (1.2) and derive the projection rule in a case of finite integral. We have already seen that in case the integral of the initial data is zero the limit solution is trivial. Therefore, we assume that the initial data $v_0(x) = f(x)$ satisfy the following properties: f is integrable and the total mass is positive

$$f \in L^1(I\!R), \quad \int f(x)\, dx > 0. \tag{9.1}$$

Let us denote this class of data by $X \subset L^1(I\!R)$. Let $X_+ = \{f \in X : f \geq 0\} = \{f \in L^1(I\!R) : f \geq 0\}$. The remaining case of negative mass can be reduced to this one by symmetry $u \mapsto -u$. We prove the following result. We denote by v_p the solution of equation (1.4) with $p > 1$ and initial data f.

Theorem 9.1 *There exists a map $P : X \mapsto X_+$ such that the limit of the approximations v_p is the maximal positive solution v of (1.2) with initial data $g = P(f)$. In order to define P we consider the coincidence set Ω of the solution $w \in W^{1,\infty}(I\!R)$, $w(x) \to 0$ as $|x| \to \infty$, of the variational inequality*

$$w_{xx} \leq f, \quad w \geq 0, \quad (w_{xx} - f)w = 0. \tag{9.2}$$

Then $g = P(f)$ is defined as $g(x) = f(x)$ in Ω, while $g(x) = 0$ for $x \notin \Omega$. In other words

$$g = f\,\chi(\{w = 0\}). \tag{9.3}$$

Therefore, P is an L^1 contraction which reduces to the identity on X_+. For all f we have $\int f = \int g$.

It is clear that $f \geq 0$ on the coincidence set, but f can have any sign on the non-coincidence set. The result reminds of the variational inequalities characterizing the solution of the famous *mesa problem*, i.e., the limit $p \to \infty$ of equation (1.4), and in fact there is a close connection. As already pointed out, there is however a major difference in the present situation: the projection does not determine a steady state but the initial data of an evolution. On the other hand, we start with data of any sign and force the solution to have a sign.

1. The proof will be performed first for a special sub-class X_1 of data f consisting of bounded and continuous functions with a finite number of sign changes, which besides are positive for all large $|x|$ and have finite and positive integral.

There are four steps in the proof

STEP 1. THE LIMIT IS POSITIVE. The family $\{v_p\}_p$ is uniformly bounded in $C([0,\infty) : L^1(I\!R))$ and has constant mass $\int v_p(x,t)\,dx = \int f(x)\,dx$. The family has also the property

$$\int |v_p(x+h,t) - v_p(x,t)|\,dx \le \int |f(x+h) - f(x)|\,dx,$$

hence it is equi-continuous in $L^1(I\!R)$ for every $t > 0$. We can pass to the limit and find a function v which is a weak solution of (1.2) and has equal or smaller mass. Passing to the integrated equation we have a family u_p which tends to u, an integral of v, solution of (1.1). We know that for u_p the limit is monotone, hence $v \ge 0$. According to the results of [27] the solution v has an initial trace that we call g and $g \ge 0$. It can be a nonnegative measure or a function. Now comparison with the solutions with initial data $f_+(x) = \max\{f(x), 0\}$ implies that $g(x) \le f_+(x)$, hence it is a function. Summing up, v has an initial trace

$$0 \le g(x) \le f_+(x).$$

In particular $g(x) = 0$ whenever $f(x) \le 0$. But g can be zero also at points where $f > 0$.

STEP 2. THE LIMIT IS A MAXIMAL SOLUTION. The argument relies on the known fact that when the data are positive and integrable there exists L^1 convergence of the solutions $v_p \in C([0,\infty) : L^1(I\!R)) \cap L^\infty(Q)$ of equation (1.4) to the maximal solution of equation (1.2) for every $t > 0$, cf. our previous works. In particular, the mass contained outside a ball B_R can be made small enough uniformly in $p \to 1$ by taking R large. In order to apply this result to the case of changing data we compare the solution v_p with data f with the solutions $v_{+,p}$ with data $f_+ = \max\{f, 0\}$ and $v_{-,p}$, solution with data $f_- = \max\{-f, 0\}$. Since

$$-v_{-,p} \le v_p \le v_{+,p},$$

we conclude that the family $v_p(\cdot,t)$ has uniformly small mass outside a large ball. Using the conservation of mass

$$\int v_p(x,t)\,dx = \int f(x)\,dx$$

we conclude from the convergence $v_p \to v$ the estimate

$$\int v(x,t)\,dx = \int f(x)\,dx = \int g(x)\,dx$$

for every $t > 0$, hence the maximality of v, cf. (1.5).

STEP 3. ALTERNATIVE. We want to show that there is a partition of $I\!R$ into a finite and disjoint family of intervals such that either $g = f > 0$ or $g = 0$ in each of them. The proof is based on following argument.

Lemma 9.2 *For every $a \in I\!R$ one of the following assertions holds:*

(i) $g(x) = 0$ in an interval of the form $a - r < x < a$,

(ii) $g(x) = 0$ in an interval of the form $a < x < a + r$,

(iii) there exist $\delta > 0$, $r > 0$, $\tau > 0$, and $p_0 > 1$ such that

$$v_p(x,t) > \delta \quad \text{for } |x-a| \le r,\ 0 < t < \tau$$

for $1 < p < p_0$.

The proof is the same of Caffarelli and Friedman in [9, Section 6] with minimal adaptations, hence we will omit it. Using this result, the alternative is concluded as follows. Under condition (iii) the equations would be uniformly parabolic in a neighbourhood of $(x, 0)$, for $1 < p < p_0$, hence the limit $p \to 1$ would preserve the initial data, $g = f$ for $|x - a| < r$, by standard parabolic theory. Next, we observe that, by lap number theory (intersection comparison) the number of sign changes is nonincreasing when passing from f to each solution v_p, and again does not increase in passing to the limit. On the other hand, we have the intervals where $g = 0$ but f is not zero correspond necessarily to intervals where f has negative values in some parts. Hence, the number of intervals of negativity of f bounds above the number of the intervals in which $g \neq f$.

Step 4. Variational inequality. In order to control the fast initial evolution by means of the function we introduce the function

$$w_p(x,t) = \frac{1}{p-1}\int_0^t (1 - |v|^{p-2}v)(x,s)\,ds, \tag{9.4}$$

following closely the mesa problem approach. Integrating equation (1.4) from 0 to t we have

$$\int_{I\!R}(f - v_p(t))\xi\,dx = \int_{I\!R} w_p(t)\,\xi_{xx}\,dx \tag{9.5}$$

for any test function ξ. Therefore,

$$w_{p,xx}(x,t) = f(x) - v_p(x,t)$$

in the sense of distributions in x for every $t > 0$. On the other hand, it is immediate to see that for uniformly bounded $v_p \le C$ we have an lower bound of the form

$$(1 - |v|^{p-2}v)/(p-1) \le -K(C),$$

hence $w_p \ge -K(C)t$ uniformly in $p \ge 1$. By (iii) of the previous lemma we also have $w_p(x,t) \to 0$ as $t \to 0$ uniformly in m and in x in the interior of any interval where $g > 0$. It follows that w_p is uniformly bounded. We may now pass to the limit $m \to 0$ to get $w_p \to w$ in the weak-star topology of L^∞, and then

$$w(x,t)_{xx} = f(x) - v(x,t).$$

We now pass to a second limit $t \to 0$ to get a function $w_0(x) = \lim_{t\to 0} w(x,t)$ such that

$$w_{0,xx} = f(x) - g(x). \tag{9.6}$$

Since the lower bound for $w_p(x,t)$ implies that $w_0(x) \geq 0$, the theorem is proved putting $w(x) = w_0(x)$ under the present assumptions.

Remark. It is to be noted that the Variational Inequality can be solved independently in each of the disjoint intervals $I_i = [a_i, b_i]$ in which $g = 0$ and $w > 0$. It follows in particular that

$$\int_{I_i} f\,dx = \int_{I_i} g\,dx \quad \forall i.$$

2. The proof is extended to X by continuity using the L^1 contraction property of the map $f \mapsto g$, which is easily proved. For instance, we may write the Variational Inequality in equivalent form

$$w_{xx} - f \in \beta(w),$$

where β is the maximal monotone graph defined on $[0,\infty)$ by $\beta(s) = 0$ if $s > 0$, $\beta(0) = (-\infty, 0]$, cf. Brezis [5].

10 Integrated equation with bounded data

The characterization of the previous section for the differentiated equation can be translated into a corresponding result for the integrated equation (1.1). We introduce the class Y of data $F(x)$ which are bounded, have finite limits at $\pm\infty$ and such that

$$F(\infty) > F(-\infty).$$

We denote by Y_m the subspace of monotone increasing functions. We prove the following theorem.

Theorem 10.1 *There exists a map $Q : Y \mapsto Y_m$ such that the limit of the approximations u_p with fixed initial data $F \in Y$ is the unique monotone solution u of (1.1) with initial data $G = Q(F)$ and constant values at $x = \pm\infty$. The map Q is an L^∞ contraction which reduces to the identity on Y_m. For all F we have $F(\pm\infty) = G(\pm\infty)$.*

Proof. The result follows from the previous section. Assume that $f = F_x$ satisfies all the assumptions of the first part of the proof of Theorem 9.1. Without loss of generality put $F(-\infty) = 0$. Let $g = P(f)$ and let

$$G(x) = \int_{-\infty}^{x} g(y)\,dy.$$

For general F use approximation and the property that Q is an L^∞-contraction (indeed, it is an L^p contraction for all $p \geq 1$). $\square$

Proposition 10.2 *For F regular, G is characterized as a projection by the property that G realizes the minimum of*

$$\int_{I\!R}(F-G)^2\,dx \tag{10.1}$$

among the monotone functions defined in $I\!R$.

Proof. F regular means that $f \in X_1$ of Section 9. We know that $g = f$ and $G = F$ on the complement of the intervals where $w > 0$. For $1 < p < 2$ let v_p be the solution of (1.4) and let

$$u_p(x,t) = \int_{-\infty}^{x} v_p(y,t)\,dy$$

be the solution of (1.3). Let also $w_p(x,t)$ be defined as in Theorem 9.1. It is clear from the previous theorem that G is the initial datum for the limit solution $u = \lim_{p\to 1} u_p(x,t)$. Then we have for $t > 0$ and a test function $h(x)$ with compact support

$$\begin{aligned}&\textstyle\int(F(x)-u_p(x,t))h(x)\,dx = -\int\int u_{p,t}h(x)\,dxdt = \\ &\frac{1}{p-1}\int\int_0^t(1-|v_p|^{p-2}v_p)_x h(x)\,dxdt = -\int w_p(x,t)h'(x)\,dx.\end{aligned}$$

Letting $p \to 1$ we get $\int(F(x)-u(x,t))h(x)\,dx = -\int w(x,t)h'(x)\,dx$. Finally, we let $t \to 0$ to obtain

$$\int(F(x)-G(x))h(x)\,dx = -\int w(x)h'(x)\,dx,$$

which is nonnegative under the assumption that h is nonincreasing on the intervals on which $w > 0$, hence $G' = 0$. Therefore,

$$\int_{I\!R}(F-G)\,h\,dx \geq 0 \tag{10.2}$$

for every h which is nonincreasing when G is constant. It follows that G realizes the minimum (10.1). ∎

Similar results apply when $F(\infty) < F(-\infty)$, while for $F(\infty) = F(-\infty)$ we know that the projection is trivial, cf. Section 8.

11 Examples

We discuss first an illustrative example where the calculations can be done directly.

Example 11.1 *Let $v_0(x)$ be a step function defined as $v_0(x) = A > 0$ for $|x| > r > 0$ and $v_0(x) = -B < 0$ for $|x| < r$. Then the limit solution is the solution of equation (1.2) with initial data $V_0(x) = A$ for $|x| > r_1$, $V_0(x) = 0$ for $|x| < r_1$, where $r_1 = r(A+B)/A$, so that*

$$V_0 \geq 0, \qquad \int(v_0(x)-V_0(x))\,dx = 0. \tag{11.1}$$

The limit solution conserves the mass, i.e., it is the maximal solution for its initial data.

Proof. First of all, we know that the limit solution has fixed sign and is less than A everywhere. We also know that the v_p are symmetric, increasing in $x > 0$ for fixed t, and go to A as $x \to \pm\infty$. Moreover, by the mass conservation law we get

$$\int (A - v_p(x,t))\, dx = 2r(A+B),$$

and this estimate passes to the limit, since the mass at infinity is controlled by comparison. It follows that the limit v is nonnegative, symmetric, increasing for $x > 0$, goes to A as $x \to \pm\infty$. Moreover, if $\overline{v}$ is the solution with initial data $\overline{v}_0(x) = \max\{v_0, 0\}$ then $0 \le v(x,t) \le \overline{v}(x,t) < A$. Besides,

$$\int (\overline{v}(x,t) - v(x,t))\, dx = 2rB > 0$$

for every $t > 0$. In order to compare v and V we integrate in x from $x = 0$ to obtain solutions u and U resp. of equation (1.1). It is clear that $u(x,0) \le U(x,0)$ for $x > 0$ while $u = U = 0$ for $x = 0$, $t \ge 0$. It follows that

$$u(x,t) \le U(x,t) \quad \text{for } x > 0,\ t > 0.$$

But since $U(x,t) \to 0$ as $t \to 0$ for $0 < x < r_1$ and $v \ge 0$, hence $u \ge 0$, we also have $\lim_{t\to 0} u(x,t) = 0$. We conclude that there exists the limit

$$u(x,0) = \lim_{t\to 0} u(x,t) \quad \text{and } u(x,0) = U(x,0), \tag{11.2}$$

hence by differentiation $v(x,0) = V_0(x)$. ∎

In terms of the integrated equation, the initial data of this example are antisymmetric with

$$u_0(x) = -B\,x \quad \text{for } 0 < x < r, \qquad u_0(x) = A\,(x - r_1) \quad \text{for } x > r, \tag{11.3}$$

and the limit solution has antisymmetric initial data such that

$$U_0(x) = \max\{u_0(x), 0\} \quad \text{for } x > 0. \tag{11.4}$$

Therefore, we speak about the *initial projection* as the form of the initial layer.

Example 11.2 We do not need step functions to have a simple description of the initial layer of this kind. The following situation will be of use below: we consider the differentiated equation with a continuous initial function $v_0 = f$ which is positive on a right-hand half-line, say $(0,\infty)$, and negative on the complement. Theorem 9.1 gives as initial trace of the limit v a certain projection $g \ge 0$ of f. By the properties of the lap number mentioned in the proof of the theorem the initial trace g will equal 0 on an interval $(0,a)$, $a > 0$, and will equal $f \ge 0$ on (a,∞), and a is precisely given by the mass equality

$$\int_{-\infty}^{a} f(x)\, dx = 0. \tag{11.5}$$

In this case the distance a completely describes the extent to which f^+ *collapses* under the influence of f^-.

12 Other data

II. The initial layer analysis of Section 9 can be extended to the class $X' \supset X$ of initial data $v_0(x) = f(x)$ satisfying the following properties: f is locally integrable, its negative part f^- has finite integral and the total mass is positive $\int f(x)\,dx > 0$. Thus, the case of infinite integral will also be allowed. Let $X'_+ = \{f \in X' : f \geq 0\} = \{f \in L^1_{loc}(I\!\!R) : f \geq 0\}$. We have

Theorem 12.1 *There exists a map $P : X' \mapsto X'_+$ such that the limit of the approximations v_p is the unique positive solution v of (1.2) with initial data $g = P(f)$. The map P is an L^1 contraction which reduces to the identity on X'_+.*

The proof is done by monotone extension to the whole X' of the result in L^1.

We still can consider cases where both f^+ and f^- have infinite integral like the following one, in which the divergent integrals occur on opposite sides. and the limit result is trivial.

Theorem 12.2 *Let $v_0 = f$ be a locally integrable function such that $\int_0^x f^+ dx$ diverges at $+\infty$ and converges at $-\infty$ while $\int_0^x f^- dx$ converges at ∞ and diverges at $-\infty$. Then the limit solution is trivial, $v(x,t) \equiv 0$.*

Proof. A simple example of such a situation is given by a step function defined as $v_0(x) = A$ for $x < 0$ and $v_0(x) = B$ for $x < 0$ and let $A > 0 > B$ The proof in this case is an easy corollary of Proposition 11.1, using displacement of the origin and comparison.

Before treating the general case we consider continuous initial data which are positive on a right-hand half-line and negative on the complement, as in the last example of the previous section; cutting f^- we consider the case where the negative integral is finite while the positive part has infinite integral. The calculation for a is still valid in this generality.

To return to the case where f^- is not integrable at $-\infty$, we use approximation with the above solutions and pass to the limit to show that the point a must move to $+\infty$ as a consequence of formula (11.5), hence $g \to 0$ in $I\!\!R$ and $v \equiv 0$.

For general f we proceed as follows: we assume that $\int_{-\infty}^a f^+(x)\,dx < \varepsilon$, replace f on $[a, \infty)$ by f^+, and finally smooth this function into a continuous function F such that $F \leq 0$ for $x < a$, $F \geq 0$ for $x > a$ and

$$\int_{-\infty}^{a} |F - f^-|\,dx + \int_a^\infty |F - f^+|\,dx \leq \varepsilon.$$

It follows that $\int_{I\!\!R}[f - F]_+\,dx \leq 2\varepsilon$. Now, the solution corresponding to F is trivial and the L^1 dependence (ordered contraction) implies that

$$\int [v - 0]_+\,dx \leq \int [f - F]_+\,dx \leq 2\varepsilon$$

In the limit $v \leq 0$. Since the situation is symmetric $v = 0$. $\square$

In the remaining situation, when f^+ and f^- have infinite integral on the same side (hence, highly intertwined supports), there can be any of three kinds of projections: either trivial, or nontrivial with any of the two sings. The trivial case is easy to construct by symmetry. Let us exhibit an example where the projection is not trivial

Example 12.3 *Let $\{a_n\}_{n\geq 1}$ be an increasing positive sequence such that $a_{2n} - a_{2n-1} = b < a_0$ while $a_{2n+1} - a_{2n} = c \geq a_0$. We define $v_0 = 0$ for $x \leq 0$, $v_0(x) = -1$ in $I_n = (a_{2n-1}, a_{2n}]$, and $v_0 = 1$ otherwise in $(0, \infty)$. Then the solution is nontrivial.*

Proof. The integrated function u_0 vanishes for $x \leq 0$. For $x > 0$ it is decreasing in the intervals I_n, it is increasing in the complement in $(0, \infty)$ and the minima go to infinity as proceed with $x \to \infty$. It can be considered as the monotone limit of data for which the initial projections and the corresponding solutions are known and are nontrivial and uniformly bounded (locally in x). The limit is therefore locally bounded and nontrivial. Differentiation gives the result for v.

This example is easily generalizable to functions such that the growth at infinity of the positive part is of higher order than the negative part, while f is integrable at $x = -\infty$. A similar example can be constructed when v_0 is a symmetric function with respect to $x = 0$. Then u_0 is antisymmetric. We leave the details to the reader.

13 Extensions, comments and related work

1. FAST-DIFFUSION EQUATIONS. The present non-existence results have counterparts in several dimensions, but the correspondence is not immediate. Thus, there is non-existence of integrable solutions, even under the restriction of positivity, for the analogous of the differentiated equation,

$$v_t = \nabla \cdot (|v|^{p-2} \nabla v), \tag{13.1}$$

for $p < 1$ in dimension $N = 2$ and for $p \leq 1$ if $N \geq 3$, cf. [35]. Let us repeat that in the porous-medium and fast-diffusion literature the notation $m = p - 1$ is in current use, so that $p \to 1$ means $m \to 0$. The nonexistence result of Section 3 can be essentially repeated in any dimensions $N > 1$, establishing the impossibility of changes of sign for equations form

$$v_t = \Delta(\Phi(v)), \tag{13.2}$$

where we assume the singularity assumption (H) on Φ, see Section 3. As in Lemma 3.1 we have

Lemma 13.1 *Let $\Phi(s)$ be a smooth function defined on $(0, \infty)$ with $\Phi'(s) = D(s) > 0$ $\forall s > 0$ and let $\mathcal{R}$ be the $(N+1)$-dimensional cylindrical region $\{a < |x| < b\} \times (0, T)$.*

We consider the problem

$$
(13.3) \qquad \begin{cases} v_t = \Delta(\Phi(v)) & \textit{in } \mathcal{R}, \\ v(x,0) = M & \textit{for } a \le |x| \le b, \\ v(x,t) = \varepsilon, & \textit{for } |x| = a, 0 \le t \le T, \\ v(x,t) = M & \textit{for } |x| = b, 0 \le t \le T, \end{cases}
$$

with $0 < \varepsilon < M$. *This problem has a unique classical solution* v_ε, $\varepsilon < v_\varepsilon < M$ *which is* $C^\infty(\mathcal{R})$. *If* $\Phi(0+) = -\infty$ *then as* $\varepsilon \to 0$ *we have*

$$
(13.4) \qquad \lim_{\varepsilon \to 0} v_\varepsilon(x,t) = 0
$$

uniformly inside $\mathcal{R}$. *If* $\Phi(0)$ *is finite the limit is nontrivial.*

Proof. The argument is essentially the same. Now the estimate $v_t \le 0$ implies that for every $t > 0$, $r^{N-1}\Phi'(v_\varepsilon)$ is decreasing for $a < |x| < b$. ∎

The concept of continuous weak solution defined in a rectangle with continuous boundary data has no novelties. Paralleling Corollary 3.4 we then derive

Theorem 13.2 *A continuous weak solution of the singular equation (13.2) defined in a space-time domain cannot change sign.*

Proof. If it changes sign we may assume that the domain contains a cylinder with a hole in the middle as in Lemma 13.1 in such a way that for $0 \le t \le T$ we have $v \le 0$ on the inner lateral boundary $|x| = a$ and be less than M at $|x| = b$, and v is somewhere positive in the cylinder. Comparison implies then than $v \le 0$ in the cylinder. This is a contradiction. ∎

Let us add some brief comments on the literature for positive solutions. The case $p = 1$ of (13.1) in $N = 2$ has an interesting interpretation in terms of the evolution of conformal metrics in the plane, [21], [37]. It is shown in [36] that in such case there exists a correspondence between solutions $V(s,t)$ in $N = 1$ and radially symmetric solutions $v(r,t)$ in $N = 2$, given by

$$
V(s,t) = r^2\, v(r,t), \quad s = \log(r),
$$

with appropriate correspondence of the conditions at infinity. Let us note that there exist radially symmetric solutions in $N = 2$ which are not strictly positive everywhere, because they admit an isolated singularity with $v = 0$ (or $v = \infty$) at the origin, $r = 0$. For more general classes of solutions see [29]. The interest of the equation in dimension 3 as a model in thin film dynamics has been pointed out and studied in [13]. Other authors have discussed questions of existence for singular diffusion equations, like [12]. Further information is contained in the references to these works.

2. Gradient-dependent diffusivities. On the other hand, in dimension $N > 1$ there is no direct relation between equation (13.1) and the gradient-dependent diffusion models like

$$u_t = c\nabla \cdot (|\nabla u|^{p-2}\nabla u). \tag{13.5}$$

In passing to the limit $p \to 1$ different options depend on the choice of the constant c, which can be interpreted as a measure of the speed of the evolution. In the present article we take $c = 1/(p-1) \to \infty$, which implies *fast evolution*, hence the instantaneous initial layers. The reader will easily observe that such a limit does not make sense in several dimensions as an evolution problem, since the factor $1/(p-1)$ makes some of the terms in the right-hand side limit divergent when $N > 1$. On the other hand, when we take a *slower evolution* with $c = 1$, and pass to the limit we get the equation

$$u_t = \nabla \cdot \left(\frac{\nabla u}{|\nabla u|}\right), \tag{13.6}$$

a well-known equation which has been proposed by Hardt and Zhou [24] as the *coarea gradient flow* and by Rudin et al. [30] in *image processing*. We refer for recent work to Andreu et al. [1], who define entropy solutions, and Giga and Giga [18], who treat a general class of related problems. In order to compare with our results we remark that for $N = 1$ equation (13.6) simplifies into $u_t = (\text{sign}(u_x))_x$, which means that monotone solutions do not evolve at all. For the study of non-monotone solutions we refer to [19]. Differentiating we get equation $v_t = (\text{sign}(u))_{xx}$, discussed in particular in [10]. Again, it can be said that this is a kind of slow-motion version of equation (1.2).

3. Mesa problems. Let us finally mention that evolution problems that produce an initial layer of discontinuity followed by a standard evolution process have also arisen in independent work of some authors on versions of the mesa problem of the porous medium equation when $p \to \infty$, cf. Bénilan and Igbida [8] and Gil and Quirós [20]. The limit $p \to 1$ considered here offers the standard mesa profile formation with the peculiar property of choice of sign.

4 Generation of semigroups. Our previous results can be interpreted as the existence of an evolution semigroup associated to the singular fast diffusion equations under consideration but defined only for precise subsets of the standard Lebesgue spaces. Thus, for equations (1.2) and (1.8) in the range $0 < p \leq 1$, we have a continuous L^1-contraction semigroup defined on the two convex subsets of $L^1(I\!R)$ formed by the nonnegative functions, C_+, and the nonpositive functions, C_- (hint: the use of the subset of maximal solutions allows to uniquely define the semigroup). But there is no continuous semigroup defined on larger subdomains of $L^1(I\!R)$. And the construction of solutions can be extended to $L^1_{loc}(I\!R)$. This happens also for $p \leq 0$, but in this new range there is a further known anomaly: the domains of acceptable data for the solutions do not include any integrable function, because of an effect of instantaneous extinction, a strong extension of the results about null solutions we have demonstrated in this paper for large classes changing-sign data, in particular when $\int f = 0$. We have also shown the discontinuity in the limit $p \to 1$ of the semigroups corresponding to equations (1.4) with $p > 1$. Similar comments apply to the integrated equations.

It is relevant at this stage to note the connection of the differentiated evolution equations with problems in nonlinear elliptic theory that have been studied years ago. Thus, if we try to solve the evolution problem $u_t = \Phi(u)_{xx}$ by Implicit Discretization in Time (IDT) using the famous Crandall-Liggett technique, the corresponding nonlinear elliptic equation takes the form $-\Phi(u)_{xx} + u = f$. Already in 1977 Crandall and Evans [11] showed that for a precise class of Φ's defined only for $u > 0$, which includes equations (1.2) and (1.8) in the range $0 < p \leq 1$, there are positive weak solutions of the elliptic problem with zero flux at infinity if and only if $f \in L^1(I\!R)$ and $\int f > 0$. Using the IDT method a semigroup can be constructed for those data, thus providing an alternative construction for integrable f to the results we obtain by approximation in Section 9; the existence and form of a boundary layer when f is somewhere negative was not remarked at the time. Let us mention two extensions of these elliptic results that come to the point: The existence results with zero flux at infinity (maximal solutions) were extended to dimension $N = 2$ in [34] (resulting in a different family of acceptable $\Phi's$). Solutions with nonzero flux in $N = 1$ were later constructed in [26] and apply for the Crandall-Evans family.

4 Variants of the integrated equations. Due to the fact that equations of the form $u_t = (\Phi(u_x))_x$, like (1.1), (1.3) and (1.7), depend on u through u_x and its derivatives, solving them is equivalent to solving any equation of the form

$$U_t = \Phi(U_x)_x + f(t), \tag{13.7}$$

for any integrable f, just using the change of variables $U(x,t) = u(x,t) + F(t)$, where F is a primitive of f. In this way we conclude that the equations

$$U_t = U_{xx}/|U_x|^n + f(t), \tag{13.8}$$

conserve for $n \geq 1$ the monotonicity properties established for equations (1.1) and (1.7).

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