UNIFORM EXPONENTIAL STABILITY OF THE LINEAR PERIODICAL SYSTEMS IN THE BANACH SPACE.

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Abstract.

This article is devoted to the study of linear periodic dynamical systems, possessing the property of uniform exponential stability. It is proved that if the Cauchy operator of these systems possesses certain compactness property, then the asymptotic stability implies the uniform exponential stability. The applications for different classes of linear evolution equations (ordinary linear differential equations in the space of Banach, retarded and neutral functional differential equations, some classes of evolution partial differential equations) are given.

Introduction

Let $A(t)-$ be a τ- periodic continuous $n \times n$ matrix-function. It is well-known that the following conditions are equivalent:

1. the trivial solution of equation

$$u' = A(t)u \tag{0.1}$$

is uniformly exponentially stable.

2. the trivial solution of equation (0.1) is uniformly asymptotically stable.

3. the trivial solution of equation (0.1) is asymptotical stable.

For the equations in the infinite-dimensional spaces the statements 1.-3. , how show the examples [13,16], aren't equivalent.

It is clear that in the general infinite-dimensional case the condition 1. implies 2. and 2. implies 3.. In this article we show that if the operator Cauchy of equation (0.1) satisfies some compactness condition, then the condition 3. implies 1. (see Theorem 2.4).

The applications for different classes of linear evolution equations (ordinary linear differential equations in the space of Banach, retarded and neutral functional differential equations, some classes of evolution partial differential equations) are given.

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§ 1. Linear non-autonomous dynamical systems.

Assume that X and Y are complete metric spaces, $\mathbb{R}(\mathbb{Z})$ is a group of real (integer) numbers, $\mathbb{T}=\mathbb{R}$ or $\mathbb{Z}, \mathbb{T}_{+}=\{t\in\mathbb{T}: t\geq 0\}, \mathbb{T}_{-}=\{t\in\mathbb{T}| t\leq 0\}$ and $\mathbb{C}$ the set of all complex numbers .

The system $(X,\mathbb{T}_{+},\pi)$ is called [4,5]:

- point dissipative, if there is $K\subseteq X$ such that for all $x\in X$

$$\lim_{t\to+\infty}\rho(xt,K)=0, \tag{1.1}$$

where $xt=\pi^{t}x=\pi(t,x)$;

- compactly dissipative, if the equality (1.1) takes place uniformly with respect to x on compacts from X ;

- locally dissipative, if for any point $p\in X$ there is $\delta_{p}>0$ such that the equality (1.1) takes place uniformly with respect to $x\in B(p,\delta_{p})=\{x\in X:\rho(x,p)<\delta_{p}\}$;

Denote by $(X,\mathbb{T}_{+},\pi)$ $((Y,\mathbb{T},\sigma))$ a semigroup (group) dynamical system on $X(Y)$. A triple $<(X,\mathbb{T}_{+},\pi),(Y,\mathbb{T},\sigma),h>$, where h is a homomorphism of $(X,\mathbb{T}_{+},\pi)$ onto $(Y,\mathbb{T},\sigma)$, is called a non-autonomous dynamical system.

A non-autonomous dynamical system $\langle(X,\mathbb{T}_{+},\pi),(Y,\mathbb{T},\sigma),h\rangle$ is said to be point (compactly, locally) dissipative, if the autonomous dynamical system $(X,\mathbb{T}_{+},\pi)$ is so.

Let (X,h,Y) be a locally trivial Banach fibre bundle over Y[1]. A non-autonomous dynamical system $\langle(X,\mathbb{T}_{+},\pi),(Y,\mathbb{T},\sigma),h\rangle$ is said to be linear if the mapping $\pi^{t}:X_{y}\to X_{yt}$ is linear for every $t\in\mathbb{T}_{+}$ and $y\in Y$, where $X_{y}=\{x\in X|h(x)=y\}$ and $yt=\sigma(t,y)$. Let $|\cdot|$ be some norm on the (X,h,Y), and $|\cdot|$ is co-ordinated with the metric ρ on X (that is $\rho(x_{1},x_{2})=|x_{1}-x_{2}|$ for any $x_{1},x_{2}\in X$ such that $h(x_{1})=h(x_{2})$).

Let E be a Banach space and $\varphi:\mathbb{T}_{+}\times E\times Y\mapsto E$ be a continuous mapping with properties: $\varphi(0,u,y)=u$ and $\varphi(t+\tau,u,y)=\varphi(t,\varphi(\tau,u,y),\sigma(\tau,y))$ for all $u\in E$, $y\in Y$ and $t,\tau\in\mathbb{T}_{+}$. A triplet $\langle E,\varphi,(Y,\mathbb{T},\sigma)\rangle$ is called a continuous cocycle on $(Y,\mathbb{T},\sigma)$ with fibre E.

Let $[E]$ be a Banach space of all linear continuous operators acting onto E with the operational norm and $U:\mathbb{T}_{+}\times Y\mapsto[E]$ be a mapping with properties: $U(0,y)=I$, $U(t+\tau,y)=U(t,\sigma(\tau,y))U(\tau,y)$ for all $y\in Y$ and $t,\tau\in\mathbb{T}_{+}$ and the mapping $U:\mathbb{T}_{+}\times Y\mapsto E$ $((t,y)\mapsto U(t,y)u)$ is continuous for every $u\in E$. A triplet $\langle[E],U,(Y,\mathbb{T},\sigma)\rangle$ is called a $C_{0}-$ cocycle on $(Y,\mathbb{T},\sigma)$ with fibre $[E]$.

The dynamical system $(X,\mathbb{T}_{+},\pi)$ is called [16] a skew-product system if $X=E\times Y$ and $\pi=(\varphi,\sigma)$ (i.e. $\pi(t,(u,y))=(\varphi(t,u,y),\sigma(t,y))$ for all $u\in E$, $y\in Y$ and $t,\tau\in\mathbb{T}_{+}$).

Theorem 1.1 [7,8]. *Let $\langle(X,\mathbb{T}_{+},\pi),(Y,\mathbb{T},\sigma),h\rangle$ be a linear non-autonomous dynamical system and the following conditions hold:*

1. Y is compact and minimal (i.e. $Y=H(y)=\overline{\{yt:t\in\mathbb{T}\}}$ for all $y\in Y$);

2. for any $x\in X$ there exists $C_{x}\geq 0$ such that $|xt|\leq C_{x}$ for all $t\in\mathbb{T}_{+}$;

3. the mapping $y\mapsto\|\pi_{y}^{t}\|$ is continuous, where $\|\pi_{y}^{t}\|$ is a norm of linear operator $\pi_{y}^{t}=\pi^{t}|_{X_{y}}$, for every $t\in\mathbb{T}_{+}$ or $(X,\mathbb{T}_{+},\pi)$ is a skew-product dynamical system.

Then there exists $M\geq 0$ such that the inequality

$$|\pi(t,x)|\leq M|x|$$

holds for all $t \in \mathbb{T}_+$ *and* $x \in X$.

Lemma 1.2. *Let* $\langle [E], U, (Y, \mathbb{T}, \sigma)\rangle$ *be a* C_0- *cocycle on* $(Y, \mathbb{T}, \sigma)$ *with fibre* $[E]$ *and* Y *be a compact, then the following assertions hold:*

1. for every $\ell > 0$ *there exists a positive number* $M(\ell)$ *such that* $\|U(t, y)\| \leq M(\ell)$ *for all* $t \in [0, \ell]$ *and* $y \in Y$.

2. the mapping $\varphi : \mathbb{T}_+ \times E \times Y \mapsto E \quad (\varphi(t, u, y) = U(t, y)u)$ *is continuous.*

3. there exist positive numbers N *and* ν *such that* $\|U(t, y)\| \leq Ne^{\nu t}$ *for all* $t \in \mathbb{T}_+$ *and* $y \in Y$.

Proof. Let $\ell > 0$ and $u \in E$, then there exists a positive number $M(\ell, u)$ such that $|U(t, y)u| \leq M(\ell, u)$ for all $(t, y) \in [0, \ell] \times Y$ because the mapping $(t, y) \to U(t, y)u$ is continuous. According to principle of uniformly boundedness there exist a positive number $M(\ell)$ such that $\|U(t, y)\| \leq M(\ell)$ for all $(t, y) \in [0, \ell] \times Y$.

Let now $(t_0, u_0, y_0) \in \mathbb{T}_+ \times E \times Y$ and $t_n \to t_0, u_n \to u_0$ and $y_n \to y_0$, then we have

$$|\varphi(t_n, u_n, y_n) - \varphi(t_0, u_0, y_0)| \leq$$

$$|\varphi(t_n, u_n, y_n) - \varphi(t_n, u_0, y_n)| + |\varphi(t_n, u_0, y_n) - \varphi(t_0, u_0, y_0)| =$$

$$\leq \|U(t_n, y_n)(u_n - u_0)\| + |(U(t_n, y_n) - U(t_0, y_0))u_0| \qquad (1.2).$$

In view of first statement of Lemma 1.2 there exists a positive number M such that

$$\|U(t_n, y_n)\| \leq M \qquad (1.3)$$

for all $n \in \mathbb{N}$. From inequalities (1.2) and (1.3) follows the continuity of mapping $\varphi : \mathbb{T}_+ \times E \times Y \mapsto E \quad (\varphi(t, u, y) = U(t, y)u)$.

Denote by $a = \sup\{\|U(t, y)\| : (t, y) \in [0, 1] \times Y\}$ and let $t \in \mathbb{T}_+, t = n + \tau (n \in \mathbb{N}, \tau \in [0, 1))$, then we obtain

$$\|U(t, y)\| \leq \|U(n, y\tau)\| \|U(\tau, y)\| \leq a^{n+1} \leq Ne^{\nu t}$$

for all $t \in \mathbb{T}_+$ and $y \in Y$, where $N = a$ and $\nu = \ln a$.

Theorem 1.3 [9]. *Let* $\langle (X, \mathbb{T}_+, \pi), (Y, \mathbb{T}, \sigma), h\rangle$ *be a linear non-autonomous dynamical system,* Y *be a compact , then the following conditions are equivalent:*

1. a non-autonomous dynamical system $\langle (X, \mathbb{T}_+, \pi), (Y, \mathbb{T}, \sigma), h\rangle$ *is uniformly exponentially stable, i.e. there exist two positive constants* N *and* ν *such that* $|\pi(t, x)| \leq Ne^{-\nu t}|x|$ *for all* $t \in \mathbb{T}_+$ *and* $x \in X$.

2. $\|\pi^t\| \to 0$ *as* $t \to +\infty$, *where* $\|\pi^t\| = \sup\{|\pi^t x| : x \in X, |x| \leq 1\}$.

3. a non-autonomous dynamical system $\langle (X, \mathbb{T}_+, \pi), (Y, \mathbb{T}, \sigma), h\rangle$ *is locally dissipative.*

§ 2. Exponential stable linear periodical dynamical systems.

Lemma 2.1 [10, Chapter 9]. *Let* $m : \mathbb{T}_+ \mapsto \mathbb{T}_+$ *be a positive and continuous function. If there exists a positive constant* M *such that* $m(t + s) \leq Mm(t)$ *for all* $s \in [0, 1]$ *and* $t \in \mathbb{T}_+$, *then* $\int_0^{+\infty} m(t)dt < +\infty$ *implies* $m(t) \to 0$ *as* $t \to +\infty$.

Theorem 2.2. *Let $\langle [E], U, (Y, \mathbb{T}, \sigma)\rangle$ be a C_0- cocycle on $(Y, \mathbb{T}, \sigma)$ with fibre $[E]$ and $(Y, \mathbb{T}, \sigma)$ be a periodical dynamical system (i.e. there are $y_0 \in Y$ and $\tau \in \mathbb{T}$ $(\tau > 0)$ such that $Y = \{y_0 t : 0 \leq t < \tau\}$). Then the following conditions are equivalent:*

(i)

$$\lim_{t\to+\infty} ||U(t, y_0)|| = 0. \tag{2.1}$$

(ii) there exist positive constants N and ν such that

$$||U(t, y)|| \leq Ne^{-\nu t} \tag{2.2}$$

for all $t \in \mathbb{T}_+$ and $y \in Y$.

(iii) there exists $p \geq 1$ such that

$$\int_0^{+\infty} |U(t, y_0)u|^p dt < +\infty \tag{2.3}$$

for all $u \in E$.

Proof. We remark that from equality (2.1) follows the condition

$$\lim_{n\to+\infty} \sup_{0\leq s\leq\tau} ||U(s + n\tau, y_0)|| = 0. \tag{2.4}$$

Really, in virtue of Lemma 1.2 there exists a positive constant M such that

$$||U(s, y)|| \leq M \tag{2.5}$$

for all $s \in [0, \tau]$ and $y \in Y$. Therefore

$$||U(s + n\tau, y_0)|| = ||U(s, y_0)U(n\tau, y_0)|| \leq M||U(n\tau, y_0)|| \tag{2.6}$$

for all $0 \leq s \leq \tau$ and, consequently, from (2.1) and (2.6) results the condition (2.4).

We will show that under the condition (2.4) the equality

$$\lim_{t\to+\infty} \sup_{y\in Y} ||U(t, y)|| = 0 \tag{2.7}$$

holds. In fact, let $y \in Y$ then there exists a number $s \in [0, \tau)$ such that $y = y_0 s$ and, consequently, for $t \in \mathbb{T}_+$ $(t = n\tau + \bar{t}, \bar{t} \in [0, \tau))$ we obtain

$$||U(t, y)|| = ||U(t, y_0 s)|| = ||U(n\tau + \bar{t}, y_0 s)|| =$$

$$||U((n - 1)\tau + \bar{t} + s, y_0\tau)U(\tau - s, y_0 s)|| \leq$$

$$M \max\{\sup_{0\leq s\leq\tau} ||U((n - 1)\tau + s, y_0)||, \sup_{0\leq s\leq\tau} ||U(n\tau, y_0)||\}. \tag{2.8}$$

From inequality (2.8) results the equality (2.7). For finishing the proof that (i) implies (ii) is sufficient to apply Theorem 1.3 .

The fact that (ii) implies (iii) is obvious.

Now we will prove that (iii) implies (i). Indeed, let $u \in E$ and we consider the function $m(t) = |U(t, y_0)u|^p \quad (t \geq 0)$. We note that

$$m(t+s) = |U(t+s, y_0)u|^p = |U(s, y_0 t)U(t, y_0)u| \leq$$

$$\|U(s, y_0 t)\| |U(t, y_0)u| \leq Mm(t)$$

for all $t \in \mathbb{T}_+$ and $s \in [0,1]$, where $M = \sup\limits_{0 \leq s \leq 1, y \in Y} \|U(s,y)\|$. In virtue of Lemma 2.1 $m(t) \to 0$ as $t \to +\infty$ and, consequently,

$$\lim_{t \to +\infty} |U(t, y_0)u|^p = 0 \tag{2.9}$$

for all $u \in E$. Let now $y \in Y$, then there exists $s \in [0, \tau)$ such that $y = y_0 s$ and for $t \geq \tau - s$ we have

$$U(t,y)u = U(t, y_0 s)u = U(t - \tau + s, y_0)U(\tau - s, y_0 s)u. \tag{2.10}$$

From equalities (2.9) and (2.10) results that

$$\lim_{t \to +\infty} |U(t, y)u|^p = 0 \tag{2.11}$$

for all $u \in E$ and $y \in Y$. According to Theorem 1.2 there exists a positive number M such that $\|U(t,y)\| \leq M$ for all $t \in \mathbb{T}_+$ and $y \in Y$. Let $t > 0$ and $u \in E$, then we obtain

$$t|U(t, y_0)u|^p = \int_0^t |U(t, y_0)u|^p ds \leq \int_0^t |U(t-s, y_{0s})|^p |U(s, y_0)u|^p ds \leq$$

$$M^p \int_0^t |U(s, y_0)u|^p ds \leq M^p \int_0^{+\infty} |U(s, y_0)u|^p ds = C_u$$

for all $t \geq 0$. In virtue of principle of uniformly boundedness there exist a positive number C such that

$$t\|U(t, y_0)\|^p \leq C$$

for all $t > 0$ and, consequently

$$\|U(t, y_0)\|^p \leq Ct^{-\frac{1}{p}} \to 0$$

as $t \to +\infty$. This completes the proof of theorem.

The operator $U(\tau, y_0)$ is called operator of monodromy for τ- periodic cocycle $U(t,y)$. The number $0 \neq \lambda \in \mathbb{C}$ is called multiplicator of operator of monodromy $U(\tau, y_0)$ if there exists $u_0 \in E \quad (u_0 \neq 0)$ such that $U(\tau, y_0)u_0 = \lambda u_0$ (or, what is the same, $U(t+\tau, y_0)u_0 = \lambda U(t, y_0)u_0$ for all $t \in \mathbb{T}_+$).

Remark 2.3. a. *We remark that the condition (2.1) and the equality*

$$\lim_{n\to+\infty} \|U(n\tau, y_0)\| = 0. \tag{2.12}$$

are equivalent.

Really from (2.1) results (2.12). We will show that (2.12) implies (2.1). Let now $t = n\tau + s, 0 \le s < \tau$, then $U(t, y_0) = U(s + n\tau, y_0) = U(s, y_0)U(n\tau, y_0)$ and, consequently,

$$\|U(t, y_0)\| \le \max_{0\le s\le\tau} \|U(s, y_0)\|\|U(n\tau, y_0)\|. \tag{2.13}$$

From conditions (2.12) and (2.13) results (2.1).

b. *We observe that the condition (2.2) and the inequality*

$$\|U(t, y_0)\| \le N_1 e^{-\nu_1 t} \quad (\forall t \in \mathbb{T}_+) \tag{2.14}$$

are equivalent, where N_1 and ν_1 are some positive constants.

Indeed, from inequality (2.14), taking into account the equality (2.10), we obtain (2.2).

c. *The condition (2.12) is satisfied if and only if $\sigma(U(\tau, y_0)) \subset \mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, where $\sigma(U(\tau, y_0))$ is a spectrum of operator of monodromy $U(\tau, y_0)$.*

In fact, from equality (2.2) results that $r_{U(\tau,y_0)} = \lim_{n\to+\infty} \sup(\|U(n\tau, y_0))\|)^{\frac{1}{n}} \le e^{-\nu} < 1$, because $U^n(\tau, y_0) = U(n\tau, y_0)$. If $\gamma = r_{U(\tau,y_0)} < 1$, then for all $\varepsilon > 0$ there exists a $n(\varepsilon) \in \mathbb{N}$ such that $(\|U(n\tau, y_0)\|)^{\frac{1}{n}} \le \gamma + \varepsilon$ for all $n \ge n(\varepsilon)$ and, consequently, $\|U(n\tau, y_0)\| \le (\gamma + \varepsilon)^n$ for all $n \ge n(\varepsilon)$. Thus $\|U(n\tau, y_0)\| \to 0$ as $n \to +\infty$.

Theorem 2.4. *Let $\langle [E], U, (Y, \mathbb{T}, \sigma)\rangle$ be a C_0- cocycle on $(Y, \mathbb{T}, \sigma)$ with fibre $[E]$, $(Y, \mathbb{T}, \sigma)$ be a periodical dynamical system and $U(\tau, y_0)$ be asymptotically compact (i.e. if the sequence $\{u_n\} \subseteq E$ is bounded, $k_n \to +\infty$ $(k_n \in \mathbb{N})$ and $\{U(k_n\tau, y_0)u_n\}$ is bounded, then the sequence $\{U(k_n\tau, y_0)u_n\}$ is precompact). Then the following conditions are equivalent:*

(i) the equality (2.1) holds.

(ii) the equality

$$\lim_{t\to+\infty} |U(t, y_0)u| = 0 \tag{2.15}$$

holds for all $u \in E$.

Proof. It is evidently that (i) implies (ii).

Under the conditions of Theorem 2.4 the discrete dynamical system (E, P), generated by mapping $P = U(\tau, y_0) : E \mapsto E$ is asymptotically compact because $P^n = U(n\tau, y_0)$. From condition (2.15) it follows that the discrete dynamical system is bounded, i.e. $\sup\{\|P^n\| : n \in \mathbb{Z}_+\} < +\infty$ (see the proof of Theorem 2.2). Thus under the conditions of Theorem 2.4 and condition (2.15) the discrete dynamical system is bounded, point dissipative and asymptotically compact. According to Theorem 2.3 [6] the dynamical system (P, E) is locally dissipative and, consequently, $\lim_{n\to+\infty} \|P^n\| = 0$, i.e. $\lim_{n\to+\infty} \|U(n\tau, y_0)\| = 0$. Taking account the Remark 2.3 we obtain the equality (2.15). The Theorem is proved.

§ 3. Some classes of linear uniformly exponentially stable periodical differential equations.

Let Λ be a complete metric space of linear operators that act on Banach space E and $C(\mathbb{R},\Lambda)$ be a space of all continuous operator-functions $A : \mathbb{R} \to \Lambda$ equipped with open-compact topology and $(C(\mathbb{R},\Lambda),\mathbb{R},\sigma)$ be a dynamical system of shifts on $C(\mathbb{R},\Lambda)$.

3.1. Ordinary linear differential equations. Let $\Lambda = [E]$ and consider the linear differential equation

$$u' = \mathcal{A}(t)u \quad , \tag{3.1}$$

where $\mathcal{A} \in C(\mathbb{R},\Lambda)$. Along with equation (3.1), we shall also consider its $H-$class, that is, the family of equations

$$v' = \mathcal{B}(t)v \quad , \tag{3.2}$$

where $\mathcal{B} \in H(\mathcal{A}) = \overline{\{\mathcal{A}_s : s \in \mathbb{R}\}}, \mathcal{A}_s(t) = \mathcal{A}(t+s) \quad (t \in \mathbb{R})$ and the bar denotes closure in $C(\mathbb{R},\Lambda)$. Let $\varphi(t,u,\mathcal{B})$ be the solution of equation (3.2) that satisfies the condition $\varphi(0,v,\mathcal{B}) = v$. We put $Y = H(\mathcal{A})$ and denote the dynamical system of shifts on $H(\mathcal{A})$ by $(Y,\mathbb{R},\sigma)$, then the triple $\langle (X,\mathbb{R}_+,\pi),(Y,\mathbb{R},\sigma),h\rangle$ is a linear non-autonomous dynamical system, where $X = E \times Y, \pi = (\varphi,\sigma) \quad$ (i.e. $\quad \pi((v,\mathcal{B}),\tau) = (\varphi(\tau,v,\mathcal{B}),\mathcal{B}_\tau)$ and $h = pr_2 : X \to Y$).

Lemma 3.1[2,3]. *(i) The mapping $(t,u,\mathcal{A}) \mapsto \varphi(t,u,\mathcal{A})$ of $\mathbb{R} \times E \times C(\mathbb{R},[E])$ to E is continuous, and*

(ii) the mapping $\mathcal{A} \mapsto U(\cdot,\mathcal{A})$ of $C(\mathbb{R},[E])$ to $C(\mathbb{R},[E])$ is continuous, where $U(\cdot,\mathcal{A})$ is the Cauchy operator [11] of equation (3.1).

Theorem 3.2. *Let $\mathcal{A} \in C(\mathbb{R},\Lambda)$ be $\tau-$ periodic (i.e. $\mathcal{A}(t+\tau) = \mathcal{A}(t)$ for all $t \in \mathbb{T}$), then the following conditions are equivalent:*

1. a trivial solution of equation (3.1) is uniformly exponentially stable, i.e. there exist positive numbers N and ν such that $\|U(t,\mathcal{A})U(\tau,\mathcal{A})^{-1}\| \le Ne^{-(t-\tau)}$ for all $t \ge \tau$.

2. there exist positive numbers N and ν such that $\|U(t,\mathcal{B})U(\tau,\mathcal{B})^{-1}\| \le Ne^{-(t-\tau)}$ for all $t \ge \tau$ and $\mathcal{B} \in H(\mathcal{A}) = \{\mathcal{A}_s : s \in [0,\tau)\}$, where $\mathcal{A}_s(t) = \mathcal{A}(t+s) \quad (t \in \mathbb{T})$.

3. $\lim\limits_{t\to+\infty} \|U(t,\mathcal{A})\| = 0$.

4. there exists $p \ge 1$ such that

$$\int_0^{+\infty} |U(t,\mathcal{A})u|^p dt < +\infty$$

for all $u \in E$.

Proof. Applying Theorem 2.2 to non-autonomous system $\langle (X,\mathbb{R}_+,\pi),(Y,\mathbb{R},\sigma),h\rangle$, generated by equation (3.1) we obtain the equivalence of conditions 2., 3. and 4. According to Lemma 3 [3] the conditions 1. and 2. are equivalent. The theorem is proved.

Theorem 3.3. *Let $\mathcal{A} \in C(\mathbb{R}, \Lambda)$ be $\tau-$ periodic and $U(\tau, \mathcal{A})$ be asymptotically compact, then the following conditions are equivalent:*

1. a trivial solution of equation (3.1) is uniformly exponentially stable, i.e. there exist positive numbers N and ν such that $\|U(t,\mathcal{A})U(\tau,\mathcal{A})^{-1}\| \le Ne^{-(t-\tau)}$ for all $t \ge \tau$.

2. $\lim_{t\to+\infty} |U(t,\mathcal{A})u| = 0$ for every $u \in E$.

Proof. Applying Theorem 2.4 to non-autonomous system $\langle (X, \mathbb{R}_+, \pi), (Y, \mathbb{R}, \sigma), h\rangle$ generated by equation (3.1), we obtain the equivalence of conditions 1. and 2.. The theorem is proved.

3.2. Partial linear differential equations. Let Λ be some complete metric space of linear closed operators acting into Banach space E (for example $\Lambda = \{A_0 + B | B \in [E]\}$, where A_0 is a closed operator that acts on E). We assume that the following conditions are fulfilled for equation (3.1) and its $H-$ class (3.2):

a. for any $v \in E$ and $\mathcal{B} \in H(\mathcal{A})$ equation (3.2) has exactly one solution defined on $\mathbb{R}_+$ and satisfies the condition $\varphi(0, v, \mathcal{B}) = v$;

b. the mapping $\varphi : (t, v, \mathcal{B}) \to \varphi(t, v, \mathcal{B})$ is continuous in the topology of $\mathbb{R}_+ \times E \times C(\mathbb{R}; \Lambda)$;

Under the above assumptions the equation (3.1) generates a linear non-autonomous dynamical system $\langle (X, \mathbb{R}_+, \pi), (Y, \mathbb{R}, \sigma), h\rangle$, where $X = E \times Y, \pi = (\varphi, \sigma)$ and $h = pr_2 : X \to Y$. Applying the results from § 2 to this system, we will obtain the analogous assertions for different classes of partial differential equations.

We will consider examples of partial differential equations which satisfy the above conditions a. and b.

Example 3.1. A closed linear operator $\mathcal{A} : D(\mathcal{A}) \mapsto E$ with dense domain $D(\mathcal{A})$ is said [13] to be sectorial if one can find a $\varphi \in (0, \frac{\pi}{2})$, an $M \ge 1$, and a real number a such that the sector

$$S_{a,\varphi} = \{\lambda : \varphi \le |arg(\lambda - a)| \le \pi, \lambda \ne a\}$$

lies in the resolvent set $\rho(\mathcal{A})$ of $\mathcal{A}$ and $\|(\lambda I - \mathcal{A})^{-1}\| \le M|\lambda - a|^{-1}$ for all $\lambda \in S_{a,\varphi}$. If A is a sectorial operator, then there exists an $a_1 > 0$ such that $Re\sigma(A + a_1 I) > 0$ $(\sigma(A) = \mathbb{C} \setminus \rho(A))$. Let $A_1 = A + a_1 I$. For $0 < \alpha < 1$, one defines the operator [13]

$$A_1^{-\alpha} = \frac{sin\pi\alpha}{\pi} \int_0^{+\infty} \lambda^{-\alpha}(\lambda I + A_1)^{-1} d\lambda,$$

which is linear, bounded, and one-to-one. Set $E^\alpha = D(A_1^\alpha)$, and let us equip the space E^α with the norm $|u|\alpha = |A_1^\alpha u|, E^0 = E, X^1 = D(A)$. Then E^α is a Banach space with the norm $|\cdot|_\alpha$, and is densely continuously embedded in E. If the operator A admits a compact resolvent, then the embedding $E^\alpha \to E^\beta$ is compact for $\alpha > \beta \ge 0$ [13]. An important class of sectorial operators is formed by elliptic operators [13,14].

Consider the differential equation

$$u' = (\mathcal{A}_0 + \mathcal{A}(t))u, \tag{3.3}$$

where $\mathcal{A}_0$ is a sectorial operator that does not depend on $t \in \mathbb{R}$, and $\mathcal{A} \in C(\mathbb{R}, [E])$. The results of [13] imply that equation (3.3) satisfies conditions a. and b.

Under the above assumptions the equation (3.3) generates a linear non-autonomous dynamical system $\langle (X, \mathbb{R}_+, \pi), (Y, \mathbb{R}, \sigma), h \rangle$, where $Y = H(\mathcal{A})$ $X = E \times Y, \pi = (\varphi, \sigma)$ and $h = pr_2 : X \to Y$. Applying the results from § 2 to this system, we will obtain the following results.

Theorem 3.4. *Let $\mathcal{A}_0-$ be a sectorial operator and $\mathcal{A} \in C(\mathbb{R}, \Lambda)$ be $\tau-$ periodic , then the following conditions are equivalent:*

1. a trivial solution of equation (3.3) is uniformly exponentially stable, i.e. there exist positive numbers N and ν such that $\|U(t, \mathcal{A}_0 + \mathcal{A}) U(\tau, \mathcal{A}_0 + \mathcal{A})^{-1}\| \leq Ne^{-(t-\tau)}$ for all $t \geq \tau$.

2. there exist positive numbers N and ν such that $\|U(t, \mathcal{A}_0 + \mathcal{B}) U(\tau, \mathcal{A}_0 + \mathcal{B})^{-1}\| \leq Ne^{-(t-\tau)}$ for all $t \geq \tau$ and $\mathcal{B} \in H(\mathcal{A})$.

3. $\lim\limits_{t \to +\infty} \|U(t, \mathcal{A}_0 + \mathcal{A})\| = 0$.

4. there exists $p \geq 1$ such that

$$\int_0^{+\infty} |U(t, \mathcal{A}_0 + \mathcal{A})u|^p dt < +\infty$$

for all $u \in E$.

5. $\sigma(U(\tau, \mathcal{A}_0 + \mathcal{A})) \subset \mathbb{D}$.

Theorem 3.5. *Let $\mathcal{A}_0-$ be a sectorial operator with compact resolvent and $\mathcal{A} \in C(\mathbb{R}, \Lambda)$ be $\tau-$ periodic, then the following conditions are equivalent:*

1. a trivial solution of equation (3.3) is uniformly exponentially stable.

2. $\lim\limits_{t \to +\infty} |U(t, \mathcal{A}_0 + \mathcal{A})u| = 0$ for every $u \in E$.

3. $|\lambda| < 1$ for every multiplicator λ of operator of monodromy $U(\tau, \mathcal{A}_0 + \mathcal{A})$.

Proof. Since the sectorial operator $\mathcal{A}_0$ admits a compact resolvent, then in view of Lemma 7.2.2 [13] the operator $U(\tau, \mathcal{A}_0 + \mathcal{A})$ is compact and, consequently (see,for example [17, p.391-396]), every $0 \neq \lambda \in \sigma(U(\tau, \mathcal{A}_0 + \mathcal{A}))$ is a multiplicator for operator of monodromy $U(\tau, \mathcal{A}_0 + \mathcal{A})$. Applying Theorem 2.4 (see also Remark 2.3) to non-autonomous system $\langle (X, \mathbb{R}_+, \pi), (Y, \mathbb{R}, \sigma), h \rangle$ generated by equation (3.3), we obtain the equivalence of conditions 1., 2. and 3. The theorem is proved.

3.3. Linear functional-differential equations.

Let $r > 0, C([a, b], \mathbb{R}^n)$ be a Banach space of all continuous functions $\varphi : [a, b] \to \mathbb{R}^n$ with sup-norm. If $[a, b] = [-r, 0]$, then we put $\mathcal{C} = C([-r, 0], \mathbb{R}^n)$. Let $\sigma \in \mathbb{R}, \alpha \geq 0$ and $u \in C([\sigma - r, \sigma + \alpha], \mathbb{R}^n)$. For any $t \in [\sigma, \sigma + \alpha]$ we define $u_t \in \mathcal{C}$ by equality $u_t(\theta) = u(t + \theta), -r \leq \theta \leq 0$. Denote by $\mathfrak{A} = \mathfrak{A}(C, \mathbb{R}^n)$ a Banach space of all linear continuous operators acting from $\mathcal{C}$ into $\mathbb{R}^n$, equipped by operational norm. Consider the equation

$$u' = \mathcal{A}(t)u_t \quad , \tag{3.4}$$

where $\mathcal{A} \in C(\mathbb{R}, \mathfrak{A})$. We put $H(\mathcal{A}) = \overline{\{\mathcal{A}_\tau : \tau \in \mathbb{R}\}}, \mathcal{A}_\tau(t) = \mathcal{A}(t + \tau)$ and the bar denotes closure in the topology of uniform convergence on every compact from $\mathbb{R}$.

Along with equation (3.4) we also consider the family of equations

$$u' = \mathcal{B}(t)u_t \quad , \tag{3.5}$$

where $\mathcal{B} \in H(\mathcal{A})$. Let $\varphi_t(v, \mathcal{B})$ be a solution of equation (3.5) with condition $\varphi_0(v, \mathcal{B}) = v$ defined on $\mathbb{R}_+$. We put $Y = H(\mathcal{A})$ and denote by $(Y, \mathbb{R}, \sigma)$ a dynamical system of shifts on $H(\mathcal{A})$. Let $X = C \times Y$ and $\pi = (\varphi, \sigma)$ a dynamical system on X, defined by equality $\pi(\tau, (v, \mathcal{B})) = (\varphi_\tau(v, \mathcal{B}), \mathcal{B}_\tau)$. A non-autonomous dynamical system $\langle (X, \mathbb{R}_+, \pi), (Y, \mathbb{R}, \sigma), h \rangle$ $(h = pr_2 : X \to Y)$ is linear. The following assertion takes place.

Lemma 3.6[8]. *Let $H(\mathcal{A})$ be compact in $C(\mathbb{R}, \mathfrak{A})$, then the non-autonomous dynamical system $\langle (X, \mathbb{R}_+, \pi), (Y, \mathbb{R}, \sigma), h \rangle$ generated by equation (3.4) is completely continuous, i.e. for every bounded set $A \subset X$ there exists a positive number ℓ such that $\pi^\ell A$ is precompact.*

Theorem 3.7. *Let $\mathcal{A}$ be $\tau-$ periodic. Then the following assertions are equivalent:*

1. *a trivial solution of equation (3.4) is uniformly exponentially stable.*
2. $\lim\limits_{t \to +\infty} |U(t, \mathcal{A})u| = 0$ *for every $u \in E$.*
3. *all multiplicators λ of operator of monodromy $U(\tau, \mathcal{A})$ satisfy $|\lambda| < 1$.*

Proof. Let $\langle (X, \mathbb{R}_+, \pi), (Y, \mathbb{R}, \sigma), h \rangle$ be a linear non-autonomous dynamical system, generated by equation (3.4). According to Lemma 3.6 this system is completely continuous and, consequently, there exists a number $k \in \mathbb{N}$ such that $U^k(\tau, y_0) = U(k\tau, y_0)$ is precompact. In virtue of theory of Riesz-Schauder (see for example [17, p.391-395]) every $0 \neq \lambda \in \sigma(U(\tau, \mathcal{A}))$ is a multiplicator of operator of monodromy $U(\tau, \mathcal{A})$. To finish the proof it is sufficiently to refer to Theorems 2.2 and Remark 2.3.

Consider the neutral functional differential equation

$$\frac{d}{dt} D u_t = \mathcal{A}(t) u_t \quad , \tag{3.6}$$

where $\mathcal{A} \in C(\mathbb{R}, \mathfrak{A})$ and $D \in \mathfrak{A}$ is nonatomic at zero operator [12, p.67]. As well as in the case of equation (3.4), the equation (3.6) generates a linear non-autonomous dynamical system $\langle (X, \mathbb{R}_+, \pi), (Y, \mathbb{R}, \sigma), h \rangle$, where $X = C \times Y, Y = H(\mathcal{A})$ and $\pi = (\varphi, \sigma)$. The following statement holds.

Lemma 3.8[8]. *Let $H(\mathcal{A})$ be compact and the operator D is stable, i.e. the zero solution of homogeneous difference equation $Dy_t = 0$ is uniformly asymptotically stable. Then a linear non-autonomous dynamical system $\langle (X, \mathbb{R}_+, \pi), (Y, \mathbb{R}, \sigma), h \rangle$, generated by equation (3.6), is asymptotically compact.*

Theorem 3.9. *Let $\mathcal{A} \in C(\mathbb{R}, \mathfrak{A})$ be $\tau-$ periodic and D is stable, then the following assertions are equivalent:*

1. *a trivial solution of equation (3.6) is uniformly exponentially stable.*
2. $\lim\limits_{t \to +\infty} |U(t, \mathcal{A})u| = 0$ *for every $u \in E$.*
3. $|\lambda| < 1$ *for every multiplicator λ of operator of monodromy $U(\tau, \mathcal{A})$.*

Proof. Let $\langle (X, \mathbb{R}_+, \pi), (Y, \mathbb{R}, \sigma), h \rangle$ be a linear non-autonomous dynamical system, generated by equation (3.6). According to Lemma 3.8 this system is asymptotically compact. . According to results of [12, Chapter 12] every $0 \neq \lambda \in \sigma(U(\tau, y_0))$ is a multiplicator of operator of monodromy $U(\tau, y_0)$. To finish the proof of Theorem 3.8 it is sufficiently to refer to Theorems 2.4 and Remark 2.3. The theorem is proved.

Remark 3.10. *The equivalence of conditions 1. and 3. in Theorem 3.5 (Theorem 3.7, Theorem 3.9) was proved in [13, p.219] (resp. in [12, p.233], [12, p.365]).*

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