The mixed Cauchy-Dirichlet problem for the heat equation in a plane angle in spaces of hölder continuous functions

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1 Introduction

The study of boundary value problems and of initial boundary value problems in domains with singularities is of great interest in applications. For a well written review of the subject covering the "state of the art" up to the beginning of the eighties, we mention [14]. In particular, elliptic boundary value problems in domains with conical or dihedral have been extensively studied, starting from Kondratiev's famous paper [13]. More recently, the monographs [8], [19] and [15] have appeared. Grisvard considered almost only problems in traditional L^p or hölder continuous spaces in plane domains, while the russian school, on the lines of Kondratiev, developed a rich theory in spaces with functions with weights. This approach seems to be more convenient for operators with nonconstant coefficients if one wants to put in light the loss of regularity of the solutions near the singularities of the boundary.

Mixed parabolic problems have been less studied. Azzam and Kreiszig (see [3], [4]) studied the smoothness properties of the solution of a mixed problem with Dirichlet and oblique derivarive boundary conditions in angles, obtaining quite precise (but not optimal) estimates of the derivatives of the solution with hölder continuous data in the variables (x,t). More recently, Solonnikov ([21]) obtained maximal regularity results for the Neumann problem in weighted Sobolev-Slobodetsky spaces and weighted hölder spaces in a dihedral angle. Analogous results for Dirichlet boundary conditions were given without explicit proof. Other problems in spaces with weights were considered, for example, in [6] and [5].

Coming back to more traditional spaces, J. O. Adeyeye studied in [2] the generation of an analytic operator by the Laplace operator with various boundary conditions in L^p spaces for $1 < p < +\infty$ in a polygon. The case $p = +\infty$ was considered in [9]. Finally J. O. Adeyeye characterized the real interpolation spaces between the domain of the Laplace operator with Dirichlet boundary conditions and $L^p(\Omega)$ for Ω polygonal.

The aim of this paper is to study the classical mixed Cauchy-Dirichlet problem for the heat equation in a plane angle. Following essentialy Grisvard's approach, we shall not consider spaces with weights, but rather we shall work in spaces of hölder continuous functions, even of negative order (they will be carefully defined in the following), on the lines of [18]. The results obtained for these spaces with low regularity can be extended by simple perturbation arguments to the case of problems with variable coefficients and allow to generalize many of the results of [18] to domains with polygonal structure. We go to explain the content of the paper more in detail.

In the second section we consider spaces of hölder continuous functions (even of negative order) in a plane angle, extending some results which are well known in the case of smooth boundary.

In the third section we study elliptic boundary value problems depending on a parameter. The main result in this direction are theorems 3.1 (case $-2 < \sigma < 0$) and 3.3 (case $\sigma > 0$). These results extend to a plane angle assertions which are known in the case of a smooth boundary (see [11] and [12]). The main difference is that we have to add to the usual space of solutions $C^{2+\sigma}(\Omega)$ a finite dimensional space of singular solutions. We have called $\tilde{C}^{2+\sigma}(\Omega)$ the resulting space. We have considered also estimates depending on a parameter in the space of bounded uniformly continuous functions $C(\Omega)$ (see 3.3) and characterized the real interpolation spaces $(C(\Omega), D(A))_{\theta,\infty}$, where A is the natural realization of the Laplace operator with Dirichlet boundary conditions in $C(\Omega)$.

The previous results are used in the fourth and final section to obtain maximal regularity results for the mixed Cauchy-Dirichlet problem. These results are contained in theorem 4.1 ($\sigma < 0$) and in theorem 4.2. This latest theorem is a plane angle version of the main result of [12]. Theorem 4.2 allows to give (together with a maximal regularity result due to E. Sinestrari) a generalized extension of the classical $C^{1+\frac{\sigma}{2},2+\sigma}$ maximal regularity result for parabolic problems and, for certain values of σ, gives a new proof of some of the results of [21] concerning Dirichlet boundary conditions.

We conclude this introduction with some notations we shall use in the following.

We shall indicate with $\mathbf{Z}$, $\mathbf{N}$, $\mathbf{N}_0$, $\mathbf{R}$, $\mathbf{C}$ the sets of, respectively, integers, positive integers,nonnegative integers, real and complex numbers. We shall often confuse $\mathbf{R}^2$ with $\mathbf{C}$.

If $\sigma \in \mathbf{R}$, we shall indicate with $[\sigma]$ the integer part of σ and with $\{\sigma\}$ the difference $\sigma - [\sigma]$.

If A is a subset of a topological space X we shall indicate with $\overline{A}$ the topological closure of A in X.

If O is an open subset of $\mathbf{R}^n$, we shall indicate with $\mathcal{D}'(O)$ the space of distributions in O. If α is a multiindex and $u \in \mathcal{D}'(O)$, we shall indicate with $\partial^\alpha u$ the corresponding derivative in the sense of distributions.

If u is a tempered distribution, we shall indicate with $\hat{u}$ or $\mathcal{F}u$ its Fourier transform.

If X and Y are Banach spaces, we shall indicate with $\mathcal{L}(X, Y)$ the Banach space of linear bounded operators from X to Y.

If A is a linear operator in a Banach space X, we shall indicate with $\rho(A)$ its resolvent set.

If X is a Banach space and I a set, $B(I; X)$ will indicate the set of functions of domain I which are bounded with values in X.

If X and Y are compatible Banach spaces and $\theta \in]0, 1[$, we shall indicate with $(X, Y)_{\theta,\infty}$ the corresponding interpolation space (see [17], ch. 2). If $X \subseteq Y \subseteq Z$ and $\theta \in]0, 1[$, we shall say that $Y \in J_\theta(X, Z)$ if there exists $C > 0$ such that, for every $x \in X$,

$$||x||_Y \leq C||x||_Z^{1-\theta}||x||_X^\theta.$$

We shall say that $Y \in K_\theta(X, Z)$ if it is continuously embedded into $(X, Z)_{\theta,\infty}$.

If O is an open subset of $\mathbf{R}^n$, we shall indicate with $H^1(O)$ and with $H^{-1}(O)$ the usual Sobolev spaces appearing in the variational theory of elliptic boundary value problems (see, for example, [19], ch. 2).

$C(\alpha, \beta, ...)$ will stand for a positive constant depending on $\alpha, \beta, ...$

2 Spaces of hölder continuous functions

Let O be an open subset in $\mathbf{R}^n$, with $n \in \mathbf{N}$. If $k \in \mathbf{N}_0$, we indicate with $\mathcal{C}^k(O)$ the linear space of complex valued functions of domain O which are continuous together with their derivatives of order less or equal to k. We put $\mathcal{C}^\infty(O) := \cap_{k\in\mathbf{N}_0}\mathcal{C}^k(O)$. We indicate with $C(O)$ the space of complex valued uniformly continuous and bounded functions of domain O. If σ is a positive integer, we set

$$C^\sigma(O) := \{f \in C(O) | \partial^\alpha f \in C(O) \quad if \quad |\alpha| \leq \sigma\}. \tag{2.1}$$

Moreover, we set, for $f \in C(O)$,

$$||f||_{0,O} := \sup_{z\in O} |f(z)|,$$

for $f \in C^\sigma(O)$ for some $\sigma \in \mathbf{N}$,

$$||f||_{\sigma,O} := \max_{|\alpha|\leq\sigma} ||\partial^\alpha f||_{0,O}.$$

If $\sigma \in]0,1[$ and $f \in C(O)$, we put

$$[f]_{\sigma,O} := \sup_{z,v\in O, z\neq v} |z-v|^{-\sigma}|f(z) - f(v)|,$$

if $\sigma > 0$ and $\sigma \notin \mathbf{N}$,

$$||f||_{\sigma,O} := \max\{||f||_{[\sigma],O}, \max_{|\alpha|=[\sigma]} [\partial^\alpha f]_{\{\sigma\},O}\}. \tag{2.2}$$

As we are dealing with uniformly continuous functions, they are continuosly extensible with their derivatives of order less or equal to $[\sigma]$ up to the boundary. We shall identify them with their continuous extension.

We shall need also spaces of functions with domain which is the boundary ∂O of an open subset O of $\mathbf{R}^2$. We shall consider the following cases:

a) ∂O is a line in $\mathbf{R}^2$; $C^\sigma(\partial O)$ can be easily defined through an affine diffeomorphism;

b) ∂O is a bounded smooth submanifold of $\mathbf{R}^2$ lying on one side of O; in this case the spaces $C^\sigma(\partial O)$ for $\sigma \geq 0$ can be naturally defined by local charts (see [22]);

c) ∂O is a smooth submanifold of $\mathbf{R}^2$ lying on one side of O and union of a bounded smooth manifold with a finite family of open unbounded subsets of lines; this case can be reduced to *a)* and *b)* through a partition of unity.

In the previous cases the boundary was smooth. As we want to study problems in plane angles, we have to work in domains which have a singularity. We introduce some notations: let $\omega \in]0, 2\pi[-\{\pi\}$; we set

$$\Omega := \{z = \rho e^{i\theta} \in \mathbf{R}^2 | \rho > 0, 0 < \theta < \omega\}. \tag{2.3}$$

In the following ω and Ω will be exclusively used in the previous sense.

For g complex valued function of domain $\partial\Omega$, we define

$$g_0, g_1 :]0, +\infty[\to \mathbf{C},$$

$$g_0(\rho) := g(\rho), g_1(\rho) := g(\rho e^{i\omega}).$$

Then, we consider the following further cases:

d) $O = \Omega$. For $\sigma \geq 0$, we shall say that $g \in C^\sigma(\partial\Omega)$ if g_0 and g_1 belong to $C^\sigma(]0, +\infty[)$ and $g_0(0) = g_1(0)$. If $g \in C^\sigma(\partial\Omega)$, we set

$$||g||_{\sigma,\partial\Omega} := \max\{||g_j||_{\sigma,]0,+\infty[} | j = 0, 1\}.$$

e) O is an open bounded subset of $\mathbf{R}^2$ whose boundary is the union of an open neighbourhood of $(0,0)$ in $\partial\Omega$ with an open subset of the boundary of an open set of the type *b)*. In this case we can define $C^\sigma(\partial O)$ and $||.||_{\sigma,\partial O}$ through a partition of unity, using the definitions in *b)* and *d)*.

In each of the previous cases we shall endow $C^\sigma(\partial O)$ with a natural norm $||.||_{\sigma,\partial O}$.

Obviously, if $f \in C^\sigma(\Omega)$, then $f_{|\partial\Omega} \in C^\sigma(\partial\Omega)$ for every $\sigma \geq 0$. On the other hand,

Proposition 2.1 *Let O be an open subset of $\mathbf{R}^2$ whose boundary satisfies one of the conditions* a)-e)*; then there exists $\mathcal{E} \in \mathcal{L}(C(\partial\Omega), C(\Omega))$ such that for every $g \in C(\partial\Omega)$ $\mathcal{E}g_{|\partial\Omega} = g$.*

Moreover, for every $\sigma > 0$ the restriction of $\mathcal{E}$ to $C^\sigma(\partial\Omega)$ is bounded with values in $C^\sigma(\Omega)$ and, if $g \in C(\partial\Omega) \cap H^{\frac{1}{2}}(\partial\Omega)$, then $\mathcal{E}g \in C(\Omega) \cap H^1(\Omega)$.

Proof See [8], 6.2.

Now we extend the definition of $C^\sigma(O)$ to the case $\sigma < 0$:

Definition 2.1 *Let $\sigma \in \mathbf{R}$, $\sigma < 0$, O an open subset of $\mathbf{R}^n$; we define*

$$C^\sigma(O) := \{f \in \mathcal{D}'(O) | f = \sum_{|\alpha| \leq |[\sigma]|} \partial^\alpha f_\alpha, f_\alpha \in C^{\{\sigma\}}(O)\}.$$

If $f \in C^\sigma(O)$, we put

$$||f||_{\sigma,O} := \inf\{\sum_{|\alpha| \leq |[\sigma]|} ||f_\alpha||_{\{\sigma\},O}, f = \sum_{|\alpha| \leq |[\sigma]|} \partial^\alpha f_\alpha, f_\alpha \in C^{\{\sigma\}}(O)\}.$$

One can easily verify that $||.||_{\sigma,O}$ so defined is a norm which makes of $C^\sigma(O)$ a Banach space. In case $O = \mathbf{R}^n$, we have that, if $\sigma \notin \mathbf{Z}$, then $C^\sigma(\mathbf{R}^n)$ coincides with the Besov space $B^\sigma_{\infty,\infty}(\mathbf{R}^n)$ (see [22] and also [11], in particular proposition 1.7) with equivalent norms.

In the specific case $O = \Omega$, owing to [8], 6.2, every element f of $C^\sigma(\Omega)$ with $\sigma \geq 0$ can be extended to an element of $C^\sigma(\mathbf{R}^2)$. So, if $\sigma \in \mathbf{R} - \mathbf{Z}$ and we set

$$B^\sigma_{\infty,\infty}(\Omega) := \{f_{|\Omega} | f \in B^\sigma_{\infty,\infty}(\mathbf{R}^2)\},$$

we have

$$B^\sigma_{\infty,\infty}(\Omega) = C^\sigma(\Omega). \tag{2.4}$$

and the restriction operator is bounded from $B^\sigma_{\infty,\infty}(\mathbf{R}^2)$ to $C^\sigma(\Omega)$.

We consider now the existence of a bounded extension operator from $C^\sigma(\Omega)$ to $C^\sigma(\mathbf{R}^2)$:

Proposition 2.2 *Let* $r \in \mathbf{N}$. *Then there exists a linear operator* $\mathcal{P}$ *from* $\bigcup_{\sigma > -r-1} C^\sigma(\Omega)$ *to* $\bigcup_{\sigma > -r-1} C^\sigma(\mathbf{R}^2)$ *such that:*

a) *for every* $f \in \bigcup_{\sigma > -r-1} C^\sigma(\Omega)$ $\mathcal{P}f_{|\Omega} = f$;

b) *if* $-r-1 < \sigma < r$, σ *is not a negative integer and* $f \in C^\sigma(\Omega)$, *then* $\mathcal{P}f \in C^\sigma(\mathbf{R}^2)$; *moreover,* $\mathcal{P}_{|C^\sigma(\Omega)} \in \mathcal{L}(C^\sigma(\Omega), C^\sigma(\mathbf{R}^2))$.

Proof In this proof we put $\Omega_1 := \{z \in \mathbf{R}^2 | x < 0\}$, $\Omega_2 := \{z \in \mathbf{R}^2 | x > 0\}$, $\Omega_3 := \{z \in \mathbf{R}^2 | y < 0\}$, $\Omega_4 := \{z \in \mathbf{R}^2 | y > 0\}$.

In [22] 2.9 the result is proved in case $\omega = \pi$. In this case $\mathcal{P}$ is such that, if $supp(f) \subseteq \Omega \cap \overline{\Omega}_1$, even $supp(\mathcal{P}f) \subseteq \overline{\Omega}_1$. To handle the general case, owing to linear diffeomorphisms, it suffices to consider the cases $\omega = \frac{\pi}{2}$ and $\omega = \frac{3\pi}{2}$.

In case $\omega = \frac{\pi}{2}$, let $f \in \bigcup_{\sigma > -r-1} C^\sigma(\Omega)$ and $\tilde{f} \in \bigcup_{\sigma > -r-1} C^\sigma(\Omega_2)$ such that $\tilde{f}_{|\Omega} = f$. Owing to the case $\omega = \pi$, $\tilde{f}$ can be extended to an element g of $\bigcup_{\sigma > -r-1} C^\sigma(\mathbf{R}^2)$ and the restriction of g to Ω_4 does not depend on the choice of $\tilde{f}$. If $f \in C^\sigma(\Omega)$ for some $\sigma \in]-r-1, r[$ which is not a negative integer, then we can choose $\tilde{f} \in C^\sigma(\Omega_2)$, so that the restriction of g to Ω_4 belongs to $C^\sigma(\Omega_4)$. Applying now again the case $\omega = \pi$ to the restriction of g to Ω_4, we obtain the result in case $\omega = \frac{\pi}{2}$.

We consider now the case $\omega = \frac{3\pi}{2}$. Let $f \in C^\sigma(\Omega)$, with $-r-1 < \sigma < r$ and σ which is not a negative integer. Then, the restriction h of f to Ω_1

belongs to $C^\sigma(\Omega_1)$. We indicate with v the extension of h to $\mathbf{R}^2$ obtained employing the result in case $\omega = \pi$. Consider now the restriction of $f - v$ to Ω_4. Its support is contained in $\overline{\Omega}_2 \cap \Omega_4$. We can extend it to an element z of $C^\sigma(\mathbf{R}^2)$ with support in $\overline{\Omega}_2$. Then $v + z$ is the desired extension of f.

Remark 2.1 By local charts the result of proposition 2.2 can be extended to any open subset O of $\mathbf{R}^2$ whose boundary satisfies one of the conditions a) - e).

Let $\sigma \in \mathbf{R}, \sigma \geq 0$; we set

$$C_0^\sigma(\Omega) := \{f \in C^\sigma(\Omega) | f_{|\partial\Omega} = 0\}. \tag{2.5}$$

Clearly $C_0^\sigma(\Omega)$ is a closed subspace of $C^\sigma(\Omega)$.

Corollary 2.1 *Let σ_0 and σ_1 be real numbers which are not negative integers with $\sigma_0 < \sigma_1$, $\theta \in]0,1[$, $\sigma := (1-\theta)\sigma_0 + \theta\sigma_1$; then,*

(I) if $\sigma \notin \mathbf{Z}$,

$$(C^{\sigma_0}(\Omega), C^{\sigma_1}(\Omega))_{\theta,\infty} = C^\sigma(\Omega)$$

with equivalent norms;

(II) if σ is not a negative integer, there exists $C > 0$ such that for every $f \in C^{\sigma_1}(\Omega)$

$$||f||_{\sigma,\Omega} \leq C ||f||_{\sigma_0,\Omega}^{1-\theta} ||f||_{\sigma_1,\Omega}^{\theta}.$$

(III) Let $0 \leq \sigma_0 < \sigma_1$, $\theta \in]0,1[$ such that $\sigma := (1-\theta)\sigma_0 + \theta\sigma_1 \notin \mathbf{N}$. Then,

$$(C^{\sigma_0}(\partial\Omega), C^{\sigma_1}(\partial\Omega))_{\theta,\infty} = C^\sigma(\partial\Omega).$$

(IV) Let $0 \leq \sigma_0 < \sigma_1$, $\theta \in]0,1[$. Then, there exists $C > 0$ such that for every $f \in C^{\sigma_1}(\partial\Omega)$

$$||f||_{\sigma,\partial\Omega} \leq C ||f||_{\sigma_0,\partial\Omega}^{1-\theta} ||f||_{\sigma_1,\partial\Omega}^{\theta}.$$

(V) Let $0 \leq \sigma_0 < \sigma_1$, $\theta \in]0,1[$ such that $\sigma := (1-\theta)\sigma_0 + \theta\sigma_1 \notin \mathbf{N}$; then

$$(C_0^{\sigma_0}(\Omega), C_0^{\sigma_1}(\Omega)_{\theta,\infty} = C_0^\sigma(\Omega).$$

(VI) Let $\sigma_0, \sigma_1, \sigma$ be as in (V); then

$$(C^{\sigma_0}(\Omega), C_0^{\sigma_1}(\Omega)_{\theta,\infty} = C_0^\sigma(\Omega).$$

Proof The results corresponding to (I) and (II) with $\mathbf{R}^n$ replacing Ω are well known (see [22] 2.4.2, [11] proposition 1.6). The case we are interested in follows from the case of $\mathbf{R}^2$, (2.4) and proposition 2.2.
Concerning (III), the trace operator is surjective from $C^{\sigma_j}(\Omega)$ to $C^{\sigma_j}(\partial\Omega)$. This gives the inclusion

$$C^{\sigma}(\partial\Omega) \subseteq (C^{\sigma_0}(\partial\Omega), C^{\sigma_1}(\partial\Omega))_{\theta,\infty}.$$

The opposite inclusion follows from proposition 2.1.

(IV) follows from (III) if $\sigma \notin \mathbf{N}$ and can be easily deduced from the well known version of (II) in case Ω is replaced by a halfline.
We show (V): let $f \in (C_0^{\sigma_0}(\Omega), C_0^{\sigma_1}(\Omega))_{\theta,\infty}$; then $f \in (C^{\sigma_0}(\Omega), C^{\sigma_1}(\Omega))_{\theta,\infty}$ $= C^{\sigma}(\Omega)$ by (I). Moreover, for every $t > 0$ and $j \in \{0,1\}$ there exists $v_j(t) \in C_0^{\sigma_j}(\Omega)$ such that

$$f = v_0(t) + v_1(t).$$

It follows that for every $t > 0$

$$f_{|\partial\Omega} = v_0(t)_{|\partial\Omega} + v_1(t)_{|\partial\Omega} = 0.$$

On the other hand, let $f \in C_0^{\sigma}(\Omega)$. By (I) for every $t > 0$ and $j \in \{0,1\}$ there exists $v_j(t) \in C^{\sigma_j}(\Omega)$ such that

$$f = v_0(t) + v_1(t),$$

and

$$||v_0(t)||_{\sigma_0,\Omega} \leq Ct^{\theta}, ||v_1(t)||_{\sigma_1,\Omega} \leq Ct^{\theta-1},$$

with $C > 0$ independent of t. Set $u_j(t) := v_j(t) - \mathcal{E}v_j(t)_{|\partial\Omega}$ for $j \in \{0,1\}, t > 0$. As for every $t > 0$

$$\mathcal{E}v_0(t)_{|\partial\Omega} + \mathcal{E}v_1(t)_{|\partial\Omega} \equiv 0,$$

we have that for every $t > 0$, applying proposition 2.1, $f = u_0(t) + u_1(t), u_j(t)$ $\in C_0^{\sigma_j}(\Omega)$, $||u_0(t)||_{\sigma_0,\Omega} \leq C't^{\theta}$, $||u_1(t)||_{\sigma_1,\Omega} \leq C't^{\theta-1}$, with $C' > 0$ independent of t.

We prove (VI): the inclusion from right to left follows from (V). On the other hand let $f \in (C^{\sigma_0}(\Omega), C_0^{\sigma_1}(\Omega)_{\theta,\infty}$. Then $f \in C^{\sigma}(\Omega)$ by (I). Moreover, for every $t > 0$ there exist $v_0(t) \in C^{\sigma_0}(\Omega)$, $v_1(t) \in C_0^{\sigma_1}(\Omega)$, $C > 0$ such that $f = v_0(t) + v_1(t)$ for every $t > 0$, $||v_0(t)||_{\sigma_0,\Omega} \leq Ct^{\theta}, ||v_1(t)||_{\sigma_1,\Omega} \leq Ct^{\theta-1}$. We have $f_{|\partial\Omega} = v_0(t)_{|\partial\Omega}$ for every $t > 0$, so that $||f_{|\partial\Omega}||_{\sigma_0,\partial\Omega} \leq Ct^{\theta}$ for every $t > 0$, which implies $f_{|\partial\Omega} = 0$.

We shall use the following result concerning Fourier multipliers:

Theorem 2.1 *Let $n \in \mathbf{N}$, $\alpha, s \in \mathbf{R}$. Then there exist $N \in \mathbf{N}$ and $c > 0$ such that, if $m \in \mathcal{C}^{\infty}(\mathbf{R}^n)$ and*

$$||m||_{N,s} := \max_{|\beta| \leq N} \sup_{\tau \in \mathbf{R}^n} (1+|\tau|)^{s+|\beta|} |\partial_\tau^\beta m(\tau)| < +\infty,$$

for every $f \in B^{\alpha}_{\infty,\infty}(\mathbf{R}^n)$

$$||\mathcal{F}^{-1}(m\mathcal{F}f)||_{B^{\alpha+s}_{\infty,\infty}(\mathbf{R}^n)} \leq c||m||_{N,s}||f||_{B^{\alpha}_{\infty,\infty}(\mathbf{R}^n)}.$$

Proof See [11], proposition 1.5.

Proposition 2.3 *Let O be an open bounded subset of $\mathbf{R}^2$, lying on one side of its boundary ∂O, which is a submanifold of $\mathbf{R}^n$ of class $\mathcal{C}^{\infty}$ and dimension 1. Let $\sigma \in \mathbf{R} - \mathbf{Z}$, $\sigma > -2$; then for every $f \in C^{\sigma}(O)$, $g \in C^{2+\sigma}(\partial O)$ there exists a unique $u \in C^{2+\sigma}(O)$ such that*

$$\left\{ \begin{array}{c} \Delta u(z) = f(z) \quad z \in O, \\ u_{|\partial O} = g. \end{array} \right. \tag{2.6}$$

If $f \in C^{\sigma}(O) \cap H^1(O)$ and $g \in C^{2+\sigma}(\partial O) \cap H^{\frac{1}{2}}(partialO)$, then $u \in C^{2+\sigma}(O) \cap H^1(O)$.

Proof It follows from [11], proposition 2.9.

Proposition 2.4 *Let O be an open subset of $\mathbf{R}^2$ whose boundary satisfies one of the conditions a)-e), $\sigma \in \mathbf{R} - \mathbf{Z}$. Then for every $f \in C^{\sigma}(O)$ there exists a sequence $(f_k)_{k \in \mathbf{N}}$ in $C^{\infty}(O)$ which is bounded in $C^{\sigma}(O)$ and such that for every $\epsilon > 0$, if $\sigma - \epsilon$ is not a negative integer,*

$$\lim_{k \to \infty} ||f_k - f||_{\sigma-\epsilon, O} = 0.$$

Proof Owing to remark 2.1, it suffices to consider the case $O = \mathbf{R}^2$. Let $\phi \in \mathcal{D}(\mathbf{R}^2)$ be such that $\int_{\mathbf{R}^2} \phi(z)dz = 1$. For $k \in \mathbf{N}$ set

$$f_k := \mathcal{F}^{-1}(\hat{\phi}(\frac{\cdot}{k})\hat{f}).$$

It is easily seen that the Fourier multipliers $\phi(\frac{\cdot}{k})$ are such that for every $N \in \mathbf{N}$

$$||\hat{\phi}(\frac{\cdot}{k})||_{N,0} \leq C(N).$$

So the sequence $(f_k)_{k \in \mathbf{N}}$ is bounded in $C^{\sigma}(\mathbf{R}^2)$, owing to theorem 2.1. By [11], proposition 1.18, $(f_k)_{k \in \mathbf{N}}$ is with values in $C^{\infty}(\mathbf{R}^2)$ and converges to f in $B^{\sigma-\epsilon}_{\infty,q}(\mathbf{R}^2)$ for every $q < +\infty$. So the conclusion follows from (say) $B^{\sigma-\epsilon}_{\infty,1}(\mathbf{R}^2) \subseteq C^{\sigma-\epsilon}(\mathbf{R}^2)$ (see also [22], 2.3). .

3 Elliptic problems in an angle

In order to study elliptic problems in Ω, we consider certain elliptic problems in a strip: we set

$$S := \mathbf{R}\times]0,\omega[.$$

Of course, we indicate with Δ the classical Laplace operator.

If g is a map of domain ∂S, we set, for $t \in \mathbf{R}$,

$$g_0(t) := g(t,0), g_1(t) := g(t,\omega).$$

Lemma 3.1 *Consider the problem*

$$\left\{ \begin{array}{lr} \Delta u(t,\theta) + 2\sigma\frac{\partial u}{\partial t}(t,\theta) + \sigma^2 u(t,\theta) = 0, & (t,\theta) \in S, \\ u(t,\theta) = g(t,\theta), & (t,\theta) \in \partial S \end{array} \right. \tag{3.1}$$

with $\sigma \in \mathbf{R}$, $\frac{\omega\sigma}{\pi} \notin \mathbf{Z}$. Let $\eta > 0$, $\eta \notin \mathbf{N}$. Then, if g_0 and g_1 belong to $C^\eta(\mathbf{R})$, problem (3.1) has a solution u belonging to $C^\eta(S)$.

Proof Using the Fourier transform with respect to the $t-$variable, we get

$$\left\{ \begin{array}{lr} \partial_\theta^2 \hat{u}(\tau,\theta) - \tau^2 \hat{u}(\tau,\theta) + 2i\sigma\tau\hat{u}(\tau,\theta) + \sigma^2\hat{u}(\tau,\theta) = 0, & (\tau,\theta) \in S, \\ \hat{u}(\tau,0) = \hat{g}_0(\tau), \hat{u}(\tau,\omega) = \hat{g}_1(\tau), & \tau \in \mathbf{R}. \end{array} \right. \tag{3.2}$$

Problem (3.2) has the solution

$$\hat{u}(\tau,\theta) = \hat{K}_0(\tau,\theta)\hat{g}_0(\tau) + \hat{K}_1(\tau,\theta)\hat{g}_1(\tau), \tag{3.3}$$

where, for $(\theta,\tau) \in \mathbf{R}\times]0,\omega[$,

$$\hat{K}_0(\tau,\theta) = \frac{\sinh((\omega-\theta)(\tau - i\sigma))}{\sinh(\omega(\tau - i\sigma))}, \tag{3.4}$$

$$\hat{K}_1(\tau,\theta) = \frac{\sinh(\theta(\tau - i\sigma))}{\sinh(\omega(\tau - i\sigma))}. \tag{3.5}$$

Observe that, owing to the assumptions made on σ

$$\sinh(\omega(\tau - i\sigma)) \neq 0$$

for every $\tau \in \mathbf{R}$. It is also easily seen that for every $\theta \in]0,\omega[$ and $j \in \{0,1\}$ $\hat{K}_j(.,\theta) \in \mathcal{S}(\mathbf{R})$, so that, if we define

$$K_j(.,\theta) := \mathcal{F}_\tau^{-1}\hat{K}_j(.,\theta),$$

we have that $K_j(.,\theta) \in \mathcal{S}(\mathbf{R})$.

We put

$$u(t,\theta) := \int_{\mathbf{R}} K_0(t-s,\theta) g_0(s) ds + \int_{\mathbf{R}} K_1(t-s,\theta) g_1(s) ds.$$

Then, it is easily seen that $u \in \mathcal{C}^\infty(S)$ and solves the first equation in (3.1). Moreover,

$$\hat{K}_0(\tau,\theta) = \exp(-\theta sgn(\tau)(\tau - i\sigma)) + r(\tau,\theta) \tag{3.6}$$

with

$$|\frac{\partial^k r}{\partial \tau^k}(\tau,\theta)| \leq C(k,\sigma) exp((\theta - 2\omega)|\tau|) \tag{3.7}$$

for every $k \in \mathbf{N} \cup \{0\}$ if $|\tau| \geq 1$. It follows from (3.6) and (3.7) that for every $k \in \mathbf{N} \cup \{0\}$, $(\tau,\theta) \in \mathbf{R} \times [0,\omega]$

$$|\frac{\partial^k \hat{K}_0}{\partial \tau^k}(\tau,\theta)| \leq C(k,\sigma)(1+|\tau|)^{-k}. \tag{3.8}$$

In fact, for $|\tau| \geq 1$

$$|\frac{\partial^k (\exp(-\theta sgn(\tau)(\tau - i\sigma))}{\partial \tau^k}| = (\theta|\tau|)^k \exp(-\theta|\tau|)|\tau|^{-k}$$

$$\leq C(k)(1+|\tau|)^{-k}.$$

Analogous estimates hold for K_1.

Now we verify that $u \in C^\eta(S)$. We start by considering the case $\eta \in]0,1[$. Owing to theorem 2.1 and (3.8), the map $\theta \to u(.,\theta)$ belongs to $B([0,\omega]; C^\eta(\mathbf{R}))$. Moreover, for $\theta \in]0,\omega[$,

$$\partial_\theta u(t,\theta) = \mathcal{F}_\tau^{-1}(\partial_\theta \hat{K}_0(.,\theta)\hat{g}_0) + \mathcal{F}_\tau^{-1}(\partial_\theta \hat{K}_1(.,\theta)\hat{g}_1)$$

with

$$\partial_\theta \hat{K}_0(\tau,\theta) = -\frac{\cosh((\omega-\theta)(\tau - i\sigma))}{\sinh(\omega(\tau - i\sigma))}(\tau - i\sigma),$$

$$\partial_\theta \hat{K}_1(\tau,\theta) = \frac{\cosh(\theta(\tau - i\sigma))}{\sinh(\omega(\tau - i\sigma))}(\tau - i\sigma).$$

It is easily seen that for every $\theta \in [0,\omega]$ $\partial_\theta \hat{K}_0(\tau,\theta)$ and $\partial_\theta \hat{K}_1(\tau,\theta)$ satisfy the assumptions of theorem 2.1 with $s = -1$ uniformly in θ. It follows from theorem 2.1 that $\theta \to \partial_\theta u(.,\theta)$ belongs to $B([0,\omega]; C^{\eta-1}(\mathbf{R}))$ and so $\theta \to u(.,\theta)$ is lipschitz with values in $C^{\eta-1}(\mathbf{R})$). As $C(\mathbf{R}) \in J_{1-\eta}(C^{\eta-1}(\mathbf{R}), C^\eta(\mathbf{R}))$, we

have that $\theta \to u(.,\theta)$ belongs to $C^\eta([0,\omega];C(\mathbf{R}))$. Moreover, it is clear that $u(.,0) = g_0$ and $u(.,\omega) = g_1$.
So the case $\eta \in]0,1[$ is complete. Assume the result true for some $\eta > 0$, $\eta \notin \mathbf{N}$. We extend it to $\eta + 1$. Let g_0 and g_1 belong to $C^{\eta+1}(\mathbf{R})$. Then $u \in C^\eta(S)$. We have to show that even $\partial_t u$ and $\partial_\theta u$ are in $C^\eta(S)$. In fact, if $\theta \in [0,\omega]$,

$$\partial_t u(.,\theta) = K_0(.,\theta) * \partial_t g_0 + K_1(.,\theta) * \partial_t g_1$$

and

$$\partial_\theta u(.,\theta) = H_0(.,\theta) * h_0 + H_1(.,\theta) * h_1$$

with

$$H_0(.,\theta) = \mathcal{F}_\tau^{-1}(-\frac{\cosh((\omega-\theta)(\tau - i\sigma))}{\sinh(\omega(\tau - i\sigma))}),$$

$$H_1(.,\theta) = \mathcal{F}_\tau^{-1}(-\frac{\cosh(\theta(\tau - i\sigma))}{\sinh(\omega(\tau - i\sigma))}),$$

$$h_j = \mathcal{F}_\tau^{-1}((\tau - i\sigma)\hat{g}_j)$$

for $j = 0,1$. Owing to theorem 2.1, h_0 and h_1 belong to $C^\eta(\mathbf{R})$ and one can verify that H_0 and H_1 satisfy estimates of the form of (3.8). So even $\partial_t u$ and $\partial_\theta u$ are in $C^\eta(S)$.

Given an open subset O of $\mathbf{R}^n$ and $\sigma \geq 0$, we put

$$P^\sigma(O) := \{f \in C^\sigma(O) | |z|^{|\alpha|-\sigma}\partial_z^\alpha f \text{ is bounded in } O - \{0\} \text{ if } |\alpha| \leq \sigma\}.$$

The following characterization of $P^\sigma(O)$ is very elementary:

Lemma 3.2 *Let $\sigma \geq 0$ and O such that $0 \in \overline{O}$ and for every $z \in O$ and $t \in]0,1]$ $tz \in O$. Then*

$$P^\sigma(O) = \{f \in C^\sigma(O) | \partial_z^\alpha f(0) = 0 \text{ if } |\alpha| < \sigma\}.$$

$P^\sigma(O)$ is a closed subspace of $C^\sigma(O)$.

If $r > 0$, we set

$$\Omega_r := \{z \in \Omega | |z| < r\},$$

$$S_0 :=]-\infty, 0[\times]0,\omega[.$$

We have:

Lemma 3.3 *(I) Let $f \in P^\sigma(\Omega_1)$ with $\sigma > 0, \sigma \notin \mathbf{N}$. Define, for $(t,\theta) \in S_0$*

$$\phi(t,\theta) := \exp(-\sigma t) f(\exp(t+i\theta)).$$

Then the map $f \to \phi$ is a linear and topological isomorphism between $P^\sigma(\Omega_1)$ and $C^\sigma(S_0)$. The inverse isomorphism is the map $\phi \to |z|^\sigma \phi(\log(.))$, with $\log(\exp(t+i\theta)) = t + i\theta$ for every $(t,\theta) \in S_0$.

(II) Let $g \in P^\sigma(]0,1[)$ with $\sigma \geq 0$. Set, for $t < 0$,

$$\psi_0(t) := \exp(-\sigma t) g(\exp(t)).$$

Then $\psi \in C^\sigma(]-\infty, 0[)$.

Proof Elementary.

Corollary 3.1 *Let $\sigma > 0, \sigma \notin \mathbf{N}, \frac{\sigma\omega}{\pi} \notin \mathbf{N}$, $g_0, g_1 \in P^\sigma(]0,1[$; then there exists $v \in C^\sigma(\Omega_1)$ such that*

$$\left\{ \begin{array}{cc} \Delta v = 0 & \textit{in } \Omega_1, \\ v(\rho,0) = g_0(\rho), v(\rho e^{i\omega}) = g_1(\rho), & \rho \in]0,1[\end{array} \right. \tag{3.9}$$

Proof Set, for $j = 0, 1$,$t < 0$,

$$h_j(t) := \exp(-\sigma t) g_j(\exp(t)).$$

By lemma 3.3(II) $h_j \in C^\sigma(]-\infty, 0[)$ and, owing to [22] 2.9, it can be extended to an element $\tilde{h}_j \in C^\sigma(\mathbf{R})$. Consider the problem

$$\left\{ \begin{array}{cc} \Delta v + 2\sigma \frac{\partial v}{\partial t} + \sigma^2 v = 0 & \text{in } S \\ v(t,0) = \tilde{h}_0(t), v(t,\omega) = \tilde{h}_1(t), & t \in \mathbf{R}. \end{array} \right. \tag{3.10}$$

Owing to the assumptions made on ω and σ and lemma 3.1, this problem has a solution $v \in C^\sigma(S)$. If we define, for $z \in \Omega_1$,

$$u(z) := |z|^\sigma v(\log z),$$

we can easily verify that u is a solution of (3.9) and belongs also to $P^\sigma(\Omega_1)$ by lemma 3.3(I).

Lemma 3.4 *Let*

$$g_0(\rho) = \sum_{k=0}^{l} c_k \rho^k, g_1(\rho) = \sum_{k=0}^{l} d_k \rho^k$$

be polynomials of degree less or equal to l, with $l \in \mathbf{N} \cup \{0\}$. Then the following conditions are equivalent:

(I) there exists a harmonic polynomial p of degree less or equal to l such that

$$p(\rho, 0) = g_0(\rho), p(\rho e^{i\omega}) = g_1(\rho) \tag{3.11}$$

for every $\rho \geq 0$;

(II) for every $k \in \mathbf{N} \cup \{0\}, k \leq l$ such that $\sin(k\omega) = 0$ one has

$$d_k = c_k cos(k\omega). \tag{3.12}$$

Proof If $p(x,y) = \sum_{k=0}^{l} \sum_{i=0}^{k} c_{k-i,i} x^{k-i} y^i$, p is harmonic if and only if

$$c_{k-i,i} = (-1)^{\frac{i}{2}} \binom{k}{i} c_{k,0} \quad \text{for } k \geq 2 \text{ and } i \text{ even}, \tag{3.13}$$

$$c_{k-i,i} = \frac{(-1)^{\frac{i-1}{2}}}{k} \binom{k}{i} c_{k-1,1} \quad \text{for } k \geq 2 \text{ and } i \text{ odd}. \tag{3.14}$$

Imposing the first condition in (3.11), we obtain from (3.13) and (3.14)

$$p(x,y) = c_0 + c_1 x + c_{0,1} y + \sum_{k=2}^{l} \Big(\sum_{\substack{0 \leq i \leq k, \\ i \text{ even}}} (-1)^{\frac{i}{2}} \binom{k}{i} c_k x^{k-i} y^i$$

$$+ \sum_{\substack{0 \leq i \leq k, \\ i \text{ odd}}} \frac{(-1)^{\frac{i-1}{2}}}{k} \binom{k}{i} c_{k-1,1} x^{k-i} y^i \Big)$$

$$= c_0 + \sum_{k=1}^{l} c_k Re\{(x+iy)^k\} + \sum_{k=1}^{l} \frac{c_{k-1,1}}{k} Im\{(x+iy)^k\}.$$

It follows

$$p(\rho e^{i\omega}) = c_0 + \sum_{k=1}^{l} c_k \rho^k \cos(k\omega) + \sum_{k=1}^{l} \frac{c_{k-1,1}}{k} \rho^k \sin(k\omega),$$

which implies the statement.

Lemma 3.5 *Let $l \in \mathbf{N}$, with $l \geq 2$, and $u \in C^l(\Omega_1)$ be such that*

$$\Delta u(z) = f(z), z \in \Omega_1, \tag{3.15}$$

$$u(\rho, 0) = g_0(\rho), u(\rho e^{i\omega}) = g_1(\rho), \rho \in]0,1[. \tag{3.16}$$

Then $f \in C^{l-2}(\Omega_1)$, g_0 and g_1 belong to $C^l(]0,1[)$ and for every $m \in \mathbf{N}$ such that $2 \leq m \leq l$ and $\sin(m\omega) = 0$ one has

$$\sum_{r=0}^{m-2}(\sum_{\substack{r+2 \leq j \leq m, \\ j-r \text{ even}}} (-1)^{\frac{j-r}{2}} \binom{m}{j} \cos^{m-j}(\omega)\sin^j(\omega)) \frac{\partial^{m-2} f}{\partial x^{m-2-r}\partial y^r}(0,0)$$

$$= \cos(m\omega) g_0^{(m)}(0) - g_1^{(m)}(0). \tag{3.17}$$

Proof The first statement is obvious.

Concerning the second, one has, using (3.15) and (3.16), with elementary computations,

$$g_1^{(m)}(0) == \cos(m\omega) g_0^{(m)}(0) + \sin(m\omega) \frac{\partial^m u}{\partial x^{m-1}\partial y}(0,0)$$

$$- \sum_{r=0}^{m-2}(\sum_{\substack{r+2 \leq j \leq m, \\ j-r \text{ even}}}$$

$$(-1)^{\frac{j-r}{2}} \binom{m}{j} \cos^{m-j}(\omega)\sin^j(\omega)) \frac{\partial^{m-2} f}{\partial x^{m-2-r}\partial y^r}(0,0).$$

This implies (3.17).

Let now $k \in \mathbf{N}$. If $\frac{k\pi}{\omega} \notin \mathbf{N}$, we set, for $0 < \rho < 1, 0 < \theta < \omega$,

$$\phi_k(\rho e^{i\theta}) = \rho^{\frac{k\pi}{\omega}} \sin(\frac{k\pi\theta}{\omega}). \tag{3.18}$$

We observe that, as $\frac{k\pi}{\omega} \notin \mathbf{N}$, $\phi_k \in C^{\frac{k\pi}{\omega}}(\Omega_1) - \bigcup_{\epsilon>0} C^{\frac{k\pi}{\omega}+\epsilon}(\Omega_1)$.

Lemma 3.6 *Let $u \in C(\Omega_1) + H^1(\Omega_1)$ be such that*

$$\begin{cases} \Delta u = 0 & in \Omega_1, \\ u_{|\partial\Omega_1 \cap \partial\Omega} = 0. \end{cases} \tag{3.19}$$

Let $\sigma > 0$; then u is of the form

$$u(\rho e^{i\theta}) = \sum_{k\in\mathbf{N}, \frac{k\pi}{\omega}<\sigma, \frac{k\pi}{\omega}\notin\mathbf{N}} c_k \phi_k(\rho e^{i\theta}) + v(\rho e^{i\theta}), \tag{3.20}$$

with $c_k \in \mathbf{C}$ and $v \in C^\sigma(\Omega_1)$.

Proof See [8] 4.4.2.

We define now ϕ_k in case $\frac{k\pi}{\omega} \in \mathbf{N}$. Observe that, as we have excluded the case $\omega = \pi$, $\frac{k\pi}{\omega} \geq 2$. If $0 < \rho < 1$ and $0 < \theta < \omega$, we set, if $\frac{k\pi}{\omega} = m$, with $m \in \mathbf{N}$, $m \geq 2$,

$$\phi_k(\rho e^{i\theta}) = \frac{\sin^2(\omega)}{2(m-1)\omega}\rho^m \ln\rho \sin(m\theta)$$

$$+\frac{\rho^m}{2(m-1)}[\sin((m-1)\theta)\sin(\theta) + \frac{\sin^2(\omega)}{\omega}\theta\cos(m\theta)]. \tag{3.21}$$

Lemma 3.7 *Let $k \in \mathbf{N}$ be such that $m := \frac{k\pi}{\omega} \in \mathbf{N}$, $m \geq 2$; then*
(I) $\phi_k \in \bigcap_{\sigma<m} C^\sigma(\Omega_1) - C^m(\Omega_1)$;
(II) $\Delta\phi_k = f_k$ in Ω_1, where

$$f_k(\rho e^{i\theta}) = \rho^{m-2}\cos((m-2)\theta);$$

(III) if $0 < \rho < 1$,

$$\phi_k(\rho, 0) = \phi_k(\rho e^{i\omega}) = 0;$$

(IV) f_k is a polynomial and the compatibility condition (3.17) is not satisfied.

Proof The statements (I)-(III) can be easily checked. Concerning (IV), we have

$$\sum_{r=0}^{m-2}\Big(\sum_{\substack{r+2\leq j\leq m,\\ j-r \text{ even}}} (-1)^{\frac{j-r}{2}}\binom{m}{j}\cos^{m-j}(\omega)\sin^j(\omega)\Big)\frac{\partial^{m-2}f_k}{\partial x^{m-2-r}\partial y^r}(0,0)$$

$$= \frac{(m-2)!}{2}\sum_{\substack{1\leq j\leq m,\\ j \text{ even}}}\binom{m}{j} j\cos^{m-j}(\omega)(i\sin(\omega))^j. \tag{3.22}$$

If $\cos(\omega) = 0$, then $\omega \in \{\frac{\pi}{2}, \frac{3\pi}{2}\}$. As $\sin(m\omega) = 0$, necessarily m is even. So (3.22) equals $\frac{(m-2)!}{2} m(i\sin(\omega))^m$ which is different from 0.

If $\cos(\omega) \neq 0$, then (3.22) equals

$$\frac{(m-2)!}{2}\cos^m(\omega) \sum_{\substack{1 \le j \le m, \\ j \text{ even}}} \binom{m}{j} j(i\tan(\omega))^j$$

$$= \frac{(m-2)!}{2} m\cos^m(\omega)\tan(\omega) Re\{i(1+i\tan\omega)^{m-1}\}$$

$$= -\frac{(m-2)!}{2} m\sin(\omega)\sin((m-1)\omega)$$

$$= \frac{(m-2)!}{2} m\sin^2(\omega)\cos(m\omega) \neq 0.$$

Proposition 3.1 *(I) Let $\sigma > -2$, $\sigma \notin \mathbf{Z}$, $\frac{(\sigma+2)\omega}{\pi} \notin \mathbf{N}$; next, let $f \in C^\sigma(\Omega)$, $g \in C^{\sigma+2}(\partial\Omega)$. Consider problem*

$$\begin{cases} \Delta u = f \ \ in \ \Omega_1 \\ (u-g)_{|\partial\Omega\cap\partial\Omega_1} = 0 \end{cases} \tag{3.23}$$

Then (3.23) has a solution of the form

$$u = v + \sum_{\substack{k \in \mathbf{N} \\ \frac{k\pi}{\omega} < 2+\sigma, \ \frac{k\pi}{\omega} \in \mathbf{N}}} c_k \phi_k,$$

with $v \in C^{2+\sigma}(\Omega_1)$, $c_k \in \mathbf{C}$ for every k;

(II) if, moreover, for every $k \in \mathbf{N}$ such that $\frac{k\pi}{\omega} \in \mathbf{N}$ and $\frac{k\pi}{\omega} < 2+\sigma$ the compatibility condition (3.17) is satisfied with $m = \frac{k\pi}{\omega}$, then problem (3.23) has a solution in $C^{2+\sigma}(\Omega_1)$;

(III) if $u \in C(\Omega_1) + H^1(\Omega)$ and solves problem (3.23), then it is of the form

$$u = v + \sum_{\substack{k \in \mathbf{N} \\ \frac{k\pi}{\omega} < 2+\sigma}} c_k \phi_k,$$

with $v \in C^{2+\sigma}(\Omega_1)$.

Proof Assume first that $\sigma > 0$ and there exists $k \in \mathbf{N}$ such that $\frac{k\pi}{\omega} < 2+\sigma$ and $\frac{k\pi}{\omega} \in \mathbf{N}$. Then, by lemma 3.7(IV), there exists $c_k \in \mathbf{C}$ such that, if we replace f with $f - c_k f_k$, condition (3.17) with $m = \frac{k\pi}{\omega}$is satisfied. In the remaining part of the proof we shall assume that condition (3.17) is satisfied for every $k \in \mathbf{N}$ such that $\frac{k\pi}{\omega} < 2 + \sigma$ and $\frac{k\pi}{\omega} \in \mathbf{N}$, with $m = \frac{k\pi}{\omega}$.

We observe now that it suffices to consider the case $f \equiv 0$; in fact, otherwise, by proposition 2.3, we can extend f to an element $\tilde{f}$ belonging to $C^{\sigma}(B_1)$; by proposition 2.3 there exists $\tilde{u} \in C^{2+\sigma}(B_1)$ such that $\Delta\tilde{u} = \tilde{f}$ in B_1. Subtracting the restriction of $\tilde{u}$ to Ω_1 one is reduced to the case $f \equiv 0$. This is the case we shall consider in the remaining part of the proof.

Observe now that, by lemma 3.4, there exists a harmonic polynomial p such that, for every $\rho \geq 0$,

$$p(\rho, 0) = \sum_{k \in \mathbf{N_0}, k < 2+\sigma} \frac{g_0^{(k)}(0)}{k!}\rho^k, p(\rho e^{i\omega}) = \sum_{k \in \mathbf{N_0}, k < 2+\sigma} \frac{g_1^{(k)}(0)}{k!}\rho^k.$$

Subtracting p, one gets (I) and (II) from corollary 3.1. (III) follows immediately from (I) and lemma 3.6.

We introduce now certain notations and definitions: first of all, we fix $\chi \in \mathcal{D}(\mathbf{R}^2)$ such that $\chi(z) = 1$ for z in some neighbourhood of $(0, 0)$. Next, we set, for $k \in \mathbf{N}$,

$$R_k(z) := \phi_k(z)\chi(z). \tag{3.24}$$

Then,

Definition 3.1 *Let O be an open subset of $\mathbf{R}^2$ such that $\Omega_1 \subseteq O \subseteq \Omega$; we set, for $\sigma > 0$,*

$$\tilde{C}^{\sigma}(O) := \begin{cases} C^{\sigma}(O) \bigoplus V(\{R_k | k \in \mathbf{N}, \frac{k\pi}{\omega} < \sigma\}) & if \quad \sigma \notin \mathbf{N}, \\ C^{\sigma}(O) \bigoplus V(\{R_k | k \in \mathbf{N}, \frac{k\pi}{\omega} \leq \sigma\}) & if \quad \sigma \in \mathbf{N}. \end{cases}$$

If $u \in \tilde{C}^{\sigma}(O)$ and $u = v + \sum_{\frac{k\pi}{\omega} < \sigma} c_k R_k$ ($u = v + \sum_{\frac{k\pi}{\omega} \leq \sigma} c_k R_k$ if $\sigma \in \mathbf{N}$), we set

$$||u||^*_{\sigma,O} := ||v||_{\sigma,O} + \sum_{\frac{k\pi}{\omega} < \sigma} |c_k|,$$

with an obvious modification if $\sigma \in \mathbf{N}$.

Observe that the coefficients c_k in definition 3.1 are uniquely determined.

Remark 3.1 With the notations introduced in definition 3.1 a careful check of the proof of proposition 3.1 shows that the solution u of problem (3.23) can be chosen as depending linearly and continuously on f and g.

Observe also that, if $u \in \tilde{C}^{2+\sigma}(O)$ for some $\sigma > 0$, then $\Delta u \in C^\sigma(O)$ and $u_{|\partial O} \in C^{2+\sigma}(\partial O)$. If $\sigma < \frac{\pi}{\omega}$, then $\tilde{C}^\sigma(O) = C^\sigma(O)$. This implies that, if $u \in \tilde{C}^{2+\sigma}(O)$, for every $\lambda \in \mathbf{C}$ $(\lambda - \Delta)u \in C^\sigma(O)$.

Lemma 3.8 *Let O be an open subset of $\mathbf{R}^2$ satisfying the conditions of definition 3.1; assume, moreover, that:*

(c) O is bounded;

(d) $\partial O - \{(0,0)\}$ is a $\mathcal{C}^\infty$-submanifold of $\mathbf{R}^2$ and O lies on one side of ∂O.

Let $\sigma \in]-2, \frac{\pi}{\omega}[$, $\sigma \notin \mathbf{Z}$, $\frac{(\sigma+2)\omega}{\pi} \notin \mathbf{N}$. Consider the problem

$$\begin{cases} (\lambda - \Delta)u = f \text{ in } O, \\ u_{|\partial O} = g, \end{cases} \tag{3.25}$$

with $\lambda \in \mathbf{C}-]-\infty, 0[$, $f \in C^\sigma(O), g \in C^{2+\sigma}(\partial O)$.

Then problem (3.25) has a unique solution u belonging to $\tilde{C}^{2+\sigma}(O)$.

Proof We start by showing the following a priori estimate: there exists $C > 0$ such that for every $u \in \tilde{C}^{2+\sigma}(O)$

$$||u||^*_{2+\sigma,O} \leq C[||\Delta u||_{\sigma,O} + ||u_{|\partial O}||_{2+\sigma,\partial O} + ||u||_{\sigma,O}]. \tag{3.26}$$

If u vanishes in some neighbourhood of $(0,0)$, then $u \in C^{2+\sigma}(O)$ and the result is a consequence of [11], proposition 2.9. Therefore, it suffices to prove (3.26) for u vanishing if $|z| \geq \frac{1}{2}$. Owing to proposition 3.1 and remark 3.1, there exists $u_1 \in \tilde{C}^{2+\sigma}(\Omega_1)$ and depending linearly and continuously on Δu and $u_{|\partial O}$ such that

$$\begin{cases} \Delta u_1 = \Delta u \text{ in } \Omega_1, \\ (u_1 - u)_{\partial\Omega\cap\partial\Omega_1} = 0 \end{cases}$$

Consider now $v := u - u_1$. By lemma 3.6 and the open mapping theorem, if $0 < \sigma' < \frac{\pi}{\omega}$, the norms $||.||_{\sigma',\Omega_1}$ and $||.||^*_{2+\sigma,\Omega_1}$ are equivalent in $\{v \in C(\Omega_1) | \Delta v = 0, v_{|\partial\Omega\cap\partial\Omega_1} = 0\}$. It follows that

$$||u||^*_{2+\sigma,O} \leq C(\sigma)[||u_1||^*_{2+\sigma,O} + ||v||^*_{2+\sigma,O}]$$

$$\leq C(\sigma,\sigma')[||\Delta u||_{\sigma,O} + ||u_{|\partial O}||_{2+\sigma,\partial O} + ||v||_{\sigma',\Omega_1}]. \tag{3.27}$$

Choose σ' satisfying the further condition $\sigma' \in]\sigma, 2+\sigma[$ and pick σ'' such that $\sigma' < \sigma'' < \frac{\pi}{\omega}$. Then, by 2.1(II) one has

$$||v||_{\sigma',\Omega_1} \leq C||v||_{\sigma,\Omega_1}^{1-\frac{\sigma'-\sigma}{\sigma''-\sigma}} ||v||_{\sigma'',\Omega_1}^{\frac{\sigma'-\sigma}{\sigma''-\sigma}}$$

$$\leq C(\epsilon)||v||_{\sigma,\Omega_1} + \epsilon||v||_{\sigma'',\Omega_1}$$

$$\leq \epsilon||u||^*_{2+\sigma,O} + C(\epsilon)[||u||_{\sigma,O} + ||u_1||^*_{2+\sigma,O}]$$

$$\leq \epsilon||u||^*_{2+\sigma,O} + C(\epsilon)[||u||_{\sigma,O} + ||\Delta u||_{\sigma,O} + ||u_{|\partial O}||_{2+\sigma,\partial O}]$$

for every $\epsilon > 0$. Using this estimate in (3.27) and choosing $\epsilon > 0$ sufficiently small, we obtain (3.26).

We pass now to prove the result. First of all we observe that, owing to 2.1, it suffices to consider the case $g \equiv 0$. If $f \in H^{-1}(O)$, one can prove by variational methods (see for example [19] ch 2) that problem 3.26 with $g \equiv 0$ has a unique solution $u \in H^1_0(O)$. If $\sigma > -1$, then $f \in H^{-1}(O)$. As in this case $\tilde{C}^{2+\sigma}(O) \subseteq H^1(O)$, we have certainly uniqueness for $\sigma > -1$. To get existence (again in case $\sigma > -1$) we show that the variational solution u of (3.26) with $g \equiv 0$ belongs in fact to $\tilde{C}^{2+\sigma}(O)$. This is a consequence of 3.1(III) if u vanishes in some neighbourhood of $(0,0)$. In general, by Sobolev's embedding theorem (see [22], 2.7.1) $u \in \bigcap_{\eta<0} C^{\eta}(O)$, so that $\Delta u = \lambda u - f \in C^{\sigma}(O)$ if $-1 < \sigma < 0$. Assume that $-1 < \sigma < 0$. By proposition 3.1 and lemma 3.7 the restriction of u to Ω_1 belongs to $\tilde{C}^{2+\sigma}(\Omega_1)$. Let v be an element of $\tilde{C}^{2+\sigma}(\Omega)$ coinciding with u in some neighbourhood of $(0,0)$. Subtracting to u v, we are reduced to the case that u vanishes in some neighbourhood of $(0,0)$, so that the result is proved if $-1 < \sigma < 0$. Let $0 < \sigma < 1$. Applying the case $-1 < \eta < 0$, we can again say that $\lambda u - f \in C^{\sigma}(O)$ and the same argument used previously gives the conclusion if $0 < \sigma < \min\{1, \frac{\pi}{\omega}\}$. Iterating the procedure, one has the result if $\sigma > -1$.

It remains the case $-2 < \sigma < -1$. We start by showing that, if $u \in \tilde{C}^{2+\sigma}(\Omega)$ and solves (3.25) with $f \equiv 0$ and $g \equiv 0$, then $u \equiv 0$. To this aim, it suffices to show that $u \in \tilde{C}^{3+\sigma}(O)$. This can be seen with the previous argument, first proving that the restriction of u to Ω_1 belongs to $\tilde{C}^{3+\sigma}(\Omega_1)$, then considering the case that u vanishes in some neighbourhood of $(0,0)$. To conclude we have only to show that problem (3.25) with $g \equiv 0$ has a solution in $\tilde{C}^{2+\sigma}(O)$ for every $f \in C^{\sigma}(O)$. First we show the following refinement of (3.26):there exists $C > 0$ such that for every $u \in \tilde{C}^{2+\sigma}(O)$

$$||u||^*_{2+\sigma,O} \leq C[||(\Delta-\lambda)u||_{\sigma,O} + ||u_{|\partial O}||_{2+\sigma,\partial O}. \qquad (3.28)$$

To prove (3.28), we argue by contradiction: assume that it is not true. So there exists a sequence $(u_k)_{k\in\mathcal{N}}$ in $\tilde{C}^{2+\sigma}(O)$ such that $||u_k||^*_{2+\sigma,O} = 1$ for every $k \in \mathcal{N}$ and

$$||(\Delta-\lambda)u||_{\sigma,O} + ||u_{|\partial O}||_{2+\sigma,\partial O} \to 0(k \to +\infty). \qquad (3.29)$$

Passing (if necessary) to a subsequence and applying Ascoli-Arzelá's theorem, we can assume that there exists $u \in \tilde{C}^{2+\sigma}(O)$ such that for every $\epsilon > 0$

$$||u_k - u||^*_{2+\sigma-\epsilon} \to 0(k \to +\infty). \qquad (3.30)$$

(3.29) implies that $(\lambda-\Delta)u = 0$ and $u_{|\partial O} = 0$, so that necessarily $u \equiv 0$. So for every $\epsilon > 0$ we have

$$||u_k||^*_{2+\sigma-\epsilon,O} \to 0(k \to +\infty).$$

So, applying (3.26), we conclude that

$$||u_k||^*_{2+\sigma,O} \to 0(k \to +\infty)$$

in contradiction with $||u_k||^*_{2+\sigma,O} = 1$ for every k. Let now $f \in C^\sigma(O)$. Then, by proposition 2.4, there exists a sequence f_k in $C^{\sigma+1}(O)$ which is bounded in $C^\sigma(O)$ and converges to f in $C^{\sigma-\epsilon}(O)$ for every $\epsilon > 0$. As we have already proved our statement for $\sigma+1$, we know that for every k there exists $u_k \in \tilde{C}^{3+\sigma}(O)$ such that $(\lambda-\Delta)u_k = f_k$ and $u_{|\partial O} = 0$. Owing to (3.28), the sequence $(u_k)_{k\in\mathcal{N}}$ is bounded in $\tilde{C}^{2+\sigma}(O)$ and so we can assume it converging to some element $u \in \tilde{C}^{2+\sigma}(O)$ in $\tilde{C}^{2+\sigma-\epsilon}(O)$ for every $\epsilon > 0$. It is immediately seen that u is the desired solution.

Lemma 3.9 *Let O be an open subset of $\mathbf{R}^2$ with ∂O smooth submanifold and O lying on one side of ∂O, $\sigma > -2$ and $\sigma \notin \mathbf{Z}$. Assume that there exists $A > 0$ such that*

$$O \cap \{z \in \mathbf{R}^2 | |z| > A\} = \Omega \cap \{z \in \mathbf{R}^2 | |z| > A\}.$$

Consider problem (3.25) with $f \in C^\sigma(O)$, $g \in C^{\sigma+2}(\partial O)$.

Then, for every $\theta_0 \in [0,\pi[$ there exists $R > 0$ such that, if $\lambda \neq 0$, $|Arg(\lambda)| \leq \theta_0$ and $|\lambda| \geq R$, (3.25) has a unique solution $u \in C^{2+\sigma}(O)$. Moreover, if $f \in H^{-1}(O)$ and $g \in H^{\frac{1}{2}}(\partial O)$, problem (3.25) has a unique solution u in $H^1(O)$.

Proof With the same methods of [11] one can show the following a priori estimate: there exists $C > 0$ such that for every $u \in C^{\sigma+2}(O)$

$$||u||_{2+\sigma,O} \le C(||\Delta u||_{\sigma,O} + ||u_{|\partial O}||_{2+\sigma,\partial O} + ||u||_{\sigma,O}). \tag{3.31}$$

An analogous estimate can be shown replacing O with $O \times \mathbf{R}$ (which is an open subset of $\mathbf{R}^3$). So, with the same method of [11] proposition 2.15, using a well known technique due to Agmon, we obtain, for $|\lambda|$ sufficiently large and $\sigma \in]-2,0[$,

$$|\lambda| ||u||_{\sigma,O} + ||u||_{2+\sigma,O} \le C(||(\lambda - \Delta)u||_{\sigma,O} + ||u_{|\partial O}||_{2+\sigma,\partial O} + |\lambda|^{\frac{2+\sigma}{2}} ||u_{|\partial O}||_{0,\partial O}). \tag{3.32}$$

Estimate 3.32 gives uniqueness for $|\lambda|$ sufficiently large.

Concerning existence, owing to 2.1, it suffices to consider the case $g \equiv 0$. By [11] proposition 2.3 and lemma 2.8, the statement of the lemma is true if $O = \mathbf{R}^2_+$. We set

$$O_1 := \{z \in \mathbf{R}^2 | y > 0\},$$

$$O_2 := \{z \in \mathbf{R}^2 - \{(0,0)\} | arg(z) \cap]\omega - \pi, \omega[\ne \emptyset\}$$

and let O_0 be open, bounded, with smooth boundary, such that

$$O = O_0 \cup ((O_1 \cup O_2) \cap O).$$

For $j \in \{0,1,2\}$ and $\phi \in C^\sigma(O_j)$ let $S_j(\lambda)\phi$ be the solution in $C^{\sigma+2}(O_j)$ of

$$\begin{cases} (\lambda - \Delta)v = \phi \text{ in } O_j, \\ v_{|\partial O_j} = 0. \end{cases} \tag{3.33}$$

For $j = 0,1,2$ let $\phi_j \in C^\infty(O)$ with $supp(\phi_0) \subseteq O_0$, for $j = 1,2$, $supp(\phi_j) \subseteq O \cap O_j$ and $\sum_{j=0}^2 \phi_j(z)^2 = 1$ for every $z \in O$. We look for a solution u of 3.25 of the form

$$u = \sum_{j=0}^{2} \phi_j S_j(\lambda)(\phi_j h), \tag{3.34}$$

for some $h \in C^\sigma(O)$. Then,

$$(\lambda - \Delta)u = h - 2\sum_{j=0}^{2} \nabla\phi_j \cdot \nabla S_j(\lambda)(\phi_j h) - \sum_{j=0}^{2} (\Delta\phi_j) S_j(\lambda)(\phi_j h)$$

$$:= h - T(\lambda)h.$$

We take h in such a way that $h - T(\lambda)h = f$. This is possible if $|\lambda|$ is sufficiently large. In fact, owing to (3.32), if $\sigma < 0$ we have, for $|\lambda|$ sufficiently large and $h \in C^{\sigma}(O)$,

$$||T(\lambda)h||_{\sigma,O} \leq C \sum_{j=0}^{2} ||S_j(\lambda)(\phi_j h)||_{\sigma+1,O} \leq C|\lambda|^{-\frac{1}{2}}||h||_{\sigma,O},$$

so that $I - T(\lambda)$ is invertible in $C^{\sigma}(O)$ if $|\lambda|$ is sufficiently large. Therefore in this case we have a unique solution $u \in C^{\sigma+2}(O)$ for every $f \in C^{\sigma}(O)$ for $\sigma \in]-2, 0[-\{-1\}$. If $f \in C^{\sigma+1}(O)$ for some $\sigma \in]-1, 0[$, then $h = T(\lambda)h + f \in C^{\sigma+1}(O)$, so that u, which is of the form (3.38), belongs to $C^{\sigma+3}(O)$, owing to the corresponding result in the case that O is a half-space or is bounded. Iterating the method, one gets the general statement in the case of $C^{\sigma}(O)$.

The case of $f \in H^{-1}(O)$ and $g \in H^{\frac{1}{2}}(\partial O)$ can be treated reducing oneself to the case $g \equiv 0$ by well known theorems of trace (see [8],1.5) and then applying a standard variational argument.

Lemma 3.10 *Let $\sigma \in]-2, \frac{\pi}{\omega}[-\mathbf{Z}$, such that $\frac{(\sigma+2)\omega}{\pi} \notin \mathbf{N}$; let $u \in \tilde{C}^{\sigma'}(\Omega)$, for some $\sigma' > 0$. Assume that $\Delta u \in C^{\sigma}(\Omega)$ and $u_{|\partial\Omega} \in C^{\sigma+2}(\partial\Omega)$.*

Then $u \in \tilde{C}^{\sigma+2}(\Omega)$.

Proof The result is obvious if $\sigma + 2 \leq \sigma'$. So we assume $\sigma + 2 > \sigma'$. By proposition 2.1, we can suppose that $u_{|\partial\Omega} \equiv 0$.

Let O be an open subset satisfying the assumptions of lemma 3.9 such that $\Omega = \Omega_1 \cup O$. For $j = 1, 2$ take $\phi_j \in C^{\infty}(\Omega)$ such that $supp(\phi_1) \subseteq \Omega_1$, $supp(\phi_2) \subseteq O$ and $\phi_1(z) + \phi_2(z) = 1$ for every $z \in \Omega$. Then,

$$\left\{ \begin{array}{c} \Delta(\phi_1 u) = \phi_1 \Delta u + 2 \nabla \phi_1 \cdot \nabla u + (\Delta\phi_1)u \text{ in } \Omega_1, \\ \phi_1 u_{|\partial\Omega_1} = 0, \end{array} \right. \tag{3.35}$$

Assume first that $\sigma' \geq \sigma + 1$. Then $\Delta(\phi_1 u) \in C^{\sigma}(\Omega_1)$ (observing that $\Delta(\phi_1 u) = \Delta u$ in some neighbourhood of $(0, 0)$), so that $\phi_1 u \in \tilde{C}^{\sigma+2}(\Omega)$ by proposition 2.1 (III). Moreover, for every $\lambda \in \mathbf{C}$

$$\left\{ \begin{array}{c} (\lambda - \Delta)(\phi_2 u) = \phi_2(\lambda - \Delta)u - 2 \nabla \phi_2 \cdot \nabla u - (\Delta\phi_2)u \text{ in } O, \\ \phi_2 u_{|\partial O} = 0, \end{array} \right. \tag{3.36}$$

Assuming again $\sigma' \geq \sigma+1$, we obtain $(\lambda-\Delta)(\phi_2 u) \in C^{\sigma}(\Omega_2)$. Choosing $\lambda \in \mathbf{R}^+$ sufficiently large and applying lemma 3.9, we have the result if $\sigma' \geq \sigma+1$.

Suppose $\sigma \leq \sigma' < \sigma+1$. Then $\sigma > -1$. If $\frac{(\sigma+1)\omega}{\pi} \notin \mathbf{N}$, we can apply what we have already obtained replacing σ with $\sigma-1$, we have that $u \in \tilde{C}^{\sigma+1}(\Omega)$ and the conclusion follows. Assume that $\frac{(\sigma+1)\omega}{\pi} \in \mathbf{N}$. We can apply the previous arguments replacing σ with $\sigma-1-\epsilon$ for every $\epsilon > 0$ sufficiently small and obtain that $u \in \bigcap_{0<\epsilon\leq\sigma+1} \tilde{C}^{\sigma+1-\epsilon}(\Omega)$. From here we have $u \in \bigcap_{0<\epsilon\leq\sigma+1} \tilde{C}^{\sigma+2-\epsilon}(\Omega)$. In particular $u \in \tilde{C}^{\sigma+1}(\Omega)$. This gives again the desired conclusion. Iterating this argument, one can prove the result in general.

Lemma 3.11 *Let $\lambda \in \mathbf{C}-]-\infty,0], \sigma \in]-2,\frac{\pi}{\omega}[-\mathbf{Z}$ such that $\frac{(2+\sigma)\omega}{\pi} \notin \mathbf{N}$. Let $f \in C^{\sigma}(\Omega) \cap H^{-1}(\Omega)$ and consider the variational solution u of*

$$\begin{cases} (\lambda-\Delta)u = f \text{ in } \Omega, \\ u_{|\partial\Omega} = 0. \end{cases} \tag{3.37}$$

Then $u \in \tilde{C}^{\sigma+2}(\Omega)$.

Proof We adopt the notations of the proof of lemma 3.10.

As $L^2(\mathbf{R}^2) \subseteq \bigcap_{\sigma'<-1} C^{\sigma'}(\mathbf{R}^2)$, we have that $u \in \bigcap_{\sigma'<0} C^{\sigma'}(\Omega)$. So, as

$$-\Delta(\phi_1 u) =$$

$$= \phi_1 f - 2 \bigtriangledown \phi_1 \cdot \bigtriangledown u - (\Delta\phi_1)u - \lambda\phi_1 u,$$

we have $-\Delta u \in C^{\min\{\sigma,\sigma'\}}(\Omega)$ for every $\sigma' < -1$. It follows from proposition 3.1(III) that $\phi_1 u \in \tilde{C}^{2+\min\{\sigma,\sigma'\}}(\Omega)$ for every $\sigma' \in]-2,-1[$.

Following the same basic idea and employing lemma 3.9 instead of proposition 3.1(III), we have that $\phi_2 u \in \tilde{C}^{2+\min\{\sigma,\sigma'\}}(\Omega)$ for every $\sigma' \in]-2,-1[$.

At this point we can apply lemma 3.10 and get the conclusion.

Theorem 3.1 *Consider the problem*

$$\begin{cases} (\lambda-\Delta)u = f \text{ in } \Omega, \\ u_{|\partial\Omega} = g, \end{cases} \tag{3.38}$$

with $\lambda \in \mathbf{C}-]-\infty,0], f \in C^{\sigma}(\Omega), g \in C^{2+\sigma}(\partial\Omega)$, *with* $\sigma \in]-2,\frac{\pi}{\omega}[-\mathbf{Z}$ *and* $\frac{(2+\sigma)\omega}{\pi} \notin \mathbf{N}$.

Then such a problem has a unique solution u *in* $\tilde{C}^{\sigma+2}(\Omega)$.

Proof Let $\lambda = R^2 \exp(i\theta)$ with $\theta \in]-\pi,\pi[$.

We start by showing uniqueness if R is sufficiently large. Then, let $u \in \tilde{C}^{\sigma+2}(\Omega)$ solving (3.38) with $f \equiv 0$ and $g \equiv 0$. By lemma 3.10, as $\Delta u = \lambda u$, we can assume $\sigma > 0$. So, employing again the notations introduced in the proof of lemma 3.10, we can say that $\phi_1 u \in H^1(\Omega)$. Moreover, $(\lambda - \Delta)(\phi_2 u) \in H^{-1}(O)$, as it belongs to $C^{\sigma}(O)$ and has compact support, because $\phi_2(z) \equiv 1$ for $|z| \geq 1$. It follows from lemma 3.9 that, if R is sufficiently large, $\phi_2 u \in H^1(O)$, so that $u \in H^1(\Omega)$. From the uniqueness of the variational solution, we conclude that $u \equiv 0$.

We show the existence of a solution if R is sufficiently large and $\sigma > -1$. As usual, we can consider only the case $g \equiv 0$. If R is sufficiently large, by lemma 3.9 there exists $v \in C^{\sigma+2}(O)$ such that

$$\left\{ \begin{array}{c} (\lambda - \Delta)v = f \text{ in } O, \\ v_{|\partial O} = 0. \end{array} \right. \tag{3.39}$$

Subtracting $\phi_2 v$ to u, one is reduced to the case $f(z) \equiv 0$ for $|z| > 1$. But, as $\sigma > -1$, in this case $f \in H^{-1}(\Omega)$. So problem (3.38) has a variational solution in $H^1(\Omega)$. Such solution belongs to $\tilde{C}^{\sigma+2}(\Omega)$ in force of lemma 3.11.

We show the existence of a solution if R is sufficiently large and $-2 < \sigma < -1$ (again in case $g \equiv 0$). Let O_1 be an open subset of Ω satisfying the assumptions of lemma 3.8 and such that $O \cup O_1 = \Omega$. Let ψ_1 and ψ_2 belong to $C^{\infty}(\Omega)$ such that $\psi_1(z)^2 + \psi_2(z)^2 = 1$ for every $z \in \Omega$, $supp(\psi_1) \subseteq O_1$, $supp(\psi_2) \subseteq O$. By lemma 3.8 there exists $u_1 \in \tilde{C}^{\sigma+2}(O_1)$ such that

$$\left\{ \begin{array}{c} (\lambda - \Delta)u_1 = \psi_1 f \text{ in } O_1, \\ u_{1|\partial O_1} = 0. \end{array} \right. \tag{3.40}$$

Owing to lemma 3.9, if R is sufficiently large,there exists $u_2 \in C^{\sigma+2}(O)$ such that

$$\left\{ \begin{array}{c} (\lambda - \Delta)u_2 = \psi_2 f \text{ in } O, \\ u_{2|\partial O} = 0. \end{array} \right. \tag{3.41}$$

Subtracting $\psi_1 u_1 + \psi_2 u_2$ from the unknown function u, one replaces f with an element of $C^{\sigma+1}(\Omega)$, so that one can apply the corresponding result for $\sigma > -1$.

Finally, we prove existence and uniqueness if R is "small". Let $R_0 > 0$ be such that problem (3.38) with $\mu := R_0^2 exp(i\theta)$ replacing λ has a unique solution for every $f \in C^\sigma(\Omega)$ and $g \in C^{\sigma+2}(\partial\Omega)$. Then it is easily seen that u solves (3.38) if and only if $v := u(\frac{R_0}{R}\cdot)$ solves

$$\begin{cases} (\mu - \Delta)v = (\frac{R_0}{R})^2 f(\frac{R_0}{R}\cdot) \text{ in } \Omega, \\ v_{|\partial\Omega} = g(\frac{R_0}{R}\cdot). \end{cases} \tag{3.42}$$

Definition 3.2 *Let $\sigma \geq 0$; we set*

$$\tilde{C}_0^\sigma(\Omega) := \{u \in \tilde{C}^\sigma(\Omega) | u_{|\partial\Omega} = 0\}.$$

Corollary 3.2 *Let $0 < \sigma_0 < \sigma_1 < 2 + \frac{\pi}{\omega}$, $\theta \in]0,1[$, $\sigma := (1-\theta)\sigma_0 + \theta\sigma_1$. Assume that $\{\sigma_0, \sigma_1, \sigma, \frac{\sigma_0\omega}{\pi}, \frac{\sigma_1\omega}{\pi}, \frac{\sigma\omega}{\pi}\} \cap \mathbf{N}$ is empty. Then,*

(I)

$$(\tilde{C}^{\sigma_0}(\Omega), \tilde{C}^{\sigma_1}(\Omega))_{\theta,\infty} = \tilde{C}^\sigma(\Omega);$$

(II)

$$(\tilde{C}_0^{\sigma_0}(\Omega), \tilde{C}_0^{\sigma_1}(\Omega))_{\theta,\infty} = \tilde{C}_0^\sigma(\Omega).$$

Proof It follows from corollary 2.1 and theorem 3.1, by the interpolation property.

Theorem 3.2 *Let $\sigma \in]-2,0[$, $\sigma \neq -1$, such that $\frac{(2+\sigma)\omega}{\pi} \notin \mathbf{N}$, $\lambda \in \mathbf{C}-]-\infty,0]$, $|\lambda| \geq 1$ and $|Arg(\lambda)| \leq \theta_0 < \pi$. Then for every $f \in C^\sigma(\Omega), g \in C^{\sigma+2}(\partial\Omega)$ problem (3.38) has a unique solution $u \in \tilde{C}^{\sigma+2}(\Omega)$. Moreover,*
(I) for every $\sigma' \in [0, 2+\sigma]$ such that either $\sigma' < \frac{\pi}{\omega}$ or $\sigma' \notin \mathbf{N}$ and $\frac{\sigma'\omega}{\pi} \notin \mathbf{N}$ there exists $C(\theta_0, \sigma') > 0$ such that

$$||u||^*_{\sigma',\Omega} \leq C(\theta_0,\sigma')|\lambda|^{\frac{\sigma'-\sigma}{2}-1}[||f||_{\sigma,\Omega} + ||g||_{2+\sigma,\partial\Omega} + |\lambda|^{1+\frac{\sigma}{2}}||g||_{0,\partial\Omega}];$$

(II) for every $\sigma' \in [\sigma, 2+\sigma]-\{-1\}$ such that $\sigma' \leq \frac{\pi}{\omega}$ there exists $C(\theta_0,\sigma') > 0$ such that

$$||u||_{\sigma',\Omega} \leq C(\theta_0,\sigma')|\lambda|^{\frac{\sigma'-\sigma}{2}-1}[||f||_{\sigma,\Omega} + ||g||_{2+\sigma,\partial\Omega} + |\lambda|^{1+\frac{\sigma}{2}}||g||_{0,\partial\Omega}];$$

Let $\sigma \in]0, \min\{\frac{\pi}{\omega}, 2\}[$, $\sigma \neq 1$, such that $\frac{(2+\sigma)\omega}{\pi} \notin \mathbf{N}$, $\lambda \in \mathbf{C}-]-\infty,0]$, $|\lambda| \geq 1$ and $|Arg(\lambda)| \leq \theta_0 < \pi$. Then for every $g \in C^{\sigma+2}(\partial\Omega)$ problem (3.38) with $f \equiv 0$ has a unique solution $u \in \tilde{C}^{\sigma+2}(\Omega)$. Moreover, *(III) for every $\sigma' \in [0, 2+\sigma]$ such that either $\sigma' < \frac{\pi}{\omega}$ or $\sigma' \notin \mathbf{N}$ and $\frac{\sigma'\omega}{\pi} \notin \mathbf{N}$ there exists $C(\theta_0, \sigma') > 0$ such that*

$$||u||^*_{\sigma',\Omega} \leq C(\theta_0, \sigma')|\lambda|^{\frac{\sigma'-\sigma}{2}-1}[||g||_{2+\sigma,\partial\Omega} + |\lambda|^{1+\frac{\sigma}{2}}||g||_{0,\partial\Omega}];$$

(IV) for every $\sigma' \in [\sigma, 2+\sigma]$ such that $\sigma' \leq \frac{\pi}{\omega}$ and $\sigma' < \frac{\pi}{\omega}$ if $\frac{\pi}{\omega} \in \mathbf{N}$ there exists $C(\theta_0, \sigma') > 0$ such that

$$||u||_{\sigma',\Omega} \leq C(\theta_0, \sigma')|\lambda|^{\frac{\sigma'-\sigma}{2}-1}[||g||_{2+\sigma,\partial\Omega} + |\lambda|^{1+\frac{\sigma}{2}}||g||_{0,\partial\Omega}];$$

(V) there exists $C(\theta_0) > 0$ such that

$$||\Delta u||_{0,\Omega} \leq C(\theta_0)|\lambda|^{-\frac{\sigma}{2}}[||g||_{2+\sigma,\partial\Omega} + |\lambda|^{1+\frac{\sigma}{2}}||g||_{0,\partial\Omega}].$$

Proof In each case the existence of the solution u follows from theorem 3.1. We prove the estimates depending on the parameter λ.

We start by considering the case $f \equiv 0$. Assume σ and σ' satisfying the conditions indicated in (II) or (IV) and, moreover, $\sigma' \geq 0$ and let $\lambda = r^2\exp(i\theta)$, with $r \geq 1$ and $|\theta| \leq \theta_0 < \pi$. If $r = 1$ by the compactness of $\{\mu \in \mathbf{C} | |\mu| = 1, |Arg(\mu)| \leq \theta_0\}$, we have

$$||u||_{\sigma',\Omega} \leq C(\theta_0)||g||_{2+\sigma,\partial\Omega}.$$

If, more generally, $r \geq 1$, it is easy to verify that $u(z) = u_0(rz)$, with

$$\begin{cases} (\exp(i\theta) - \Delta)u_0 = 0 \text{ in } \Omega, \\ u_{0|\partial\Omega} = g(\frac{\cdot}{r}. \end{cases} \tag{3.43}$$

It follows

$$\begin{aligned} ||u||_{\sigma',\Omega} &\leq C(\sigma')r^{\sigma'}||u_0||_{\sigma',\Omega} \\ &\leq C(\theta_0, \sigma')r^{\sigma'}||g(\frac{\cdot}{r})||_{2+\sigma,\partial\Omega} \\ &\leq C(\theta_0, \sigma')r^{\sigma'}(\sum_{j=0}^{2+[\sigma]} r^{-j}||g||_{j,\partial\Omega} + r^{-2-\sigma}||g||_{2+\sigma,\partial\Omega}) \\ &\leq C(\theta_0, \sigma')r^{\sigma'}[r^{-2-\sigma}||g||_{2+\sigma,\partial\Omega} + ||g||_{0,\partial\Omega}], \end{aligned}$$

applying corollary 2.1. So we have (II) and (IV) in case $\sigma' \geq 0$ and $f \equiv 0$.

A similar argument gives (V).

Let now $\sigma' \in]-2, \min\{\frac{\pi}{\omega} - 2, \sigma\}[-\{-1\}$ and, again, $f \equiv 0$. From the previous case

$$||u||_{\sigma',\Omega} = |\lambda|^{-1}||\Delta u||_{\sigma',\Omega} \leq C|\lambda|^{-1}||u||_{2+\sigma',\Omega}$$

$$\leq C(\theta_0, \sigma')|\lambda|^{\frac{\sigma'-\sigma}{2}-1}[||g||_{2+\sigma,\partial\Omega} + |\lambda|^{1+\frac{\sigma}{2}}||g||_{0,\partial\Omega}].$$

Applying now corollary 2.1 (II), we obtain (II) and (IV) in case $f \equiv 0$.

We show (I) and (III) in case $\sigma' = 2 + \sigma$ and $f \equiv 0$. In fact, using theorem 3.1 and the previous case,

$$||u||^*_{2+\sigma,\Omega} \leq C(||(1-\Delta)u||_{\sigma,\Omega} + ||g||_{2+\sigma,\partial\Omega})$$

$$\leq C(|\lambda|||u||_{\sigma,\Omega} + ||g||_{2+\sigma,\partial\Omega})$$

$$\leq C(\theta_0, 2+\sigma)(||g||_{2+\sigma,\partial\Omega} + |\lambda|^{\frac{1+\sigma}{2}}||g||_{0,\partial\Omega}).$$

Employing now the previous cases and corollary 3.2, we obtain (I) and (III) in case $f \equiv 0$. Now we remove this condition in case $\sigma < 0$: let $\mathcal{P}$ be the operator introduced in proposition 2.2 and consider the problem

$$(\lambda - \Delta)v = \mathcal{P}f \text{ in } \mathbf{R}^2. \tag{3.44}$$

Using the Fourier transform and the arguments of [11],one can easily verify that problem (3.44) has a unique solution $v \in C^{\sigma+2}(\mathbf{R}^2)$. Moreover, for every $\sigma' \in [\sigma, 2+\sigma] - \{-1\}$

$$||v||_{\sigma',\mathbf{R}^2} \leq C(\theta_0, \sigma')|\lambda|^{\frac{\sigma'-\sigma}{2}-1}||\mathcal{P}f||_{\sigma,\mathbf{R}^2} \leq C(\theta_0, \sigma')|\lambda|^{\frac{\sigma'-\sigma}{2}-1}||f||_{\sigma,\Omega}. \tag{3.45}$$

It follows, for σ' satisfying the conditions of (II), using the fact that $(\lambda - \Delta)(u - v_{|\Omega}) = 0$ and employing (3.45),

$$||u||_{\sigma',\Omega} \leq ||v||_{\sigma',\mathbf{R}^2} + ||u - v_{|\Omega}||_{\sigma',\Omega}$$

$$\leq C(\theta_0, \sigma')|\lambda|^{\frac{\sigma'-\sigma}{2}-1}[||f||_{\sigma,\Omega} + ||g||_{2+\sigma,\partial\Omega} + |\lambda|^{1+\frac{\sigma}{2}}||g||_{0,\partial\Omega}$$

$$+||v_{|\partial\Omega}||_{2+\sigma,\partial\Omega} + |\lambda|^{1+\frac{\sigma}{2}}||v_{|\partial\Omega}||_{0,\partial\Omega}]$$

$$\leq C(\theta_0, \sigma')|\lambda|^{\frac{\sigma'-\sigma}{2}-1}[||f||_{\sigma,\Omega} + ||g||_{2+\sigma,\partial\Omega} + |\lambda|^{1+\frac{\sigma}{2}}||g||_{0,\partial\Omega}].$$

The same argument works to prove (I).

Definition 3.3 *We set*

$$D(A) := \bigcap_{0<\sigma<2} \{u \in C_0^\sigma(\Omega) | \Delta u \in C(\Omega)\},$$
$$Au = \Delta u \text{ for } u \in D(A).$$

To prove the main properties of A, we shall use the following

Lemma 3.12 *Let X be a Banach space, B a closed operator in X such that for some $R > 0$ $[R, +\infty[\subseteq \rho(B)$ and $||(\lambda - B)^{-1}||_{\mathcal{L}(X)} = O(\lambda^{-1})$ for $\lambda \to +\infty$. Then,*

(I) let Y be a Banach space such that $D(B) \subseteq Y \subseteq X$ with continuous embeddings and $\theta \in]0,1[$. Assume that $||(\lambda - B)^{-1}||_{\mathcal{L}(X,Y)} = O(\lambda^{\theta-1})$ $(\lambda \to +\infty)$. Then $Y \in J_\theta(X, D(B))$;

(II) let $D_\theta(B) := (X, D(B))_{\theta,\infty}$ and $D_{1+\theta}(B) := \{x \in D(B) : Bx \in D_\theta(B)\}$. We set, for $x \in D_{1+\theta}(B)$,

$$||x||_{D_{1+\theta}(B)} := \max\{||x||_{D_\theta(B)}, ||Bx||_{D_\theta(B)}\}.$$

Then $D(B) \in J_{\frac{1-\theta}{2}}(D_\theta(B), D_{1+\theta}(B)) \cap K_{\frac{1-\theta}{2}}(D_\theta(B), D_{1+\theta}(B))$.

Proof (I) follows from [2], theorem 5.2.

We show (II): we have from [7], corollary 3.1 that $D(B) \in J_{\frac{1}{2}}(X, D(B^2))$ $\cap K_{\frac{1}{2}}(X, D(B^2))$. It follows from the reiteration property that

$$D_\theta(B) = (X, D(B^2))_{\frac{\theta}{2},\infty}$$

and, as $D_{1+\theta}(B) = (D(B), D(B^2))_{\theta,\infty}$,

$$D_{1+\theta}(B) = (X, D(B^2))_{\frac{1+\theta}{2},\infty}.$$

Then (II) follows again by reiteration.

We have:

Theorem 3.3 *Consider the operator A in $C(\Omega)$; then:*

(I) $\mathbf{R} -]-\infty, 0] \subseteq \rho(A)$;

(II) for every $\theta_0 \in [0, \pi[$ there exists $C(\theta_0) > 0$ such that for every $\lambda \in \mathbf{C}$ with $|\lambda| \geq 1$ and $|Arg(\lambda)| \leq \theta_0$

$$||(\lambda - A)^{-1}||_{\mathcal{L}(C(\Omega))} \leq C(\theta_0)|\lambda|^{-1};$$

(III) for every $\theta \in]0,1[- \{\frac{1}{2}\}$ such that $\frac{2\theta\omega}{\pi} \notin \mathbf{N}$ one has

$$(C(\Omega), D(A))_{\theta,\infty} = \tilde{C}_0^{2\theta}(\Omega).$$

Proof (I) follows from theorem 3.1, as $C(\Omega) \subseteq C^{\sigma}(\Omega)$ for every $\sigma < 0$, $\sigma \notin \mathbf{Z}$.

(II) follows immediately from the identity

$$(\lambda - A)^{-1} f = |\lambda|^{-1}((\frac{\lambda}{|\lambda|} - A)^{-1} f(|\lambda|^{-\frac{1}{2}}\cdot))(|\lambda|^{\frac{1}{2}}\cdot). \tag{3.46}$$

We show (III). Clearly $C_0^2(\Omega) \subseteq D(A)$. It follows from corollary 2.1(VI) that

$$C_0^{2\theta}(\Omega) \subseteq (C(\Omega), C_0^2(\Omega))_{\theta,\infty} = (C(\Omega), D(A))_{\theta,\infty}.$$

Moreover, it is clear that for every $k \in \mathbf{N}$ $R_k \in D(A)$. It follows that

$$\tilde{C}_0^{2\theta}(\Omega) \subseteq (C(\Omega), D(A))_{\theta,\infty}.$$

Assume that $2\theta < \frac{\pi}{\omega}$. Then $D(A) \subseteq C^{2\theta}(\Omega)$ and from (3.46) we draw, for $\lambda \in \mathbf{R}, \lambda \geq 1$,

$$||(\lambda - A)^{-1} f||_{2\theta,\Omega} \leq C\lambda^{\theta-1}||f||_{0,\Omega},$$

with $C > 0$ independent of λ and f. It follows from lemma 3.12(I) that, if $\theta < \frac{\pi}{2\omega}$, $C_0^{2\theta}(\Omega) \in J_\theta(C(\Omega), D(A))$. We conclude, applying corollary 2.1(VI) and the reiteration property that, if $\theta < \frac{\pi}{2\omega}$, then (III) holds. Let now $\theta_0 \in]0, \min\{1, \frac{\pi}{2\omega}\}[-\{\frac{1}{2}\}$. Set $\sigma := 2\theta_0 - 2$, $D(A_\sigma) = C_0^{2\theta_0}(\Omega)$, $A_\sigma u := \Delta u$ for every $u \in D(A_\sigma)$. Then $D(A_\sigma^2)$ is the domain of the part of A in $(X, D(A))_{\theta_0,\infty}$. By lemma 3.12(II) we have $D(A) \in J_{\frac{|\sigma|}{2}}(D(A_\sigma), D(A_\sigma^2)) \cap K_{\frac{|\sigma|}{2}}(D(A_\sigma), D(A_\sigma^2))$. Let now $\theta_0 < \theta < 1$. Then, by the reiteration property,

$$\begin{aligned}(C(\Omega), D(A))_{\theta,\infty} &= (D(A_\sigma), D(A))_{\frac{\theta-\theta_0}{1-\theta_0},\infty} \\ &= (D(A_\sigma), D(A_\sigma^2))_{\theta-\theta_0,\infty} \\ &\subseteq (\tilde{C}_0^{2\theta_0}(\Omega), \tilde{C}_0^{2(1+\theta_0)}(\Omega))_{\theta-\theta_0,\infty} \\ &= \tilde{C}_0^{2\theta}(\Omega)\end{aligned}$$

by corollary 3.2(II).

Proposition 3.2 *Let* $\sigma \in]-2, 0[-\{-1\}$ *be such that* $\frac{(2+\sigma)\omega}{\pi} \notin \mathbf{N}$ *and* $\theta \in]0, 1[$; *then,*

(I) if $\sigma + 2\theta < 0$ *and* $\sigma + 2\theta \neq -1$, $(C^\sigma(\Omega), \tilde{C}_0^{\sigma+2}(\Omega))_{\theta,\infty} = C^{\sigma+2\theta}(\Omega)$;

(II) if $\sigma + 2\theta > 0$, $\sigma + 2\theta \neq 1$ *and* $\frac{(\sigma+2\theta)\omega}{\pi} \notin \mathbf{N}$, $(C^\sigma(\Omega), \tilde{C}_0^{\sigma+2}(\Omega))_{\theta,\infty} = \tilde{C}_0^{\sigma+2\theta}(\Omega)$.

Proof We have already seen in the proof of theorem 3.3(III) that $D(A) \in J_{\frac{|\sigma|}{2}}(D(A_\sigma), D(A_\sigma^2))$. It follows that

$$C(\Omega) \in J_{\frac{|\sigma|}{2}}(C^\sigma(\Omega), \tilde{C}_0^{2+\sigma}(\Omega)) \cap K_{\frac{|\sigma|}{2}}(C^\sigma(\Omega), \tilde{C}_0^{2+\sigma}(\Omega)).$$

Then the result follows from the reiteration property, corollary 2.1(I) and theorem 3.3(III).

Proposition 3.3 *Let $\sigma \in]0, \min\{\frac{\pi}{\omega}, 2\}$, $\sigma \neq 1$, $\frac{(2+\sigma)\omega}{\pi} \notin \mathbf{N}$. Consider problem (3.38), with $f \in C^\sigma(\Omega), g \in C^{2+\sigma}(\partial\Omega)$, $\lambda \in \mathbf{C} -]-\infty, 0]$. Then there exists a unique solution $u \in \tilde{C}^{2+\sigma}(\Omega)$. Moreover, for every $\theta_0 \in [0, \pi[, \sigma' \in [\sigma, 2+\sigma]$ with either $\sigma' < \frac{\pi}{\omega}$ or $\sigma' \notin \mathbf{N}$ and $\frac{\sigma'\omega}{\pi} \notin \mathbf{N}$ there exists $C(\theta_0, \sigma') > 0$ such that, if $|Arg(\lambda)| \leq \theta_0$ and $|\lambda| \geq 1$,*

$$||u||^*_{\sigma',\Omega} \leq C(\theta_0, \sigma')|\lambda|^{\frac{\sigma'-\sigma}{2}-1}[||f||_{\sigma,\Omega} + |\lambda|^{\frac{|\sigma|}{2}}||f_{|\partial\Omega}||_{0,\partial\Omega}$$

$$+||g||_{2+\sigma,\partial\Omega} + |\lambda|^{1+\frac{\sigma}{2}}||g||_{0,\partial\Omega}]. \tag{3.47}$$

Finally, there exists $C(\theta_0) > 0$ such that

$$||\Delta u||_{0,\Omega} \leq C(\theta_0)|\lambda|^{-\frac{\sigma}{2}}[||f||_{\sigma,\Omega} + |\lambda|^{\frac{|\sigma|}{2}}||f_{|\partial\Omega}||_{0,\partial\Omega}]. \tag{3.48}$$

Proof In case $f \equiv 0$ estimate (3.47) was proved in theorem 3.2 (III)-(IV)-(V). So we consider only the case $g \equiv 0$. By (3.46), if $\lambda = |\lambda|\exp(i\theta)$, with $|\lambda| \geq 1$ and $|\theta| \leq \theta_0 < \pi$, we have

$$||u||_{\sigma,\Omega} \leq C(\theta_0, \sigma)|\lambda|^{-1}(||f||_{\sigma,\Omega} + |\lambda|^{\frac{\sigma}{2}}||f||_{0,\Omega})$$

from which

$$\begin{aligned}||u||^*_{2+\sigma,\Omega} &\leq C(||\Delta u||_{\sigma,\Omega} + ||u||_{\sigma,\Omega}) \\ &\leq C(|\lambda|||u||_{\sigma,\Omega} + ||f||_{\sigma,\Omega}) \\ &\leq C(\theta_0, 2+\sigma)(||f||_{\sigma,\Omega} + |\lambda|^{\frac{\sigma}{2}}||f||_{0,\Omega}).\end{aligned}$$

By interpolation, we get for every σ' in the statement:

$$||u||^*_{\sigma',\Omega} \leq C(\theta_0, \sigma')|\lambda|^{\frac{\sigma'-\sigma}{2}-1}[||f||_{\sigma,\Omega} + |\lambda|^{\frac{\sigma}{2}}||f||_{0,\Omega}). \tag{3.49}$$

In the same way we can obtain

$$||\Delta u||_{0,\Omega} \leq C(\theta_0)|\lambda|^{-\frac{\sigma}{2}}(||f||_{\sigma,\Omega} + |\lambda|^{\frac{\sigma}{2}}||f||_{0,\Omega}). \tag{3.50}$$

We want to replace in (3.49) and in (3.50) $||f||_{0,\Omega}$ with $||f_{|\partial\Omega}||_{0,\partial\Omega}$. So we consider first the case $f \in C_0^\sigma(\Omega)$. By theorem 3.3(II)-(III), we have in this case

$$||u||_{\sigma,\Omega} \leq C(\theta_0,\sigma)|\lambda|^{-1}||f||_{\sigma,\Omega}$$

and, arguing as before, for σ' as in the statement,

$$||u||^*_{\sigma',\Omega} \leq C(\theta_0,\sigma')|\lambda|^{\frac{\sigma'-\sigma}{2}-1}||f||_{\sigma,\Omega}. \tag{3.51}$$

By lemma 3.12(II) one has also

$$||\Delta u||_{0,\Omega} \leq C(\theta_0)|\lambda|^{-\frac{\sigma}{2}}||f||_{\sigma,\Omega}. \tag{3.52}$$

To treat the general case, decompose f in the form $f = (f - \mathcal{E}f_{|\partial\Omega}) + \mathcal{E}f_{|\partial\Omega}$ and put $v := (\lambda - A)^{-1}(f - \mathcal{E}f_{\partial\Omega})$, $z := (\lambda - A)^{-1}\mathcal{E}f_{|\partial\Omega}$. Then, using (3.51) and (3.49),

$$||u||^*_{\sigma',\Omega} \leq ||v||^*_{\sigma',\Omega} + ||z||^*_{\sigma',\Omega}$$

$$\leq C(\theta_0,\sigma')|\lambda|^{\frac{\sigma'-\sigma}{2}-1}(||f - \mathcal{E}f_{|\partial\Omega}||_{\sigma,\Omega} + ||\mathcal{E}f_{|\partial\Omega}||_{\sigma,\Omega} + |\lambda|^{\frac{\sigma}{2}}||\mathcal{E}(f_{|\partial\Omega}||_{0,\Omega})$$

$$\leq C(\theta_0,\sigma')|\lambda|^{\frac{\sigma'-\sigma}{2}-1}(||f||_{\sigma,\Omega} + |\lambda|^{\frac{\sigma}{2}}||f_{|\partial\Omega})||_{0,\Omega}).$$

Analogously, applying (3.50) and (3.52), one gets (3.48).

4 Parabolic problems in an angle

In this section we shall consider the mixed parabolic problem

$$\begin{cases} \partial_t u - \Delta_z u = f(t,z), & (t,z) \in [0,T] \times \Omega, \\ u(t,z') = g(t,z'), & (t,z') \in [0,T] \times \partial\Omega, \\ u(0,z) = u_0(z), & z \in \Omega. \end{cases} \tag{4.1}$$

We fix σ' and σ such that $-2 < \sigma' < \sigma < 0$, σ and σ' both different from -1 with $\frac{(\sigma+2)\omega}{\pi} \notin \mathbf{N}$ and look for conditions ensuring the existence of a solution $u \in C([0,T];\tilde{C}^{2+\sigma'}(\Omega)) \cap B([0,T];\tilde{C}^{2+\sigma}(\Omega)) \cap C^1([0,T];C^{\sigma'}(\Omega))$ with $\partial_t u \in B([0,T];C^\sigma(\Omega))$. We start with a lemma:

Lemma 4.1 *Let $\sigma \in]-2,0[-\{-1\}$, with $\frac{(2+\sigma)\omega}{\pi} \notin \mathbf{N}$; then $C(\Omega) \in J_{\frac{|\sigma|}{2}}$ $(C^\sigma(\Omega), \tilde{C}^{2+\sigma}(\Omega))$.*

Proof The result is a particular case of corollary 2.1(II) if $2+\sigma < \frac{\pi}{\omega}$. If $2+\sigma > \frac{\pi}{\omega}$, fix $\sigma' \in]0, \frac{\pi}{\omega}[$. Then $C(\Omega) \in J_{\frac{|\sigma|}{\sigma'+|\sigma|}}(C^{\sigma}(\Omega), C^{\sigma'}(\Omega))$, that is, there exists $C>0$ such that for every $u \in C^{\sigma'}(\Omega)$

$$||u||_{0,\Omega} \leq C||u||_{\sigma}^{1-\frac{|\sigma|}{\sigma'+|\sigma|}} ||u||_{\sigma',\Omega}^{\frac{|\sigma|}{\sigma'+|\sigma|}}. \tag{4.2}$$

By theorem 3.3(III) and the reiteration theorem, there exists $C>0$ such that for every $v \in \tilde{C}_0^{2+\sigma}(\Omega)$

$$||v||_{\sigma',\Omega} \leq C||v||_{0,\Omega}^{1-\frac{\sigma'}{2+\sigma}} ||v||_{2+\sigma,\Omega}^{*\,\frac{\sigma'}{2+\sigma}}.$$

We have

$$||u||_{\sigma',\Omega} \leq ||u - \mathcal{E}u_{|\partial\Omega}||_{\sigma',\Omega} + ||\mathcal{E}u_{|\partial\Omega}||_{\sigma',\Omega}$$

The first summand can be majorized with

$$C||u - \mathcal{E}u_{|\partial\Omega}||_{0,\Omega}^{1-\frac{\sigma'}{2+\sigma}} ||u - \mathcal{E}u_{|\partial\Omega}||_{2+\sigma,\Omega}^{*\,\frac{\sigma'}{2+\sigma}}$$

$$\leq C||u||_{0,\Omega}^{1-\frac{\sigma'}{2+\sigma}} ||u||_{2+\sigma,\Omega}^{*\,\frac{\sigma'}{2+\sigma}}.$$

Moreover, owing to corollary 2.1(IV),

$$||\mathcal{E}u_{|\partial\Omega}||_{\sigma',\Omega} \leq C||u_{|\partial\Omega}||_{\sigma',\partial\Omega}$$

$$\leq C||u_{|\partial\Omega}||_{0,\partial\Omega}^{1-\frac{\sigma'}{2+\sigma}} ||u_{|\partial\Omega}||_{2+\sigma,\partial\Omega}^{\frac{\sigma'}{2+\sigma}}$$

$$\leq C||u||_{0,\Omega}^{1-\frac{\sigma'}{2+\sigma}} ||u||_{2+\sigma,\Omega}^{*\,\frac{\sigma'}{2+\sigma}}.$$

We conclude that $C^{\sigma'}(\Omega) \in J_{\frac{\sigma'}{2+\sigma}}(C(\Omega), \tilde{C}^{2+\sigma}(\Omega))$.

Inserting the inequality that follows from this fact in (4.2), one gets the result.

Lemma 4.2 *Let σ' and σ such that $-2 < \sigma' < \sigma < 0$, σ and σ' both different from -1 with $\frac{(\sigma+2)\omega}{\pi} \notin \mathbf{N}$, $u \in C([0,T]; \tilde{C}^{2+\sigma'}(\Omega)) \cap B([0,T]; \tilde{C}^{2+\sigma}(\Omega)) \cap C^1([0,T]; C^{\sigma'}(\Omega))$ with $\partial_t u \in B([0,T]; C^{\sigma}(\Omega))$. Set $f(t,.) := \partial_t u(t,.) - \Delta_z u(t,.)$, $g(t,.) := u(t,.)_{|\partial\Omega}$, $u_0 := u(0,.)$. Then,*

(I) $f \in C([0,T]; C^{\sigma'}(\Omega)) \cap B([0,T]; C^{\sigma}(\Omega))$;

(II) $g \in C([0,T]; C^{2+\sigma'}(\partial\Omega)) \cap B([0,T]; C^{2+\sigma}(\partial\Omega))$;

(III) $u_0 \in \tilde{C}^{2+\sigma}(\Omega)$;
(IV) $u_{0|\partial\Omega} = g(0,.)$;
(V) $g \in C^{1+\frac{\sigma}{2}}([0,T]; C(\partial\Omega))$.

Proof The four first assertions are clear.

To prove (V) observe that u is bounded with values in $\tilde{C}^{2+\sigma}(\Omega)$ and lipschitz continuous with values in $C^{\sigma}(\Omega)$. So (V) follows from lemma 4.1.

Theorem 4.1 *Let σ' and σ such that $-2 < \sigma' < \sigma < 0$, σ and σ' both different from -1 with $\frac{(\sigma+2)\omega}{\pi} \notin \mathbf{N}$. Assume that conditions (I)-(V) in lemma 4.2 are satisfied. Then problem (4.1) has a unique solution $u \in C([0,T]; \tilde{C}^{2+\sigma'}(\Omega)) \cap B([0,T]; \tilde{C}^{2+\sigma}(\Omega)) \cap C^1([0,T]; C^{\sigma'}(\Omega))$ with $\partial_t u \in B([0,T]; C^{\sigma}(\Omega))$.*

Proof Clearly it is not restrictive to assume that even $\frac{(2+\sigma')\omega}{\pi} \notin \mathbf{N}$. We set $D(A_{\sigma'}) := \tilde{C}_0^{2+\sigma'}(\Omega)$, $A_{\sigma'}u = \Delta u$ for $u \in D(A_{\sigma'})$. We think of $A_{\sigma'}$ as an unbounded operator in $C^{\sigma'}(\Omega)$. Then, owing to theorem 3.2(II), $\mathbf{C}-]-\infty, 0[\subseteq \rho(A_{\sigma'})$ and for every $\theta_0 \in [0,\pi[$ there exists $C(\theta_0) > 0$ such that, if $|\lambda| \geq 1$ and $|Arg(\lambda)| \leq \theta_0$, $||(\lambda - A_{\sigma'})^{-1}||_{\mathcal{L}(C^{\sigma'}(\Omega))} \leq C(\theta_0)|\lambda|^{-1}$. From the theory of analytic semigroups (possibly not strongly in 0), we obtain the uniqueness of a solution (see, for example, [17], proposition 4.1.2).

To show existence, we start by considering the case $u_0 = 0$ and $g \equiv 0$. In this case we have $f \in C([0,T]; C^{\sigma'}(\Omega)) \cap B([0,T]; (C^{\sigma'}(\Omega), D(A_{\sigma'}))_{\frac{\sigma-\sigma'}{2},\Omega})$ (owing to proposition 3.2(I)). Then the result follows from [17], theorem 3.4.8. We consider now the case $f \equiv 0$. Subtracting the constant function $t \to u_0$, we can limit ourselves to assume that even $u_0 = 0$. So we want to find a solution of problem (4.1) in case $f \equiv 0$, $u_0 = 0$, $g \in C([0,T]; \tilde{C}^{2+\sigma'}(\partial\Omega)) \cap B([0,T]; \tilde{C}^{2+\sigma}(\partial\Omega)) \cap C^{1+\frac{\sigma}{2}}([0,T]; C(\partial\Omega)), g(0,.) = 0$. The argument is quite similar to that used (for example) in [10]. So we shall just sketch the proof.

If $\lambda \in \mathbf{C}-]-\infty, 0[$ and $g \in C^{2+\sigma'}(\partial\Omega)$, we indicate with $N(\lambda)g$ the solution in $\tilde{C}^{2+\sigma'}(\Omega)$ of (3.38), with $f \equiv 0$. If $t > 0$ and Γ is the counterclockwise oriented boundary of $\{\lambda \in \mathbf{C} - \{0\} | |\lambda| \geq 1, |Arg(\lambda)| \leq \theta_0\}$, for some fixed $\theta_0 \in]\frac{\pi}{2}, \pi[$, we set

$$K(t) := \frac{1}{2\pi i}\int_{\Gamma} \exp(\lambda t) N(\lambda) d\lambda, \qquad (4.3)$$

Owing to theorem 3.2 (I)-(II), $K \in \mathcal{C}^{\infty}(\mathbf{R}^+; \mathcal{L}(C^{2+\sigma'}(\partial\Omega), \tilde{C}^{2+\sigma'}(\Omega))) \cap \mathcal{C}^{\infty}(\mathbf{R}^+; \mathcal{L}(C^{2+\sigma}(\partial\Omega), \tilde{C}^{2+\sigma}(\Omega)))$. Moreover, there exists $C > 0$ such that for every $g \in C^{2+\sigma'}(\partial\Omega)$, for every $t \in]0,T]$

$$||K(t)g||_{\sigma',\Omega} \leq C(||g||_{2+\sigma',\partial\Omega} + t^{-1-\frac{\sigma'}{2}}||g||_{0,\partial\Omega}), \tag{4.4}$$

$$t^{\frac{\sigma'-\sigma}{2}+1}||K(t)g||^*_{2+\sigma',\Omega} + ||K(t)g||_{\sigma,\Omega} \leq C(||g||_{2+\sigma,\partial\Omega} + t^{-1-\frac{\sigma}{2}}||g||_{0,\partial\Omega}). \tag{4.5}$$

Applying Cauchy's theorem, it is easily seen that for every $t > 0$

$$K(t)g_{|\partial\Omega} = 0. \tag{4.6}$$

Observe also that

$$\Delta K(t)g = \frac{1}{2\pi i}\int_\Gamma \exp(\lambda t)\lambda N(\lambda)gd\lambda = K'(t)g. \tag{4.7}$$

Concerning K', we shall use the following estimate, again a consequence of theorem 3.2(I)-(II):

$$||K'(t)g||_{\sigma',\Omega} \leq Ct^{-1}(||g||_{2+\sigma',\partial\Omega} + t^{-1-\frac{\sigma'}{2}}||g||_{0,\partial\Omega}). \tag{4.8}$$

We introduce also operator $K^{(-1)}(t)$, defined as follows:

$$K^{(-1)}(t) = \int_0^t K(s)ds. \tag{4.9}$$

Observe that, owing to (4.5), the integral in (4.9) converges at least in $\mathcal{L}(C^{2+\sigma'}(\partial\Omega), C^{\sigma'}(\Omega))$. Using, however, the alternative representation

$$K^{(-1)}(t) = \frac{1}{2\pi i}\int_\Gamma \exp(\lambda t)\lambda^{-1}N(\lambda)d\lambda, \tag{4.10}$$

we have that $K^{(-1)} \in \mathcal{C}^\infty(\mathbf{R}^+; \mathcal{L}(C^{2+\sigma'}(\partial\Omega), \tilde{C}^{2+\sigma'}(\Omega)) \cap \mathcal{C}^\infty(\mathbf{R}^+; \mathcal{L}(C^{2+\sigma}$ $(\partial\Omega), \tilde{C}^{2+\sigma}(\Omega))$ and

$$||K^{(-1)}(t)g||^*_{2+\sigma',\Omega} \leq Ct^{\frac{\sigma-\sigma'}{2}}(||g||_{2+\sigma,\partial\Omega} + t^{-1-\frac{\sigma}{2}}||g||_{0,\partial\Omega}), \tag{4.11}$$

$$||K^{(-1)}(t)g||^*_{2+\sigma,\Omega} \leq C(||g||_{2+\sigma,\partial\Omega} + t^{-1-\frac{\sigma}{2}}||g||_{0,\partial\Omega}). \tag{4.12}$$

¿From formula (4.10) one gets also

$$\Delta K^{(-1)}(t)g = K(t)g \tag{4.13}$$

and

$$K^{(-1)}(t)g_{|\partial\Omega} = g. \tag{4.14}$$

We set

$$u(t) := \int_0^t K(t-s)g(s)ds.$$

Observe that, by (4.4), u belongs, at least, to $C([0,T];C^{\sigma'}(\Omega))$. We verify that u is the solution we are looking for. Observe, first of all, that

$$u(t) = \int_0^t K(t-s)[g(s)-g(t)]ds + K^{(-1)}(t)g(t).$$

The assumptions made on g, (4.4) and (4.11) give

$$||K(t-s)[g(s)-g(t)]||^*_{2+\sigma',\Omega} \leq C(t-s)^{\frac{\sigma-\sigma'}{2}-1},$$

$$||K^{(-1)}(t)g(t)||^*_{2+\sigma',\Omega} \leq C(t-s)^{\frac{\sigma-\sigma'}{2}}.$$

These estimates imply that $u \in C([0,T];\tilde{C}^{2+\sigma'}(\Omega))$. Using (4.8) and (4.14), we obtain

$$u(t)_{|\partial\Omega} = g(t)$$

for every $t \in [0,T]$. Moreover, owing to (4.7) and (4.13),

$$\Delta u(t) = \int_0^t K'(t-s)g(s)ds + K(t)g(t). \tag{4.15}$$

Set now, for $\epsilon \in]0,T[$, $t \in [\epsilon, T]$,

$$u_\epsilon(t) := \int_0^{t-\epsilon} K(t-s)g(s)ds.$$

Clearly $u_\epsilon \in C^1([\epsilon,T];C^{\sigma'}(\Omega))$ and, for $t \in [\epsilon,T]$,

$$u'_\epsilon(t) = K(\epsilon)g(t-\epsilon) + \int_0^{t-\epsilon} K'(t-s)g(s)ds$$

$$= K(t)g(t) + K(\epsilon)[g(t-\epsilon)-g(t)] + \int_0^{t-\epsilon} K'(t-s)[g(s)-g(t)]ds.$$

It is easily seen, using the properties of g,(4.4) and (4.8), that

$$\lim_{\epsilon\to 0} ||u'_\epsilon(t) - K(t)g(t) - \int_0^t K'(t-s)[g(s)-g(t)]ds||_{\sigma',\Omega} = 0$$

uniformly for $t \in [\delta, T]$ for every $\delta \in]0, T[$. It follows that, for $t \in]0, T]$,

$$u'(t) = \Delta u(t).$$

This identity can be easily extended by continuation to $t = 0$. To conclude, it remains to show that $u \in B([0,T]; \tilde{C}^{2+\sigma}(\Omega))$ and $u' \in B([0,T]; C^{\sigma}(\Omega))$. Evidently the second fact is a consequence of the first. Observe now that, owing to theorem 3.1, it is equivalent to show that $\Delta u \in B([0,T]; C^{\sigma}(\Omega))$. First of all, from (4.15) and using (4.5),

$$||K(t)g(t)||_{\sigma,\Omega} \leq C$$

with $C > 0$ independent of $t \in]0, T]$. Next, by proposition 3.2(I), $C^{\sigma}(\Omega) = (C^{\sigma'}(\Omega), D(A_{\sigma'}))_{\frac{\sigma-\sigma'}{2},\infty}$, where the meaning of $A_{\sigma'}$ is clear. Owing to theorem 3.2(II) and theorem 3.1 in [7], it suffices to show that there exists $C > 0$ such that for every $\xi \geq 0$, for every $t \in [0, T]$

$$\xi^{\frac{\sigma-\sigma'}{2}} ||(A_{\sigma'} - 1)(\xi + 1 - A_{\sigma'})^{-1} \Delta (\int_0^t K(t-s)g(s)ds)||_{\sigma',\Omega} \leq C.$$

This is a consequence of the previous estimates and of the identity

$$(A_{\sigma'} - 1)(\xi + 1 - A_{\sigma'})^{-1} \Delta (\int_0^t K(t-s)[g(s) - g(t)]ds)$$

$$= \frac{1}{2\pi i} \int_0^t (\int_{\Gamma} \frac{\lambda(\lambda-1)}{\xi + 1 - \lambda} \exp(\lambda(t-s)) N(\lambda)[g(s) - g(t)] d\lambda) ds.$$

We consider now the case $\sigma > 0$ and look for conditions garanteeing the existence of a solution u belonging to $C^1([0,T]; C(\Omega)) \cap C([0,T]; \tilde{C}^2(\Omega)) \cap B([0,T]; \tilde{C}^{2+\sigma}(\Omega))$ with $\partial_t u \in B([0,T]; C^{\sigma}(\Omega))$. We start by looking for necessary conditions:

Lemma 4.3 *Let* $\sigma \in]0, \min\{\frac{\pi}{\omega}, 2\}[-\{1\}$ *be such that* $\frac{(2+\sigma)\omega}{\pi} \notin \mathbf{N}$, $u \in C^1([0,T]; C(\Omega)) \cap C([0,T]; \tilde{C}^2(\Omega)) \cap B([0,T]; \tilde{C}^{2+\sigma}(\Omega))$ *with* $\partial_t u \in B([0,T]; C^{\sigma}(\Omega))$. *Set* $f(t,.) := \partial_t u - \Delta_z u$, $g(t,.) := u(t,.)_{\partial\Omega}$, $u_0 := u(0,.)$. *Then,*

(I) $f \in C([0,T]; C(\Omega)) \cap B([0,T]; C^{\sigma}(\Omega))$;

(II) $g \in C([0,T]; C^2(\partial\Omega)) \cap B([0,T]; C^{2+\sigma}(\partial\Omega)) \cap C^1([0,T]); C(\partial\Omega))$, $\partial_t g \in B([0,T]; C^{\sigma}(\partial\Omega))$;

(III) $u_0 \in \tilde{C}^{2+\sigma}(\Omega)$;

(IV) $u_{0\partial\Omega} = g(0,.)$;

(V) $\partial_t g(0,.) - f(0,.)_{\partial\Omega} = \Delta u_{0\partial\Omega}$;

(VI) $\partial_t g - f_{\partial\Omega} \in C^{\frac{\sigma}{2}}([0,T]; C(\partial\Omega))$.

Proof The conditions (I)-(V) are clear.

Concerning (VI), observe that $\partial_t g - f_{|\partial\Omega} = \Delta u_{|\partial\Omega}$. We have

$$u = \mathcal{E}g + (u - \mathcal{E}g).$$

By proposition 2.1 $\mathcal{E}g$ is lipschitz continuous with values in $C^\sigma(\Omega)$ and bounded with values in $C^{2+\sigma}(\Omega)$. It follows from corollary 2.1(IV) that $\mathcal{E}g \in C^{\frac{\sigma}{2}}([0,T]; C^2(\Omega))$, so that $\Delta\mathcal{E}g \in C^{\frac{\sigma}{2}}([0,T]; C(\Omega))$. Consider now $u - \mathcal{E}g$. This mapping has the same regularity of u and, further, $(u - \mathcal{E}g)_{|\partial\Omega} = 0$. From lemma 4.1 we have that $C(\Omega) \in J_{1-\frac{\sigma}{2}}(C^{\sigma-2}(\Omega), C^\sigma(\Omega))$. This implies that $D(A) \in J_{1-\frac{\sigma}{2}}(C_0^\sigma(\Omega), \tilde{C}_0^{2+\sigma}(\Omega))$. We conclude that $(u - \mathcal{E}g \in C^{\frac{\sigma}{2}}([0,T]; D(A))$, which implies $\Delta(u - \mathcal{E}g) \in C^{\frac{\sigma}{2}}([0,T]; C(\Omega))$.

Theorem 4.2 *Let $\sigma \in]0, \min\{\frac{\pi}{\omega}, 2\} - \{1\}$ be such that $\frac{(2+\sigma)\omega}{\pi} \notin \mathbf{N}$. Assume that f, g and u_0 satisfy the conditions (I)-(VI) in lemma 4.3. Then problem (4.1) has a unique solution $u \in\in C^1([0,T]; C(\Omega)) \cap C([0,T]; \tilde{C}^2(\Omega)) \cap B([0,T]; \tilde{C}^{2+\sigma}(\Omega))$ with $\partial_t u \in B([0,T]; C^\sigma(\Omega))$.*

Proof It is very similar (with obvious modifications, using theorem 3.3 and proposition 3.3) to the proof of theorem 2 in [12].

We conclude generalizing one of the most classical results in parabolic equations (see, for example, [16] IV, theorem 5.1).

Let $\sigma \in]0, \min\{2, \frac{\pi}{\omega}\}[-1$, such that $\frac{(2+\sigma)\omega}{\pi} \notin \mathbf{N}$. We set $Q := [0,T] \times \Omega$ and

$$\begin{aligned} &\tilde{C}^{1+\frac{\sigma}{2}, 2+\sigma}(Q) \\ &:= \{u \in C^{1+\frac{\sigma}{2}}([0,T]; C(\Omega)) \cap B([0,T]; \tilde{C}^{2+\sigma}(\Omega)) | \partial_t u \in B([0,T]; C^\sigma(\Omega))\}. \end{aligned} \tag{4.16}$$

We shall give necessary and sufficient conditions in order that problem (4.1) have a solution $u \in \tilde{C}^{1+\frac{\sigma}{2}, 2+\sigma}(Q)$. We recall the following result, due to E. Sinestrari (see [20]):

Theorem 4.3 *Let E be a Banach space, A a linear operator in E whose resolvent $\rho(A)$ contains $\{\lambda \in \mathbf{C} | |\lambda| \geq R, Re\lambda \geq 0\}$, for some $R \geq 0$, and there exists $M > 0$ such that, for every λ in the previous set, $||(\lambda - A)^{-1}||_{\mathcal{L}(E)} \leq M|\lambda|^{-1}$. Consider the Cauchy problem*

$$\left\{ \begin{array}{rl} u'(t) = Au(t) + f(t), & t \in [0,T], \\ u(0) = u_0. & \end{array} \right. \tag{4.17}$$

Then, if, for some $\alpha \in]0,1[$, $f \in C^{\alpha}([0,T];E)$, $u_0 \in D(A)$ and $Au_0 + f(0) \in (E, D(A))_{\alpha,\infty}$, problem (4.17) has a unique solution $u \in C^{1+\alpha}([0,T]; E) \cap C^{\alpha}([0,T]; D(A))$.

Theorem 4.4 *Let $\sigma \in]0, \min\{\frac{\pi}{\omega}, 2\} - \{1\}$. Then the following conditions are necessary and sufficient in order that problem (4.1) have a solution $u \in \tilde{C}^{1+\frac{\sigma}{2}, 2+\sigma}(Q)$:*

(I) $f \in C^{\frac{\sigma}{2}}([0,T]; C(\Omega)) \cap B([0,T]; C^{\sigma}(\Omega))$;

(II) $g \in C^{1+\frac{\sigma}{2}}([0,T]; C(\partial\Omega)) \cap B([0,T]; C^{2+\sigma}(\partial\Omega))$, $\partial_t g \in B([0,T]; C^{\sigma}(\partial\Omega))$;

(III) $u_0 \in \tilde{C}^{2+\sigma}(\Omega)$;

(IV) $u_{0|\partial\Omega} = g(0,.)$;

(V) $\partial_t g(0,.) - f(0,.)_{|\partial\Omega} = \Delta u_{0|\partial\Omega}$.

Proof Concerning the necessity, the only nontrivial point is that $\Delta u \in C^{\frac{\sigma}{2}}([0,T]; C(\Omega))$. Observe that, if $u \in \tilde{C}^{1+\frac{\sigma}{2}, 2+\sigma}(Q)$, then $u \in C^1([0,T]; C(\Omega)) \cap C([0,T]; \tilde{C}^2(\Omega)) \cap B([0,T]; \tilde{C}^{2+\sigma}(\Omega))$ and $\partial_t u \in B([0,T]; C^{\sigma}(\Omega))$. In the proof of lemma 4.3 it was shown that a function with these properties is such that $\Delta u \in C^{\frac{\sigma}{2}}([0,T]; C(\Omega))$.

Concerning the sufficiency, observe that the assumptions (I)-(V) imply the assumptions (I)- (VI) of lemma 4.3. So, by theorem 4.2, there exists a unique solution $u \in C^1([0,T]; C(\Omega)) \cap C([0,T]; \tilde{C}^2(\Omega)) \cap B([0,T]; \tilde{C}^{2+\sigma}(\Omega))$ with $\partial_t u \in B([0,T]; C^{\sigma}(\Omega))$. To conclude, it remains only to show that $u \in C^{1+\frac{\sigma}{2}}([0,T]; C(\Omega))$. Subtracting $\mathcal{E}g$ from u, one is reduced to the case $g \equiv 0$. But now, owing to theorem 3.3, we can apply theorem 4.3 and get the conclusion.

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