A priori estimates for a singularly perturbed mixed order boundary value problem

R. Denk

NWF I-Mathematik, Universität Regensburg, D-93040 Regensburg, Germany

and L. Volevich[1]

Keldysh Institute of Applied Mathematics, Russian Acad. Sci., Miusskaya sqr. 4, 125047 Moscow, Russia

Abstract. In this paper we study mixed-order (Douglis–Nirenberg) boundary value problems which depend on a real parameter but which are not elliptic with parameter in the sense of Agmon–Agranovich–Vishik. Using the method of the Newton polygon, we are able to prove a priori estimates for the solutions of such problems in corresponding Sobolev spaces. For the related singularly perturbed problem the boundary layer structure of the solutions is described. As an application of the a priori estimate, we obtain new estimates for a transmission problem studied by Faierman [7].

1. Introduction

The aim of this paper is to study mixed order systems of partial differential operators with block structure of the form

$$A(x,D,\lambda) = \begin{pmatrix} A_{11}(x,D) & A_{12}(x,D) \\ A_{21}(x,D) & A_{22}(x,D) - \lambda I \end{pmatrix} \tag{1.1}$$

depending on the real parameter λ and acting on a compact manifold M with boundary ∂M. The system (1.1) is assumed to have Douglis–Nirenberg structure and will be supplemented with general boundary conditions. Note that the operator (1.1) may be considered as some mixture of a parameter-independent Douglis–Nirenberg system and a λ-dependent system. If the matrix $A_{11}(x,D)$ were replaced by $A_{11}(x,D) - \lambda I$ we would obtain a parameter-dependent mixed order system as considered in [4], [12], [13] and other papers.

There are several reasons for studying the (non-standard) operator matrix (1.1). First of all, we want to mention the investigation of the resolvent of a Douglis–Nirenberg system on a manifold with boundary. In general, it is not possible to

[1]The work of the second author is supported by RFBR grant No 0001-00387.

assign to such a system a parameter-dependent quasi-homogeneous principal symbol. If, for instance, we consider a 2×2 Douglis–Nirenberg system

$$\begin{pmatrix} A_{11}(x,D) - \lambda & A_{12}(x,D) \\ A_{21}(x,D) & A_{22}(x,D) - \lambda \end{pmatrix} \tag{1.2}$$

with $\operatorname{ord} A_{11} > \operatorname{ord} A_{22}$, there is no definite "weight" of the parameter λ. Therefore, it is not possible to apply the theory of parameter-ellipticity developed by Agmon [1] and Agranovich–Vishik [3]. The aim of the investigation of boundary value problems connected with (1.2) is to prove unique solvability and *uniform* (with respect to λ) estimates for the solution for large λ. For scalar equations and systems of constant order, the analog results follow from the Agmon–Agranovich–Vishik theory. For general mixed order systems, this problem still seems to be open (see [4], [12], [13] for partial results).

To address this problem in its full generality, one can use the concept of the Newton polygon which turned out to be very fruitful both in the theory of parameter-dependent Douglis–Nirenberg systems on closed manifolds and in the theory of singularly perturbed operator pencils (see, e.g., [4], [5], [6]). The basic idea of the Newton polygon approach is to assign to the parameter λ various weights and to obtain for each weight a quasi-homogeneous symbol of the operator. If in (1.2) we assign to λ the weight $\operatorname{ord} A_{22}$, the quasi-homogeneous symbol with respect to this weight will be

$$\begin{pmatrix} A_{11}(x,\xi) & A_{12}(x,\xi) \\ A_{21}(x,\xi) & A_{22}(x,\xi) - \lambda \end{pmatrix}$$

which is exactly the principal quasi-homogeneous symbol of (1.1). For each weight of the parameter we obtain some operator with quasi-homogeneous principal symbol. So we are led to the question under which conditions we can prove uniform estimates for the solutions of (1.1).

A second reason for studying operators of the form (1.1) is given by the close connection of the parameter-dependent operator (1.1) to singularly perturbed boundary value problems (cf. [8], [14], [16]). Let us consider the equation $A(x,D,\lambda)u = 0$ with A being given by (1.1), supplemented with appropriate boundary conditions. Setting $\lambda = \varepsilon^{-1}$ and multiplying the second block row in (1.1) by ε, we obtain the system

$$\begin{aligned} A_{11}(x,D)u_1 + A_{22}(x,D)u_2 &= 0\,, \\ \varepsilon\big(A_{21}(x,D)u_1 + A_{22}(x,D)u_2\big) - u_2 &= 0\,. \end{aligned} \tag{1.3}$$

Here the question is under which conditions on the boundary operators there exist estimates on the solution which are *uniform* with respect to ε for $\varepsilon \to 0$. Moreover, one wants to find an asymptotic expansion in ε for the solution and construct solutions of boundary layer type. As the equations (1.3) are only a re-formulation of the equation $A(x,D,\lambda)u = 0$, the results on (1.1) immediately imply results on singularly perturbed boundary value problems of the form (1.3).

Finally, we want to mention the question of spectral asymptotics for boundary value problems and transmission problems with indefinite weight functions, cf., e.g., [2] and the references therein. In the case where the weight function vanishes identically in some subdomain, a model transmission problem appears which has been studied in a recent paper by Faierman [7]. Again one of the main goals is to obtain uniform estimates on the solutions of the transmission problem, as this finally leads to spectral asymptotics (see [2]).

Summarizing, we can see that in all these applications the question is to find conditions, in particular on the boundary operators, which ensure estimates on the solutions which are uniform with respect to the parameter. It should be possible to formulate these conditions similarly to the classical (parameter-independent or parameter-dependent in the sense of Agmon–Agranovich–Vishik) conditions of Shapiro–Lopatinskii type. As in the theory of parameter-ellipticity, the estimates should hold in appropriate parameter-dependent Sobolev spaces, and so the second question is to define spaces which are suitable for operators of the form (1.1). Speaking more accurately we endow the traditional Sobolev space with parameter-dependent norm adjusted to treat our problems. Here the main difficulty lies in the definition of the boundary Sobolev spaces (boundary parameter-dependent norms).

A first result in this direction is given in the paper [6] where scalar operator pencils are considered which are polynomial with respect to the parameter λ but where the highest term with respect to λ contains an operator of positive order. (Note that, roughly speaking, the determinant of the symbol of the system (1.1) is of this form.) The case of a system of the form (1.1) is more complicated than the case of a scalar pencil depending polynomially on λ, due to the fact that now the solution consists of several components which are connected by the boundary conditions and the operator $A(x, D, \lambda)$ and which belong to different parameter-dependent Sobolev spaces.

The present paper solves the problem described above for the system (1.1) by defining the notion of weak parameter-ellipticity and proving an a priori estimate for the corresponding weakly parameter-elliptic boundary value problems (with general boundary conditions). The notion of weak parameter-ellipticity and the norms in the a priori estimates are defined in terms of the Newton polygon connected with the boundary value problem, continuing the program started in the papers [4]–[6]. It seems that the Newton polygon approach provides the appropriate tool for the investigation and understanding of non-standard boundary value problems like (1.1) and others.

The paper is organized as follows. In Section 2 we state the precise assumptions on the structure of the operator (1.1) and the boundary operators, and define the notion of weak parameter-ellipticity for such boundary value problems. In Section 3 we prove the basic ODE estimate for the so-called fundamental system of solutions of the corresponding model problem in the half space. These estimates are re-written in terms of the Newton polygon in Section 4, where also basic definitions for Sobolev spaces connected with the Newton polygon can be found. The ODE estimates will be the essential step for the proof of a priori estimates in Section 5. In Section 6 we apply these results to transmission problems, and in Section 7 we discuss the formal asymptotic solution for the related singularly perturbed problem.

2. The Newton polygon and weakly parameter-elliptic boundary value problems

As mentioned in the introduction, let $A(x,D,\lambda)=(a_{ij}(x,D))_{i,j=1,\dots N}$ be an $N\times N$-operator matrix acting on a compact manifold M with boundary ∂M. We assume that this operator has Douglis–Nirenberg structure, i.e. there exist nonnegative integer numbers s_j and t_j for $j=1,\dots,N$ such that

$$\operatorname{ord} a_{ij}(x,D)\le s_i+t_j \quad (i,j=1,\dots,N)\,. \tag{2.1}$$

The operator $a_{ij}(x,D)$ is assumed to be of the form

$$a_{ij}(x,D)=\sum_{|\alpha|\le s_i+t_j} a_{ij\alpha}(x)D^\alpha$$

with infinitely smooth scalar coefficients. The principal symbol of $a_{ij}(x,D)$ is defined as

$$a_{ij}^{(0)}(x,\xi)=\sum_{|\alpha|=s_i+t_j} a_{ij\alpha}(x)\xi^\alpha\,.$$

Note that $a_{ij}^{(0)}=0$ if the order of a_{ij} is less than s_i+t_j. Here and in the following, we use the standard multi-index notation $D^\alpha=(-i)^{|\alpha|}(\frac{\partial}{\partial x_1})^{\alpha_1}\cdots(\frac{\partial}{\partial x_n})^{\alpha_n}$ and $\xi^\alpha=\xi_1^{\alpha_1}\cdot\dots\cdot\xi_n^{\alpha_n}$.

We suppose that there exist numbers N_1 and N_2 with $N_1+N_2=N$ such that

$$\begin{aligned} s_1+t_1=s_2+t_2=\dots=s_{N_1}+t_{N_1}&=\quad 2\mu\,,\\ s_{N_1+1}+t_{N_1+1}=\dots=s_N+t_N&=2m-2\mu \end{aligned} \tag{2.2}$$

with integers $m>\mu>0$. Accordingly we write the operator matrix $A(x,D,\lambda)$ in the form

$$A(x,D,\lambda)=\begin{pmatrix} A_{11}(x,D) & A_{12}(x,D)\\ A_{21}(x,D) & A_{22}(x,D)-\lambda I_{N_2}\end{pmatrix},$$

where we have set

$$\begin{aligned} A_{11} &= (a_{ij}(x,D))_{i,j=1,\dots,N_1},\\ A_{12} &= (a_{ij}(x,D))_{i=1,\dots,N_1,\, j=N_1+1,\dots,N},\\ A_{21} &= (a_{ij}(x,D))_{i=N_1+1,\dots,N,\, j=1,\dots,N_1},\\ A_{22} &= (a_{ij}(x,D))_{i,j=N_1+1,\dots,N}. \end{aligned}$$

Here I_{N_2} stands for the $N_2 \times N_2$-dimensional unit matrix. The principal symbol of the operator $A(x, D, \lambda)$ is given by

$$A^{(0)}(x, \xi, \lambda) := \begin{pmatrix} A_{11}^{(0)}(x, \xi) & A_{12}^{(0)}(x, \xi) \\ A_{21}^{(0)}(x, \xi) & A_{22}^{(0)}(x, \xi) - \lambda I_{N_2} \end{pmatrix}$$

with $A_{ij}^{(0)}$ denoting the principal symbols of A_{ij}. For reasons of homogeneity, we will replace λ in the following by $q^{2m-2\mu}$; for simplicity, we will write $A(x, D, q)$ instead of $A(x, D, q^{2m-2\mu})$.

We assume that we have a matrix

$$B(x, D) = \Big(b_{ij}(x, D)\Big)_{\substack{i=1,\dots,N_1\mu+N_2(m-\mu) \\ j=1,\dots,N}} \tag{2.3}$$

of boundary operators of the form

$$b_{ij}(x, D)u = \Big(\sum_{|\beta| \le m_i + t_j} b_{ij\beta}(x) D^\beta u\Big)\Big|_{\partial M}$$

of order $\le m_i + t_j$ with coefficients $b_{ij\beta} \in C^\infty(\overline{M})$. Here m_i are integer numbers, and we additionally assume that

$$m_1 \le m_2 \le \ldots \le m_{N_1\mu} < m_{N_1\mu+1} \le \ldots \le m_{N_1\mu+N_2(m-\mu)}\,. \tag{2.4}$$

In the following, we will denote the number of boundary conditions by

$$R := N_1\mu + N_2(m - \mu) \tag{2.5}$$

and we will use the abbreviations

$$R_1 := N_1\mu, \quad R_2 := N_2(m - \mu)\,. \tag{2.6}$$

The aim of this paper is to find a priori estimates for the solutions $u = \begin{pmatrix} u_1 \\ \vdots \\ u_N \end{pmatrix}$ of the boundary value problem

$$A(x, D, q)u = f = \begin{pmatrix} f_1 \\ \vdots \\ f_N \end{pmatrix}, \quad B(x, D)u = g = \begin{pmatrix} g_1 \\ \vdots \\ g_R \end{pmatrix}. \tag{2.7}$$

We will assume that this boundary value problem satisfies an ellipticity condition called weak parameter-ellipticity. First we define this term for the operator A.

Definition 2.1. The symbol $A(x^0,\xi,q)$ is called weakly parameter-elliptic (with parameter $q\in[0,\infty)$) at the point $x^0\in\overline{M}$ if the inequality

$$|\det A^{(0)}(x^0,\xi,q)|\geq C\,|\xi|^{2R_1}\,(q+|\xi|)^{2R_2}\quad(\xi\in\mathbb{R}^n\,,\ q\in[0,\infty)) \tag{2.8}$$

holds with a constant C not depending on ξ or q. The operator $A(x,D,q)$ and its symbol are called weakly parameter-elliptic in $\overline{M}$ if (2.8) holds for every $x^0\in\overline{M}$. For operators acting in $\mathbb{R}^n$ the analog definition holds.

Remark 2.2. a) If A is weakly parameter-elliptic in $\overline{M}$ then the constant C in (2.8) may be assumed to be independent of x^0, too (by continuity and compactness).

b) The right-hand side of (2.8) has a close connection to the so-called Newton polygon corresponding to the symbol $\det A(x^0,\xi,q)$. This polygon is defined as the convex hull of the set

$$\{(0,0),(0,2R_2),(2R_1,2R_2),(2R_1+2R_2,0)\}$$

(cf. Figure 1). The points $(2R_1,2R_2)$ and $(2R_1+2R_2,0)$ of this polygon correspond to the expressions $|\xi|^{2R_1}q^{2R_2}$ and $|\xi|^{2R_1+2R_2}$. The sum of these two terms is equivalent to the right-hand side of (2.8). We will come back to this point of view in Section 4.

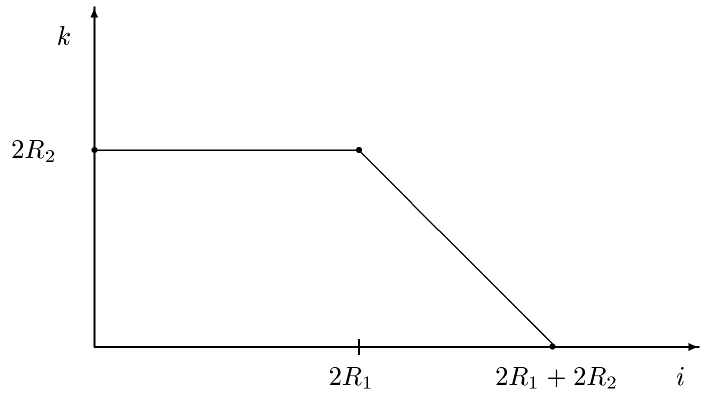

Fig. 1. The Newton polygon $N_{(2R_1,2R_2)}$.

c) Denote

$$P(x,\xi,q):=\det A^{(0)}(x,\xi,q)\,.$$

For a fixed x this is a polynomial in (ξ,q) of order $2R=2\mu N_1+2(m-\mu)N_2=2R_1+2R_2$ and of order $2R_2$ with respect to q. Then we can write this polynomial in the form

$$P(x,\xi,q)=P_{2R}(x,\xi)+q\,P_{2R-1}(x,\xi)+\cdots+q^{2R_2}\,P_{2R_1}(x,\xi),$$

where P_j are polynomials in ξ of order j. It follows from the definition of the determinant that

$$P_{2R}(x,\xi) = \det A^{(0)}(x,\xi,0), \quad P_{2R_1}(x,\xi) = (-1)^{N_2} \det A_{11}^{(0)}(x,\xi)\,.$$

Now we obtain from Lemma 3.2 in [5] the following equivalent conditions for weak parameter-ellipticity.

Lemma 2.3. *The operator A is weakly parameter-elliptic in $\overline{M}$ if and only if the following conditions are satisfied:*

(i) *For all $x^0 \in \overline{M}$ the matrix operator $A(x,D,0)$ is elliptic in the sense of Douglis-Nirenberg.*

(ii) *For all $x^0 \in \overline{M}$ the matrix operator $A_{11}(x,D)$ is elliptic in the sense of Douglis-Nirenberg.*

(iii) $\det A^{(0)}(x^0,\xi,q) \neq 0$ *for all $x^0 \in \overline{M}$, $\xi \in \mathbb{R}^n \backslash \{0\}$ and $q \in (0,\infty)$.*

If A is weakly parameter-elliptic and $x^0 \in \partial M$ is fixed we rewrite A in local coordinates corresponding to x^0, i.e. we choose a coordinate system in a neighbourhood of x^0 such that locally ∂M is given by the equation $x_n = 0$ and M is given by $x_n > 0$. We will write $x = (x', x_n)$ for the variables and (ξ', τ) for the dual variables. Due to 2.3 (iii), the polynomial $\det A^{(0)}(x^0,\xi',\tau,q)$ has no real roots τ for $\xi' = (\xi_1,\dots,\xi_{n-1}) \neq 0$. If $n > 2$ this implies that this polynomials has exactly R roots in $\mathbb{C}_+ := \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$ with R defined in (2.5); for $n = 2$ we will assume this in the following. In local coordinates corresponding to $x^0 \in \partial M$ we define (cf. [4])

$$Q(x^0,\tau) := \tau^{-2R_1} \det A^{(0)}(x^0,0,\tau,1) \quad (\tau \in \mathbb{R})\,. \tag{2.9}$$

As already mentioned in the introduction, the boundary value problem (2.7) may equivalently be formulated as a singularly perturbed boundary value problem. To introduce the polynomial Q defined above, might look unusual in elliptic theory, but in fact it is a basic notion in singular perturbation theory. The condition of the next definition first appeared in the paper of Vishik–Lyusternik [16] on singular perturbations and was called there the condition of regular degeneration. Let us emphasize here that this is an example of the usefulness of changing the point of view between large parameter problems and singular perturbations. Note that according to (2.8) the polynomial (2.9) has no real roots, its roots belong to $\mathbb{C}_+$ and to $\mathbb{C}_-$.

Definition 2.4. We say that $A(x,\xi',\tau,q)$ satisfies the Vishik–Lyusternik condition if for each $x^0 \in \partial M$ the polynomial $Q(x^0,\tau)$ has exactly R_2 roots in $\mathbb{C}_+$.

Remark 2.5. In the important special case $N = 2$ and $N_1 = N_2 = 1$ the condition of Definition 2.4 is satisfied automatically if A is weakly parameter-elliptic. To see this, we note that in this case $A_{ij}^{(0)}(x^0,0,\tau)$ is a (scalar) homogeneous polynomial in

τ of degree s_i+t_j. Thus we can write $A^{(0)}_{ij}(x^0,0,\tau)=a_{ij}\tau^{s_i+t_j}$ with $a_{ij}\in\mathbb{C}$. Due to weak parameter-ellipticity, we have

$$\det A^{(0)}(x^0,0,\tau,1)\neq 0 \quad \text{for all } \tau\in\mathbb{R}\backslash\{0\}\,.$$

Therefore $Q(x^0,\tau)=(a_{11}a_{22}-a_{12}a_{21})\tau^{2R_2}-a_{11}$ has no real roots and exactly R_2 roots in $\mathbb{C}_+$.

Now we want to formulate the analog of the standard Shapiro–Lopatinskii condition in the case of the boundary value problem (1.1), (2.3) where we have a more delicate dependence on the parameter than in the classical case of the Agmon–Agranovich–Vishik theory (for results on parameter-elliptic Douglis–Nirenberg systems, see also [15]). For this we consider $x^0\in\partial M$ and the corresponding local system of coordinates. According to the block form of the matrix A we write $v(t)=\begin{pmatrix} v^{(1)}(t)\\ v^{(2)}(t)\end{pmatrix}$, where $v^{(1)}$ consists of the first N_1 components of the vector v and $v^{(2)}$ consists of the other N_2 components. Accordingly we write

$$B(x,D)=\begin{pmatrix} B_{11}(x,D) & B_{12}(x,D)\\ B_{21}(x,D) & B_{22}(x,D)\end{pmatrix}$$

where B_{ij} are matrix differential operators and their sizes are, correspondingly, $R_1\times N_1, R_1\times N_2, R_2\times N_1$ and $R_2\times N_2$. We also set

$$\begin{aligned} B_1(x,D) &:= \big(B_{11}(x,D)\,,\quad B_{12}(x,D)\big)\,,\\ B_2(x,D) &:= \big(B_{21}(x,D)\,,\quad B_{22}(x,D)\big)\,. \end{aligned}$$

Definition 2.6. The boundary value problem (A,B) of the form (1.1), (2.3) is called weakly parameter-elliptic (with parameter in $[0,\infty)$) if the following conditions are satisfied:

(i) $A(x,D,q)$ is weakly parameter-elliptic in $\overline{M}$.

(ii) For every $x^0\in\partial M$, $\xi'\in\mathbb{R}^{n-1}\backslash\{0\}$, $q\in[0,\infty)$ and every $g=(g_1,\dots,g_R)\in\mathbb{C}^R$ the problem

$$A^{(0)}(x^0,\xi',D_t,q)\begin{pmatrix} w^{(1)}(t)\\ w^{(2)}(t)\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}\quad (t>0)\,, \tag{2.10}$$

$$B^{(0)}(x^0,\xi',D_t)\begin{pmatrix} w^{(1)}(t)\\ w^{(2)}(t)\end{pmatrix}\Bigg|_{t=0} = g\,, \tag{2.11}$$

$$w^{(i)}(t)\ \to\ 0 \quad \text{for } t\to\infty \quad (i=1,2)$$

has a unique solution. Here we have set $D_t=-i\frac{\partial}{\partial t}$.

(iii) For every $x^0 \in \partial M$, $\xi' \in \mathbb{R}^{n-1}\backslash\{0\}$ and every $h^{(1)} \in \mathbb{C}^{R_1}$ the problem

$$\begin{aligned} A_{11}^{(0)}(x^0,\xi',D_t)w^{(1)}(t) &= 0 \quad (t>0)\,, &(2.12)\\ B_{11}^{(0)}(x^0,\xi',D_t)w^{(1)}(t)\Big|_{t=0} &= h^{(1)}\,, &(2.13)\\ w^{(1)}(t) &\to 0 \quad \text{for } t\to\infty \end{aligned}$$

has a unique solution.

(iv) For every $x^0 \in \partial M$ and every vector $h^{(2)} \in \mathbb{C}^{R_2}$ the problem

$$\begin{aligned} A^{(0)}(x^0,0,D_t,1)\begin{pmatrix} v^{(1)}(t)\\ v^{(2)}(t)\end{pmatrix} &= \begin{pmatrix}0\\0\end{pmatrix} \quad (t>0)\,, &(2.14)\\ B_2^{(0)}(x^0,0,D_t)\begin{pmatrix} v^{(1)}(t)\\ v^{(2)}(t)\end{pmatrix}\Bigg|_{t=0} &= h^{(2)}, &(2.15)\\ v^{(i)}(t) &\to 0 \quad \text{for } t\to\infty \quad (i=1,2) \end{aligned}$$

has a unique solution.

Remark 2.7. a) Condition (iii) in Definition 2.6 is the usual Shapiro–Lopatinskii condition for the Douglis–Nirenberg system $A_{11}(x,D)$ with boundary operators given by $B_{11}(x,D)$. Similarly, if we set $q = 0$ in 2.6 (ii), we obtain the usual (parameter-independent) Shapiro–Lopatinskii condition for the whole system $(A(x,D,0),B(x,D))$.

b) For $\xi' = 0$ condition 2.6 (ii), in general, is not satisfied, even for positive q. This is the essential difference if we compare this definition with the definition of elllipticity with parameter due to Agmon–Agranovich–Vishik.

Due to homogeneity it is sufficient to consider (ii) on the hemisphere $\{(\xi',q) \in \mathbb{R}^{n-1}\times[0,\infty) : |\xi'|^2+q^2=1\}$. Contrary to the Agmon–Agranovich–Vishik condition (satisfied for the points of the closed hemisphere), 2.6 (ii) is satisfied on the hemisphere with punctured point $(0,1)$. Conditions (iii) and (iv) give in some sense a completion for this point. We want to explain this point of view.

Consider the transformation

$$(\xi',q) \longmapsto \left(\frac{\xi'}{(|\xi'|^2+q^2)^{1/2}}\,,\ \frac{q}{(|\xi'|^2+q^2)^{1/2}}\right)$$

which maps every $(\xi',q) \in \mathbb{R}^{n-1}\times[0,\infty)$ with $|\xi'|+q \neq 0$ to a point on the hemisphere described above. Under this transformation, the point $(0,1)$ appears both as the image of $(0,q)$ with finite $q>0$ and as the limit of the images of points (ξ',q) with $q\to+\infty$.

Condition 2.6 (iv), where we have $\xi'=0$, corresponds to the first group of points, i.e. to the images of the points $(0,q)$. Now let us consider $q\to\infty$. Setting $\lambda=\varepsilon^{-1}$ and considering homogeneous right-hand side, we obtain the system (1.3) with small parameter. Taking the limit $\varepsilon \to 0$, we get $u_2 = 0$ in (1.3). Thus we have to solve the system

$$A_{11}(x, D)u_1 = 0$$

with an overdetermined system of boundary conditions. If we take only the first R_1 boundary conditions, we obtain a problem for which condition 2.6 (iii) guarantees solvability. To satisfy all boundary conditions, we need condition 2.6 (iv) which, due to Vishik–Lyusternik theory, allows us to add to the above solution a combination of boundary layers.

We will come back to the fact that condition (iv) is connected with the existence of boundary layers in Section 7.

c) The conditions 2.6 (ii)–(iv) may be formulated in an algebraic way, cf. [17]. For instance, condition 2.6 (iv) is equivalent to the following condition:

(iv′) Fix $x^0 \in \partial M$ and let $\gamma^{(2)}$ be a contour in $\mathbb{C}_+$ enclosing all zeros of $Q(x^0, \cdot)$ with positive imaginary part. Then the rank of the matrix

$$\int_{\gamma^{(2)}} B_2^{(0)}(x^0, 0, \tau)\left[A^{(0)}(x^0, 0, \tau, 1)\right]^{-1} \left(I_N, \tau I_N, \dots, \tau^{2m-1} I_N\right) d\tau$$

equals R_2 (i.e. the rank of this matrix is maximal).

3. An estimate for the solutions of the ordinary differential equation system

The aim of this section is to find, under the assumption of weak parameter-ellipticity, estimates for the solutions of (2.10)–(2.11). So we assume throughout this section that A satisfies the Vishik–Lyusternik condition and that (A, B) satisfies the conditions of Definition 2.6. We fix $x^0 \in \overline{M}$ and write the boundary value problem in coordinates corresponding to x^0. We set $A(\xi, q) := A^{(0)}(x^0, \xi, q)$, $B(\xi) := B^{(0)}(x^0, \xi)$ and $Q(\tau) := Q(x^0, \tau)$.

Lemma 3.1. *Let $\gamma^{(1)}$ denote a contour in $\mathbb{C}_+$ enclosing for $|\xi'| = 1$ all zeros of* $\det A_{11}(\xi', \cdot)$ *with positive imaginary part and let $\gamma^{(2)}$ be a contour in $\mathbb{C}_+$ enclosing all zeros of Q in $\mathbb{C}_+$. Then there exists a $q_0 > 0$ and a numbering $\tau_1(\xi', q), \dots, \tau_R(\xi', q)$ of the zeros of* $\det A(\xi', \cdot, q)$ *with positive imaginary part such that for all $q \geq q_0$ and $|\xi'| = 1$ we have*

(i) *$\gamma^{(1)}$ encloses $\tau_1(\xi', q), \dots, \tau_{R_1}(\xi', q)$,*

(ii) *$\gamma^{(2)}$ encloses $q^{-1}\, \tau_{R_1+1}(\xi', q), \dots, q^{-1}\, \tau_R(\xi', q)$.*

Proof. As already mentioned in Remark 2.2 c), the determinant $P(\xi',\tau,q) = \det A(\xi',\tau,q)$ is weakly parameter-elliptic in the sense of [5]. The behaviour of the zeros of this polynomial is described in Lemma 3.5 of [5]. As a result, using an appropriate numbering, the set $\{\tau_1(\xi',q),\dots,\tau_{R_1}(\xi',q)\}$ tends to the set $\{\tau_1^0(\xi'),\dots,\tau_{R_1}^0(\xi',q)\}$ of all zeros of $\det A_{11}(\xi',\cdot)$ with positive imaginary part, and $q^{-1}\tau_j(\xi',q)\to\tau_j^1$ for $j=R_1+1,\dots,R$, where $\tau_{R_1+1}^1,\dots,\tau_R^1$ are the zeros of Q in $\mathbb{C}_+$. From this we already obtain (i) and (ii). □

Denote by $\mathfrak{M}(\xi',q)$ the finite-dimensional space of solutions of the homogeneous equation

$$A(\xi',D_t,q)v(t)=0 \quad (t>0)\,,$$

and let $\mathfrak{M}_+(\xi',q)$ be the subspace of $\mathfrak{M}(\xi',q)$ consisting of solutions tending to zero as t tends to $+\infty$. The dimension of this space is R. According to Lemma 3.1 this space can be represented as a direct sum

$$\mathfrak{M}_+(\xi',q)=\mathfrak{M}_+^{(1)}(\xi',q)\dotplus\mathfrak{M}_+^{(2)}(\xi',q)\,,$$

where the first space on the right-hand side corresponds to the zeros bounded as $q\to\infty$ and the other subspace corresponds to the other zeros. Note that for $q\ge q_0$

$$\dim\mathfrak{M}_+^{(1)}(\xi',q)=R_1,\quad \dim\mathfrak{M}_+^{(2)}(\xi',q)=R_2\,.$$

Lemma 3.2. *Suppose that condition* (iii) *of Definition* 2.6 *is satisfied. Then there exists $q_0>0$ such that for $|\xi'|=1$ and $q\ge q_0$ the following statements hold.*

(i) *The problem on the half-line*

$$\begin{aligned} A(\xi',D_t,q)v(t) &= 0 \quad (t>0)\,, && (3.1)\\ B_1(\xi',D_t)v(0) &= h^{(1)}\,, && (3.2)\\ v(t) &\in \mathfrak{M}_+^{(1)}(\xi',q) && (3.3)\end{aligned}$$

is uniquely solvable for arbitrary $h^{(1)}\in\mathbb{C}^{R_1}$.

(ii) *There exists an $N\times R_1$ rectangular matrix $N^{(1)}(\xi',\tau,q)$, polynomial with respect to τ and bounded for $|\xi'|=1, q\ge q_0$ and $\tau\in\gamma^{(1)}$, such that*

$$\frac{1}{2\pi i}\int_{\gamma^{(1)}} B_1(\xi',\tau)A^{-1}(\xi',\tau,q)N^{(1)}(\xi',\tau,q)d\tau = I_{R_1}\,,$$

where $\gamma^{(1)}$ is defined in Lemma 3.1.

Proof. (i). Similarly to condition (iv′) in Remark 2.7, a necessary and sufficient condition for unique solvability of the problem (3.1)–(3.3) can be written in the following form (see [17]): the rank of the matrix

$$\Lambda(\xi',q):=\frac{1}{2\pi i}\int_{\gamma^{(1)}} B_1(\xi',\tau)A^{-1}(\xi',\tau,q)(I_N,\tau I_N,\dots,\tau^{2m-1}I_N)d\tau$$

is equal to R_1. First we show that we have

$$A^{-1}(\xi',\tau,q)=\begin{pmatrix} A_{11}^{-1}(\xi',\tau) & 0\\ 0 & 0\end{pmatrix}+O(q^{-2(m-\mu)})\,. \tag{3.4}$$

To see this, we rewrite the matrix $A(\xi,q)$ in the form

$$A(\xi,q)=\begin{pmatrix} A_{11}(\xi) & 0\\ 0 & -q^{m-\mu}I_{N_2}\end{pmatrix}\Big[I_N+H(\xi,q)\Big]\begin{pmatrix} I_{N_1} & 0\\ 0 & q^{m-\mu}I_{N_2}\end{pmatrix},$$

where

$$H(\xi,q):=\begin{pmatrix} 0 & q^{-(m-\mu)}A_{11}^{-1}(\xi)A_{12}(\xi)\\ -q^{-(m-\mu)}A_{21}(\xi) & -q^{-2(m-\mu)}A_{22}(\xi)\end{pmatrix}.$$

From this and the condition $m>\mu$ it follows that the inverse matrix $A^{-1}(\xi,q)=:$ $G(\xi,q)$ exists for large q and is equal to

$$\begin{aligned} G(\xi,q)&=\begin{pmatrix} I_{N_1} & 0\\ 0 & q^{-(m-\mu)}I_{N_2}\end{pmatrix}\Big[I_N-H+O\big(q^{-2(m-\mu)}\big)\Big]\begin{pmatrix} A_{11}^{-1}(\xi) & 0\\ 0 & -q^{-(m-\mu)}I_{N_2}\end{pmatrix}\\ &=\begin{pmatrix} A_{11}^{-1}(\xi) & 0\\ 0 & 0\end{pmatrix}+O(q^{-2(m-\mu)})\,.\end{aligned}$$

From (3.4) we obtain

$$\lim_{q\to\infty}\Lambda(\xi',q)=\frac{1}{2\pi i}\int_{\gamma^{(1)}}\big(B_{11}(\xi',\tau)A_{11}^{-1}(\xi',\tau),0\big)(I_N,\tau I_N,\dots,\tau^{2m-1}I_N)d\tau\,.$$

Denote by $M(\xi',q)$ the $R_1\times R_1$ sub-matrix of $\Lambda(\xi',q)$ consisting of the columns of $\Lambda(\xi',q)$ with indices

$$kN+j\quad(k=0,\dots,\mu-1;\ j=1,\dots,N_1)\,.$$

Then we obtain

$$M(\xi'):=\lim_{q\to\infty}M(\xi',q)=\frac{1}{2\pi i}\int_{\gamma^{(1)}}B_{11}(\xi',\tau)A_{11}^{-1}(\xi',\tau)(I_{N_1},\dots,\tau^{\mu-1}I_{N_1})d\tau\,.$$

According to condition (iii) of Definition 2.6 the matrix $M(\xi')$ is nonsingular. It follows from (3.4) that for $q\geq q_0$, where q_0 is large enough, the matrix $M(\xi',q)$ is also nonsingular, which proves (i).

To prove (ii) it suffices to define (compare [17]) the $N\times R_1$-dimensional matrix $N^{(1)}(\xi',\tau,q)$ as

$$N^{(1)}(\xi',\tau,q):=\begin{pmatrix} N_1^{(1)}(\xi',\tau,q)\\ 0\end{pmatrix}$$

with the $N_1 \times R_1$-dimensional matrix $N_1^{(1)}(\xi', \tau, q)$ being given by

$$N_1^{(1)}(\xi', \tau, q) = (I_{N_1}, \tau I_{N_1}, \ldots, \tau^{\mu-1} I_{N_1}) M^{-1}(\xi', q) .$$

$\square$

Lemma 3.3. *Suppose that condition* (iv) *of Definition* 2.6 *is satisfied. Then there exists* $q_0 > 0$ *such that for* $|\xi'| = 1$ *and* $q \geq q_0$ *the following statements hold.*

(i) *The problem on the half-line*

$$\begin{aligned} A(\xi', D_t, q)\, v(t) &= 0 \quad (t > 0) , && (3.5)\\ B_2(\xi', D_t)\, v(0) &= h^{(2)} , && (3.6)\\ v(t) &\in \mathfrak{M}_+^{(2)}(\xi', q) && (3.7) \end{aligned}$$

is uniquely solvable for arbitrary $h^{(2)} \in \mathbb{C}^{R_2}$.

(ii) *There exists an* $N \times R_2$ *rectangular matrix* $N^{(2)}(\xi', \tau, q)$, *polynomial with respect to* τ *and bounded for* $|\xi'| = 1, q \geq q_0$ *and* $\tau \in \gamma^{(2)}$ *such that*

$$\frac{1}{2\pi i} \int_{\gamma^{(2)}} B_2\Big(\frac{\xi'}{q}, \tau\Big) A^{-1}\Big(\frac{\xi'}{q}, \tau, 1\Big) N^{(2)}(\xi', \tau, q) d\tau = I_{R_2},$$

where $\gamma^{(2)}$ *is the contour defined in Lemma 3.1.*

Proof. (i). As in Lemma 3.2 we shall use the criterion from [17] for unique solvability of the problem on the half-line. This criterion can be trivially modified: the matrices $\tau^l I_N$ can be replaced by $(c\tau)^l I_N$ with arbitrary real $c \neq 0$.

To prove (i) we check that the rank of the matrix

$$\frac{1}{2\pi i} \int_{\gamma^{(2)}(q)} B_2(\xi', \tau) A^{-1}(\xi', \tau, q) \Big(I_N, \frac{\tau}{q} I_N, \ldots, \Big(\frac{\tau}{q}\Big)^{2m-1} I_N \Big) d\tau$$

is equal to R_2. Here $\gamma^{(2)}(q) := \{q\tau : \tau \in \gamma^{(2)}\}$. As in the proof of Lemma 3.2 we set $G(\xi, q) = A^{-1}(\xi, q)$. The element $g_{ij}(\xi, q)$ of the matrix G is a homogeneous function in (ξ, q) of degree $-s_j - t_i$. Therefore, after the change of variables $\tau = q\tilde{\tau}$ in the integral above, we obtain the matrix

$$\begin{pmatrix} q^{m_{R_1+1}} & & \\ & \ddots & \\ & & q^{m_R} \end{pmatrix} \Lambda(\xi', q) \begin{pmatrix} q^{1-s_1} & & \\ & \ddots & \\ & & q^{1-s_N} \end{pmatrix}$$

where

$$\Lambda(\xi', q) = \frac{1}{2\pi i} \int_{\gamma^{(2)}} B_2\Big(\frac{\xi'}{q}, \tau\Big) A^{-1}\Big(\frac{\xi'}{q}, \tau, 1\Big) (I_N, \tau I_N, \ldots, \tau^{2m-1} I_N) d\tau .$$

Obviously we have to check that the rank of this matrix in the neighbourhood of $q=+\infty$ is equal to R_2. According to condition (iv) of Definition 2.6 the rank of the matrix

$$\Lambda(0,1):=\frac{1}{2\pi i}\int_{\gamma^{(2)}} B_2(0,\tau)A^{-1}(0,\tau,1)(I_N,\tau I_N,\dots,\tau^{2m-1}I_N)d\tau$$

is equal to R_2. For $\tau\in\gamma^{(2)}$ and $|\xi'|=1$ the matrix-valued function $B_2(\xi'/q,\tau)$ $A^{-1}(\xi'/q,\tau,1)$ uniformly tends to $B_2(0,\tau)A^{-1}(0,\tau,1)$ as $q\to+\infty$. This proves (i).

To prove (ii), we choose an $R_2\times R_2$ dimensional invertible sub-matrix of $\Lambda(0,1)$ consisting of the columns, say, with indices $j_1,\dots,j_{R_2}$. Then for large q the sub-matrix of $\Lambda(\xi',q)$ consisting of the same columns $j_1,\dots,j_{R_2}$ is invertible, too. In other words, there exists a $2mN\times R_2$-matrix $\tilde N^{(2)}(\xi',q)$, such that $\Lambda(\xi',q)\tilde N^{(2)}(\xi',q)=I_{R_2}$. Now it remains to set

$$N^{(2)}(\xi',q):=(I_N,\tau I_N,\dots,\tau^{2m-1}I_N)\tilde N^{(2)}(\xi',q)\,.$$

□

Lemma 3.4. *Let* $w(t)=\begin{pmatrix} w^{(1)}(t,\xi',q)\\ w^{(2)}(t,\xi',q)\end{pmatrix}$ *be the solution of* (2.10)–(2.11). *Then for* $|\xi'|=1$ *and sufficiently large* q *the function* w *can be written in the form*

$$\begin{aligned} w(t,\xi',q)&=\frac{1}{2\pi i}\int\limits_{\gamma^{(1)}} A^{-1}(\xi',\tau,q)N^{(1)}(\xi',\tau,q)e^{it\tau}\,d\tau\cdot\psi^{(1)}(\xi',q)\\ &+\begin{pmatrix} q^{-t_1} & & \\ & \ddots & \\ & & q^{-t_N}\end{pmatrix}\frac{1}{2\pi i}\int\limits_{\gamma^{(2)}} A^{-1}\Big(\frac{\xi'}{q},\tau,1\Big)N^{(2)}(\xi',\tau,q)e^{itq\tau}d\tau\cdot\psi^{(2)}(\xi',q)\end{aligned}\tag{3.8}$$

where $\gamma^{(1)}$ *and* $\gamma^{(2)}$ *are given in Lemma* 3.1 *and* $N^{(1)}$, $N^{(2)}$ *are given in Lemma* 3.2 *and* 3.3, *respectively. The vectors* $\psi^{(1)}(\xi',q)\in\mathbb{C}^{R_1}$ *and* $\psi^{(2)}(\xi',q)\in\mathbb{C}^{R_2}$ *are given by*

$$\begin{pmatrix}\psi^{(1)}(\xi',q)\\ \psi^{(2)}(\xi',q)\end{pmatrix}=M(\xi',q)\,g$$

with some matrix $M(\xi',q)=\Big(M_{ij}(\xi',q)\Big)_{i,j=1,\dots,R}$ *satisfying*

$$|M_{ij}(\xi',q)|\le\begin{cases} C\,, & i\le R_1\,,\quad j\le R_1\,,\\ C\,q^{m_{R_1}-m_j}\,, & i\le R_1\,,\quad j>R_1\,,\\ C\,q^{-m_{R_1+1}}\,, & i>R_1\,,\quad j\le R_1\,,\\ C\,q^{-m_j}\,, & i>R_1\,,\quad j>R_1\,.\end{cases}\tag{3.9}$$

Proof. We are following the scheme of [6], Lemmas 3.1–3.3, which uses an idea of Frank [8]. We define w by (3.8) with unknown vectors $\psi^{(i)}(\xi',q)$. First we note that applying the ordinary differential operator $A(\xi', D_t, q)$ to each of the two integrals on the right-hand side of (3.8) gives zero. For the first integral it is obvious. As for the second integral, we note that

$$A(\xi', D_t, q) \begin{pmatrix} q^{-t_1} & & \\ & \ddots & \\ & & q^{-t_N} \end{pmatrix} = \begin{pmatrix} q^{s_1} & & \\ & \ddots & \\ & & q^{s_N} \end{pmatrix} A\Big(\frac{\xi'}{q}, \frac{1}{q} D_t, 1\Big) .$$

To compute $\psi^{(i)}$, we apply the boundary operators to (3.8) and obtain

$$\begin{aligned} &B_1(\xi', D_t) w(t, \xi', q) \\ &= \psi^{(1)}(\xi', q) + \Delta_1 \frac{1}{2\pi i} \int\limits_{\gamma^{(2)}} B_1\Big(\frac{\xi'}{q}, \tau\Big) A^{-1}\Big(\frac{\xi'}{q}, \tau, 1\Big) N^{(2)}(\xi', \tau, q)\, d\tau\, \psi^{(2)}(\xi', q) \end{aligned}$$

with

$$\Delta_1 := \begin{pmatrix} q^{m_1} & & \\ & \ddots & \\ & & q^{m_{R_1}} \end{pmatrix} .$$

Here we used Lemma 3.2 and the homogeneity of b_{ij}. In the same way we obtain, using Lemma 3.3,

$$\begin{aligned} &B_2(\xi', D_t) w(t, \xi', q) \\ &= \frac{1}{2\pi i} \int\limits_{\gamma^{(1)}} B_2(\xi', \tau) A^{-1}(\xi', \tau, q) N^{(1)}(\xi', \tau, q) d\tau\, \psi^{(1)}(\xi', q) + \Delta_2\, \psi^{(2)}(\xi', q) \end{aligned}$$

with

$$\Delta_2 := \begin{pmatrix} q^{m_{R_1+1}} & & \\ & \ddots & \\ & & q^{m_R} \end{pmatrix} .$$

Therefore, condition (2.11) leads to the linear equation system

$$\begin{pmatrix} I_{R_1} & \Delta_1 H_{12} \\ H_{21} & \Delta_2 \end{pmatrix} \begin{pmatrix} \psi^{(1)} \\ \psi^{(2)} \end{pmatrix} = g \,, \tag{3.10}$$

where we have set

$$\begin{aligned} H_{12} &:= \frac{1}{2\pi i} \int\limits_{\gamma^{(2)}} B_1\Big(\frac{\xi'}{q}, \tau\Big) A^{-1}\Big(\frac{\xi'}{q}, \tau, 1\Big) N^{(2)}(\xi', \tau, q)\, d\tau \,, \\ H_{21} &:= \frac{1}{2\pi i} \int_{\gamma^{(1)}} B_2(\xi', \tau) A^{-1}(\xi', \tau, q) N^{(1)}(\xi', \tau, q)\, d\tau \,. \end{aligned}$$

As it is shown in the proof of Lemma 3.5 in [6], for large q the matrix in (3.10) is invertible (here we use $m_{R_1} < m_{R_1+1}$, see (2.4)) and its inverse is given by

$$M = \begin{pmatrix} G_1 & -G_1\Delta_1 H_{12}\Delta_2^{-1} \\ -G_2\Delta_2^{-1}H_{21} & G_2\Delta_2^{-1} \end{pmatrix}$$

with $G_1 := (I_{R_1} - \Delta_1 H_{12}\Delta_2^{-1}H_{21})^{-1}$ and $G_2 := (I_{R_2} - \Delta_2^{-1}H_{21}\Delta_1 H_{12})^{-1}$. As the matrices G_1 and G_2 are bounded for sufficiently large q, the estimate (3.9) follows from this. (We remark that in the situation of Lemma 3.5 in [6] the additional condition $m_j \geq 0$ was assumed. However, from the proof of this lemma it is easily seen that this additional condition is not necessary; the essential condition is $m_{R_1} < m_{R_1+1}$.) □

For $j = 1, \ldots, R$, denote by $w_j = \begin{pmatrix} w_{j1}(t,\xi',q) \\ \vdots \\ w_{jN}(t,\xi',q) \end{pmatrix}$ the solution of (2.10)–(2.11) with g replaced by the j-th unit vector $e_j \in \mathbb{C}^R$.

Theorem 3.5. a) *For $i = 1, \ldots, N_1$, $j = 1, \ldots, R$ and $l = 0, 1, 2, \ldots$ we have*

$$\|D_t^l w_{ji}(\cdot,\xi',q)\|_{L_2(\mathbb{R}_+)} \leq C \begin{cases} |\xi'|^{l-m_j-t_i-1/2}, & l \leq m_{R_1+1}+t_i, \quad j \leq R_1, \\ |\xi'|^{m_{R_1+1}-m_j}(q+|\xi'|)^{l-m_{R_1+1}-t_i-1/2}, & l > m_{R_1+1}+t_i, \quad j \leq R_1, \\ |\xi'|^{l-m_{R_1}-t_i-1/2}(q+|\xi'|)^{m_{R_1}-m_j}, & l \leq m_{R_1}+t_i, \quad j > R_1, \\ (q+|\xi'|)^{l-m_j-t_i-1/2}, & l > m_{R_1}+t_i, \quad j > R_1, \end{cases} \tag{3.11}$$

for all $\xi' \in \mathbb{R}^{n-1}\setminus\{0\}$ and $q \in [0,\infty)$.

b) *For $i = N_1+1, \ldots, N$, $j = 1, \ldots, R$ and $l = 0, 1, 2, \ldots$ we have*

$$\|D_t^l w_{ji}(\cdot,\xi',q)\|_{L_2(\mathbb{R}_+)} \leq C \begin{cases} |\xi'|^{l-m_j-1/2}(q+|\xi'|)^{-t_i}, & l \leq m_{R_1+1}, \quad j \leq R_1, \\ |\xi'|^{m_{R_1+1}-m_j}(q+|\xi'|)^{l-m_{R_1+1}-t_i-1/2}, & l > m_{R_1+1}, \quad j \leq R_1, \\ |\xi'|^{l-m_{R_1}-1/2}(q+|\xi'|)^{m_{R_1}-m_j-t_i}, & l \leq m_{R_1}, \quad j > R_1, \\ (q+|\xi'|)^{l-m_j-t_i-1/2}, & l > m_{R_1}, \quad j > R_1, \end{cases} \tag{3.12}$$

for all $\xi' \in \mathbb{R}^{n-1}\setminus\{0\}$ and $q \in [0,\infty)$.

Proof. a) First assume $i \leq N_1$. Due to unique solvability of (2.10)–(2.11) we have for $\xi' \neq 0$ and $q \in [0,\infty)$

$$w_{ji}(t,\xi',q) = |\xi'|^{-m_j-t_i} w_{ji}\left(|\xi'|t, \frac{\xi'}{|\xi'|}, \frac{q}{|\xi'|}\right)$$

and, consequently,

$$\|D_n^l w_{ji}(\cdot,\xi',q)\|_{L_2(\mathbb{R}_+)} = |\xi'|^{l-m_j-t_i-1/2}\Big\|D_n^l w_{ji}\Big(\cdot,\frac{\xi'}{|\xi'|},\frac{q}{|\xi'|}\Big)\Big\|_{L_2(\mathbb{R}_+)}.$$

Therefore it is sufficient to show that for $|\xi'|=1$ and for all $q\in[0,\infty)$ the following estimate holds:

$$\begin{aligned}&\|D_t^l w_{ji}(\cdot,\xi',q)\|_{L_2(\mathbb{R}_+)}\\ &\quad\le\; C\begin{cases}1\,, & l\le m_{R_1+1}+t_i\,,\quad j\le R_1\,,\\ \tilde q^{\,l-m_{R_1+1}-t_i-1/2}\,, & l> m_{R_1+1}+t_i\,,\quad j\le R_1\,,\\ \tilde q^{\,m_{R_1}-m_j}\,, & l\le m_{R_1}+t_i\,,\quad j> R_1\,,\\ \tilde q^{\,l-m_j-t_i-1/2}\,, & l> m_{R_1}+t_i\,,\quad j> R_1\,,\end{cases}\end{aligned}\tag{3.13}$$

where $\tilde q := \max\{1,q\}$. For $|\xi'|=1$ and $q\in[0,q_0]$ this is true due to continuity and compactness, using condition (ii) of Definition 2.6. So let q be sufficiently large. We use the representation (3.8) for $g=e_j$. Considering $D_t^l w_j$ instead of w_j, we obtain an additional factor τ^l in the integral over $\gamma^{(1)}$ and an additional factor $q^l\tau^l$ in the integral over $\gamma^{(2)}$. Integrating $D_n^l w_j$ with respect to t, we obtain

$$\|D_t^l w_{ji}\|_{L_2(\mathbb{R}_+)} \le C\left(|\psi^{(1)}(\xi',q)| + q^{l-t_i-1/2}|\psi^{(2)}(\xi',q)|\right)\tag{3.14}$$

where we used the fact that we may choose $\gamma^{(1)}$ and $\gamma^{(2)}$ with positive distance from the real axis and $|\cdot|$ stands for the standard norm in $\mathbb{C}^{R_1}$ and $\mathbb{C}^{R_2}$, respectively.

Now the inequalities (3.9) lead to

$$\|D_t^l w_{ji}\|_{L_2(\mathbb{R}_+)} \le C\begin{cases} O(1)+O(q^{l-t_i-1/2-m_{R_1+1}})\,, & j\le R_1\,,\\ O(q^{m_{R_1}-m_j})+O(q^{l-t_i-1/2-m_j})\,, & j>R_1\,,\end{cases}$$

which implies (3.13).

b) The proof for the case $i>N_1$ can be made in exactly the same way with one difference. In part a) we used the fact that the elements of $G(\xi',\tau,q)=A^{-1}(\xi',\tau,q)$ are $O(1)$ for fixed (ξ',τ) and large q. According to (3.4) the elements $g_{ij}(\xi',\tau,q)$ of this matrix for $i>N_1$ are $O(q^{-2(m-\mu)})=O(q^{-2(s_i+t_i)})$, and for such i we can replace (3.14) by the stronger estimate

$$\|D_t^l w_{ji}\|_{L_2(\mathbb{R}_+)} \le C\left(|q^{-t_i}\psi^{(1)}(\xi',q)| + q^{l-t_i-1/2}|\psi^{(2)}(\xi',q)|\right)$$

which leads to the desired estimate. □

4. The Newton polygon and corresponding Sobolev spaces

As it was already mentioned in Remark 2.2 b), the definition of weak parameter-ellipticity has a close connection to the Newton polygon which corresponds to the determinant of the symbol of A. Moreover, the a priori estimates below will use Sobolev spaces which appear in the Newton polygon approach. In the present section, we briefly recall some main concepts and results of this approach and re-write the estimate of Theorem 3.5 in terms of the Newton polygon. For a more detailed exposition, the reader is referred to [9] and [4], [5].

Let $\mathbf{r} = (r_1, r_2)$ be a tuple of real numbers. Then we define the weight function $\Xi_{\mathbf{r}} : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}$ by

$$\Xi_{\mathbf{r}}(\xi, q) := (1 + |\xi|)^{r_1} \ (q + |\xi|)^{r_2} \,. \tag{4.1}$$

Similarly, we define the function $\Phi_{\mathbf{r}}$ for $\mathbf{r} \in \mathbb{R}^2$ by

$$\Phi_{\mathbf{r}}(\xi, q) := |\xi|^{r_1} \ (q + |\xi|)^{r_2} \,. \tag{4.2}$$

If r_1 and r_2 are integer positive numbers, these functions allow a geometrical interpretation which we want to describe now. We define for positive integers r_1, r_2 the Newton polygon $N_{\mathbf{r}}$ as the convex hull of the set $\{(0,0), (0, r_2), (r_1, r_2), (r_1 + r_2, 0)\}$ (cf. also Figure 1). It is easily seen that we have the equivalence

$$\Xi_{\mathbf{r}}(\xi, q) \approx \sum_{(i,k) \in N_{\mathbf{r}} \cap \mathbb{Z}^2} |\xi|^i q^k$$

where the sign $\approx$ means that the quotient of the left-hand side and the right-hand side is bounded from below and from above by positive constants not depending on ξ or q. Similarly, $\Phi_{\mathbf{r}}(\xi, q) \approx \sum_{(i,k)} |\xi|^i q^k$ where now the sum runs over all integer points on the edge connecting (r_1, r_2) and $(r_1 + r_2, 0)$. (This edge is in some sense the leading part of the Newton polygon $N_{\mathbf{r}}$.)

Now let r_1 and r_2 again be arbitrary real numbers. Connected with the weight function $\Xi_{\mathbf{r}}$, we endow the (classical) Sobolev space $H^{r_1+r_2}(\mathbb{R}^n)$ with the parameter-dependent norm $\|\cdot\|_{\mathbf{r}}$ defined by

$$\|u\|_{\mathbf{r}} := \|F^{-1} \Xi_{\mathbf{r}} F u\|_{L_2(\mathbb{R}^n)} \,. \tag{4.3}$$

Here F denotes the Fourier transform. We will write $H_{\mathbf{r}}(\mathbb{R}^n)$ for $H^{r_1+r_2}(\mathbb{R}^n)$ endowed with this norm, omitting for sake of brevity the parameter q in the notation for $H_{\mathbf{r}}$ and $\|\cdot\|_{\mathbf{r}}$. If $\mathbb{R}^n$ is replaced by $\mathbb{R}^{n-1} = \{(x', x_n) \in \mathbb{R}^n : x_n = 0\}$ we use $\Xi_{\mathbf{r}}(\xi', q) := \Xi_{\mathbf{r}}(\xi', 0, q)$ in the definition of the norm $\|\cdot\|_{\mathbf{r}, \mathbb{R}^{n-1}}$ in $H^{r_1+r_2}(\mathbb{R}^{n-1})$. On the half space $\mathbb{R}^n_+ := \{x = (x', x_n) \in \mathbb{R}^n : x_n > 0\}$ and on the manifold M and its boundary ∂M the analog norms may be defined by standard methods (cf. also [18]); the resulting norms will be denoted by $\|\cdot\|_{\mathbf{r}, \mathbb{R}^n_+}$ etc.

If we consider the traces of functions in $H_{\mathbf{r}}(\mathbb{R}^n_+)$ on the boundary $\mathbb{R}^{n-1}$, the question is how to define parameter-dependent norms on the Sobolev spaces on the boundary for which the trace operator $\gamma_0 \colon u \mapsto u(\cdot, 0)$ is continuous with norm bounded by a constant which does not depend on q. To answer this question, we define for $\mathbf{r} = (r_1, r_2) \in \mathbb{R}^2$ and $a \in \mathbb{R}$ the functions

$$\Xi_{\mathbf{r}}^{(-a)}(\xi, q) = \begin{cases} (1+|\xi|)^{r_1-a}(q+|\xi|)^{r_2}, & a \le r_1, \\ (q+|\xi|)^{r_1+r_2-a}, & a > r_1, \end{cases} \tag{4.4}$$

and

$$\Phi_{\mathbf{r}}^{(-a)}(\xi, q) = \begin{cases} |\xi|^{r_1-a}(q+|\xi|)^{r_2}, & a \le r_1, \\ (q+|\xi|)^{r_1+r_2-a}, & a > r_1. \end{cases} \tag{4.5}$$

The norm connected with the function $\Xi_{\mathbf{r}}^{(-a)}$ will be denoted by $\|\cdot\|_{\mathbf{r}}^{(-a)}$. For the space $H^{r_1+r_2-a}$ endowed with this norm we will use the notation $H_{\mathbf{r}}^{(-a)}$.

Again these functions can be interpreted geometrically. For positive integers r_1 and r_2 and for $0 \le a \le r_1 + r_2$ the function $\Xi_{\mathbf{r}}^{(-a)}$ corresponds to the Newton polygon which is constructed from $N_{\mathbf{r}}$ by a shift of length a to the left parallel to the horizontal axis.

The following theorem describes the trace spaces of the Sobolev space $H_{\mathbf{r}}(\mathbb{R}^n_+)$ in the sense described above. This is one of the main results in the theory of Sobolev spaces defined by Newton polygons; in the case where r_1 and r_2 are positive integers, this result follows from Theorem 2.9 in [5]. However, it is not difficult to see that the proof used in [5] works for arbitrary real r_1, r_2 (with $r_1 + r_2 > \frac{1}{2}$), too.

Theorem 4.1. *Let $\mathbf{r} \in \mathbb{R}^2$ with $r_1 + r_2 > \frac{1}{2}$. For every $q_0 > 0$ and every $l \in \mathbb{Z}$ with $0 \le l \le r_1 + r_2 - \frac{1}{2}$ there exists a constant $C > 0$, independent of u and q, such that*

$$\|\gamma_0 D_n^l u\|_{\mathbf{r},\mathbb{R}^{n-1}}^{(-l-1/2)} \le C\|u\|_{\mathbf{r},\mathbb{R}^n_+}$$

holds for all $u \in H_{\mathbf{r}}(\mathbb{R}^n_+)$ and $q \ge q_0$.

Now we are going to re-write the ODE estimate from Theorem 3.5 in terms of the Newton polygon. In the a priori estimates below the norms for the i-th component will be of different structure for $i \le N_1$ and $i > N_1$, respectively. To obtain a common notation for these cases, we introduce the tuple

$$\mathbf{e}_i := \begin{cases} (1,0), & \text{if } 1 \le i \le N_1, \\ (0,1), & \text{if } N_1 + 1 \le i \le N. \end{cases} \tag{4.6}$$

With this definition we obtain

$$\Phi_{\mathbf{r}+t_i\mathbf{e}_i}(\xi, q) = \begin{cases} |\xi|^{r_1+t_i}(q+|\xi|)^{r_2}, & \text{if } i \le N_1, \\ |\xi|^{r_1}(q+|\xi|)^{r_2+t_i}, & \text{if } i > N_1. \end{cases} \tag{4.7}$$

Theorem 4.2. *Assume that A satisfies the Vishik–Lyusternik condition and that (A,B) is weakly parameter-elliptic. Let $r_1 + r_2 \geq m_R + 1/2$ and $r_1 \in [m_{R_1} + 1/2, m_{R_1+1} + 1/2]$ be real numbers. Then for $j = 1,\dots,R$ and $l = 0,1,2,\dots$ the inequality*

$$\|D_t^l w_{ji}(\cdot,\xi',q)\|_{L_2(\mathbb{R}_+)} \leq C\, \frac{\Phi_{\mathbf{r}}^{(-m_j-1/2)}(\xi',q)}{\Phi_{\mathbf{r}+t_i\mathbf{e}_i}^{(-l)}(\xi',q)} \tag{4.8}$$

holds for all $\xi' \in \mathbb{R}^{n-1}\backslash\{0\}$ and $q \in [0,\infty)$.

Proof. First assume that $i \leq N_1$ and $r_1 < m_{R_1+1} + 1/2$. We have to compare the right-hand side of (4.8) with the right-hand side of (3.11). Using (4.5) and the fact that $m_j + t_i + 1/2 \leq r_1 + t_i$ if and only if $j \leq R_1$ we obtain

$$\frac{\Phi_{\mathbf{r}}^{(-m_j-1/2)}(\xi',q)}{\Phi_{\mathbf{r}+t_i\mathbf{e}_i}^{(-l)}(\xi',q)} = \begin{cases} |\xi'|^{l-m_j-t_i-1/2}, & l \leq r_1+t_i, \quad j \leq R_1, \\ |\xi'|^{r_1-m_j-1/2}(q+|\xi|)^{l-t_i-r_1}, & l > r_1+t_i, \quad j \leq R_1, \\ |\xi'|^{l-t_i-r_1}(q+|\xi|)^{r_1-m_j-1/2}, & l \leq r_1+t_i, \quad j > R_1, \\ (q+|\xi'|)^{l-m_j-t_i-1/2}, & l > r_1+t_i, \quad j > R_1, \end{cases} \tag{4.9}$$

First let $j \leq R_1$. If $l \leq r_1 + t_i$ we have $l \leq m_{R_1+1} + t_i$, and the first line in (3.11) has to be applied. If $l > r_1 + t_i$, the quotient of the right-hand sides of (3.11) and (4.9) equals

$$\begin{cases} \left(\dfrac{|\xi'|}{q+|\xi'|}\right)^{l-t_i-r_1}, & l \leq m_{R_1+1} + t_i\,, \\ \left(\dfrac{|\xi'|}{q+|\xi'|}\right)^{1/2+m_{R_1+1}-r_1}, & l > m_{R_1+1} + t_i\,. \end{cases}$$

In each case this quotient is not greater than 1 due to the condition on r_1.

Similarly, for $j > R_1$ the quotient of the right-hand sides of (3.11) and (4.9) equals

$$\begin{cases} \left(\dfrac{|\xi'|}{q+|\xi'|}\right)^{r_1-m_{R_1}-1/2}, & l \leq m_{R_1} + t_i\,, \\ \left(\dfrac{|\xi'|}{q+|\xi'|}\right)^{r_1+t_i-l}, & m_{R_1} + t_i < l \leq r_1 + t_i\,, \\ 1\,, & l > r_1 + t_i \end{cases}$$

which proves (4.8) for $r_1 \in [m_{R_1} + 1/2, m_{R_1+1} + 1/2)$.

Now let us consider the case $i \leq N$ and $r_1 = m_{R_1+1} + \frac{1}{2}$. Define the index k by

$$m_{R_1+1} = m_{R_1+2} = \ldots = m_k < m_{k+1}\,.$$

Then $m_j + \frac{1}{2} \leq r_1$ holds if and only if $j \leq k$. Computing the right-hand side of (4.8), we see that for $j \leq k$ we obtain the first and second line on the right-hand side of

(4.9). But for $j = R_1 + 1, \dots, k$ we have $m_j + \frac{1}{2} = r_1$, and consequently for these values of j there is no difference between the first and the third line and between the second and the fourth line in (4.9). Therefore, (4.9) holds in the case $r_1 = m_{R_1+1} + \frac{1}{2}$, too. The remainder of the proof is the same as in the case $r_1 < m_{R_1+1} + \frac{1}{2}$.

Finally, the proof for $i > N_1$ can be made similarly, now using Theorem 3.5 b) instead of Theorem 3.5 a). $\square$

5. A priori estimates

In this section we will prove a priori estimates for operators of the form (1.1), (2.3) with constant coefficients and without lower order terms in $\mathbb{R}^n$ and $\mathbb{R}^n_+$, respectively (model problems). For the remainder of this section, we fix a tuple $\mathbf{r} = (r_1, r_2)$ of real numbers with $r_1 + r_2 \geq m_R + 1$ and $r_1 \in [m_{R_1} + 1/2, m_{R_1+1} + 1/2]$. For simplicity, let us also assume that $r_1 + r_2$ is an integer number; in this case we obtain an easy description (see (5.13) below) for the norms appearing in the a priori estimate. For the estimates below, we have to keep in mind the definitions (4.1), (4.4) and (4.6).

First let $A = A(D, q)$ be a model operator of the form (1.1) acting in $\mathbb{R}^n$.

Lemma 5.1. *For $q \geq q_0 > 0$, the operator*

$$A(D,q)\colon\ \prod_{i=1}^{N} H_{\mathbf{r}+t_i\mathbf{e}_i}(\mathbb{R}^n) \to \prod_{i=1}^{N} H_{\mathbf{r}-s_i\mathbf{e}_i}(\mathbb{R}^n)$$

is continuous. Here the norm in the space $H_{\mathbf{r}}(\mathbb{R}^n)$ is defined in (4.3), (4.1) *and continuity has to be understood in the sense that A is a bounded operator between these spaces whose norm can be estimated by a constant $C = C(q_0)$ independent of q.*

Proof. Due to the Douglis–Nirenberg structure, the parameter-independent operator

$$A_{ij}(D,0)\colon H_{\mathbf{r}+t_j\mathbf{e}_j}(\mathbb{R}^n) \to H_{\mathbf{r}-s_i\mathbf{e}_i}(\mathbb{R}^n) \tag{5.1}$$

is continuous. Indeed, if $j \leq N_1$, then the left space is $H_{r_1+t_j, r_2}$ which is mapped into $H_{r_1-s_i, r_2}$. According to the obvious embedding

$$H_{r_1-a, r_2}(\mathbb{R}^n) \subset H_{r_1, r_2-a}(\mathbb{R}^n) \quad (a \geq 0) \tag{5.2}$$

we obtain the boundedness of (5.1). In the case $j > N_1$ the left space is H_{r_1, r_2+t_j} which is mapped into $H_{r_1-s_i-t_j, r_2+t_j}$. Using again (5.2), we see the continuity of (5.1). The same is true for the multiplication operator $q^{2m-2\mu}$. $\square$

Theorem 5.2. *Let $A(D,q)$ be weakly parameter-elliptic in the sense of Definition* 2.1. *Then for $q \geq q_0 > 0$ the a priori estimate*

$$\sum_{i=1}^{N} \|u_i\|_{\mathbf{r}+t_i\mathbf{e}_i} \leq C\Big(\sum_{i=1}^{N} \|f_i\|_{\mathbf{r}-s_i\mathbf{e}_i} + \sum_{i=1}^{N_1} q^{r_2}\|u_i\|_{L_2} + \sum_{i=N_1+1}^{N} q^{r_2+t_i}\|u_i\|_{L_2}\Big)$$

holds for $f = A(D,q)u$.

Proof. We take an arbitrary vector $c \in \mathbb{C}^N$, pose $d = A(\xi,q)c$ and prove the inequality

$$\sum_{i=1}^N \Phi_{\mathbf{r}+t_i\mathbf{e}_i}(\xi,q)\ |c_i| \le C \sum_{j=1}^N \Phi_{\mathbf{r}-s_j\mathbf{e}_j}(\xi,q)\ |d_j|. \tag{5.3}$$

As in the proof of Lemma 3.2, we set $G(\xi,q) = (g_{ij}(\xi,q)) = A^{-1}(\xi,q)$. We already know that $g_{ij}(\xi,q)$ is homogeneous in (ξ,q) of degree $-s_j - t_i$. We write

$$g_{ij}(\xi,q) = (-1)^{i+j}(\det A(\xi,q))^{-1} \det \left[A(\xi,q)^{ji}\right],$$

where the $(N-1)\times(N-1)$ matrix A^{ji} is obtained by canceling the j-th row and i-th column in matrix A. Using the condition of weak parameter-ellipticity and the homogeneity of the elements of the matrix $A(\xi,q)$, it is not difficult to obtain the following estimate (cf. also the proof of Proposition 3.10 in [4])

$$|g_{ij}(\xi,q)| \le C \begin{cases} |\xi|^{-s_j-t_i}, & i,j \le N_1\,, \\ |\xi|^{t_j-t_i}\,(q+|\xi|)^{-s_j-t_j}, & i \le N_1, j > N_1\,, \\ |\xi|^{s_i-s_j}\,(q+|\xi|)^{-s_i-t_i}, & i > N_1, j \le N_1\,, \\ |\xi|^{s_i+t_j}(q+|\xi|)^{-s_i-t_i-s_j-t_j}, & i,j > N_1, i \ne j\,. \\ (q+|\xi|)^{-s_j-t_j}, & i = j > N_1. \end{cases} \tag{5.4}$$

Let us first consider the case $i \le N_1$. In this case the i-term in the sum on the left-hand side of (5.3) equals $|\xi|^{r_1+t_i}(q+|\xi|)^{r_2}|c_i|$. Using (5.4), we see that the i-th term is not greater than

$$\begin{aligned} &\sum_{j=1}^N |\xi|^{r_1+t_i}(q+|\xi|)^{r_2}|g_{ij}(\xi,q)|\ |d_j| \\ &\quad\le\ C\Big(\sum_{j=1}^{N_1}|\xi|^{r_1-s_j}(q+|\xi|)^{r_2}|d_j| + \sum_{j=N_1+1}^{N}|\xi|^{r_1+t_j}(q+|\xi|)^{r_2-s_j-t_j}|d_j|\Big) \\ &\quad\le\ C\sum_{j=1}^N \Phi_{\mathbf{r}-s_j\mathbf{e}_j}(\xi,q)|d_j|\,. \end{aligned}$$

In the same way we obtain that for $i > N_1$ the i-th term in the sum on the left-hand side of (5.3) is not greater than

$$\begin{aligned} &\sum_{j=1}^N |\xi|^{r_1}(q+|\xi|)^{r_2+t_i}|g_{ij}(\xi,q)|\ |d_j| \\ &\quad\le\ C\Big(\sum_{j=1}^{N_1}|\xi|^{r_1+s_i-s_j}(q+|\xi|)^{r_2-s_i}|d_j| + \sum_{j=N_1+1}^{N}|\xi|^{r_1+s_i+t_j}(q+|\xi|)^{r_2-s_i-s_j-t_j}|d_j|\Big) \end{aligned}$$

$$
\begin{aligned}
&\leq C\Big(\sum_{j=1}^{N_1}|\xi|^{r_1-s_j}(q+|\xi|)^{r_2}|d_j|+\sum_{j=N_1+1}^{N}|\xi|^{r_1}(q+|\xi|)^{r_2-s_j}|d_j|\Big)\\
&= C\sum_{j=1}^{N}\Phi_{\mathbf{r}-s_j\mathbf{e}_j}(\xi,q)|d_j|\,,
\end{aligned}
$$

which finishes the proof of (5.3).

For $|\xi|\geq 1$ we have $|\xi|\approx 1+|\xi|$ and we may replace in (5.3) Φ by Ξ. For $|\xi|\leq 1$ we have

$$
\Xi_{\mathbf{r}+t_i\mathbf{e}_i}(\xi,q)\leq C\begin{cases} q^{r_2}, & \text{if } i\leq N_1,\\ q^{r_2+t_i}, & \text{if } i>N_1.\end{cases}
$$

Therefore, we obtain for all $\xi\in\mathbb{R}^n$ the inequality

$$
\sum_{i=1}^{N}\Xi_{\mathbf{r}+t_i\mathbf{e}_i}(\xi,q)\ |c_i|\leq C\Big(\sum_{j=1}^{N}\Xi_{\mathbf{r}-s_j\mathbf{e}_j}(\xi,q)\ |d_j|+\sum_{i=1}^{N_1}q^{r_2}|c_i|+\sum_{i=N_1+1}^{N}q^{r_2+t_i}|c_i|\Big)\,. \tag{5.5}
$$

Now it remains to replace in (5.5) the numbers c_i by $Fu_i(\xi)$ and d_j by $Ff_j(\xi)$, take the square of each side and integrate with respect to $\xi\in\mathbb{R}^n$ to finish the proof of the theorem. (Here and in the following we tacitly use the fact that in finite-dimensional spaces all norms are equivalent and thus we may replace terms like $(\sum_i|c_i|)^2$ by $\sum_i|c_i|^2$ etc.) □

Now let us consider the model problem in the half space $\mathbb{R}^n_+$, i.e. operators A,B of the form (1.1), (2.3) with constant coefficients and without lower order terms.

Lemma 5.3. *The operator*

$$
(A,B)\colon\ \prod_{i=1}^{N}H_{\mathbf{r}+t_i\mathbf{e}_i}(\mathbb{R}^n_+)\to\prod_{i=1}^{N}H_{\mathbf{r}-s_i\mathbf{e}_i}(\mathbb{R}^n_+)\times\prod_{j=1}^{R}H_{\mathbf{r}}^{(-m_j-1/2)}(\mathbb{R}^{n-1})
$$

is continuous.

Proof. We fix an extension operator E from $\mathbb{R}^n_+$ to $\mathbb{R}^n$, being continuous from $L_2(\mathbb{R}^n_+)$ to $L_2(\mathbb{R}^n)$ and from $H_{\mathbf{r}+t_i\mathbf{e}_i}(\mathbb{R}^n_+)$ to $H_{\mathbf{r}+t_i\mathbf{e}_i}(\mathbb{R}^n)$. (Such an operator can be defined using the standard Hestenes–Seeley construction.) Let $u\in\prod_{i=1}^{N}H_{\mathbf{r}+t_i\mathbf{e}_i}(\mathbb{R}^n_+)$ and $f:=Au$. Setting $Ef:=AEu$, we obtain due to Lemma 5.1 and the definition of the norm in the half space the inequality

$$
\sum_{i=1}^{N}\|f_i\|_{\mathbf{r}-s_i\mathbf{e}_i,\mathbb{R}^n_+}\leq C\sum_{i=1}^{N}\|(Ef)_i\|_{\mathbf{r}-s_i\mathbf{e}_i,\mathbb{R}^n}\leq C\sum_{i=1}^{N}\|(Eu)_i\|_{\mathbf{r}+t_i\mathbf{e}_i,\mathbb{R}^n}
$$

$$\leq C \sum_{i=1}^{N} \|u_i\|_{\mathbf{r}+t_i\mathbf{e}_i,\mathbb{R}^n_+}.$$

This proves the continuity of the operator A.

For the continuity of B we note, repeating the argument of Lemma 5.1,

$$B_{kj}(D)\big(H_{\mathbf{r}+t_j\mathbf{e}_j}(\mathbb{R}^n_+)\big) \subset H_{\mathbf{r}+t_j\mathbf{e}_j-(t_j+m_k)\mathbf{e}_1}(\mathbb{R}^n_+) \subset H_{\mathbf{r}-m_k\mathbf{e}_1}(\mathbb{R}^n_+).$$

According to Theorem 4.1, we have

$$\gamma_0 B_{kj}(D)\big(H_{\mathbf{r}+t_j\mathbf{e}_j}(\mathbb{R}^n_+)\big) \subset H^{(-1/2)}_{\mathbf{r}-m_k\mathbf{e}_1}(\mathbb{R}^{n-1}).$$

But due to definition, the space on the right-hand side coincides with the space $H^{(-m_k-1/2)}_{\mathbf{r}}(\mathbb{R}^{n-1})$. □

Theorem 5.4. *Let $A(D,q)$, $B(D)$ be of the form (1.1), (2.3) with constant coefficients and without lower order terms, acting on the half-space $\mathbb{R}^n_+$. Assume that A satisfies the Vishik–Lyusternik condition and that (A,B) is weakly parameter-elliptic. Then for $q \geq q_0 > 0$ the a priori estimate*

$$\begin{aligned}\sum_{i=1}^{N} \|u_i\|_{\mathbf{r}+t_i\mathbf{e}_i,\mathbb{R}^n_+} \leq C\bigg(&\sum_{i=1}^{N} \|f_i\|_{\mathbf{r}-s_i\mathbf{e}_i,\mathbb{R}^n_+} + \sum_{j=1}^{R} \|g_j\|^{(-m_j-1/2)}_{\mathbf{r},\mathbb{R}^{n-1}} \\ &+ \sum_{i=1}^{N_1} q^{r_2}\|u_i\|_{L_2(\mathbb{R}^n_+)} + \sum_{i=N_1+1}^{N} q^{r_2+t_i}\|u_i\|_{L_2(\mathbb{R}^n_+)}\bigg)\end{aligned} \tag{5.6}$$

holds, where we have set $Au = f$ and $Bu = g$.

Proof. We will follow a standard scheme in elliptic theory (cf. also [5]).

(i) *Reduction to the case $f = 0$.* Fixing a cut-off function $\psi \in C^\infty(\mathbb{R}^n)$ with $\psi(\xi) = 1$ for $|\xi| \leq 1$ and $\psi(\xi) = 0$ for $|\xi| \geq 2$, we write u in the form

$$u = u' + u'' + v \tag{5.7}$$

with

$$u' := [\psi(D)Eu]\Big|_{\mathbb{R}^n_+} \quad \text{and} \quad u'' := [(1-\psi(D))G(D,q)Ef]\Big|_{\mathbb{R}^n_+}.$$

Here E is the extension operator from the proof of Lemma 5.3 and $G(\xi,q) := A^{-1}(\xi,q)$; we use the pseudodifferential notation $\psi(D) := F^{-1}\psi(\xi)F$.

To estimate u', we note that for $q \geq q_0$ we have

$$\begin{aligned}
\sum_{i=1}^{N} \|u_i'\|_{\mathbf{r}+t_i\mathbf{e}_i,\mathbb{R}^n_+} &\leq C \sum_{i=1}^{N} \|\psi(D)(Eu)_i\|_{\mathbf{r}+t_i\mathbf{e}_i,\mathbb{R}^n} \\
&\leq C\Big(\sum_{i=1}^{N_1} q^{r_2}\|(Eu)_i\|_{L_2(\mathbb{R}^n)} + \sum_{i=N_1+1}^{N} q^{r_2+t_i}\|(Eu)_i\|_{L_2(\mathbb{R}^n)}\Big) \qquad (5.8)\\
&\leq C\Big(\sum_{i=1}^{N_1} q^{r_2}\|u_i\|_{L_2(\mathbb{R}^n_+)} + \sum_{i=N_1+1}^{N} q^{r_2+t_i}\|u_i\|_{L_2(\mathbb{R}^n_+)}\Big),
\end{aligned}$$

where we used the fact that $\psi(D)$ is infinitely smoothing.

To estimate u'', it is sufficient to consider

$$\sum_{i=1}^{N} \|(1-\psi(D))(G(D,q)Ef)_i\|_{\mathbf{r}+t_i\mathbf{e}_i,\mathbb{R}^n}\,.$$

For this we use (5.3) with $d = F(Ef)(\xi)$ and $c = G(\xi,q)d$, remarking that now we may replace $\Phi(\xi,q)$ in (5.3) by $\Xi(\xi,q)$ as $1-\psi(\xi)$ vanishes for $|\xi| \leq 1$. We obtain, integrating with respect to ξ,

$$\sum_{i=1}^{N} \|u_i''\|_{\mathbf{r}+t_i\mathbf{e}_i,\mathbb{R}^n_+} \leq C\sum_{i=1}^{N} \|(Ef)_i\|_{\mathbf{r}-s_i\mathbf{e}_i,\mathbb{R}^n} \leq C\sum_{i=1}^{N} \|f_i\|_{\mathbf{r}-s_i\mathbf{e}_i,\mathbb{R}^n}\,. \qquad (5.9)$$

It remains to estimate $v = u - u' - u''$ which is a solution of

$$\begin{aligned}
A(D,q)\,v &= 0 \quad \text{in } \mathbb{R}^n_+\,,\\
B(D)\,v &= h \quad \text{on } \mathbb{R}^{n-1}
\end{aligned} \qquad (5.10)$$

with $h := g - B(u'+u'')$. Because of Lemma 5.3, the norm of $B(u'+u'')$ in the space $\prod_{j=1}^{R} H_{\mathbf{r}}^{(-m_j-1/2)}(\mathbb{R}^{n-1})$ is not greater than a constant times

$$\sum_{i=1}^{N} \big(\|u_i'\|_{\mathbf{r}+t_i\mathbf{e}_i,\mathbb{R}^n_+} + \|u_i''\|_{\mathbf{r}+t_i\mathbf{e}_i,\mathbb{R}^n_+}\big)\,.$$

Moreover,

$$\begin{aligned}
&\sum_{i=1}^{N_1} q^{r_2}\|v_i\|_{L_2(\mathbb{R}^n_+)} + \sum_{i=N_1+1}^{N} q^{r_2+t_i}\|v_i\|_{L_2(\mathbb{R}^n_+)}\\
&\qquad\qquad \leq \sum_{i=1}^{N_1} q^{r_2}\|u_i\|_{L_2(\mathbb{R}^n_+)} + \sum_{i=N_1+1}^{N} q^{r_2+t_i}\|u_i\|_{L_2(\mathbb{R}^n_+)}
\end{aligned}$$

$$+\sum_{i=1}^{N}\|u_i'\|_{\mathbf{r}+t_i\mathbf{e}_i,\mathbb{R}^n_+}+\sum_{i=1}^{N}\|u_i''\|_{\mathbf{r}+t_i\mathbf{e}_i,\mathbb{R}^n_+}\,,$$

and from (5.8) and (5.9) we see that we have to prove

$$\begin{aligned}\sum_{i=1}^{N}\|v_i\|_{\mathbf{r}+t_i\mathbf{e}_i,\mathbb{R}^n_+}\le C\Big(&\sum_{j=1}^{R}\|h_j\|^{(-m_j-1/2)}_{\mathbf{r},\mathbb{R}^{n-1}}\\&+\sum_{i=1}^{N_1}q^{r_2}\|v_i\|_{L_2(\mathbb{R}^n_+)}+\sum_{i=N_1+1}^{N}q^{r_2+t_i}\|v_i\|_{L_2(\mathbb{R}^n_+)}\Big)\,,\end{aligned}\tag{5.11}$$

i.e. we may assume without restriction that $f=0$.

(ii) *Proof for the case* $f=0$. Let v be a solution of (5.10). We have the equivalences

$$\Xi_{\mathbf{r}+t_i\mathbf{e}_i}(\xi,q)\approx\Phi_{\mathbf{r}+t_i\mathbf{e}_i}(\xi,q)+\begin{cases}q^{r_2}, & \text{if } i\le N_1,\\ q^{r_2+t_i}, & \text{if } i>N_1.\end{cases}\tag{5.12}$$

and

$$\Phi_{\mathbf{r}+t_i\mathbf{e}_i}(\xi,q)\approx\Big[\sum_{l=0}^{r_1+r_2+t_i}\xi_n^{2l}\big(\Phi^{(-l)}_{\mathbf{r}+t_i\mathbf{e}_i}(\xi',q)\big)^2\Big]^{1/2}$$

(cf. also [6], Proof of Theorem 4.2). Therefore,

$$\begin{aligned}\sum_{i=1}^{N}\|v_i\|_{\mathbf{r}+t_i\mathbf{e}_i,\mathbb{R}^n_+}\approx&\sum_{i=1}^{N_1}q^{r_2}\|v_i\|_{L_2(\mathbb{R}^n_+)}+\sum_{i=N_1+1}^{N}q^{r_2+t_i}\|v_i\|_{L_2(\mathbb{R}^n_+)}\\&+\sum_{i=1}^{N}\Big[\sum_{l=0}^{r_1+r_2+t_i}\int_{\mathbb{R}^{n-1}}\|D_n^l(F'v_i)(\xi',\cdot)\|^2_{L_2(\mathbb{R}_+)}\big(\Phi^{(-l)}_{\mathbf{r}+t_i\mathbf{e}_i}(\xi',q)\big)^2d\xi'\Big]^{1/2}.\end{aligned}\tag{5.13}$$

We take in (5.10) partial Fourier transform with respect to $\xi'\in\mathbb{R}^{n-1}$ and use unique solvability due to condition 2.6 (ii) for $\xi'\ne 0$. We obtain

$$(F'v)(x_n,\xi',q)=\sum_{j=1}^{R}(F'h_j)(\xi')w_j(x_n,\xi',q)\,.$$

Due to Theorem 4.2, we have

$$\|D_n^l(F'v_i)(\cdot,\xi',q)\|^2_{L_2(\mathbb{R}_+)}\big(\Phi^{(-l)}_{\mathbf{r}+t_i\mathbf{e}_i}(\xi',q)\big)^2\le C\sum_{j=1}^{R}|(F'h_j)(\xi')|^2\big(\Phi_{\mathbf{r}}^{(-m_j-1/2)}(\xi',q)\big)^2\,.$$

Integrating this with respect to $\xi' \in \mathbb{R}^{n-1}$ we obtain from (5.13) the inequality

$$\sum_{i=1}^{N} \|v_i\|_{\mathbf{r}+t_i\mathbf{e}_i,\mathbb{R}^n_+} \le C\Big(\sum_{i=1}^{N_1} q^{r_2} \|v_i\|_{L_2(\mathbb{R}^n_+)} + \sum_{i=N_1+1}^{N} q^{r_2+t_i} \|v_i\|_{L_2(\mathbb{R}^n_+)} + \sum_{j=1}^{R} \|h_j\|^{(-m_j-1/2)}_{\mathbf{r},\mathbb{R}^{n-1}} \Big)$$

which finishes the proof of the theorem. □

Note that due to Lemma 5.1 and Lemma 5.3, the a priori estimates in Theorems 5.2 and 5.4, respectively, are two-sided.

Now let us come back to operators of the form (1.1) with boundary operators (2.3), acting on a smooth compact manifold M with smooth boundary ∂M, as described in Section 2. Choosing a finite number of coordinate systems, we obtain in local coordinates operators of the same form acting in $\mathbb{R}^n$ and $\mathbb{R}^n_+$, respectively. Roughly speaking, the fact that the a priori estimates above for the model operators are two-sided allows us to obtain a priori estimates for operators with variable coefficients. Altogether, we arrive at the following theorem for which the proof demands no new idea (cf. also the proof of Theorem 5.6 in [5]):

Theorem 5.5. *Let the operators A, B of the form* (1.1), (2.3) *act on a smooth compact manifold M with smooth boundary ∂M. Assume that the operator $A(x, D, q)$ satisfies the Vishik–Lyusternik condition and that the boundary value problem (A, B) is weakly parameter-elliptic in the sense of Definition* 2.6. *Then there exists a constant $q_0 > 0$ and a constant $C > 0$, independent of q and of u, such that the a priori estimate*

$$\sum_{i=1}^{N} \|u_i\|_{\mathbf{r}+t_i\mathbf{e}_i,M} \le C\Bigg(\sum_{i=1}^{N} \|f_i\|_{\mathbf{r}-s_i\mathbf{e}_i,M} + \sum_{j=1}^{R} \|g_j\|^{(-m_j-1/2)}_{\mathbf{r},\partial M} + \sum_{i=1}^{N_1} q^{r_2} \|u_i\|_{L_2(M)} + \sum_{i=N_1+1}^{N} q^{r_2+t_i} \|u_i\|_{L_2(M)} \Bigg) \tag{5.14}$$

holds, where we have set $f = Au$ and $g = Bu$.

Remark 5.6. Looking through the proofs of the preceding results, we see that the assumptions may be weakened in the following sense: the operator $A_{22}(x, D) - q^{2m} I_{N_2}$ can be replaced by an operator $A_{22}(x, D, q)$ which is elliptic with parameter in the sense of Agmon–Agranovich–Vishik. Note that this makes sense here as all entries on the diagonal of the Douglis–Nirenberg system A_{22} have the same order and thus the parameter has a definite weight in this operator matrix.

In this case the results above hold with almost literally the same proofs.

6. Application to transmission problems

In the theory of elliptic boundary value problems with indefinite weight functions a transmission problem arises where the operator is elliptic in one part of the region and elliptic with parameter in the other part. Such a problem has been treated in a recent paper of Faierman [7] to which we also refer the reader for further references. In the present paper we show how to apply the results of Section 4 to such problems. The a priori estimates obtained differ from those in [7] and seem to be new. Moreover, in [7] only the canonical transmission conditions (see below) are treated, while our results allow us to find conditions on general transmission conditions which imply the validity of a priori estimates. For simplicity, we restrict ourselves to the case of differential operators with constant coefficients and without lower order terms, acting in $\mathbb{R}^n$. The case of variable coefficients can be treated by the well-known method of localization, see also [7], where even non-smooth coefficients are handled.

So we assume that we have given a transmission problem of the form

$$\begin{aligned} A_1(D)u_1(x) &= f_1(x) \quad (x_n < 0)\,, && (6.1)\\ A_2(D,q)u_2(x) &= f_2(x) \quad (x_n > 0)\,, && (6.2)\\ B_{j1}(D)u_1(x',0) + B_{j2}(D)u_2(x',0) &= g_j(x') \quad (j = 1,\ldots,2m)\,. && (6.3)\end{aligned}$$

Here A_1 and A_2 are scalar partial differential operators of the same order $2m$ with constant coefficients and B_{j1}, B_{j2} are boundary differential operators of order m_j for $j = 1,\ldots,2m$. We assume that these operators coincide with their principal parts and that

$$m_1 \leq \cdots \leq m_m < m_{m+1} \leq \cdots \leq m_{2m}\,. \tag{6.4}$$

We will assume that the following conditions hold:

(A1) The operator $A_1(D)$ is elliptic, i.e.

$$A_1(\xi) \neq 0 \quad \text{for } \xi \in \mathbb{R}^n \backslash \{0\}\,.$$

(A2) The operator $A_2(D,q)$ is elliptic with parameter in the sense of Agmon–Agranovich–Vishik along the ray $[0,\infty)$, i.e.

$$A_2(\xi,q) \neq 0 \quad \text{for } (\xi,q) \in \mathbb{R}^n \times [0,\infty) \text{ with } |\xi| + q > 0\,.$$

(A3) For any $\xi' \in \mathbb{R}^{n-1}\backslash\{0\}$, $q \in [0,\infty)$ and $g = (g_1,\ldots,g_{2m}) \in \mathbb{C}^{2m}$ the transmission problem on the real line

$$\begin{aligned} A_1(\xi',D_t)w_1(t) &= 0 \quad (t < 0)\,,\\ A_2(\xi',D_t,q)w_2(t) &= 0 \quad (t > 0)\,,\\ \Big[B_{j1}(\xi',D_t)w_1(t) + B_{j2}(\xi',D_t)w_2(t)\Big]\Big|_{t=0} &= g_j \quad (j = 1,\ldots,2m)\,, && (6.5)\\ w_1(t) &\to 0 \quad (t \to -\infty)\,,\\ w_2(t) &\to 0 \quad (t \to +\infty)\end{aligned}$$

has a unique solution.

(A4) The boundary value problem $(A_1, B_{11}, \dots, B_{1m})$ in $\mathbb{R}^n_-$ satisfies the classical Shapiro–Lopatinskii condition.

(A5) For any $q \in [0, \infty)$ and $(h_{m+1}, \dots, h_{2m}) \in \mathbb{C}^m$ the ordinary differential equation system on $\mathbb{R}_+$

$$\begin{aligned} A_2(0, D_t, 1)v_2(t) &= \;0 \quad (t > 0)\,, \\ B_{j2}(0, D_t)v_2(t)\Big|_{t=0} &= h_j \quad (j = m+1, \dots, 2m) \\ v_2(t) &\to \;0 \quad (t \to +\infty) \end{aligned} \tag{6.6}$$

has a unique solution.

The following theorem is the analog of Theorem 5.4 for such transmission problems.

Theorem 6.1. *Suppose that assumptions* (A1)–(A5) *hold. Let* $\mathbf{r} = (r_1, r_2)$ *be a tuple of real numbers for which* $r_1 \in [m_m + \frac{1}{2}, m_{m+1} + \frac{1}{2}]$ *and* $r_1 + r_2 \geq m_{2m} + 1$ *is an integer. Then there exists a* $q_0 > 0$ *and a constant* C*, not depending on* q *or on* u*, such that for* $q \geq q_0$ *and for any solution* u_1, u_2 *of* (6.1)–(6.3) *the a priori estimate*

$$\begin{aligned} \|u_1\|_{\mathbf{r},\mathbb{R}^n_-} + \|u_2\|_{\mathbf{r},\mathbb{R}^n_+} \leq C\Big[&\|f_1\|_{(r_1-2m, r_2),\mathbb{R}^n_-} + \|f_2\|_{(r_1, r_2-2m),\mathbb{R}^n_+} \\ &+ \sum_{j=1}^{2m} \|g_j\|^{(-m_j-1/2)}_{\mathbf{r},\mathbb{R}^{n-1}} + q^{r_2}\big(\|u_1\|_{L_2(\mathbb{R}^n_-)} + \|u_2\|_{L_2(\mathbb{R}^n_+)}\big)\Big] \end{aligned} \tag{6.7}$$

holds.

Proof. We set $v_1(x) := u_1(x', -x_n)$ and $v_2(x) := u_2(x)$ for $x \in \mathbb{R}^n_+$. Then $\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ is a solution of

$$\begin{pmatrix} A_1(D', -D_n) & 0 \\ 0 & A_2(D', D_n, q) \end{pmatrix} \begin{pmatrix} v_1(x) \\ v_2(x) \end{pmatrix} = \begin{pmatrix} f_1(x', -x_n) \\ f_2(x) \end{pmatrix} \quad (x \in \mathbb{R}^n_+),$$

$$\begin{pmatrix} B_{11}(D', -D_n) & B_{12}(D) \\ \vdots & \vdots \\ B_{2m,1}(D', -D_n) & B_{2m,2}(D) \end{pmatrix} \begin{pmatrix} v_1(x', 0) \\ v_2(x', 0) \end{pmatrix} = \begin{pmatrix} g_1(x') \\ \vdots \\ g_{2m}(x') \end{pmatrix} \quad (x' \in \mathbb{R}^{n-1})\,.$$

This is of the form (1.1), (2.3) with $N_1 = N_2 = 1$, $s_j = 2m$ and $t_j = 0$ where now (μ, m) in (2.2) has to be replaced by $(m, 2m)$. Due to Remark 2.5, the Vishik–Lyusternik condition is satisfied. The conditions of Definition 2.6 follow from the assumptions (A1)–(A5). Indeed, 2.6 (i) is a consequence of (A1) and (A2), 2.6 (ii) follows from (A3) and 2.6 (iii) is identical with (A4). To see that 2.6 (iv) holds, we have to consider the ODE on the half-line

$$\begin{aligned} A_1(0,-D_t)v_1(t) &= 0 \quad (t>0)\,, \\ A_2(0,D_t,1)v_2(t) &= 0 \quad (t>0)\,, \\ B_{j1}(0,-D_n)v_1(t)+B_{j2}(0,D_t)v_2(t)\Big|_{t=0} &= h_j \quad (j=m+1,\dots,2m)\,, \\ v_1(t),\, v_2(t) &\to 0 \quad (t\to+\infty)\,. \end{aligned} \tag{6.8}$$

As $A_1(0,-D_t)=\mathrm{const}\, D_t^{2m}$, the function $v_1(t)$ is a polynomial in t of degree not greater than $2m-1$. Because of $v_1(t)\to 0$ for $t\to+\infty$, this polynomial is identically zero. Thus (6.8) is reduced to (6.6). □

The canonical transmission conditions are given by

$$D_n^{j-1}u_1(x',0)-D_n^{j-1}u_2(x',0)=0 \quad (j=1,\dots,2m)\,.$$

This case is treated in [7]. More general, we will consider

$$D_n^{j-1}u_1(x',0)-D_n^{j-1}u_2(x',0)=g_j(x') \quad (j=1,\dots,2m)\,. \tag{6.9}$$

For these transmission condition, the conditions (A1)–(A5) hold under very natural assumptions on A_1 and A_2. Moreover, in the a priori estimate the $\|u_2\|_{L_2}$-term may be omitted, as the following result shows. Here we restrict ourselves to the case $\mathbf{r}=(r_1,r_2)=(m,m)$ which seems to be the most natural choice. Of course, corresponding results hold for more general parameters $\mathbf{r}$ instead of (m,m).

Theorem 6.2. *Suppose that the operator $A_1(D)$ is elliptic without parameter and that $A_2(D,q)$ is elliptic with parameter in the sense of Agmon–Agranovich–Vishik. Then there exist constants $q_0>0$ and $C>0$, independent of $q\ge q_0$ and u such that for every solution u of the transmission problem* (6.1), (6.2), (6.9) *the following a priori estimate holds:*

$$\begin{aligned} &\|(1+|D|)^m(q+|D|)^m u_1\|_{L_2(\mathbb{R}^n_-)}+\|(1+|D|)^m(q+|D|)^m u_2\|_{L_2(\mathbb{R}^n_-)} \\ &\le C\Bigg[\|(1+|D|)^{-m}(q+|D|)^m f_1\|_{L_2(\mathbb{R}^n_-)} \\ &\quad+\|(1+|D|)^m(q+|D|)^{-m} f_2\|_{L_2(\mathbb{R}^n_+)} \\ &\quad+\sum_{j=1}^m \|(1+|D'|)^{m-j+1/2}(q+|D'|)^m g_j\|_{L_2(\mathbb{R}^{n-1})} \\ &\quad+\sum_{j=m+1}^{2m}\|(q+|D'|)^{2m-j+1/2} g_j\|_{L_2(\mathbb{R}^{n-1})}+q^m\|u_1\|_{L_2(\mathbb{R}^n_-)}\Bigg]\,. \end{aligned} \tag{6.10}$$

Setting $g_1 = \cdots = g_{2m} = 0$ in (6.10), we get an estimate obtained by Igor Fedotov (unpublished).

Proof. a) First we show that conditions (A1)–(A5) are satisfied for the transmission problem (6.1), (6.2), (6.9). Since (A1) and (A2) hold by assumption and (A4) means that the Dirichlet problem for the scalar elliptic operator $A_1(D)$ satisfies the Shapiro–Lopatinskii condition, it remains to consider (A3) and (A5).

Starting with (A5), note that $A(0,\tau,1)$ has m roots in the upper half-plane of the complex plane. Therefore, the problem (6.6) (where now $B_{2j} = D_t^{j-1}$ for $j = m+1,\dots,2m$) has a unique solution.

To prove that condition (A3) holds, we use an argument from [2], Section 7. We denote by $\mathfrak{M}_1$ the m-dimensional subspace of all solutions of

$$A_1(\xi', D_t)z(t) = 0$$

which tend to zero for $t \to -\infty$ and choose a basis $z_1,\dots,z_m$ in $\mathfrak{M}_1$. Similarly, we fix a basis $z_{m+1},\dots,z_{2m}$ of the space $\mathfrak{M}_2$ of all solutions of

$$A_2(\xi', D_t, q)z(t) = 0$$

which tend to zero for $t \to \infty$. Then it is obvious that $\{z_1,\dots,z_{2m}\}$ is a set of solutions of the ordinary differential equation on the whole line

$$A_{1,-}(\xi', D_t)A_{2,+}(\xi', D_t, q)z(t) = 0 \tag{6.11}$$

where we have set

$$A_{1,-}(\xi',\tau) := \prod_{j=m+1}^{2m} (\tau - \tau_{1j}(\xi'))$$

and

$$A_{2,+}(\xi',\tau,q) := \prod_{j=1}^{m} (\tau - \tau_{2j}(\xi',q))\,.$$

Here τ_{1j} ($j = m+1,\dots,2m$) denote all zeros of $A_1(\xi',\cdot)$ with negative imaginary part and τ_{2j} ($j = 1,\dots,m$) stand for the zeros of $A_2(\xi',\cdot,q)$ with positive imaginary part.

Moreover, the set $\{z_1,\dots,z_{2m}\}$ is linearly independent, i.e. it forms a fundamental system for the ODE (6.11) which is of order $2m$. Indeed, in the case of linear dependence there would exist constants $c_1,\dots,c_{2m} \in \mathbb{C}$ with

$$\sum_{j=1}^{m} c_j z_j = \sum_{j=m+1}^{2m} c_j z_j\,, \tag{6.12}$$

where both sides of (6.12) are nontrivial. But as $z_j(t) \to 0$ for $t \to -\infty$ in the case $j \le m$ and $z_j(t) \to 0$ for $t \to +\infty$ in the case $j > m$, this would mean that both sides of (6.12) are bounded on the whole line, i.e. that (6.11) would have a nontrivial bounded (stable) solution. But this is impossible as all roots of A_1 and A_2 have non-vanishing imaginary part.

Now let v_1, v_2 be a solution of (6.5). Then there exist constants $d_1, \dots, d_{2m} \in \mathbb{C}$ such that

$$v_1 = \sum_{j=1}^{m} d_j z_j \quad \text{and} \quad v_2 = -\sum_{j=m+1}^{2m} d_j z_j \,.$$

The transmission conditions

$$D_t^{j-1}(v_1(t) - v_2(t))\Big|_{t=0} = g_j \quad (j = 1, \dots, 2m)$$

are equivalent to

$$\begin{pmatrix} z_1(0) & \cdots & z_{2m}(0) \\ \vdots & & \vdots \\ D_t^{2m-1} z_1(0) & \cdots & D_t^{2m-1} z_{2m}(0) \end{pmatrix} \begin{pmatrix} d_1 \\ \vdots \\ d_{2m} \end{pmatrix} = \begin{pmatrix} g_1 \\ \vdots \\ g_{2m} \end{pmatrix} . \tag{6.13}$$

As the Wronskii determinant of the fundamental system $\{z_1, \dots, z_{2m}\}$ is different from zero, the linear equation system (6.13) is uniquely solvable, which finishes the proof of (A3).

b) Due to part a) of the proof, Theorem 6.1 can be applied to the transmission problem (6.1), (6.2), (6.9). For $\mathbf{r} = (m, m)$ we obtain the estimate (6.10) but with the additional term $q^m \|u_2\|_{L_2(\mathbb{R}^n_+)}$ on the right-hand side. To finish the proof, we have to estimate this term. For this we use the fact that u_2 is a solution of

$$\begin{aligned} A_2(D,q) u_2(x) &= f_2(x) \quad (x_n > 0)\,, & (6.14) \\ D_n^{j-1} u_2(x', 0) &= -g_j(x') + D_n^{j-1} u_1(x', 0) \quad (j = 1, \dots, m)\,. & (6.15) \end{aligned}$$

Due to assumption, the operator $A_2(D,q)$ is elliptic with parameter. As the Dirichlet boundary conditions $\{D_n^{j-1} : j = 1, \dots, m\}$ are absolutely elliptic, we may apply the Agmon–Agranovich–Vishik theory to (6.14)–(6.15). We obtain (see [3], Theorem 3.1)

$$\begin{aligned} \|(q + |D|)^{2m} u_2\|_{L_2(\mathbb{R}^n_+)} \le C \Bigg(& \|f_2\|_{L_2(\mathbb{R}^n_+)} + \sum_{j=1}^{m} \|(q + |D'|)^{2m-j+1/2} g_j\|_{L_2(\mathbb{R}^{n-1})} \\ & + \sum_{j=1}^{m} \|(q + |D'|)^{2m-j+1/2} D_n^{j-1} u_1(x', 0)\|_{L_2(\mathbb{R}^{n-1})} \Bigg) . \end{aligned}$$

Due to [3], Proposition 3.1, we know that

$$\sum_{j=1}^{m} \|(q+|D'|)^{2m-j+1/2} D_n^{j-1} u_1(x',0)\|_{L_2(\mathbb{R}^{n-1})} \le C\|(q+|D|)^{2m} u_1\|_{L_2(\mathbb{R}^n_-)}\,.$$

From the last two inequalities we obtain

$$q^m\|u_2\|_{L_2(\mathbb{R}^n_+)} \le C\Bigg(\|q^{-m} f_2\|_{L_2(\mathbb{R}^n_+)} + \sum_{j=1}^{m} \|q^{-m}(q+|D'|)^{2m-j+1/2} g_j\|_{L_2(\mathbb{R}^{n-1})} + \|q^{-m}(q+|D|)^{2m} u_1\|_{L_2(\mathbb{R}^n_-)} \Bigg)\,. \tag{6.16}$$

Now we shall estimate the right-hand side of the last inequality by the right-hand side of (6.10).

Note that for $q \ge q_0 > 1$ and $\xi \in \mathbb{R}^n$

$$q^{-m}(q+|\xi|)^m(1+|\xi|)^{-m} = \Big(\frac{q+|\xi|}{q+q|\xi|}\Big)^m \le 1$$

and the first term on the right-hand side of (6.16) can be estimated by the second term on the right-hand side of (6.10). In the same way

$$\frac{q^{-m}(q+|\xi'|)^{2m-j+1/2}}{(1+|\xi'|)^{m-j+1/2}(q+|\xi'|)^m} = q^{-j+1/2}\Big(\frac{q+|\xi'|}{q+q|\xi'|}\Big)^{m-j+1/2} \le q^{-j+1/2} < 1$$

and the sum on the right-hand side of (6.16) can be estimated by by the third term in (6.10).

Now we show that for each $q_0 > 1$ we can find a constant $C(q_0)$ such that for $q \ge q_0$ the inequality

$$\|q^{-m}(q+|D|)^{2m}u_1\|_{L_2(\mathbb{R}^n_-)} \le \|(1+|D|)^m(q+|D|)^m u_1\|_{L_2(\mathbb{R}^n_-)} + C(q_0)q^m\|u_1\|_{L_2(\mathbb{R}^n_-)}$$

holds. To prove this inequality we find a constant $C(q_0)$ such that

$$q^{-m}(q+|\xi|)^{2m} \le (1+|\xi|)^m(q+|\xi|)^m + C(q_0)q^m \tag{6.17}$$

holds for $q \ge q_0 > 1$ and $\xi \in \mathbb{R}^n$. First of all note that

$$\lim_{|\xi|\to\infty} \frac{q^{-m}(q+|\xi|)^{2m}}{(1+|\xi|)^m(q+|\xi|)^m} = q^{-m} \le q_0^{-m} < 1\,.$$

From this we see that there exists $R = R(q_0)$ such that the left-hand side of (6.17) is not greater that the first term on the right-hand side of (6.17) for $|\xi| \ge R(q_0)$. For $|\xi| \le R(q_0)$ the left-hand side of (6.17) is not greater than the second term on the right-hand side if $C(q_0) = (1+\frac{R(q_0)}{q_0})^{2m}$.

Therefore, all terms on the right-hand side of (6.16) can be estimated by the right-hand side of (6.10) which finishes the proof of the theorem. □

Remark 6.3. The proof of the last theorem can also be done by repeating the proof of Theorem 5.4 for the diagonal system

$$\begin{pmatrix} A_1(D', -D_n) & 0 \\ 0 & A_2(D', -D_n, q) \end{pmatrix}.$$

In this case one has to replace the smoothing operator $\psi(D)I_2$ which appears in the decomposition (5.7) by the operator

$$\begin{pmatrix} \psi(D) & 0 \\ 0 & 1 \end{pmatrix}.$$

With this replacement we directly obtain inequality (6.10), i.e. the estimate without the term $q^m \|u_2\|_{L_2(\mathbb{R}^n_+)}$.

7. Remarks on singularly perturbed problems

Now we come back to the singularly perturbed problem (1.3). For simplicity we assume, as in the previous section, that A_{ij} is a scalar differential operator of order $2m$ with constant coefficients and without lower order terms, acting in the half-space $\mathbb{R}^n_+$. If we replace, for reasons of homogeneity, in (1.3) ε by ε^{2m}, we obtain the system

$$\begin{aligned} A_{11}(D)\,u_1 + A_{12}(D)\,u_2 &= 0 \quad \text{in } \mathbb{R}^n_+, \\ \varepsilon^{2m}(A_{21}(D)\,u_1 + A_{22}(D)\,u_2) - u_2 &= 0 \quad \text{in } \mathbb{R}^n_+. \end{aligned} \tag{7.1}$$

We assume that the boundary conditions are of the form

$$b_{j1}(D)u_1 + b_{j2}(D)u_2 = g_j \quad (j = 1, \dots, 2m) \quad \text{on } \mathbb{R}^{n-1}. \tag{7.2}$$

We define $B(D)$ by (2.3) with $N_1 = N_2 = 1$ and (μ, m) in (2.3) being replaced by $(m, 2m)$. Again we assume that $B(D)$ has constant coefficients and no lower-order terms.

Definition 7.1. The boundary value problem (7.1)–(7.2) is called weakly parameter-elliptic if $(A(D,\lambda), B(D))$ is weakly parameter-elliptic in the sense of Definition 2.6, where $A(D,\lambda)$ is given by (1.1).

Due to Remark 2.5, in this case the Vishik–Lyusternik condition is satisfied automatically. As it was already mentioned above, all results of the previous sections can equivalently be re-formulated in the context of boundary value problems with small parameter. In particular, weak parameter-ellipticity implies uniform a priori estimates for the solutions u_1, u_2 of (7.1)–(7.2) (uniform with respect to $\varepsilon \to 0$). As the necessary modifications are obvious, we do not want to formulate these results explicitly. The aim of the present section is to show the connection between the conditions of weak parameter-ellipticity and the existence and construction of boundary layers for the singularly perturbed problem (7.1)–(7.2), see also Remark 2.7 b).

Let $u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ be a solution of (7.1)–(7.2). Multiplying the second equation in (7.1) by ε^{2m}, we see that $(F'u)(t,\xi')$ is a solution of (2.10)–(2.11) with $q = \varepsilon^{-1}$ and $g := (F'h)(\xi')$. For sufficiently small ε, this solution is given by Lemma 3.4 as a sum of two integrals. As the second integral in (3.8) contains the exponential term $e^{itq\tau} = e^{it\tau/\varepsilon}$, we see that this integral describes a boundary layer. The existence of this boundary layer term is a consequence of condition 2.6 (iv), which can be seen from Lemma 3.3. The first integral in (3.8), however, is described in Lemma 3.2 which uses condition 2.6 (iii). So we can already see from these considerations that the (in some sense non-standard) conditions (iii) and (iv) in Definition 2.6 lead to the fact that the solution of the singularly perturbed problem splits into a regular part and a boundary layer part.

The remainder of this section is devoted to the problem to construct a formal asymptotic solution (FAS) of the boundary value problem (7.1)–(7.2), i.e. a formal series

$$u(x,\varepsilon) = \sum_{k=0}^{\infty} \varepsilon^k u^{(k)}(x,\varepsilon), \quad u^{(k)}(x,\varepsilon) = \begin{pmatrix} u_1^{(k)}(x,\varepsilon) \\ u_2^{(k)}(x,\varepsilon) \end{pmatrix}$$

for which the partial sums $\sum_{k=0}^{N} \varepsilon^k u^{(k)}$ satisfy (7.1)–(7.2) up to order $O(\varepsilon^N)$. We will see below that again conditions 4.2 (iii) and (iv) are essential for this construction.

Similar to the notation in [14], let us call $(A_{11}(D), B_{11}(D))$ (see (2.12)–(2.13)) the first limit problem and (2.14)–(2.15) the second limit problem. Under the condition of weak parameter-ellipticity, the first limit problem is elliptic. To avoid technical difficulties, let us assume that it is uniquely solvable, too. If this is not the case, one has to deal with kernels and co-kernels of the operator related to this boundary value problem.

Following Vishik–Lyusternik [16] (see also [11]), we seek the solution in the form

$$u(x,\varepsilon) = v(x,\varepsilon) + w(x,\varepsilon),$$

where

$$v(x,\varepsilon) = \sum_{k=0}^{\infty} \varepsilon^k v^{(k)}(x), \quad v^{(k)}(x) = \begin{pmatrix} v_1^{(k)}(x) \\ v_2^{(k)}(x) \end{pmatrix} \tag{7.3}$$

is a so-called exterior expansion and

$$w(x,\varepsilon) = \sum_{k=0}^{\infty} \varepsilon^{m+k} w^{(k)}\Big(x', \frac{x_n}{\varepsilon}\Big), \quad w^{(k)}(x) = \begin{pmatrix} w_1^{(k)}(x) \\ w_2^{(k)}(x) \end{pmatrix} \tag{7.4}$$

is a so-called interior expansion or boundary layer. Our aim is to find partial differential equations and boundary conditions determining the functions $v^{(k)}$, $w^{(k)}$. We start with the equations in the interior of the domain $\mathbb{R}^n_+$. In the following, we set

$$A(D) := A(D,0) = \begin{pmatrix} A_{11}(D) & A_{12}(D) \\ A_{21}(D) & A_{22}(D) \end{pmatrix} .$$

(i) *Differential equations for* $v^{(k)}$. Substituting (7.3) into (7.1), we obtain

$$\begin{aligned} &\left[\begin{pmatrix} 1 & 0 \\ 0 & \varepsilon^{2m} \end{pmatrix} A(D) - \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right] \sum_{k=0}^{\infty} \varepsilon^k \begin{pmatrix} v_1^{(k)} \\ v_2^{(k)} \end{pmatrix} \\ &= \sum_{k=0}^{\infty} \varepsilon^k \left[\begin{pmatrix} A_{11}(D) & A_{12}(D) \\ 0 & -1 \end{pmatrix} \begin{pmatrix} v_1^{(k)} \\ v_2^{(k)} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ A_{21}(D) & A_{22}(D) \end{pmatrix} \begin{pmatrix} v_1^{(k-2m)} \\ v_2^{(k-2m)} \end{pmatrix}\right], \end{aligned}$$

where we have set

$$v^{(k)} := 0 \quad \text{for } k = -2m, -2m+1, \dots, -1. \tag{7.5}$$

Thus we obtain the recurrence relations

$$A_{11}(D)v_1^{(k)} = -A_{12}(D)\Big(A_{21}(D) \quad A_{22}(D)\Big)v^{(k-2m)} \quad (k = 0,1,2,\dots), \tag{7.6}$$

$$v_2^{(k)} = \Big(A_{21}(D) \quad A_{22}(D)\Big)v^{(k-2m)} \quad (k = 0,1,2,\dots). \tag{7.7}$$

In order to determine $v^{(k)}$ (with starting values (7.5)), we have to impose m boundary conditions on $v_1^{(k)}$, see below.

(ii) *Differential equations for* $w^{(k)}$. To find the corresponding equations for $w^{(k)}$, we note that

$$A(D)\Big[w^{(k)}\Big(x', \frac{x_n}{\varepsilon}\Big)\Big] = \varepsilon^{-2m}\Big[A(\varepsilon D', D_n)w^{(k)}\Big]\Big(x', \frac{x_n}{\varepsilon}\Big),$$

due to homogeneity. Substituting (7.4) into (7.1), we obtain

$$\begin{aligned} &\left[\begin{pmatrix} 1 & 0 \\ 0 & \varepsilon^{2m} \end{pmatrix} A(D) - \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right] \sum_{k=0}^{\infty} \varepsilon^{k+m} w^{(k)}\Big(x', \frac{x_n}{\varepsilon}\Big) \\ &= \begin{pmatrix} \varepsilon^{-m} & 0 \\ 0 & \varepsilon^m \end{pmatrix} \sum_{k=0}^{\infty} \varepsilon^k \left(\left[A(\varepsilon D', D_n) - \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right] w^{(k)}\right)\Big(x', \frac{x_n}{\varepsilon}\Big). \end{aligned} \tag{7.8}$$

Now we expand $A(\varepsilon D', D_n)$ in a Taylor series with respect to ε,

$$A(\varepsilon D', D_n) = \sum_{\ell=0}^{2m} \varepsilon^{\ell} A^{(\ell)}(D', D_n)$$

with $A^{(0)}(D', D_n) = A(0, D_n)$ and $A^{(2m)}(D', D_n)$ being a constant complex 2×2 matrix.

Substituting this expansion into the last sum in (7.8), we see that this sum equals

$$\sum_{k=0}^{\infty} \varepsilon^k \left[\left(A^{(0)}(D) - \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right) w^{(k)} + \sum_{\ell=1}^{2m} A^{(\ell)}(D) w^{(k-\ell)} \right]$$

where we have set

$$w^{(j)} := 0 \quad (j = -2m, -2m+1, \dots, -1) . \tag{7.9}$$

Therefore we obtain the recurrence relations

$$\left[A^{(0)}(D) - \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right] w^{(k)} = - \sum_{\ell=1}^{2m} A^{(\ell)}(D) w^{(k-\ell)} \quad (k = 0, 1, 2, \dots) \tag{7.10}$$

(iii) *Boundary conditions.* Now we want to find boundary conditions for the functions $v_1^{(k)}$ and $w^{(k)}$ with $k = 0, 1, \dots$. Setting

$$b_j(D) := \begin{pmatrix} b_{j1}(D) & b_{j2}(D) \end{pmatrix} \quad (j = 1, \dots, 2m) ,$$

we trivially have

$$b_j(D) \sum_{k=0}^{\infty} \varepsilon^k v^{(k)}(x', 0) = \sum_{k=0}^{\infty} \varepsilon^k b_j(D) v^{(k)}(x', 0)$$

and, by homogeneity,

$$b_j(D) w^{(k)} \left(x', \frac{x_n}{\varepsilon} \right) \Big|_{x_n = 0} = \varepsilon^{-m_j} b_j(\varepsilon D', D_n) w^{(k)}(x', x_n) \Big|_{x_n = 0} .$$

Therefore

$$\begin{aligned} B(D) \left[\sum_{k=0}^{\infty} \varepsilon^k v^{(k)} + \sum_{k=0}^{\infty} \varepsilon^{m+k} w^{(k)} \left(x', \frac{x_n}{\varepsilon} \right) \right] \Bigg|_{x_n = 0} &= \sum_{k=0}^{\infty} \varepsilon^k B(D) v^{(k)}(x', 0) \\ + \sum_{k=0}^{\infty} \begin{pmatrix} \varepsilon^{-m_1} & & \\ & \ddots & \\ & & \varepsilon^{-m_{2m}} \end{pmatrix} & \varepsilon^{m+k} B(\varepsilon D', D_n) w^{(k)}(x', 0) . \end{aligned}$$

Again we use the Taylor expansion with respect to ε,

$$b_j(\varepsilon D', D_n) = \sum_{\ell=0}^{m_j} \varepsilon^\ell b_j^{(\ell)}(D', D_n) ,$$

where $b_j^{(0)}(D',D_n)=b_j(0,D_n)$. We receive

$$\begin{aligned} b_j(D)(v+w) &= \sum_{k=0}^{\infty}\Big[\varepsilon^k b_j(D)v^{(k)}+\varepsilon^{-m_j+m+k}\sum_{\ell=0}^{m_j}\varepsilon^\ell b_j^{(\ell)}(D)w^{(k)}\Big] \\ &= \sum_{k=\min\{m-m_j,0\}}\varepsilon^k\Big[b_j(D)v^{(k)}+\sum_{\ell=0}^{m_j}b_j^{(\ell)}(D)w^{(k+m_j-m-\ell)}\Big]. \end{aligned} \tag{7.11}$$

Note that here negative powers of ε may appear; for negative values of k in $v^{(k)}$ and $w^{(k)}$ we use (7.5) and (7.9).

From the conditions $b_j(v+w)=g_j$ for $j=1,\dots,2m$ we obtain the boundary conditions

$$\begin{aligned} & b_j(D)v^{(k)}+\sum_{\ell=0}^{m_j}b_j^{(\ell)}(D)w^{(k+m_j-m-\ell)}=\delta_{k0}\, g_j \\ &\qquad \text{for } k=\min\{m-m_j,0\},\min\{m-m_j,0\}+1,\dots \\ &\qquad \text{and } j=1,\dots,2m. \end{aligned} \tag{k,j}$$

This set of boundary conditions is numbered using the indices k and j. At the first moment it seems to be unclear how conditions (k,j) determine the functions $v^{(k)}$ and $w^{(k)}$. This is the essence of the following theorem, where we restrict ourselves, for simplicity, to the particular case that

$$\operatorname{ord} B_j = j-1 \quad (j=1,\dots,2m)\,. \tag{7.12}$$

Theorem 7.2. *Let the boundary value problem* (7.1)–(7.2) *be weakly parameter-elliptic and the first limit problem be uniquely solvable. Assume further that* (7.12) *holds. Then the recursion formulas* (7.5)–(7.7), (7.9)–(7.10) *with boundary conditions* (k,j) *uniquely determine the functions* $v^{(k)}$ *and* $w^{(k)}$ *for* $k=0,1,2\dots$ *The boundary conditions for* $v_1^{(k)}$ *and* $w^{(k)}$ *have the form*

$$B_{11}(D)v_1^{(k)}(x',0)=g^{(k)}(x') \tag{7.13}$$

and

$$B_2(0,D_n)w^{(k)}(x',0)=h^{(k)}(x')\,, \tag{7.14}$$

respectively. The right-hand side of (7.13) *contains only* $v^{(\ell)}$ *and* $w^{(\ell)}$ *with* $\ell<k$, *while the right-hand side of* (7.14) *contains only* $v^{(\ell)}$ *with* $\ell\le k$ *and* $w^{(\ell)}$ *with* $\ell<k$. *In particular, the right-hand sides can be determined recursively.*

We remark that the case where (7.12) does not hold can be treated with the same idea, but is somewhat more delicate.

Proof. Due to the recursion formulas (7.6)–(7.7) and (7.10) with starting values (7.5) and (7.9), we only have to show that the boundary condition (k,j) determine $v_1^{(k)}$ and $w^{(k)}$.

Figure 1 shows for each index pair (k,j) which appears in the above boundary condition the function for which this condition is used. (The index pairs marked with do not appear). The main question is in what order the formula (k,j) has to be applied.

$k \downarrow$ \ $j \longrightarrow$	1	2	$\cdots$	m	$m+1$	$\cdots$	$\cdots$	$2m$
$-m+1$								$w^{(0)}$
$\vdots$							$w^{(0)}$	$w^{(1)}$
-1						$w^{(0)}$	$w^{(1)}$	$\vdots$
0	$v^{(0)}$	$v^{(0)}$	$v^{(0)}$	$v^{(0)}$	$w^{(0)}$	$w^{(1)}$	$\vdots$	
1	$v^{(1)}$	$v^{(1)}$	$v^{(1)}$	$v^{(1)}$	$w^{(1)}$	$\vdots$		
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$			

Figure 1: Usage of formula (k,j).

We will use the formulas (k,j) with $j \leq m$ as boundary conditions for $v^{(k)}$ and formulas $(k-j+1-m,j)$ with $j=m+1,\ldots,2m$ as boundary conditions for $w^{(k)}$. We have to show the statement on the right-hand sides of (7.13) and (7.14).

Let us assume that in step k we already know the functions $v^{(\ell)}$ and $w^{(\ell)}$ with $\ell < k$. We want to find $v^{(k)}$ and $w^{(k)}$. First let $j \leq m$. Then condition (k,j) contains the functions

$$v^{(k)}, w^{(k+j-m-1)}, w^{(k+j-m-2)}, \ldots, w^{(k-m)}\,.$$

As $k+j-m-1<k$ and $v_2^{(k)}$ is defined by (7.7) which only contains $v^{(k-2m)}$, the

only unknown function in condition (k,j) is $v_1^{(k)}$. We get (7.13) with

$$g_j^{(k)} := \delta_{k0}\, g_j - b_{j2}(D)u_2^{(k)} - \sum_{\ell=0}^{j-1} b_j^{(\ell)}(D)w^{(k+j-1-m-\ell)} \quad (j=1,\dots,m)\,.$$

Due to the condition of unique solvability of the first limit theorem, the function $v_1^{(k)}$ is uniquely determined by (7.6) and (7.13).

Now let $j>m$. The boundary condition $(k-j+1-m,j)$ contains the functions

$$w^{(k)}, w^{(k-1)}, \dots, w^{(k-j+1)} \text{ and } v^{(k-j+m+1)}.$$

As $k-j+m+1 \le k$ and as we already know $v^{(\ell)}$ for $\ell \le k$, this gives m boundary conditions for $w^{(k)}$. Due to unique solvability of the second limit problem, the function $w^{(k)}$ is determined by these boundary conditions, and we can continue with step $k+1$. The boundary conditions for $w^{(k)}$ have the form (7.14) with

$$h_j^{(k)} = \delta_{k-j+m+1,0}\, g_j - b_j(D)v^{(k-j+m+1)} - \sum_{\ell=1}^{j-1} b_j^{(\ell)}(D)w^{(k-\ell)} \quad (j=m+1,\dots,2m)\,.$$

Summarizing, we see that if we use formula (k,j) as boundary conditions for $v^{(k)}$ and $w^{(k)}$ in the way indicated in Figure 1, the right-hand sides of (7.13) and (7.14) can be determined recursively. □

Remark 7.3. Due to the Theorem 7.2, the terms in the asymptotic expansion are given as solutions of the first and second limit problem. In particular, condition 2.6 (iv), i.e. unique solvability of the second limit problem, leads to the boundary layer terms, while condition 2.6 (iii), i.e. the first limit problem, corresponds to the exterior expansion. This demonstrates that conditions (iii) and (iv) are in some sense very natural conditions, if they are considered from the point of view of singular perturbations.

References

[1] Agmon, S.: On the eigenfunctions and on the eigenvalues of general elliptic boundary value problems. *Comm. Pure Appl. Math.* **15** (1962), 119-147.

[2] Agranovich, M., Denk, R., Faierman, M.: Weakly smooth nonselfadjoint spectral elliptic boundary problems. In M. Demuth et al. (eds.): Spectral Theory, Microlocal Analysis, Singular Manifolds. Math. Top. **14** (1997), 138–199.

[3] Agranovich, M. S., Vishik, M. I.: Elliptic problems with parameter and parabolic problems of general form (Russian). *Uspekhi Mat. Nauk* **19** (1964), No. 3, 53-161. English transl. in *Russian Math. Surv.* **19** (1964), No. 3, 53-157.

[4] Denk, R., Mennicken, R., Volevich, L.: The Newton polygon and elliptic problems with parameter. *Math. Nachr.* **192** (1998), 125-157.

[5] Denk, R., Mennicken, R., Volevich, L.: Boundary value problems for a class of elliptic operator pencils. *Keldysh Inst. Appl. Math. Preprint* **58** (1998).

[6] Denk, R., Mennicken, R., Volevich, L.: On elliptic operator pencils with general boundary conditions. *Keldysh Inst. Appl. Math. Preprint* **37** (1999).

[7] Faierman, M.: A transmission problem for elliptic equations involving a parameter and a weight. To appear in *Glas. Mat. Ser. III.*

[8] Frank, L.: Coercive singular perturbations. I. A priori estimates. *Ann. Mat. Pura Appl.* (4) **119** (1979), 41–113.

[9] Gindikin, S. G., Volevich, L. R.: *The method of Newton's polyhedron in the theory of partial differential equations.* Math. Appl. (Soviet Ser.) **86**, Kluwer Academic, Dordrecht, 1992.

[10] Gindikin, S. G., Volevich, L. R.: *Mixed problem for partial differential equations with quasihomogeneous principal part.* Transl. Math. Monogr. **147**, Amer. Math. Soc., Providence, RI, 1996.

[11] Il'in, A. M.: *Matching of asymptotic expansions of solutions of boundary value problems* (Russian). "Nauka", Moscow, 1989. English transl. as Transl. Math. Monogr. **102**, Amer. Math. Soc., Providence, RI, 1992.

[12] Kozhevnikov, A.: Asymptotics of the spectrum of Douglis–Nirenberg elliptic operators on a closed manifold. Math. Nachr. **182** (1996), 261–293.

[13] Kozhevnikov, A.: Parameter-ellipticity for mixed-order systems elliptic in the sense of Petrovskii. To appear in *Commun. Appl. Anal.*

[14] Nazarov, S. A.: The Vishik–Lyusternik method for elliptic boundary value problems in regions with conic points. I. The problem in a cone (Russian). *Sibirsk. Mat. Zh.* **22** (1981), No. 4, 142-163. English transl. in *Siberian Math. J.* **22** (1982), 594-611.

[15] Roitberg, Y: *Elliptic boundary value problems in the spaces of distributions.* Math. Appl. **384**, Kluwer Academic, Dordrecht, 1996.

[16] Vishik, M. I., Lyusternik, L. A.: Regular degeneration and boundary layer for linear differential equations with small parameter (Russian). *Uspehi Mat. Nauk (N.S.)* **12** (1957), No. 5 (77), 3-122. English transl. in *Amer. Math. Soc. Transl.* (2) **20** (1962), 239-364.

[17] Volevich, L. R.: Solvability of boundary value problems for general elliptic systems (Russian). *Mat. Sb.* **68** (1965), No. 3, 373-416. English transl. in *Amer. Math. Soc. Transl., Ser.* 2, **67**, 182-225 (1968).

[18] Volevich, L. R., Paneah, B. P.: Some spaces of generalized functions and embedding theorems (Russian). *Uspehi Mat. Nauk* **20** (1965), No. 1 (121), 3-74. English transl. in *Russian Math. Surv.* **20** (1964), No. 1, 1-73.