Characterizations of Qualitative Properties of C-regularized Semigroups in Terms of Their Generators

by

Miao Li

A Dissertation
for the Degree of Doctor of Science

Sichuan University

May 2000

Acknowledgement

I wish to express my sincere gratitude to my advisor, Professor Fa-Lun Huang, for his excellent guidance, constant encouragement, and being a continuous source of new ideas. I could not have hoped for a better teacher.

I also wish to take this opportunity to thank Professor Ralph deLaubenfels for helpful discussions and comments on Chapter 2. I am grateful to Professor Quan Zheng for reading my manuscripts and addressing the question at the end of Section 1.3. And I thank Xiao-Hui Gu and Ling-Hong Xie for a fruitful collaboration.

Finally, I address my deepest thanks to my parents and my sister for their understanding and encouragement during these years.

Abstract

A generalization of strongly continuous semigroups, C-regularized semigroups, has recently received much attention (see [DP, dL4]). However, the main concern is the generation theorem ([dL4]), not its fundamental structural characterizations. In the present thesis we concentrate on some special properties of C-regularized semigroups, such as contractivity, almost periodicity, asymptotic almost periodicity, ergodicity, norm continuity and stability. These qualitative properties have been studied extensively for strongly continuous semigroups (see, e.g. [HP, Pa, Zhe]); on the other hand, there are few results for C-regularized semigroups. However, these properties are basically important since the properties of the semigroups correspond to the same properties of the solutions of the abstract Cauchy problems. Also, it is noted that C-regularized semigroups are different from strongly continuous semigroups in essence. In investigating C-regularized semigroups, some new skills will be employed and some new results will be obtained.

A main technique we will use is the Hille-Yosida space. The Hille-Yosida space for an operator A (on X) is a maximal continuously embedded subspace of X on which A generates a strongly continuous semigroup. deLaubenfels [dL3] introduced it into the theory of C-regularized semigroups and used it to connect a C-regularized semigroup with a strongly continuous semigroup. But this interpolation space was only considered of theoretical value before. We find in our research that the technique of the Hille-Yosida space can greatly simplify the proofs; by making use of this space, we can characterize the qualitative properties of a C-regularized semigroup by its generator.

Our work not only generalizes those related results for strongly continuous semigroups, but also is a supplement to the theory of C-regularized semigroups.

Contents

Introduction

It is well known that the theory of semigroups of operators has developed rapidly since the generation theorem was founded by Hille and Yosida in 1948. Now it is an important branch of mathematics and has its applications to many other fields such as differential operators, abstract Cauchy problems, the theory of control, ergodicity, potential theory and mathematical physics.

But, the main work still focuses on strongly continuous semigroups. And by now we have an almost satisfactory system. However, the applications of strongly continuous semigroups to PDEs are limited since the generators must have dense domains. In order to deal with the cases while the generators satisfy weaker conditions, it is necessary to have other classes of semigroups. Motivated by the abstract Cauchy problem

$$\frac{d}{dt}u(t) = Au(t)(t \geq 0),\ u(0) = x$$

a generalization of strongly semigroups, C-regularized semigroups, has recently received much attention (see [DP, dL4]). Generating a C-regularized semigroup corresponds to the abstract Cauchy problem having a unique solution, whenever $x = Cy$, for some y in the domain of A. It is well known that the class of operators that generate C-regularized semigroups is much larger than the class of operators that generate strongly continuous semigroups. The notion of C-semigroup was introduced by Da Prato (1967), E.B. Davies and M.M. Pang (1989) independently. The basic results of the theory of C-regularized semigroups can be found in the book of R. deLaubenfels [dL4].

In the present thesis we concentrate on some special properties of C-regularized semigroups, such as contractivity, almost periodicity, ergodicity, norm continuity and stability. These qualitative properties have been studied extensively for strongly continuous semigroups; on the other hand, there are few results for C-regularized semigroups. However, these properties are basically important since certain property of the semigroups implies the same thing of the solutions of the abstract Cauchy problems. Also, we note that C-regularized semigroups are different from strongly continuous semigroups in essence. In investigating C-regularized semigroups, we need to employ some new skills and thus obtain some new results.

A main technique we will use is the Hille-Yosida space. This is a maximal continuously imbedded subspace of X on which A generates a strongly continuous semigroup. S. Kantorovitz [Ka] first introduced the Hille-Yosida space for a closed operator A with $(0, \infty) \subset \rho(A)$, on which the restriction of A generates a strongly continuous semigroups. R. deLaubenfels [dL3] extended it to more general cases that A has no eigenvalues in $(0, \infty)$, and used it to connect C-regularized semigroup with strongly continuous semigroup.

We now sketch the main results of this thesis. In Chapter 1 we characterize contraction C-regularized semigroups. A complete characterization is given in Section 1.2 by using the Hille-Yosida space. And when $C(D(A))$ is dense in $R(C)$, one of the conditions can be removed. Some special cases are presented in Section 1.3 which guarantee the denseness of $C(D(A))$ in $R(C)$. In particular, we show if $R(C)$ is dense, then A generates a contraction C-regularized semigroup if and only if it is a generator of a contraction strongly continuous semigroup. Section 1.4 provides a way to get a contraction regularized semigroup from every bounded regularized semigroup, and a counterexample is given to answer the question asked at the end of Section 1.3 in the negative.

Chapter 2 is devoted to the almost periodicity and asymptotically almost periodicity of C-regularized semigroups. In Section 2.2 the almost periodicity of C-regularized semigroup is characterized. Our result extends recent results by Q. Zheng and L. Liu [ZL] for the case of $R(C)$ being dense in X. The key fact here is that a solution of the abstract Cauchy problem is almost periodic in the Hille-Yosida space if and only if it is almost periodic in X. This idea can also be applied to study the asymptotic almost periodicity of C-regularized semigroups, which is the topic of Section 2.3.

In Chapter 3, we define the ergodicity of C-regularized semigroups and then characterize the generators. An analogue of strong ergodicity theorem for strongly continuous semigroup is proved in Section 3.2. The uniform ergodicity is considered in Section 3.4. We demonstrate by an example that the condition that $R(A)$ is closed is no longer necessary for the uniform ergodicity. By considering the ergodicity in the Hille-Yosida space, we define the orbital ergodicity in Section 3.3.

We study the norm continuity of C-regularized semigroups on Banach spaces in Section 4.1. Several equivalent conditions are given. Section 4.2 deals with the norm continuity on Hilbert spaces. We show if a loss of regularity is accepted, then the norm continuity of the regularized semigroup can be derived from certain conditions on the C-resolvents of its generator.

Chapter 5 is concerned with stability of C-regularized semigroups. In Section 5.1, we give two characterizations on the C-resolvents of the generator of an exponentially bounded C-regularized semigroup. By making use of the Hille-Yosida space, a Tauberian theorem [BNR2] and the relations between the spectrum of the generator on its Hille-Yosida space and that on X, we are able to give some sufficient stability conditions. Our results generalize those corresponding for strongly continuous semigroups (see, cf.[AB, Ba, Hu2]).

Chapter 1

Characterizations of contraction C-regularized semigroups

The starting point of this chapter is to try to give an answer to the question asked by R. deLaubenfels in [dL3 , Open question 6.10]: Does there exists an analogue of the Lumer-Phillips theorem for C-regularized semigroups? Since the Lumer-Phillips theorem characterizes the generators of contraction strongly continuous semigroups, this gives us the motivation to make a suitable definition for the contractions of C-regularized semigroups and then characterize the generators.

On the other hand, many work has been done to generalize the Hille-Yosida theorem to C-regularized semigroups. Earlier, Davies and Pang [DP] gave a characterization of an exponentially bounded C-regularized semigroup under the assumption that $R(C)$ is dense in X. Later, Tanaka and Miyadera [TM1] generalized their results to the case of $R(C)$ not dense, they gave a sufficient and necessary condition for a closed linear operator with dense domain to be the generator of an exponentially bounded C-regularized semigroup. After defining the contraction C-regularized semigroup, we are also interested in characterizing the generator by the Hille-Yosida type theorem. Here the main difficulty we meet with is that the generator may be not densely defined, we choose the Hille-Yosida space to give an additional condition on the generator.

This chapter is organized as follows. Section 1.1 is devoted to some preliminaries on C-regularized semigroups. In Section 1.2 we characterize the generators of contraction C-regularized semigroups in general cases, and under the assumption that $C(D(A))$ is dense in $R(C)$, the characterization can be simplified. Section 1.3 deals with several special cases of $\rho(A) \neq \emptyset$ or $\overline{R(C)} = X$, we obtain both Hille-Yosida theorem and Lumer-Phillips theorem in such cases. This means that we partly give the answer to the question mentioned above in the affirmative. In Section 1.4 we show, for every bounded C-regularized semigroup, an extrapolation space can be constructed such that the extension of the regularized semigroup on it is a contraction one. Our method is different from that of deLaubenfels'[dL3, dL4]. By using this type of space, we can answer the question asked at the end of Section 1.3 in the negative.

Throughout this thesis, X will be a Banach space. The space of all bounded linear operators on X will be denoted by $B(X)$, and C will always be an injective operator in $B(X)$. For an operator A, we will write $D(A)$ for its domain, $R(A)$ for its range and $\rho(A)$ for its resolvent set, and we will denote by $\overline{E}$ for the closure of a subspace of X, E. If Y is a subspace of X, then the restriction of A on Y, $A|_Y$, is defined by $A|_Y(x) = Ax$ for every $x \in D(A|_Y)$ with $D(A|_Y) = \{x \in Y : Ax \in Y\}$.

1.1 Preliminaries

First, we recall the definition of C-regularized semigroups.

Definition 1.1.1 A strongly continuous family $(T(t))_{t\geq 0} \subset B(X)$ is called a *C-regularized semigroup* if $T(t+s)C = T(t)T(s)$ for $t, s \geq 0$ and $T(0) = C$. $(T(t))_{t\geq 0}$ is *exponentially bounded* if there exist $M < \infty$ and $\omega \in R$ such that $||T(t)|| \leq Me^{\omega t}$; $(T(t))_{t\geq 0}$ is *bounded* if ω can be chosen as zero.

The *generator of* $(T(t))_{t\geq 0}$, A , is defined by

$$Ax = C^{-1}\left[\lim_{t\downarrow 0}\frac{1}{t}(T(t)x - Cx)\right]$$

with

$$D(A) = \left\{x \in X : \lim_{t\downarrow 0}\frac{1}{t}(T(t)x - Cx) \text{ exists and is in } R(C)\right\}.$$

The complex number λ is in $\rho_C(A)$, the *C-resolvent set of* A, if $(\lambda - A)$ is injective and $R(C) \subseteq R(\lambda - A)$. The complement of $\rho_C(A)$ is denoted by $\sigma_C(A)$.

Next, let us recall the definition of solution and mild solution.

Definition 1.1.2 The *strong solution* (or simply *solution*) of the abstract Cauchy problem

$$\frac{d}{dt}u(t) = Au(t),\ \forall t \geq 0; \qquad u(0) = x \tag{1.1}$$

is the function $u(t)$ satisfying $u(t) \in C([0,\infty), [D(A)]) \cap C^{-1}([0,\infty), X)$ and (1.1).

The mild solution of (1.1) is the function $u(t)$ satisfying $u(t) \in C([0,\infty), X)$ and $\int_0^t u(s)ds \in D(A), \forall t \geq 0$,

$$A(\int_0^t u(s)ds) = u(t) - x(t \geq 0), \qquad u(0) = x.$$

We collect the basic properties of C-regularized semigroups and the generators in the following; see [dL4] for the proofs and more details.

Lemma 1.1.3 *Suppose A generates a C-regularized semigroup $(T(t))_{t\geq 0}$. Then*

(a) *A is a closed linear operator with $\overline{D(A)} \supseteq R(C)$;*

(b) *$\forall x \in D(A)$ and $t \geq 0$, $T(t)x \in D(A)$ with $AT(t)x = T(t)Ax$;*

(c) *$\forall x \in D(A)$ and $t \geq 0$, $T(t)x$ is differentiable and $\frac{d}{dt}T(t)x = T(t)Ax$;*

(d) *$\forall x \in X,\ t \geq 0,\ T(t)x = Cx + A\int_0^t T(s)xds$, which implies $T(\cdot)x$ is a mild solution of the abstract Cauchy problem* (1.1).

(e) *$A = C^{-1}AC$, where $D(C^{-1}AC) = \{x \in X : Cx \in D(A)$* and *$ACx \in R(C)\}$;*

(f) *$\forall x \in C(D(A))$, there is a unique solution of* (1.1) *given by $u(t,x) = T(t)C^{-1}x$;*

(g) *$\forall x \in R(C)$, there is a unique mild solution of* (1.1) *given by $u(t,x) = T(t)C^{-1}x$;*

(h) *If $||T(t)|| \leq Me^{\omega t}$, then $(\omega, \infty) \subseteq \rho_C(A)$. For every $r \in \mathbf{C}$ with $Rer > \omega$ and $n \in \mathbf{N}$, $D((r-A)^{-n}) \supseteq R(C)$ and*

$$(r-A)^{-n}C = \frac{1}{(n-1)!}\int_0^\infty t^{n-1}e^{-rt}T(t)dt, \tag{1.2}$$

which implies $||(r-\omega)^n(r-A)^{-n}C|| \leq M$.

Next we need to introduce the Hille-Yosida space for an operator, for the details we refer to [dL4].

Definition 1.1.4 Suppose A has no eigenvalues in $(0,\infty)$. The *Hille-Yosida space for A,* Z_0, is defined by

$$Z_0 = \{x \in X : \text{The Cauchy problem (1.1) has a bounded uniformly continuous mild solution } u(\cdot,x)\}$$

with

$$||x||_{Z_0} = \sup\{||u(t,x)||;\ t \geq 0\} \text{ for } x \in Z_0.$$

And the *weak Hille-Yosida space for A,* Y, is the space defined by

$$Y = \{x \in X : x \in R((s-A)^n) \text{ for } \forall\, s > 0, n \in \mathbf{N} \text{ with } ||x||_Y = \sup\{s^n||(s-A)^{-n}x||; s > 0, n+1 \in \mathbf{N}\} < \infty\}.$$

The relation between Z_0 and Y is as follows.

Lemma 1.1.5 *Suppose A has no eigenvalues in $(0, \infty)$, Z_0 and Y are defined as above. Then*

(a) *Z_0 and Y are Banach spaces;*

(b) *$||x|| \leq ||x||_{Z_0}$, $\forall x \in Z_0$, so Z_0 is continuously imbedded into X;*

(c) *$Z_0 \subset Y$ and $||x||_{Z_0} = ||x||_Y$ for all $x \in Z_0$;*

(d) *Z_0 is the closure, in Y, of $D(A|_Y)$;*

(e) *$A|_{Z_0}$ generates a contraction strongly continuous semigroup $(T(t))_{t\geq 0}$ on Z_0 and $T(t)x = u(t,x)$, $\forall x \in Z_0$, where $u(t,x)$ is the mild solution of (1.1);*

(f) *Z_0 is maximal unique in the sense that if W is continuously imbedded into X and $A|_W$ is the generator of a bounded strongly continuous semigroup, then $W \hookrightarrow Z_0$.*

Lemma 1.1.6[dL4] If A generates a bounded strongly uniformly continuous C-regularized semigroup $(T(t))_{t\geq 0}$, then $R(C) \subset Z_0$ and $A|_{Z_0}$ generates a contraction strongly continuous semigroup $S(t) = C^{-1}T(t)$ and

$$Z_0 = \{x \in X | t \to C^{-1}T(t)x \text{ is bounded and uniformly continuous } \}$$

with

$$\|x\|_{Z_0} = \sup_{t\geq 0} \|C^{-1}T(t)x\|.$$

1.2 C-regularized semigroups of contractions

A C-regularized semigroup $(T(t))_{t\geq 0}$ is *of contractions* if $||T(t)x|| \leq ||Cx||$ for $t \geq 0$ and $x \in X$. In this section, we give the characterizations of the generators of contraction C-regularized semigroups. We start with the following

Proposition 1.2.1 *Suppose A generates a contraction C-regularized semigroup, then*

(a) *$(0, \infty) \subseteq \rho_C(A)$, and for $\lambda > 0, n \in \mathbf{N}$ and $x \in X$, $R(C) \subseteq R((\lambda - A)^n)$*

$$\lambda^n ||(\lambda - A)^{-n}Cx|| \leq ||Cx||;$$

(b) *for every* $x \in D(A)$, *there exists an* $x^* \in F(Cx)$, *that is ,* $x^* \in X^*, ||x^*|| = ||Cx||$ *and* $x^*(Cx) = ||Cx||^2$, *such that*

$$\mathrm{Re}\langle CAx, x^*\rangle \leq 0,$$

where $\langle x, x^*\rangle$ *denotes the value of* x^* *at* x.

Proof. (a) follows directly from Lemma 1.1.3(h).

Let $x \in D(A)$ and $x^* \in F(Cx)$. Then

$$\mathrm{Re}\langle \frac{T(t)x - Cx}{t}, x^*\rangle \leq \mathrm{Re}\langle \frac{T(t)x}{t}, x^*\rangle - \frac{||Cx||^2}{t} \leq 0 \quad for \;\; t > 0,$$

hence

$$\mathrm{Re}\langle CAx, x^*\rangle = \lim_{t\downarrow 0} \mathrm{Re}\langle \frac{T(t)x - Cx}{t}, x^*\rangle \leq 0.$$

This is (b). □

Remark 1.2.2 If an operator A with $CA \subseteq AC$ satisfies (b), we call A is C-*dissipative.* Similar to the proof of [Pa, Chapter 1, Theorem 4.2], we can prove that A is C-dissipative if and only if $||(\lambda - A)Cx|| \geq \lambda||Cx||$, $\forall x \in D(A)$ and $\lambda > 0$. Note that if $\lambda||(\lambda - A)^{-1}Cx|| \leq ||Cx||$ for all $x \in X$, then for $x \in D(A)$,

$$||(\lambda - A)Cx|| = ||C(\lambda - A)x|| \geq ||\lambda(\lambda - A)^{-1}C(\lambda - A)x|| = \lambda||Cx||.$$

Using the Hille-Yosida space, we can completely characterize the generators.

Theorem 1.2.3 *Let* A *be an operator on* X. *Then* A *generates a contraction* C*-regularized semigroup if and only if* A *satisfies*

(a) $A = C^{-1}AC$;

(b) $(0, \infty) \subseteq \rho_C(A), R(C) \subseteq R((\lambda - A)^n)$ *and* $\lambda^n||(\lambda - A)^{-n}Cx|| \leq ||Cx||$ *for* $\lambda > 0, n \in \mathbf{N}$ *and* $x \in X$;

(c) *for some* $\lambda > 0$, *the Hille-Yosida space for* $A - \lambda I$, *denoted by* Z_λ, *contains* $R(C)$.

Proof. For the necessity, Lemma 1.1.3(e) and Proposition 1.2.1 imply (a) and (b). It remains to show (c). Let $\lambda > 0$, define $S(t) = e^{-\lambda t}T(t)$ for $t \geq 0$. Thus $(S(t))_{t\geq 0}$ is a bounded uniformly strongly continuous C-regularized semigroup, generated by $A - \lambda I$. By Lemma 1.1.6, $R(C)$ is contained in the Hille-Yosida space for $A - \lambda I$, i.e.,Z_λ.

Conversely, let $A_\lambda = A|_{Z_\lambda}$. By Lemma 1.1.5, $A_\lambda - \lambda I$ generates a strongly continuous semigroup of contractions, $(e^{t(A_\lambda-\lambda I)})_{t\geq 0}$, on $(Z_\lambda, ||\cdot||_{Z_\lambda})$, which implies $(e^{tA_\lambda})_{t\geq 0}$ is also a strongly continuous semigroup on $(Z_\lambda, ||\cdot||_{Z_\lambda})$.

For $t \geq 0$, define $W(t) : X \to X$ by $W(t) = e^{tA_\lambda}C$, we show that $(W(t))_{t\geq 0}$ is a C-regularized semigroup generated by A.

In fact, by (a), $CA \subseteq AC$, so that C commutes with e^{tA_λ} for $t \geq 0$. Thus

$$W(t+s)Cx = e^{(t+s)A_\lambda}C^2x = e^{tA_\lambda}Ce^{sA_\lambda}Cx = W(t)W(s)x$$

that is, $W(t+s)C = W(t)W(s)$.

Moreover, if $x \in D(A)$, then $Cx \in Z_\lambda \cap D(A)$ with $ACx = CAx \in Z_\lambda$, so that $Cx \in D(A_\lambda)$, which implies that $e^{tA_\lambda}Cx$ is differentiable and

$$Ae^{tA_\lambda}Cx = A_\lambda e^{tA_\lambda}Cx = e^{tA_\lambda}A_\lambda Cx = e^{tA_\lambda}CAx,$$

hence $W(t)x \in D(A)$ with $AW(t)x = W(t)Ax$. So $(W(t))_{t\geq 0}$ is generated by an extension of A. To show A is the generator, we only need to prove that A is closed. It is exactly as in the proof of [ZL, Lemma 2.2].

Finally, by (b) and the exponential formulas for C-regularized semigroups, we have

$$||W(t)x|| = \lim_{n\to\infty} ||(1 - \frac{t}{n}A)^{-n}Cx|| \leq ||Cx||,$$

so that $(W(t))_{t\geq 0}$ is of contractions. □

Condition (c) in Theorem 1.2.3 seems to be difficult to check, but in the case of $C(D(A))$ dense in $R(C)$, it can be omitted.

Theorem 1.2.4 *Suppose $C(D(A))$ is dense in $R(C)$. Then A generates a contraction C-regularized semigroup if and only if A satisfies (a) and (b) in Theorem 1.2.3.*

Proof. We only need to show the sufficiency.

If $x \in D(A)$, then $Cx \in D(A)$ with $ACx = CAx$ by (a). By (b), $R(C) \subseteq Y$, the weak Hille-Yosida space for A, since Z_0 is the closure of $D(A|_Y)$ in Y by Lemma 1.1.5, we have $Cx \in Z_0$.

For every $x \in X$, there exists a sequence $\{x_n\} \subset D(A)$, such that $Cx_n \to Cx$, in X. Moreover, for $n, m \in \mathbf{N}$, by Lemma 1.1.5,

$$||Cx_n - Cx_m||_{Z_0} = ||Cx_n - Cx_m||_Y \leq ||Cx_n - Cx_m||,$$

so that $\{Cx_n\}$ is a Cauchy sequence in Z_0, which implies $R(C) \subseteq Z_0$. So Theorem 1.2.4 follows from Theorem 1.2.3. □

From the proof above we know that in this case we can choose $\lambda = 0$ in Theorem 1.2.3(c).

Note that if $D(A)$ is dense in X, then $C(D(A))$ is dense in $R(C)$. However, [dL2, Example 6.2] gave an example of a C-regularized semigroup whose generator A is not densely defined while $C(D(A))$ is dense in $R(C)$.

1.3 Special cases

In this section, we make some applications of the results from the preceding section. First we give a sufficient condition that $C(D(A))$ is dense in $R(C)$.

Lemma 1.3.1 *Suppose that A generates an exponentially bounded C-regularized semigroup and there exists a suquence $\{\lambda_n\} \subset \rho(A)$ such that $\lambda_n \to +\infty$, then $C(D(A))$ is dense in $R(C)$.*

Proof. Let $\lambda \in \rho(A)$, then $\forall x \in X$, $(\lambda - A)^{-1}Cx = C(\lambda - A)^{-1}x \in C(D(A))$. An estimation using Eq.(1.2) yields that $\lambda(\lambda - A)^{-1}Cx \to Cx$ as $\lambda \to +\infty$. So that $\lambda_n(\lambda_n - A)^{-1}Cx \to Cx (n \to \infty)$, and since $\lambda_n(\lambda_n - A)^{-1}Cx \in C(D(A))$, our result holds. $\square$

And the next lemma will be needed in the sequel.

Lemma 1.3.2 *Let A be a closed linear operator with $CA \subseteq AC$. Suppose $0 \neq \lambda \in \rho_C(A)$ and $\lambda||(\lambda - A)^{-1}Cx|| \leq ||Cx||$, $\forall x \in X$. Then $R(\lambda - A) \supseteq \overline{R(C)}$, and $\lambda||(\lambda - A)^{-1}x|| \leq ||x||$ for all $x \in \overline{R(C)}$.*

Proof. Let $x \in \overline{R(C)}$. There exists a sequence $\{x_n\} \subset X$ such that $Cx_n \to x$ as $n \to \infty$. Define $x'_n = (\lambda - A)^{-1}Cx_n$, thus

$$||x'_n - x'_m|| = ||(\lambda - A)^{-1}C(x_n - x_m)|| \leq \frac{1}{\lambda}||C(x_n - x_m)||$$

for $n, m \in \mathbf{N}$, so $\{x'_n\}$ is a Cauchy sequence. Suppose $x'_n \to x_0 \in X$ as $n \to \infty$. Since $(\lambda - A)x'_n = Cx_n$ and A is closed, it follows that $x_0 \in D(A)$ and $(\lambda - A)x_0 = x$. Moreover,

$$\begin{aligned} \lambda||(\lambda - A)^{-1}x|| = \lambda||x_0|| = \lim_{n\to\infty} ||x'_n|| &= \lim_{n\to\infty} \lambda||(\lambda - A)^{-1}Cx_n|| \\ &\leq \lim_{n\to\infty} ||Cx_n|| = ||x||, \end{aligned}$$

as desired. $\square$

Now we can apply Theorem 1.2.4 to the case of $\rho(A) \neq \emptyset$. It is remarked that since $\rho(A) \neq \emptyset, CA \subseteq AC$ implies $A = C^{-1}AC$.

Theorem 1.3.3 *Let A be an operator on X. Suppose that $(0, \infty) \subseteq \rho(A)$. Then A generates a contraction C-regularized semigroup if and only if A satisfies*

(*a*) $CA \subseteq AC$;

(*b*) $\lambda||(\lambda - A)^{-1}Cx|| \leq ||Cx||$ *for* $\lambda > 0$ *and* $x \in X$;

(*c*) $C(D(A))$ *is dense in* $R(C)$.

Proof. Theorem 1.2.3 and Lemma 1.3.2 imply the necessity.

Conversely, we define an operator B on X by

$$D(B) = \{Cx : x \in D(A)\}, \quad Bx = Ax \ \ for \ \ x \in D(B).$$

So that $D(B) = C(D(A))$ and $R(B) \subseteq R(C)$. For $\lambda > 0$, since $(\lambda - A)^{-1}Cx = C(\lambda - A)^{-1}x$, so $R(\lambda - B) \supseteq R(C)$ and $\lambda||(\lambda - B)^{-1}Cx|| = \lambda||(\lambda - A)^{-1}Cx|| \leq ||Cx||$. From Remark 1.2.2, we know that B is dissipative on $(\overline{R(C)}, ||\cdot||)$. Since $\overline{D(B)} = \overline{C(D(A))} = \overline{R(C)}$, by [Pa, Chapter 1, Theorem 4.3], B is closable in $\overline{R(C)}$ (hence in X), and the closure of B in $(\overline{R(C)}, ||\cdot||)$ (or X), $\overline{B}$, is dissipative on $(\overline{R(C)}, ||\cdot||)$. By Lemma 1.3.2, $R(\lambda - \overline{B}) = \overline{R(C)}$ for $\lambda > 0$. Therefore, the Lumer-Phillips theorem for strongly continuous semigroup implies that $\overline{B}$ generates a contraction strongly continuous semigroup, $(S(t))_{t \geq 0}$, on $(\overline{R(C)}, ||\cdot||)$. Define $T(t) : X \to X$ by $T(t) = S(t)C$. Thus $(T(t))_{t \geq 0}$ is a C-regularized semigroup of contractions on X. For $x \in D(A)$,

$$\frac{T(t)x - Cx}{t} = \frac{S(t)Cx - Cx}{t} \to BCx = CAx,$$

so that an extension of A is the generator, and since $\rho(A) \neq \emptyset$, it is exactly A. □

Remark 1.3.4 (a) The conditions (a)-(c) in Theorem 1.3.3 are equivalent to (a), (c) and $(b)'$ A is C-dissipative. In fact, by Remark 1.2.2, $(b)'$ implies that $||\lambda(\lambda - A)Cx|| \geq \lambda||Cx||(\lambda > 0, x \in D(A))$. Since $(0, \infty) \subseteq \rho(A)$, for $\lambda > 0$,

$$||Cx|| = ||(\lambda - A)C(\lambda - A)^{-1}x|| \geq \lambda||C(\lambda - A)^{-1}x|| = \lambda||(\lambda - A)^{-1}Cx||,$$

which is (b).

(b) In [dL2, Theorem 3.3], it is claimed that if $\rho(A) \neq \emptyset$ and A generates a C-regularized semigroup of $O(e^{\omega t})$, then $(\omega, \infty) \subset \rho(A)$. However, there appears to be a gap in the argument, because it fails to prove that, if C^{-1} and $(r - A)$ both have resolvents that commute, then $C^{-1}(r - A) = (r - A)C^{-1}$. Here is a counterexample, suggested by deLaubenfels himself in his letter. Take $X = BC([0, \infty))$, the space of all bounded continuous functions on $[0, \infty)$ with supremum norm. Define $(Af)(s) = -sf(s)$ with $D(A) = \{f \in X, \ Af \in X\}$ and $(Cf)(s) = \frac{s}{1+s}f(s)$ for $s \geq 0$. Then $\sigma(A)$(the spectrum of A)$=(-\infty, 0]$, and

$C^{-1}(\lambda - A) = (\lambda - A)C^{-1}$ for all $\lambda \in \rho(A)$. It is obvious that the function $f(s) = \frac{1}{1+s}$ is in $D(C^{-1}A)$ but is not in $D(AC^{-1})$. Thus $C^{-1}A \neq AC^{-1}$. We do not know whether the claimed result remains true.

Let us consider the case when $\rho(A) \neq \emptyset$. Let $C = (r - A)^{-n}$, where $r \in \rho(A)$ and $n \in \mathbf{N} \cup \{0\}$. From [ZL, Lemma 6.1], we know $\rho_C(A) = \rho(A)$. Since $R(C) = D(A^n)$ and $C(D(A)) = D(A^{n+1})$, as a direct consequence of Theorem 1.3.3, we have

Corollary 1.3.5 *Suppose $r \in \rho(A) \neq \emptyset$, let $C = (r - A)^{-n}$, $n \in \mathbf{N} \cup \{0\}$. Then the following statements are equivalent.*

(a) A generates a contraction C-regularized semigroup ;

(b) A satisfies

(i) $(0, \infty) \subseteq \rho_C(A)$,

(ii) $\forall x \in D(A^n)$ and $\lambda > 0$, $||\lambda(\lambda - A)^{-1}x|| \leq ||x||$,

(iii) $D(A^{n+1})$ is dense in $D(A^n)$;

(c) A satisfies (i), (iii) and

$(ii)'$ A is C-dissipative.

In the case of $\overline{R(C)} = X$, the generator of a contraction C-regularized semigroup is in fact the generator of a contraction strongly continuous semigroup.

Theorem 1.3.6 *Suppose $\overline{R(C)} = X$. Then the following assertions are equivalent:*

(a) A generates a contraction C-regularized semigroup $(T(t))_{t\geq 0}$;

(b) A generates a contraction strongly continuous semigroup $(S(t))_{t\geq 0}$ and $CA \subseteq AC$;

(c) A satisfies

(i) A is closed and $CA \subseteq AC$,

(ii) $(0, \infty) \subseteq \rho_C(A)$ and $\lambda||(\lambda - A)^{-1}Cx|| \leq ||Cx||$ for $\lambda > 0, x \in X$,

(iii) $D(A)$ is dense;

(d) A satisfies (i), (iii) and

$(ii)'$ $(0, \infty) \subseteq \rho_C(A)$ and A is C-dissipative.

Proof. $(a) \Rightarrow (c)$ and $(c) \Rightarrow (d)$ are obvious.

$(c) \Rightarrow (b)$. By Lemma 1.3.2, (ii) implies $R(\lambda - A) = X$ and $\lambda||(\lambda - A)^{-1}x|| \leq ||x||$ for $\lambda > 0$ and $x \in X$, applying the Hille-Yosida theorem for strongly continuous semigroups to A gives (b).

$(b) \Rightarrow (a)$. By defining $T(t) = S(t)C$, it is not hard to show that $(T(t))_{t\geq 0}$ is a contraction C-regularized semigroup generated by A.

$(d) \Rightarrow (c)$. Since A is C-dissipative, $\forall x \in D(A)$, we have $||(\lambda - A)Cx|| \geq$

$\lambda||Cx||$, so that for $x \in R(\lambda - A)$,

$$||Cx|| = ||(\lambda - A)C(\lambda - A)^{-1}x|| \geq \lambda||C(\lambda - A)^{-1}x|| = \lambda||(\lambda - A)^{-1}Cx||.$$

Since $R(\lambda - A) \supseteq R(C)$, which is dense in X, a similar proof as that of Lemma 1.3.2 will do. $\Box$

Consider when B generates a strongly continuous semigroup $(S(t))_{t\geq 0}$ of contractions on $\overline{R(C)}$ and $CB \subseteq BC$. Define $T(t) = S(t)C(t \geq 0)$, we get a contraction C-regularized semigroup $(T(t))_{t\geq 0}$ on X. Suppose A is the generator. It is not hard to verify that $B = A|_{\overline{R(C)}}$. For the converse, in the case of Theorem 1.3.3, we know it is true.

It is natural to ask the following question: Suppose A is the generator of a contraction C-regularized semigroup on a Banach space X. Does there exists a restriction of A, A', which is the generator of a contraction strongly continuous semigroup on $\overline{R(C)}$?

We will answer this question in the negative in the following section by constructing an extrapolation space for a bounded C-regularized semigroup.

1.4 The extrapolation space and a counterexample

It is natural that all bounded strongly continuous semigroups are strongly uniformly continuous. However, this is far from obvious as to bounded C-regularized semigroups.

We do not know whether all bounded C-regularized semigroups are strongly uniformly continuous, but for every bounded C-regularized semigroup $(T(t))_{t\geq 0}$ on X, we can construct an extrapolation space X_s for it, such that the extension of $(T(t))_{t\geq 0}$ to X_s, $(T_s(t))_{t\geq 0}$, is a strongly uniformly continuous contraction C_s-semigroup on X_s, where C_s is the extension of C to X_s. Our methods are different from deLaubenfels' ([dL2, dL4]), while he put more additional assumptions on the C-regularized semigroups.

Let $(T(t))_{t\geq 0}$ be a bounded C-regularized semigroup on X with generator A. So that there exists a constant $M > 0$ such that $\|T(t)\| \leq M$ for all $t \geq 0$. For each $x \in X$, define $\|x\|_s = \sup\limits_{t\geq 0} \|T(t)x\|$, hence

$$\|Cx\| \leq \|x\|_s \leq M\|x\|. \tag{1.3}$$

Since C is injective, $\|\cdot\|_s$ is a norm on X. Denote by X_s the completion of X, with respect to the norm $\|\cdot\|_s$. Extend $T(t)$ to X_s by defining $T_s(t)y = \lim\limits_{n\to\infty} T(t)x_n$ for all $t \geq 0$, with the limit taken in X, whenever $\{x_n\}$ is a sequence in X converging to y, in X_s. We also denote by C_s the extension of C to X_s. It is not hard to see that $(T_s(t))_{t\geq 0}$ is bounded on X_s and C_s is injective.

Theorem 1.4.1 *Let X_s, $T_s(t)$, C_s be as above. Then*

(a) for all $t \geq 0$, $R(T_s(t))$ is contained in the closure of $R(T(t))$ in X. In particular, $R(T_s(t)) \subseteq X$ and $R(C_s) \subseteq \overline{R(C)}$, the closure of $R(C)$ in X.

(b) $(T_s(t))_{t\geq 0}$ is a strongly uniformly continuous contraction C_s-regularized semigroup.

Proof. (a) follows immediately from the definition of $T_s(t)$.

To prove (b), first, we show $T_s(t+s)C_s = T_s(t)T_s(s)$ for all $t, s \geq 0$. Let $y \in X_s$, there exists $\{x_n\} \subset X$ such that x_n converges to y in X_s, which means $T(t)x_n$ converges in X for all $t \geq 0$. Also, by the definition of $T_s(t)$ and (a), we have

$$\begin{aligned} C_sT_s(t+s)y &= C\lim_{n\to\infty} T(t+s)x_n = \lim_{n\to\infty} CT(t+s)x_n \\ &= \lim_{n\to\infty} T(t)T(s)x_n = T(t)\lim_{n\to\infty} T(s)x_n \\ &= T(t)T_s(s)y = T_s(t)T_s(s)y \end{aligned}$$

with the four limits taken in X.

Next, for all $x \in X$,

$$\|T(t)x\|_s = \sup_{s\geq 0} \|T(s)T(t)x\| = \sup_{s\geq 0} \|T(s+t)Cx\| \leq \|Cx\|_s,$$

therefore, $(T_s(t))_{t\geq 0}$ is of contractions since X is dense in X_s.

At last, we show that $(T_s(t))_{t\geq 0}$ is strongly uniformly continuous. Now let $y \in X$, and $\{x_n\} \subset X$ with $\|x_n - y\|_s \to 0$ as $n \to \infty$. Then

$$\begin{aligned} & \|T_s(t+h)y - T_s(t)y\|_s \\ \leq\ & \|T_s(t+h)y - T_s(t+h)x_n\|_s \\ & \quad + \|T_s(t+h)x_n - T_s(t)x_n\|_s + \|T_s(t)x_n - T_s(t)y\|_s \\ \leq\ & 2\|C_s(x_n - y)\|_s + \sup_{s\geq 0} \|T(t+s+h)Cx_n - T(t+s)Cx_n\| \\ \leq\ & 2\|C_s(x_n - y)\|_s + M\|T(h)x_n - Cx_n\|. \end{aligned}$$

We use the contractivity of $(T_s(t))_{t\geq 0}$ in the above. Note that the right side is independent of t, hence $(T_s(t))_{t\geq 0}$ is strongly uniformly continuous. □

The following theorem indicates the relation between A and the generator of $(T_s(t))_{t\geq 0}$, A_s.

Theorem 1.4.2 *Let $(T(t))_{t\geq 0}$ be a bounded C-regularized semigroup with generator A on X and $(T_s(t))_{t\geq 0}$, X_s, C_s be as in Theorem 1.4.1. Suppose A_s is the generator of $(T_s(t))_{t\geq 0}$, then*

(a) $A \subseteq A_s$;

(b) $A_s = C_s^{-1} A C_s$;

(c) $A = A_s|_X$.

Proof. (a) Suppose $x \in D(A)$, hence

$$\|\frac{T(t)x - Cx}{t} - CAx\| \to 0 \text{ as } t \to 0. \tag{1.4}$$

By (1.3), we know

$$\|\frac{T(t)x - Cx}{t} - CAx\|_s \leq M\|\frac{T(t)x - Cx}{t} - CAx\|,$$

combining with (1.4), it follows that $x \in D(A_s)$ with $A_s x = Ax$.

(b) If $y \in D(A_s)$, then

$$\|\frac{T_s(t)y - C_s y}{t} - C_s A_s y\|_s \to 0 \text{ as } t \to 0.$$

Since $R(T_s(t)) \subseteq X$, by the definition of $\|\cdot\|_s$, we have

$$\|\frac{T(h)(T_s(t)y - C_s y)}{t} - T(h)C_s A_s y\| \to 0 \text{ as } t \to 0$$

uniformly in h. Set $h = 0$, note that C_s commutes with A_s and $T_s(t)$, so

$$\|\frac{T_s(t)C_s y - C_s C_s y}{t} - C_s A_s C_s y\| \to 0 \text{ as } t \to 0. \tag{1.5}$$

Moreover, $C_s y \in D(A_s) \cap X$ and $A_s C_s y = C_s A_s y \in X$, (1.5) means

$$\|\frac{T(t)C_s y - CC_s y}{t} - CA_s C_s y\| \to 0 \text{ as } t \to 0,$$

which implies $C_s y \in D(A)$ and $AC_s y = A_s C_s y = C_s A_s y$, i.e., $A_s y = C_s^{-1} A C_s y$. So we get $A_s \subseteq C_s^{-1} A C_s$.

On the other hand, $C_s^{-1} A C_s \subseteq C_s^{-1} A_s C_s = A_s$ since A_s is the generator.

(c) If $x \in D(A_s) \cap X$ and $A_s x \in X$, then $Cx = C_s x \in D(A)$ by (b) and $ACx = A_s Cx = C_s A_s x = CA_s x$, which implies $A_s x = C^{-1} ACx$. So the claim follows from the fact that $A = C^{-1}AC$. $\square$

Now we take up the question asked in Section 1.3. If it is true, then $(\lambda - A')^{-1}\overline{R(C)} = \overline{R(C)}$ for all $\lambda > 0$. Hence it is crucial that $\overline{R(C)}$ be an invariant subspace for $(\lambda - A)^{-1}$ since $A' \subseteq A$. So one way to answer the question in the negative is to construct a contraction C-regularized semigroup with generator A, while $(\lambda - A)^{-1}$ does not leave $\overline{R(C)}$ invariant. It is more easier to construct a bounded C-regularized semigroup than a contraction one. Now the extrapolation space is a good help. By making use of it, we can obtain contraction C-regularized semigroups from bounded C-regularized semigroups.

We first give an example in which the C-regularized semigroup is bounded, but not contractive.
Example 1.4.3 Suppose X is a complex Banach space and $C_0 \in B(X)$ is injective with $\overline{R(C_0)} \neq X$. Let

$$A = \begin{pmatrix} i & C_0^{-1} \\ 0 & -i \end{pmatrix} \text{ with } D(A) = X \times R(C_0)$$

and $C = \begin{pmatrix} C_0 & 0 \\ 0 & C_0 \end{pmatrix}$. It is not hard to show that A generates a bounded C-regularized semigroup

$$T(t) = \begin{pmatrix} e^{it}C_0 & \frac{1}{2i}(e^{it} - e^{-it}) \\ 0 & e^{-it}C_0 \end{pmatrix},$$

but $(T(t))_{t\geq 0}$ is not contractive. For every $\lambda \neq 0$, if $x_2 \notin \overline{R(C_0)}$, then

$$(\lambda - A)^{-1}C\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} (\lambda - i)^{-1}C_0x_1 + (\lambda + i)^{-1}(\lambda - i)^{-1}x_2 \\ (\lambda + i)^{-1}C_0x_2 \end{pmatrix} \notin \overline{R(C)}. \quad (1.6)$$

So that $(\lambda - A)^{-1}$ does not leave $\overline{R(C)}$ invariant.

We then build a contraction C-regularized semigroup on the basis of Example 1.4.3.
Example 1.4.4 Let X, C_0, C, A, $(T(t))_{t\geq 0}$ be as in Example 1.4.3. Moreover, we assume that $0 \notin \sigma_a(C_0)$, i.e., $C_0x_n \to 0$ implies $x_n \to 0$. Constructing the extrapolation space X_s for $(T(t))_{t\geq 0}$ as in Theorem 1.4.1, we obtain a contraction C_s-regularized semigroup $(T_s(t))_{t\geq 0}$ on X_s, where $T_s(t)$ is the extension of $T(t)$ to X_s and C_s is the extension of C to X_s. And the generator, A_s, is an extension of A.

We claim, for any $x \in X$ with $x \notin \overline{R(C_0)}$, $(\lambda - A_s)^{-1}C_s\begin{pmatrix} 0 \\ x \end{pmatrix} \notin \overline{R(C_s)}$ for $\lambda > 0$, where $\overline{R(C_s)}$ is the closure of $R(C_s)$ in X_s. By (1.6), we have

$$(\lambda - A_s)^{-1}C_s\begin{pmatrix} 0 \\ x \end{pmatrix} = (\lambda - A)^{-1}C_s\begin{pmatrix} 0 \\ x \end{pmatrix} = \begin{pmatrix} (\lambda - i)^{-1}(\lambda + i)^{-1}x \\ (\lambda + i)^{-1}C_0x \end{pmatrix}.$$

Since $\begin{pmatrix} 0 \\ C_0 x \end{pmatrix} \in R(C_s)$, we only need to show $\begin{pmatrix} x \\ 0 \end{pmatrix} \notin \overline{R(C_s)}$. If $\begin{pmatrix} x \\ 0 \end{pmatrix} \in \overline{R(C_s)}$, then there exists $\left\{\begin{pmatrix} y_n^1 \\ y_n^2 \end{pmatrix}\right\} \subset R(C_s) \subseteq \overline{R(C_0)} \times \overline{R(C_0)}$ such that

$$\left\| \begin{pmatrix} y_n^1 \\ y_n^2 \end{pmatrix} - \begin{pmatrix} x \\ 0 \end{pmatrix} \right\|_s \to 0,$$

that is,

$$\left\| T(t) \begin{pmatrix} y_n^1 - x \\ y_n^2 \end{pmatrix} \right\| \to 0 \text{ as } n \to \infty \text{ uniformly in } t,$$

i.e.,

$$\left\| \begin{pmatrix} e^{it} C_0 y_n^1 - e^{it} C_0 x + (\lambda - i)^{-1} (\lambda + i)^{-1} y_n^2 \\ e^{-it} C_0 y_n^2 \end{pmatrix} \right\| \to 0.$$

So we have $e^{-it} C_0 y_n^2 \to 0$ and $e^{it} C_0 (y_n^1 - x) + (\lambda - i)^{-1} (\lambda + i)^{-1} y_n^2 \to 0$. Since $0 \notin \sigma_a(C_0)$, $e^{-it} C_0 y_n^2 \to 0$ implies $y_n^2 \to 0$, so that $e^{it} C_0 (y_n^1 - x) \to 0$, which means $y_n^1 \to x$. However, $\{y_n^1\} \subset \overline{R(C_0)}$ while $x \notin \overline{R(C_0)}$, this is a contradiction.

Therefore, $\overline{R(C_s)}$ is not an invariant space of $(\lambda - A_s)^{-1}$. Thus no restriction of A, generates a contraction strongly continuous semigroup on $\overline{R(C_s)}$.

Chapter 2

Almost periodic and asymptotically almost periodic C-regularized semigroup

Characterizations of almost periodic strongly continuous semigroups and strongly continuous groups have been studied by Bart and Goldberg [BG] in 1978. Later, Cioranescu [Ci2], Piskarev [Pi1, Pi2] and others discussed the almost periodicity of strongly continuous cosine functions. Recently, Zheng and Liu [ZL] studied the almost periodicity of C-regularized semigroups and C-regularized cosine functions under the assumption that $R(C)$ is dense.

In this chapter, we investigate the situation where $R(C)$ is allowed to be non-dense. In Section 2.2 we characterize the generator of an almost periodic C-regularized semigroup, A, via the Hille-Yosida space, Z_0. Let $\mathcal{I} \equiv span\{x \in D(A) : Ax = irx \text{ for some } r \in R\}$. We show in Theorem 2.2.4 that if A has no eigenvalues in $(0, \infty)$ and $C^{-1}AC = A$, then A generates an almost periodic C-semigroup if and only if the image of C is contained in $(Z_0)_a$, the closure of $\mathcal{I}$ in Z_0; and $(Z_0)_a$ is proved to be a maximal continuously imbedded subspace of X on which A generates an almost periodic C_0-semigroup of contractions (Theorem 2.2.6). Our results generalize those corresponding in [ZL].

If $\sigma(A) \bigcap iR$ is at most countable, then C-regularized semigroup $(T(t))_{t\geq 0}$ is almost periodic if and only if $e^{-\lambda t}T(t)x$ has uniformly convergent means for $\lambda \in \sigma(A) \bigcap iR$, $x \in X$. This is proved in Theorem 2.2.9.

On the other hand, the asymptotic almost periodicity of strongly continuous semigroup have been studied systematically in [dV, Vu, RS1, RS3]. In this chapter, we also discuss the asymptotic almost periodicity of C-regularized semigroups. By means of the Hille-Yosida space, we are able to characterize the generator. In general, We show that $(T(t))_{t\geq 0}$ is asymptotically almost periodic if and only if $(T(t))_{t\geq 0}$ is bounded, strongly uniformly continuous and $R(C) \subseteq X_{0a} + X_{0s}$, where $X_{0a} = \{x \mid x \in Z_0, S(\cdot)x \in AP(\mathbf{R}^+, Z_0)\}, X_{0s} = \{x \mid x \in Z_0, S(\cdot)x \in$

$C_0(\mathbf{R}^+, Z_0)\}$, while $(S(t))_{t\geq 0}$ is the strongly continuous semigroup generated by $A|_{Z_0}$ (Theorem 2.3.2). When $R(C)$ is dense in X, $(T(t))_{t\geq 0}$ is asymptotically almost periodic if and only if it is bounded and $R(C) \subseteq X_a + X_s$, here $X_a = \{x \mid x \in X, T(t)x \in AP(\mathbf{R}^+, X)\}$, $X_s = \{x \mid x \in X, T(t)x \in C_0(\mathbf{R}^+, X)\}$(Theorem 2.3.4). Moreover, Theorem 2.3.5 indicates that A generates an asymptotically almost periodic C-regularized semigroup if and only if $R(C) \subseteq Z_{aap}$, the maximal continuous imbedded subspace of X on which the restriction of A generates an asymptotically almost periodic strongly continuous semigroup.

Under suitable spectral conditions, we obtain a theorem for asymptotic almost periodicity of C-semigroups that is more easily testified: assume that $\sigma_C(A) \bigcap i\mathbf{R}$ is countable, then $(T(t))_{t\geq 0}$ is asymptotically almost periodic if and only if it is bounded, strongly uniformly continuous, and for each $ir \in \sigma_C(A) \bigcap i\mathbf{R}$, $x \in X$, $\lim\limits_{\lambda\to 0} \lambda \int_0^{+\infty} e^{-(\lambda+ir)t} T(t+s)x dt$ exists, uniformly for $s \geq 0$ (Theorem 2.3.6).

2.1 Almost periodic and asymptotically almost periodic functions

Definition 2.1.1 (a) A function $f \in C(J, X)$ is *almost periodic*, where $J = \mathbf{R}$ or $\mathbf{R}^+$, written $f \in AP(J, X)$, if $\forall \varepsilon > 0$, there exists $l > 0$ such that every subinterval of J of length l contains at least one τ satisfying $\|f(t+\tau) - f(t)\| \leq \varepsilon, \forall t \in J$.

(b) A function $f \in C_b(\mathbf{R}^+, X)$ is said to be *asymptotically almost periodic*, if for every $\varepsilon > 0$, there exist $l > 0$ and $M > 0$ such that every subinterval of $\mathbf{R}^+$ of length l contains at least one τ satisfying $||f(t+\tau) - f(t)|| \leq \varepsilon$ for all $t \geq M$. The space of all asymptotically almost periodic functions will be denoted by $AAP(\mathbf{R}^+, X)$.

(c) Let $(F(t))_{t\in J} \subset B(X)$ be a strongly continuous operator family, then $(F(t))_{t\in J}$ is *almost periodic* (or *asymptotically almost periodic*), if for every $x \in X$, $F(\cdot)x$ is *almost periodic* (or *asymptotically almost periodic*, respectively); $(F(t))_{t\in J}$ is periodic with *period* p if $F(t+p) = F(t), \forall t \in J$.

We collect some basic results of vector-valued almost periodic functions in the following lemma(see [ZL]).

Lemma 2.1.2 *Let* $f \in AP(\mathbf{R}, X)$*, then*

(*a*) $f(t)$ *is bounded, i.e.* $\sup_{t\in\mathbf{R}} \|f(t)\| < \infty$*;*

(*b*) *if* $g \in AP(\mathbf{R}, X)$*,* $h \in AP(\mathbf{R}, C)$*, then* $f + g$*,* $hf \in AP(\mathbf{R}, X)$*;*

(c) $\forall r,\ \alpha \in \mathbf{R},\ a_r(f) \equiv \lim_{t\to\infty} \frac{1}{t}\int_0^t e^{-irs} f(s)ds$ *exists and*

$$a_r(f) = \lim_{t\to\infty} \frac{1}{t}\int_\alpha^{\alpha+t} e^{-irs} f(s)ds;$$

(d) $\sigma(f) = \{r \in \mathbf{R}; a_r(f) \neq 0\}$ *is at most countable;*

(e) *if* $a_r(f) = 0$, $\forall r \in \mathbf{R}$, *then* $f(t) = 0$, $\forall t \in \mathbf{R}$*;*

(f) *if* $X \not\supset c_0$ *and* $g(t) = \int_0^t f(s)ds (t \in \mathbf{R})$ *is bounded, then* $g \in AP(\mathbf{R}, X)$*;*

(g) *if* $\{f_n\}_{n\in\mathbf{N}} \in AP(\mathbf{R}, X)$ *and* $\{f_n\}_{n\in\mathbf{N}}$ *converges uniformly to* f*, then* $f \in AP(\mathbf{R}, X)$*;*

(h) *if* $f'(t)$ *exists and is uniformly continuous, then* $f' \in AP(\mathbf{R}, X)$.

And for asymptotically almost periodic functions, we have

Lemma 2.1.3([DG, RS1, RS3]). *For a function* $f \in C(\mathbf{R}^+, X)$*, the following statements are equivalent:*

(a) $f \in AAP(\mathbf{R}^+, X)$.

(b) *There exist uniquely determined functions* $g \in AP(\mathbf{R}, X)$ *and* $\varphi \in C_0(\mathbf{R}^+, X)$ *such that* $f = g|_{\mathbf{R}^+} + \varphi$.

The following lemma follows immediately from Lemma 2.1.2 and 2.1.3 and Lemma 1.1.6.

Lemma 2.1.4 *Suppose* $(T(t))_{t\geq 0}$ *is an almost periodic (or asymptotically almost periodic) C-regularized semigroup with generator A. Then*

(a) $(T(t))_{t\geq 0}$ *is bounded and strongly uniformly continuous;*

(b) $R(C) \subset Z_0$*, the Hille-Yosida space for A, and* $T(t) = e^{tA|_{Z_0}}C$.

At last, we recall the following result of H. Bart and S. Goldberg [BG] for almost periodic strongly continuous semigroups and groups.

Lemma 2.1.5 (a) *Every almost periodic semigroup can be extended to be an almost periodic group.*

(b) *A group* $(T(t))_{t\in\mathbf{R}}$ *is almost periodic if and only if it is uniformly bounded and the set of eigenvectors of A, its generator, spans a subspace dense in X.*

2.2 Almost Periodicity

In this section, we discuss the almost periodicity of C-regularized semigroups and C-regularized groups. The following is the main result of this section.

Theorem 2.2.1 *Let $(T(t))_{t\geq 0}$ be a C-regularized semigroup on X with generator A. Then $(T(t))_{t\geq 0}$ is almost periodic if and only if $R(C)\subset (Z_0)_a$, the closure of $\mathcal{I}$ in Z_0, where $\mathcal{I}:=span\{x\in D(A): Ax=irx \text{ for some } r\in \mathbf{R}\}$.*

Proof. Sufficiency. Since $R(C)\subset (Z_0)_a$, for fixed $x\in X$, $\varepsilon>0$, there exist finitely many points $r_k\in\mathbf{R}$ and $x_k\in ker(ir_k-A)$ such that $\|Cx-\sum\alpha_k x_k\|_{Z_0}\leq \varepsilon$. Thus $\|e^{tA|_{Z_0}}Cx-\sum\alpha_k e^{tA|_{Z_0}}x_k\|_{Z_0}\leq\varepsilon$. But $Ax_k=ir_kx_k$, so $e^{tA|_{Z_0}}x_k=e^{ir_kt}x_k\in AP(\mathbf{R}^+,X)$, i.e., $\|e^{tA|_{Z_0}}Cx-\sum\alpha_k e^{ir_kt}x_k\|_{Z_0}\leq\varepsilon$. So we have $\|e^{tA|_{Z_0}}Cx-\sum\alpha_k e^{ir_kt}x_k\|\leq\|e^{tA|_{Z_0}}Cx-\sum\alpha_k e^{ir_kt}x_k\|_{Z_0}\leq\varepsilon$, for $t\geq 0$. Hence $T(t)x=e^{tA|_{Z_0}}Cx\in AP(\mathbf{R}^+,X)$, and so $(T(t))_{t\geq 0}$ is almost periodic.

Necessity. Define $P_rx=\lim\limits_{t\to\infty}\frac{1}{t}\int_0^t e^{-irs}T(s)xds, \forall r\in\mathbf{R}, x\in X$. Then by Lemma 2.1.2(c) and from the proof of [ZL, Theorem 2.1], we know that P_rx exists and belongs to $D(A)$ with $AP_rx=irP_rx$. Thus,

$$\begin{aligned}
T(t)P_rx &= \lim_{s\to\infty}\frac{1}{s}\int_0^s e^{-ir\tau}T(t+\tau)Cxd\tau\\
&= C\lim_{s\to\infty}\frac{1}{s}\int_t^{t+s} e^{-ir(\tau-t)}T(\tau)xd\tau\\
&= Ce^{irt}\lim_{s\to\infty}\frac{1}{s}\int_t^{t+s} e^{-ir\tau}T(\tau)xd\tau\\
&= e^{irt}CP_rx.
\end{aligned}$$

Hence, $T(t)P_rx\in R(C)$ and $C^{-1}T(t)P_rx=e^{irt}P_rx$ is bounded, and uniformly continuous. This implies $P_rx\in Z_0$ and $\{P_rx; r\in\mathbf{R}, x\in X\}\subset D(A|_{Z_0})$ with $A|_{Z_0}P_rx=irP_rx$.

For every $x\in X$, since $t\longmapsto T(t)x$ is bounded and uniformly continuous, we have that $T(t)x\in Z_0, \forall t\in\mathbf{R}^+$. Next, we show $T(t)x\in AP(\mathbf{R}^+,Z_0)$. Since $T(t)x\in AP(\mathbf{R}^+,X)$, then $\forall\varepsilon>0$, there exists $l>0$ such that every subinterval of $\mathbf{R}^+$ of length l contains at least one τ satisfying $\sup\limits_{t\in\mathbf{R}^+}\|T(t+\tau)x-T(t)x\|\leq\varepsilon$. Then

$$\begin{aligned}
\sup_{t\geq 0}\|T(t+\tau)x-T(t)x\|_{Z_0} &= \sup_{t\geq 0,s\geq 0}\|C^{-1}T(s)T(t+\tau)x-C^{-1}T(s)T(t)x\|\\
&= \sup_{t\geq 0,s\geq 0}\|T(t+s+\tau)x-T(t+s)x\|\\
&\leq \sup_{t\geq 0}\|T(t+\tau)x-T(t)x\|\leq\varepsilon
\end{aligned}$$

i.e. $T(t)x\in AP(\mathbf{R}^+,Z_0)$. If $f\in Z_0^*$ such that $f(P_rx)\equiv 0$, $\forall x\in X$, $r\in\mathbf{R}$, then $\lim\limits_{t\to\infty}\frac{1}{t}\int_0^t e^{-irs}f(T(s)x)ds=f(P_rx)\equiv 0$. But $f(T(t)x)\in AP(\mathbf{R}^+,C)$. Thus by Lemma 2.1.2(e), we get $f(T(t)x)\equiv 0, \forall t\in\mathbf{R}^+, x\in X$. In particular, $f(Cx)\equiv 0$. Therefore, $\{P_rx; r\in\mathbf{R}, x\in X\}^\perp\subset R(C)^\perp$, i.e.,

$$R(C)\subset^\perp (R(C)^\perp) \quad\subset\quad ^\perp(\{P_rx; r\in\mathbf{R}, x\in X\}^\perp)$$

$$= \quad \overline{span}\{P_r x; r \in \mathbf{R}, x \in X\} \subset (Z_0)_a = \overline{\mathcal{I}},$$

all the closures are taken in Z_0. □

Now we have the following result [ZL, Theorem 2.1] as our corollary.

Corollary 2.2.2 *If $\overline{R(C)} = X$, then $(T(t))_{t\geq 0}$ is an almost periodic C-regulaized semigroup with generator A if and only if $(T(t))_{t\geq 0}$ is bounded and $X = X_a$, where X_a is the closure of $\mathcal{I}$ in X.*

Proof. The sufficiency is obvious. For the converse, since $Z_0 \hookrightarrow X$, a Cauchy sequence $\{x_n\}$ in Z_0 is also a Cauchy sequence in X, so that $(Z_0)_a \subseteq X_a$. By Theorem 2.2.1, $R(C) \subset (Z_0)_a$, hence $R(C) \subset X_a$, taking closure on both sides yields $X = X_a$. □

Combining Theorem 2.2.1 with [dL4, Theorem 5.16], we have

Theorem 2.2.3 *Suppose A has no eigenvalues in $(0, \infty)$. Then there exists an almost periodic mild C-existence family for A if and only if $R(C) \subset (Z_0)_a$.*

Moreover, combining Theorem 2.2.1 with [dL4, Theorem 5.17] and [dSW, Corollary 3.14] gives

Theorem 2.2.4 *Suppose A is closed and has no eigenvalues in $(0, \infty)$, and $C^{-1}AC = A$. Then A generates an almost periodic C-regularized semigroup if and only if $R(C) \subset (Z_0)_a$.*

Now we investigate a special case.

Corollary 2.2.5 *If $C = (r - A)^{-n}$, for some $n \in N$, and $\mathbf{T} \equiv (T(t))_{t\geq 0}$ is a bounded strongly uniformly continuous C-regularized semigroup generated by A, then $\mathbf{T}$ is almost periodic if and only if $(S(t))_{t\geq 0}$, defined by $S(t) \equiv e^{tA|_{Z_0}}$, is almost periodic.*

Proof. From the proof of Theorem 2.2.1, we see that $(T(t))_{t\geq 0}$ is almost periodic on X implies $(T(t))_{t\geq 0}$ is also almost periodic on Z_0. Since $T(t) = S(t)(r - A)^{-n}$, by applying Lemma 2.1.2(h) n times, we get $(S(t))_{t\geq 0}$ is almost periodic. The converse holds since $T(t) = S(t)C$ and $Z_0 \hookrightarrow X$. □

The following clarifies the relations between almost periodic C-regularized semigroups and strongly continuous semigroups.

Theorem 2.2.6 *Let $(T(t))_{t\geq 0}$ be an almost periodic C-regularized semigroup with generator A. Then there exists a maximal continuously imbedded subspace of X,*

W, such that $A|_W$ generates a contraction almost periodic strongly continuous semigroup on W and $R(C) \subset W$; W is maximal-unique in the sense that if $Y \hookrightarrow X$ and $A|_Y$ generates a contraction almost periodic strongly continuous semigroup on Y, then $Y \hookrightarrow W$.

Proof. Let $(S(t))_{t\geq 0}$ be the strongly continuous semigroup generated by $A|_{Z_0}$. Since $S(t)x = e^{irt}x$, for $Ax = irx$, $S(t)$ clearly takes $\mathcal{I}$ to itself, therefore, since $S(t)$ is continuous, it takes the closure of $\mathcal{I}$ to itself, that is saying, $S(t)(Z_0)_a \subset (Z_0)_a$. Set $W = (Z_0)_a$, the first half of the result follows.

Now suppose $Y \hookrightarrow X$ and $A|_Y$ generates a contraction almost periodic strongly continuous semigroup, then $Y \hookrightarrow Z_0$, since Z_0 is maximal by Lemma 1.1.5. It follows that $(Z_0)_a$ contains the closure of span$\{x \in D(A|_Y); Ax = irx$ for some $r \in \mathbf{R}\}$ in Y, which is exactly Y, so that $Y \hookrightarrow W = (Z_0)_a$. □

Remark 2.2.7 We can consider $(Z_0)_a$ for any closed operator A with $(s - A)$ injective for $s > 0$. The results of Theorem 2.2.5 are also true, and it is not hard to see that $(Z_0)_a$ equals the set of all almost periodic orbits.

Theorem 2.2.8 *Assume that A generates a C-regularized group $(T(t))_{t\in\mathbf{R}}$. Then $(T(t))_{t\in\mathbf{R}}$ is a periodic C-regularized group with period p if and only if $\sigma_C(A) \subset (2\pi i/p)\mathbf{Z}$ and $R(C) \subset (Z_0)_a$.*

Proof. Necessity. By Lemma 2.1.2(b) and the fact that $T(p) = C$,

$$(\lambda - A)\int_0^p e^{-\lambda s}T(s)xds = (1 - e^{-\lambda p})Cx, \forall x \in X.$$

Combining this with $T(s)Ax = AT(s)x$, $\forall x \in D(A)$, we get $\sigma_C(A) \subset (2\pi i/p)Z$, while $R(C) \subset (Z_0)_a$ follows from Theorem 2.2.1.

Sufficiency. If $x \in ker(2\pi ik/p - A)$ for some $k \in \mathbf{Z}$, then $T(t)x = e^{2\pi ikt/p}Cx$, which implies $T(t+p)x = T(t)x, \forall t \in \mathbf{R}$; the same holds for every $x \in \mathcal{I}$. Since $T(t)$ is continuous in Z_0, we have $T(t+p)x = T(t)x$ for all $x \in (Z_0)_a$; in particular, $T(t+p)Cx = T(t)Cx$ for all $x \in X$ by our assumption $R(C) \subset (Z_0)_a$, therefore, since C is injective, we obtain $T(t+p) = T(t)$. □

From Lemma 2.1.5 we know that every almost periodic strongly continuous semigroup can be extended to an almost periodic group; also from [ZL, Theorem 3.1], we know that every almost periodic C-regularized semigroup can also be extended to an almost periodic C-regularized group. So we can assume that A generates an almost periodic C-regularized group.

Applying [RS3, Theorem 4.4] and the Hille-Yosida space, we obtain the following result, where we say that a function u has uniformly convergent means if

$$\lim_{R\to\infty} \frac{1}{R} \int_{a-R}^{a+R} u(s)ds$$

exists, uniformly in $a \in \mathbf{R}$.

Theorem 2.2.9 *Suppose $(T(t))_{t\in\mathbf{R}}$ is a bounded strongly uniformly continuous C-regularized group with generator A such that $\sigma(A)\bigcap i\mathbf{R}$ is at most countable, then the following assertions are equivalent.*

(a) $(T(t))_{t\in\mathbf{R}}$ is almost periodic.

(b) For $\lambda \in \sigma(A)\bigcap i\mathbf{R}, x \in X, e^{-\lambda t}T(t)x$ has uniformly convergent means.

Proof. By [ZL, Theorem 3.1], A and $-A$ generate C-regularized semigroups $(T(t))_{t\geq 0}$ and $(T(-t))_{t\geq 0}$, respectively, so that the Cauchy problem (1.1) has a bounded uniformly continuous mild solution on $\mathbf{R}$ given by $T(t)C^{-1}x$ for every $x \in R(C)$.

Suppose $(S(t))_{t\geq 0}$ is the strongly continuous semigroup generated by $A|_{Z_0}$. From the proof of Theorem 2.2.1, we know $T(t)x$ is almost periodic if and only if $S(t)Cx$ is almost periodic in Z_0. To see that (b) implies (a), by [RS3, Theorem 4.4], we only need to show that $e^{-\lambda t}S(t)Cx$ has uniformly convergent means in Z_0 for $\forall\lambda \in \sigma(A|_{Z_0})\bigcap i\mathbf{R}$. This can be achieved by a small modification of [dV, Theorem 4].

(a)⇒(b) is trivial, since $T(t)x$ and $e^{-\lambda t}T(t)x$ ($\forall\lambda \in i\mathbf{R}$) are almost periodic. □

2.3 Asymptotic almost periodicity

In order to characterize the asymptotic almost periodicity of C-regularized semigroups, we need the following result.

Lemma 2.3.1 *Assume that $(T(t))_{t\geq 0}$ is a C-regularized semigroup on X generated by A, and A has no eigenvalues in $(0,\infty)$. Suppose $T(\cdot)x : \mathbf{R}^+ \mapsto X$ is asymptotically almost periodic for some $x \in X$. Then*

(a) There exist $y \in X$ and $\varphi \in C_0(\mathbf{R}^+, X)$ such that for all $t \geq 0$, $T(t)y \in R(C)$, $C^{-1}T(\cdot)y \in AP(\mathbf{R}^+, X)$ and

$$T(t)x = C^{-1}T(t)y + \varphi(t).$$

(b) If $(T(t))_{t\geq 0}$ is strongly uniformly continuous and bounded, then there exist $y, z \in Z_0$ such that $S(\cdot)y \in AP(\mathbf{R}^+, Z_0)$, $S(\cdot)z \in C_0(\mathbf{R}^+, Z_0)$ and

$$T(t)x = S(t)y + S(t)z$$

for all $t \geq 0$, where Z_0 is the Hille-Yosida space for A, $(S(t))_{t\geq 0}$ is the strongly continuous semigroup generated by $A|_{Z_0}$.

Proof. (a) By Lemma 2.1.3, there exist uniquely determined function $h \in AP(\mathbf{R}, X)$ and $\varphi \in C_0(\mathbf{R}^+, X)$ such that $T(\cdot)x = h|_{\mathbf{R}^+} + \varphi$, and from the proof of Lemma 2.1.3 (see [RS1]) we know there exist $0 < t_n \to \infty$ such that

$$h(t) = \lim_{n\to\infty} T(t + t_n)x$$

for all $t \geq 0$. Let $y = h(0)$, then we have $\lim_{n\to\infty} T(t_n)x = y$. Therefore, for each $t \geq 0$,

$$Ch(t) = \lim_{n\to\infty} CT(t + t_n)x = T(t) \lim_{n\to\infty} T(t_n)x = T(t)y,$$

which implies $T(t)y \in R(C)$ for all $t \geq 0$ and $C^{-1}T(\cdot)y = h|_{\mathbf{R}^+} \in AP(\mathbf{R}^+, X)$, so that

$$T(t)x = C^{-1}T(t)y + \varphi(t) \text{ for all } t \geq 0.$$

(b) Let y and $\varphi(t)$ be the same as in (a). Since $C^{-1}T(\cdot)y = h|_{\mathbf{R}^+} \in AP(\mathbf{R}^+, X)$, we have that $C^{-1}T(t)y$ is uniformly continuous and bounded in $[0, +\infty)$, it follows that $y \in Z_0$ by Lemma 1.1.6. Setting $z = Cx - y$, obviously $z \in Z_0$. Hence, $S(t)y = C^{-1}T(t)y$ and

$$S(t)z = C^{-1}T(t)z = C^{-1}T(t)Cx - C^{-1}T(t)y = T(t)x - C^{-1}T(t)y = \varphi(t),$$

so that,

$$T(t)x = S(t)y + S(t)z \text{ for all } t \geq 0.$$

From the proof of Theorem 2.2.1, we know that $S(t)y \in AP(\mathbf{R}^+, X)$ implies $S(t)y \in AP(\mathbf{R}^+, Z_0)$. Also,

$$||S(t)z||_{Z_0} = \sup_{s\geq 0} ||C^{-1}T(s)S(t)z|| = \sup_{s\geq 0} ||S(t+s)z|| = \sup_{s\geq t} ||S(s)z||$$

converges to 0 as $t \to +\infty$ since $S(t)z \in C_0(\mathbf{R}^+, X)$, this yields $S(t)z \in C_0(\mathbf{R}^+, Z_0)$. □

Now we can prove the following theorem, which characterizes the asymptotic almost periodicity of C-regularized semigroups.

Theorem 2.3.2 *Let $(T(t))_{t\geq 0}$ be a C-regularized semigroup generated by A on X. Then $(T(t))_{t\geq 0}$ is asymptotically almost periodic if and only if $R(C) \subseteq X_{0a} + X_{0s}$, where $X_{0a} = \{x \mid x \in Z_0,\ S(t)x \in AP(\mathbf{R}^+, Z_0)\}$, $X_{0s} = \{x \mid x \in Z_0,\ S(t)x \in C_0(\mathbf{R}^+, Z_0)\}$ where Z_0 is the Hille-Yosida space for A, $(S(t))_{t\geq 0}$ is the strongly continuous semigroup generated by $A|_{Z_0}$.*

Proof. Necessity. First it follows from Lemma 2.1.4 that $(T(t))_{t\geq 0}$ is bounded and strongly uniformly contiuous. By Lemma 2.3.1(b), for every $x \in X$, there exist $y, z \in Z_0$ such that $S(\cdot)y \in AP(\mathbf{R}^+, Z_0), S(\cdot)z \in C_0(\mathbf{R}^+, Z_0)$ and $T(t)x = S(t)y + S(t)z$ for all $t \geq 0$. Choose $t = 0$, we obtain $Cx = y + z$, i.e., $R(C) \subseteq X_{0a} + X_{0s}$.

Sufficiency. Since $R(C) \subseteq X_{0a} + X_{0s}$, for any $x \in X$, there exist $y \in X_{0a}, z \in X_{0s}$ such that $Cx = y + z$. Hence, by Lemma 2.1.5 we have $T(t)x = S(t)Cx = S(t)y + S(t)z$, while $S(t)y \in AP(\mathbf{R}^+, Z_0)$ and $S(t)z \in C_0(\mathbf{R}^+, Z_0)$. Since Z_0 is continuously imbedded in X, it follows that $T(t)x$ is asymptotically almost periodic by Proposition 2.1.3, i.e., $(T(t))_{t\geq 0}$ is an asymptotically almost periodic C-regularized semigroup. □

From the proof of Theorem 2.2.1 and Lemma 2.3.1(b) we know that for $y, z \in Z_0$, $S(t)y \in AP(\mathbf{R}^+, X)$ and $S(t)z \in C_0(\mathbf{R}^+, X)$ if and only if $S(t)y \in AP(\mathbf{R}^+, Z_0)$ and $S(t)z \in C_0(\mathbf{R}^+, Z_0)$, respectively. Hence we have

Corollary 2.3.3 *Let $(T(t))_{t\geq 0}$ be a C-regularized semigroup generated by A on X. Then $(T(t))_{t\geq 0}$ is asymptotically almost periodic if and only if for any $x \in X$, there exist $y,\ z \in Z_0$ such that $Cx = y + z$, $C^{-1}T(t)y \in AP(\mathbf{R}^+, X)$, and $C^{-1}T(t)z \in C_0(\mathbf{R}^+, X)$, where Z_0 is the Hille-Yosida spaces for A.*

In the case of $\overline{R(C)} = X$, we have

Theorem 2.3.4 *Assume that $(T(t))_{t\geq 0}$ is a C-regularized semigroup on X and $\overline{R(C)} = X$. Then $(T(t))_{t\geq 0}$ is asymptotically almost periodic if and only if $R(C) \subseteq X_a + X_s$, where $X_a = \{x \mid x \in X, T(t)x \in AP(\mathbf{R}^+, X)\}$, $X_s = \{x \mid x \in X, T(t)x \in C_0(\mathbf{R}^+, X)\}$.*

Proof. Necessity. By Lemma 2.3.1.(a), for every $x \in X$, there exist $y \in X$ and $\varphi \in C_0(\mathbf{R}^+, X)$ such that for $\forall t \geq 0$, $T(t)y \in R(C)$, $C^{-1}T(\cdot)y \in AP(\mathbf{R}^+, X)$ and

$$T(t)x = C^{-1}T(t)y + \varphi(t).$$

It follows that $T(t)Cx = CT(t)x = T(t)y + C\varphi(t)$. Setting $z = Cx - y$, then $T(t)z = T(t)Cx - T(t)y = C\varphi(t) \in C_0(\mathbf{R}^+, X)$; on the other hand, $T(\cdot)y \in AP(\mathbf{R}^+, X)$ since C is bounded. So we have $R(C) \subseteq X_a + X_s$.

Sufficiency follows from the fact that $R(C)$ is dense in X and $AAP(\mathbf{R}^+, X)$ is closed in $C_b(\mathbf{R}^+, X)$. □

The following result clarifies the relations between the generator of an asymptotically almost periodic C-regularized semigroup and of an asymptotically almost periodic strongly continuous semigroup.

Theorem 2.3.5 *Suppose A is closed and has no eigenvalues in $(0,\infty)$, and assume that $C^{-1}AC = A$. Then A generates an asymptotically almost periodic C-regularized semigroup if and only if $R(C) \subseteq Z_{aap} \equiv (Z_0)_a + (Z_0)_s$, where $(Z_0)_a$ is defined in Theorem 2.2.1, $(Z_0)_s = \{x \mid x \in X$, the Cauchy problem (1.1) has a mild solution $u(\cdot,x) \in C_0(\mathbf{R}^+, X)\}$. And Z_{aap} is the maximal continuously imbedded subspace on which A generates an asymptotically almost periodic strongly continuous semigroup.*

Proof. Necessity holds by Lemma 2.3.1 and the relations between C-regularized semigroup and solutions of the corresponding Cauchy problem ([dL4, Theorem 3.13]).

Sufficiency. By Definition 1.1.4, we know that both $(Z_0)_a$ and $(Z_0)_s$ are contained in Z_0; so $R(C) \subseteq Z_0$. Thus by [dL4, Theorem 5.17] and [dSW, Corollary 3.14], A generates a bounded C-regularized semigroup; since all mild solutions with initial data taking from $R(C)$ are asymptotically almost periodic, so is the C-regularized semigroup.

The maximality of Z_{aap} can be proved similarly as that of Theorem 2.2.6. □

Remarks. (a) From Theorem 2.2.6 and Remark 2.2.7, we know $(Z_0)_a$ is the maximal subspace on which A generates an almost periodic strongly continuous semigroup.
(b) From the proof of Lemma 2.3.1, Theorem 2.3.2 and 2.3.5, it is clear that $(Z_0)_a = X_{0a}$, $(Z_0)_s = X_{0s}$.

When $\sigma_C(A) \bigcap i\mathbf{R}$ is countable, we obtain a result which is more easily testified for asymptotic almost periodicity of C-regularized semigroups.

Let $f : \mathbf{R}^+ \longrightarrow X$ be strongly measurable, and let $\tilde{f}$ be the Laplace transform of f :

$$\tilde{f}(z) = \int_0^{+\infty} e^{-tz} f(t)dt.$$

We assume that $\tilde{f}(z)$ exists for all z in $\mathbf{C}^+$, so $\tilde{f}$ is holomorphic in $\mathbf{C}^+$ (usually, f will be bounded). A point $\lambda = i\eta$ in $i\mathbf{R}$ is said to be a regular point for $\tilde{f}$ if there is an open neighborhood U of λ in $\mathbf{C}$ and a holomorphic function $g : U \longrightarrow X$ such that $g(z) = \tilde{f}(z)$ whenever $z \in U \bigcap \mathbf{C}^+$. The singular set E of $\tilde{f}$ is the set of all points of $i\mathbf{R}$ which are not regular points.

Theorem 2.3.6 *Let $(T(t))_{t\geq 0}$ be a C-regularized semigroup generated by A in X, $\sigma_C(A) \bigcap i\mathbf{R}$ is countable. Then the following assertions are equivalent:*

(a) $(T(t))_{t\geq 0}$ is asymptotically almost periodic.

(b) $(T(t))_{t\geq 0}$ is bounded, strongly uniformly continuous and for every $r \in \sigma_C(A) \bigcap i\mathbf{R}$, $x \in X$, $\lim\limits_{\lambda\to 0} \lambda \int_0^{+\infty} e^{-(\lambda+ir)t} T(t+s)x dt$ exists, uniformly for $s \geq 0$.

Proof. $(a) \Rightarrow (b)$ It follows from Lemma 2.1.4 that $(T(t))_{t\geq 0}$ is bounded and strongly uniformly continuous. So it remains to verify the second part of (b). By Lemma 2.3.1.(b), for any $x \in X$, there exist $y,\ z \in Z_0$ such that $S(\cdot)y \in AP(\mathbf{R}^+, Z_0)$, $S(\cdot)z \in C_0(\mathbf{R}^+, Z_0)$ and

$$T(t)x = S(t)y + S(t)z$$

for all $t \geq 0$, where Z_0 is the Hille-Yosida space for A, $(S(t))_{t\geq 0}$ is the strongly continuous semigroup generated by $A|_{Z_0}$. Therefore, $\lim_{t\to+\infty} \frac{1}{t}\int_0^t e^{-iru}S(u)ydu$, for every $r \in \mathbf{R}$, exists in Z_0 by Lemma 2.1.3 and $\lim_{t\to+\infty} \frac{1}{t}\int_0^t e^{-iru}S(u)zdu = 0$ in Z_0. It follows that

$$\lim_{t\to+\infty} \frac{1}{t}\int_0^t e^{-iru}T(u)xdu$$

exists in Z_0. Suppose the limit is Px, then given $\varepsilon> 0$, there exists $T > 0$ such that

$$||\frac{1}{t}\int_0^t e^{-iru}T(u)xdu - Px||_{Z_0} < \varepsilon$$

for $t \geq T$. Hence for $r \in \mathbf{R}$ and $\mathrm{Re}\lambda > 0$,

$$\begin{aligned}
& \sup_{s\geq 0} ||\lambda \int_0^{+\infty} e^{-(\lambda+ir)t}T(t+s)xdt - C^{-1}T(s)Px|| \\
= \ & \sup_{s\geq 0} ||\lambda C^{-1}T(s) \int_0^{+\infty} e^{-(\lambda+ir)t}T(t)xdt - C^{-1}T(s)Px|| \\
= \ & ||\lambda \int_0^{+\infty} e^{-(\lambda+ir)t}T(t)xdt - Px||_{Z_0} \\
\leq \ & ||\lambda^2 \int_0^{+\infty} e^{-\lambda t}\left(\int_0^t e^{-irs}T(s)xds - tPx\right) dt||_{Z_0} \\
\leq \ & \lambda^2 \int_T^{+\infty} e^{-\lambda t}\varepsilon dt + \lambda^2 \int_0^T e^{-\lambda t}|| \left(\int_0^t e^{-irs}T(s)xds - tPx\right) ||_{Z_0}dt \\
\leq \ & \varepsilon + \lambda^2 \int_0^T e^{-\lambda t}|| \left(\int_0^t e^{-irs}T(s)xds - tPx\right) ||_{Z_0}dt \\
\to \ & \varepsilon \ \ (\lambda \to 0).
\end{aligned}$$

Thus for each $r \in \sigma_C(A) \bigcap i\mathbf{R},\ x \in X$,

$$\lim_{\lambda\to 0} \lambda \int_0^{+\infty} e^{-(\lambda+ir)t}T(t+s)xdt$$

exists, uniformly for $s \geq 0$.

$(b) \Rightarrow (a)$. Given $x \in X$, let $f(t) = T(t)x$, we have that $f(t)$ is bounded, uniformly continuous and $\widetilde{f}(\lambda) = (\lambda - A)^{-1}Cx$ $(Re\lambda > 0)$. Let E be the singular set of $\widetilde{f}$ in $i\mathbf{R}$, then $E \subseteq \sigma_C(A) \bigcap i\mathbf{R}$, it follows that E is countable by the assumption. Moreover, for each $ir \in \sigma_C(A) \bigcap i\mathbf{R}$,

$$\lim_{\lambda\to 0} \lambda \widetilde{f}_s(\lambda + ir) = \lim_{\lambda\to 0} \lambda \int_0^{+\infty} e^{-(\lambda+ir)t}T(t+s)xdt$$

exists uniformly for $s \geq 0$, where $f_s(t) = f(s+t)$. Therefore, $f(t) = T(t)x$ is asymptotically almost periodic by [BNR2, Theorem 4.1]; so $(T(t))_{t\geq 0}$ is asymptotically almost periodic. □

Remark. The result of Theorem 2.3.6 can be deduced directly from [dV, Theorem 4] with the assumption on $\sigma_C(A)$ replaced by that on $\sigma(A)$; while with the aid of [BNR2, Theorem 4.1], the result can be improved.

Chapter 3

Ergodic theory for C-regularized semigroups

The first ergodic theorems, which concerned the long run average behavior of individual trajectories on the phase space of a dynamical system, evolved in 1931-1932. In 1943, N.Dunford introduced resolvent techniques in ergodic theory and unified most of the known mean ergodic theorems. Following this, E.Hille and R.S.Phillips obtained a unified treatment of ergodic theory for strongly continuous semigroups based on the properties of the resolvent of the generator. Their work can be found in [HP]. Further results along this line were obtained by Lin ([Lin]) and Shaw ([Sh1]).

On the other hand, a significant generalization of strongly continuous semigroups that has recently received a great deal of attention is C-regularized semigroups. An operator A generating a C-regularized semigroup corresponds to the abstract Cauchy problem (1.1) with respect to A having a mild solution for every $x \in R(C)$. In this regard, it is also interesting to discuss the ergodic properties of C-regularized semigroups. Shaw in [Sh2] gave two notions of *Abel* limit and two notions of *Cesáro* limit, and obtained some properties of the ergodic limits. However, there was no definition for the ergodicity of C-regularized semigroups. In this chapter, we give definitions of *Abel*- and *Cesáro*-ergodicity of C-regularized semigroups, which require that the operators Q_a and Q_c given in [Sh2] have X as their domains, respectively, and then characterize the generators. In the case of $C = 1$, these definitions are the same as those of strongly continuous semigroups. Our results generalize those corresponding results for strongly continuous semigroups (see [HP, Lin, Sh1]).

We first clarify the relations between ergodicity of C-regularized semigroups and that of mild solutions of (1.1) (Theorem 3.1.3). Next, we investigate strong ergodicity in Section 3.2. As we know, $X = \ker A \oplus \overline{R(A)}$ characterizes the strong *Abel*-ergodicity of a strongly continuous semigroup (cf.[Gol]). For C-regularized semigroups, we have a similar result: if $R(C)$ is dense in X, then

$R(C) \subseteq \ker A \oplus \overline{R(A)}$ implies the strong *Abel*-ergodicity (Theorem 3.2.2). And Theorem 3.2.7 allows us to remove the hypothesis that $R(C)$ is dense, but an additional condition is added. Moreover, holomorphic C-regularized semigroups are treated in Theorem 3.2.5.

The other major tool for C-regularized semigroups is the Hille-Yosida space ([dL4]), which connects C-regularized semigroups with strongly continuous semigroups. When we consider the ergodicity in the Hille-Yosida space, we find this requires that the ergodic limit converge uniformly along the orbit in the original space X. This leads to the definition of orbital ergodicity (Definition 3.3.1). We show next that for strongly continuous semigroups, the orbital ergodicity and the ergodicity are equivalent (Theorem 3.3.2); and for C-regularized semigroups, the orbital ergodicity and the ergodicity in the Hille-Yosida space are equivalent (Theorem 3.3.6). An orbitally ergodic C-regularized semigroups will be constructed in Section 3.4 (Example 3.4.3).

Uniform ergodicity is discussed in Section 3.4. First, we give a sufficient condition that $X = \ker A \oplus R(A)$ (Theorem 3.4.1). Next, we establish a counterexample (Example 3.4.3); in which the generator of a uniformly *Abel*-ergodic C-regularized semigroup has non-closed range, this indicates the essential difference between strongly continuous semigroups and C-regularized semigroups, because the characterization of uniformly ergodic strongly continuous semigroup is $R(A)$ being closed. So we must seek for other techniques to investigate the uniform ergodicity of C-regularized semigroups. By using the poles of the resolvent on the imaginary axis, we are able to obtain another characterization (Theorem 3.4.4).

In the following Section, we give some preliminaries on the definitions and some remarks about ergodicity for C-regularized semigroups.

3.1 Preliminaries on ergodicity

Definition 3.1.1 Let $(T(t))_{t\geq 0}$ be a C-regularized semigroup with generator A.

(a) $(T(t))_{t\geq 0}$ is *strongly (weakly, uniformly) Abel-ergodic* (*A-ergodic*, for short), if $(0, +\infty) \subset \rho_C(A)$ and $\lim\limits_{\lambda \searrow 0} \lambda(\lambda - A)^{-1}C := P$ exists in the strong (weak, uniform) operator topology.

(b) $(T(t))_{t\geq 0}$ is *strongly (weakly, uniformly) Cesáro-ergodic* (*C-ergodic*, for short), if $\lim\limits_{t\to\infty} \frac{1}{t}\int_0^t T(s)ds := P$ exists in the strong (weak, uniform) operator topology.

(c) $(T(t))_{t\geq 0}$ is *strongly (weakly, uniformly) ergodic* if $\lim\limits_{t\to\infty} T(t) := P$ exists in the strong (weak, uniform) operator topology.

And we call P the ergodic limit.

The proof of the following lemma is a straightforward modification of the cited results in [Gol, HP, Zhe] and therefore omitted.

Lemma 3.1.2 *Suppose $(T(t))_{t\geq 0}$ is a C-regularized semigroup. Then*

- (a) *the uniform (A-, C-)ergodicity of $(T(t))_{t\geq 0}$ implies its strong (A-, C-) ergodicity; and its strong (A-, C-)ergodicity implies its weak (A-, C-) ergodicity;*
- (b) *if $(T(t))_{t\geq 0}$ is bounded, then the uniform (strong, weak) A-ergodicity of $(T(t))_{t\geq 0}$ is equivalent to its uniform (strong, weak) C-ergodicity;*
- (c) *the uniform (strong, weak) ergodicity of $(T(t))_{t\geq 0}$ implies its uniform (strong, weak) C-ergodicity.*

It appears that the ergodicity of $(T(t))_{t\geq 0}$ doesn't give the ergodicity of mild solutions of (1.1), since the mild solutions are given by $C^{-1}T(t)$ (see [dL2]). However, when we consider the mild solutions of (1.1), we usually take the initial values from $R(C)$, so the mild solution $C^{-1}T(t)x$ is in fact $T(t)y$, where $Cy = x$. Therefore, we have

Theorem 3.1.3 $(T(t))_{t\geq 0}$ *is an (A-,C-)ergodic C-regularized semigroup with generator A if and only if for every initial value $x \in R(C)$,* (1.1) *has a mild solution, which is an (A-,C-) ergodic function in the usual sense.*

3.2 Strong Ergodicity

We collect the basic properties of the A-ergodic limits in the following

Proposition 3.2.1 *Suppose $(T(t))_{t\geq 0}$ is a bounded C-regularized semigroup generated by A and $(T(t))_{t\geq 0}$ is strongly A-ergodic. Let P be the ergodic limit. Then*
(a) $P^2 = CP = PC$;
(b) $C \ker A \subseteq R(P) \subseteq \ker A$, $C \ker P \subseteq \overline{R(A)} \subseteq \ker P$;
(c) $T(t)P = PC = CP, \forall t \geq 0$.

Proof. (a) For any $x \in X$,

$$\begin{aligned} P^2x &= \lim_{\lambda\searrow 0} \lambda(\lambda - A)^{-1}C(\lim_{\mu\searrow 0} \mu(\mu - A)^{-1}Cx) \\ &= \lim_{\lambda\searrow 0} \lim_{\mu\searrow 0} \lambda\mu(\lambda - A)^{-1}(\mu - A)^{-1}C^2x \end{aligned}$$

$$
\begin{aligned}
&= \lim_{\lambda \searrow 0} \lim_{\mu \searrow 0} \frac{\lambda\mu}{\lambda - \mu}[(\mu - A)^{-1}C^2 x - (\lambda - A)^{-1}C^2 x] \\
&= PCx.
\end{aligned}
$$

Also, it is easy to see $PCx = CPx$.

(b) First note that if $x = x_1 + x_2$, where $x_1 \in \ker A$, and there exists $y \in D(A)$ such that $Ay = x_2$, then for $\lambda > 0$,

$$
\begin{aligned}
\lambda(\lambda - A)^{-1}Cx &= \lambda(\lambda - A)^{-1}Cx_1 + \lambda(\lambda - A)^{-1}CAy \\
&= Cx_1 + \lambda^2(\lambda - A)^{-1}Cy - \lambda Cy \\
&\to Cx_1, \quad as \quad \lambda \to 0, \qquad (3.1)
\end{aligned}
$$

which implies $Px = Cx_1$, $Px_2 = 0$. Thus $C \ker A \subseteq R(P)$, $\overline{R(A)} \subseteq \ker P$.

Suppose next that $x \in R(P)$. Then there exists $y \in X$ such that $x = Py = \lim_{\lambda \searrow 0} \lambda(\lambda - A)^{-1}Cy$. Further, $\lambda A(\lambda - A)^{-1}Cy = \lambda^2(\lambda - A)^{-1}Cy - \lambda Cy \to 0$ as $\lambda \to 0$. It follows from the closedness of A that $x \in D(A)$ with $Ax = 0$. Therefore, $R(P) \subseteq \ker A$.

Finally, let $x \in \ker P$, then $A(\lambda - A)^{-1}Cx = \lambda(\lambda - A)^{-1}Cx - Cx \to -Cx$ as $\lambda \to 0$; this proves $C \ker P \subseteq \overline{R(A)}$.

(c) For $x \in X$, by Lemma 3.1.2.(c),

$$
\begin{aligned}
T(t)Px &= \lim_{s \to \infty} \frac{1}{s} \int_0^s T(t)T(r)x dr \\
&= \lim_{s \to \infty} \frac{1}{s} \int_0^s T(t + r)Cx dr \\
&= \lim_{s \to \infty} \frac{t + s}{s} \cdot \frac{1}{t + s}[\int_0^{t+s} T(r)Cx dr - \int_0^t T(r)Cx dr] \\
&= PCx = CPx.
\end{aligned}
$$

This is (c). □

In the case of $R(C)$ dense, we have the following strongly ergodic theorem for C-regularized semigroups.

Theorem 3.2.2. *Suppose $(T(t))_{t \geq 0}$ is a bounded C-regularized semigroup with generator A, and $\overline{R(C)} = X$. Then the following assertions are equivalent.*

(a) *$(T(t))_{t \geq 0}$ is strongly A-ergodic;*

(b) *$(T(t))_{t \geq 0}$ is weakly A-ergodic;*

(c) *for every $x \in X$, $\{\lambda(\lambda - A)^{-1}Cx : 0 < \lambda < 1\}$ is weakly sequentially compact, i.e., there exists a sequence $\{\lambda_n\}$, $\lambda_n \to 0$, and some $y \in X$ such that weak-$\lim_{n \to \infty} \lambda_n(\lambda_n - A)^{-1}Cx = y$;*

(*d*) $R(C) \subseteq \ker A \oplus \overline{R(A)}$.

Proof. (*a*) $\Rightarrow$ (*b*) and (*b*) $\Rightarrow$ (*c*) are obvious.

(*c*) $\Rightarrow$ (*d*). For $x \in X$, there exist $\lambda_n \to 0$ and $y \in X$, such that $y_n = \lambda_n(\lambda_n - A)^{-1}Cx \to y$ weakly, hence

$$\begin{aligned} Ay_n &= \lambda_n A(\lambda_n - A)^{-1}Cx = -\lambda_n Cx + \lambda_n^2(\lambda_n - A)^{-1}Cx \\ &= -\lambda_n Cx + \lambda_n y_n \to 0 \text{ as } n \to \infty, \end{aligned}$$

since y_n is bounded. The closedness of A implies that A is weakly closed, so that $y \in D(A)$ and $Ay = 0$.

Moreover,

$$Cx - y = weak - \lim_{n\to\infty}(Cx - y_n) = weak - \lim_{n\to\infty}(-\frac{Ay_n}{\lambda_n}),$$

hence $Cx - y \in \overline{(R(A))_w}$, the weak closure of $R(A)$. From [Ta, p.158], we know $\overline{(R(A))_w} = \overline{R(A)}$, which leads to $Cx = y + (Cx - y) \in \ker A + \overline{R(A)}$. And from (3.1) we know that the sum is a direct sum.

(*d*) $\Rightarrow$ (*a*). Since $(T(t))_{t\geq 0}$ is bounded, by Lemma 1.1.3, $R(C) \subseteq R(\lambda - A)$ and $\lambda(\lambda - A)^{-1}C$ is uniformly bounded for $\lambda > 0$. Again by (3.1), for $x = x_1 + x_2$, where $x_1 \in \ker A, x_2 \in R(A)$, we have $\lambda(\lambda - A)^{-1}Cx \to Cx_1$ as $\lambda \to 0$.

Moreover, by the assumption, $R(C) \subseteq \ker A \oplus \overline{R(A)}$ and $R(C)$ is dense, we have $\ker A + R(A)$ is dense in X; since the family of operators $\{\lambda(\lambda - A)^{-1}C\}_{\lambda>0}$ is uniformly bounded, the limit $\lim_{\lambda\searrow 0} \lambda(\lambda - A)^{-1}Cx$ exists for every $x \in X$. □

From the proof of the above theorem, one can obtain

Theorem 3.2.3 *Suppose $(T(t))_{t\geq 0}$ is a bounded C-regularized semigroup generated by A and* $\ker A + R(A)$ *is dense in X. Then $(T(t))_{t\geq 0}$ is strongly Abel-ergodic.*

Remark 3.2.4. By Lemma 3.1.2.(*b*), in the case of $(T(t))_{t\geq 0}$ bounded, the A-ergodicity and C-ergodicity are equivalent, so we can replace A-ergodic by C-ergodic in Theorem 3.2.2 and 3.2.3.

If $(T(t))_{t\geq 0}$ is holomorphic, we will see that the three definitions of ergodicity are equivalent.

Theorem 3.2.5 *Suppose $(T(t))_{t\geq 0}$ is a bounded holomorphic C-regularized semigroup generated by A on X, and $R(C)$ is dense in X. Then the following assertions are equivalent.*

(*a*) $(T(t))_{t\geq 0}$ *is strongly A-ergodic;*

(*b*) $(T(t))_{t\geq 0}$ *is strongly C-ergodic;*

(c) $(T(t))_{t\geq 0}$ *is strongly ergodic, that is,* $\lim_{t\to\infty} T(t)x$ *exists for every* $x \in X$.

Proof. It suffices to show $(a) \Rightarrow (c)$. By [dL1, Theorem 6.1], for bounded holomorphic C-regularized semigroup, we have $\lim_{t\to\infty} T(t)x = 0$ for every $x \in \overline{R(A)}$. And for $x \in \ker A$, $T(t)x \equiv Cx$. Therefore, for any $x \in \ker A \oplus \overline{R(A)}$, $\lim_{t\to\infty} T(t)x$ exists. If (a) holds, by Theorem 3.2.2, $R(C) \subset \ker A \oplus \overline{R(A)}$, so that (c) holds since $R(C)$ is dense. □

By applying Theorem 3.2.3, we also have

Theorem 3.2.6 *Suppose* $(T(t))_{t\geq 0}$ *is a bounded holomorphic* C*-regularized semigroup generated by* A*, and suppose* $\ker A + R(A)$ *is dense in* X*. Then* $(T(t))_{t\geq 0}$ *is strongly ergodic, i.e.,* $\lim_{t\to\infty} T(t)x$ *exists for every* $x \in X$.

We end this section with an ergodic theorem in the general case.

Theorem 3.2.7 *Let* $(T(t))_{t\geq 0}$ *be a bounded* C*-regularized semigroup with generator* A*. Then* $(T(t))_{t\geq 0}$ *is strongly* A*-ergodic if and only if for every* $x \in X$*, we have* $Cx = y + z$*, where* $y \in \ker A$*,* $z \in \overline{R(A)}$ *such that* $\lambda(\lambda - A)^{-1}z \to 0$ *as* $\lambda \to 0$*. And this decomposition is unique.*

Proof. Necessity. For $x \in X$, let $y = Px = \lim_{\lambda \searrow 0} \lambda(\lambda - A)^{-1}Cx$; by Proposition 3.2.1, $Px \in \ker A$, thus $(\lambda - A)Px = \lambda Px$; this implies $Px \in R(\lambda - A)$ and $\lambda(\lambda - A)^{-1}Px = Px$ for $\lambda > 0$. Next, choose $z = Cx - y$, then $z \in \bigcap_{\lambda>0} R(\lambda - A)$. From the proof $(c) \Rightarrow (d)$ of Theorem 3.2.2, we have $z = Cx - y \in \overline{R(A)}$. Hence,

$$\lambda(\lambda - A)^{-1}z = \lambda(\lambda - A)^{-1}Cx - \lambda(\lambda - A)^{-1}Px = \lambda(\lambda - A)^{-1}Cx - Px \to 0.$$

Moreover, since $\ker A$ and $\overline{R(A)}$ have only the zero element in common, this decomposition is unique.

Sufficiency. The assumption $y \in \ker A$ implies that $y \in R(\lambda - A)$ and $\lambda(\lambda - A)^{-1}y = y$. Thus

$$\lambda(\lambda - A)^{-1}Cx = \lambda(\lambda - A)^{-1}y + \lambda(\lambda - A)^{-1}z \to y, \qquad \lambda \to 0.$$

This proves our claim. □

3.3 Orbital Ergodicity

In this section, we introduce the notions of orbital ergodicity and ergodicity in Z_0, the Hille-Yosida space of A. Next we will clarify the relations between them.

Definition 3.3.1 We call $(T(t))_{t\geq 0}$ (a strongly continuous semigroup or C-regularized semigroup) *orbitally* A*-ergodic*, if $R(T(t)) \subset R(\lambda - A)$ for all $t \geq 0, \lambda > 0$, and $\lim\limits_{\lambda \searrow 0} \lambda(\lambda - A)^{-1}T(t)x$ exists uniformly in t.

$(T(t))_{t\geq 0}$ is *orbitally* C*-ergodic*, if for $x \in X$, $\lim\limits_{s\to\infty} \frac{1}{s}\int_0^s T(t+\tau)x d\tau$ exists uniformly in t.

$(T(t))_{t\geq 0}$ is *orbitally ergodic*, if for $x \in X$, $\lim\limits_{s\to\infty} T(t+s)x$ exists uniformly in t.

It is remarked that if $(T(t))_{t\geq 0}$ is bounded, then ortitally A-ergodicity and C-ergodicity are equivalent.

The following result for the orbital ergodicity of strongly continuous semigroup is easy to verify.

Theorem 3.3.2 *If* $(T(t))_{t\geq 0}$ *is a bounded strongly continuous semigroup, then* $(T(t))_{t\geq 0}$ *is orbitally (A-,C-)ergodic if and only if* $(T(t))_{t\geq 0}$ *is (A-,C-)ergodic.*

In order to relate orbital ergodicity of C-regularized semigroups with their ergodicity, we now define the ergodicity in the Hille-Yosida space, Z_0.

Definition 3.3.3 Suppose $(T(t))_{t\geq 0}$ is a bounded strongly uniformly continuous C-regularized semigroup with generator A on X. We call $(T(t))_{t\geq 0}$

(a) *uniformly (strongly, weakly)* A*-ergodic from* X *to* Z_0, if $\lim\limits_{\lambda \searrow 0} \lambda(\lambda - A|_{Z_0})^{-1}C$ exists in the uniform (strong, weak) operator topology in Z_0.

(b) *uniformly (strongly, weakly)* C*-ergodic from* X *to* Z_0, if $\lim\limits_{t\to\infty} \frac{1}{t}\int_0^t T(s)ds$ exists in the uniform (strong, weak) operator topology in Z_0.

(c) *uniformly (strongly, weakly) ergodic from* X *to* Z_0, if $\lim\limits_{t\to\infty} T(t)$ exists in the uniform (strong, weak) operator topology in Z_0.

And we know (a)$\Longleftrightarrow$(b), (c)$\Rightarrow$(b).

The following result is an analogue of Theorem 3.2.2, and the proof is also analogous. But note that here we do not assume that $R(C)$ is dense.

Theorem 3.3.4 *Suppose* $(T(t))_{t\geq 0}$ *is a bounded strongly uniformly continuous* C*-regularized semigroup generated by* A *on* X*. The following conditions are equivalent.*

(a) $(T(t))_{t\geq 0}$ *is strongly* A*-ergodic from* X *to* Z_0*;*
(b) $(T(t))_{t\geq 0}$ *is weakly* A*-ergodic from* X *to* Z_0*;*

(*c*) *for every* $x \in X$, $\{\lambda(\lambda - A|_{Z_0})^{-1}Cx : 0 < \lambda < 1\}$ *is weakly sequentially compact in* Z_0;

(*d*) $R(C) \subseteq \ker(A|_{Z_0}) + \overline{(R(A|_{Z_0}))_0}$; *here* $\overline{(\cdot)_0}$ *denotes the closure in* Z_0.

Remark 3.3.5 (a) It is easy to show that $\ker(A|_{Z_0}) = \ker A$.

(b) Since $Z_0 \hookrightarrow X$, $(T(t))_{t\geq 0}$ being strongly A-ergodic from X to Z_0 implies that it is strongly A-ergodic in X.

Theorem 3.3.6 *Suppose* $(T(t))_{t\geq 0}$ *is a bounded strongly uniformly continuous* C*-regularized semigroup. Then* $(T(t))_{t\geq 0}$ *is strongly* A*-ergodic from* X *to* Z_0 *if and only if* $(T(t))_{t\geq 0}$ *is orbitally* A*-ergodic.*

Proof. By Lemma 1.1.6,

$$\begin{aligned}
&\|\frac{1}{s}\int_0^s T(\tau)xd\tau - Px\|_{Z_0} \\
= &\sup_{t\geq 0}\|\frac{1}{s}C^{-1}T(t)\int_0^s T(\tau)xd\tau - C^{-1}T(t)Px\| \\
= &\sup_{t\geq 0}\|\frac{1}{s}\int_0^s T(t+\tau)d\tau - Px\|,
\end{aligned}$$

here Lemma 3.1.2 is applied. □

So Theorem 3.3.4.(d) gives a characterization of orbital A-ergodicity.

Many C-regularized semigroups are orbitally A-ergodic; for instance, given an A-ergodic strongly continuous semigroup $(S(t))_{t\geq 0}$, then $(CS(t))_{t\geq 0}$ is an orbitally A-ergodic C-regularized semigroup. Another example will be given in the next section.

3.4 Uniform Ergodicity

We first give a sufficient condition for uniform A-ergodicity.

Theorem 3.4.1 *Suppose* $(T(t))_{t\geq 0}$ *is a bounded* C*-regularized semigroup generated by* A *on* X*, and* $X = \ker A \oplus R(A)$*. Then* $(T(t))_{t\geq 0}$ *is uniformly* A*-ergodic.*

Proof. Since A is closed, also $\ker A$ is closed. Then by $X = \ker A \oplus R(A)$ and [Ta, p.217], $R(A)$ is closed.

By the open mapping theorem (cf.[Ta, p.213]), there is some $m > 0$ such that each $x \in R(A)$ is equal to Ay for some $y \in D(A)$ with $\|y\| \leq m\|x\|$.

Next, we claim that $R(A)$ is invariant under $R_\lambda \equiv \lambda(\lambda - A)^{-1}C$. In fact, for

$x = Ay \in R(A)$,

$$R_\lambda x = \lambda(\lambda - A)^{-1}Cx = \lambda(\lambda - A)^{-1}CAy = AR_\lambda y \in R(A).$$

Moreover, given $x \in R(A)$, we can choose $y \in D(A)$ satisfying $Ay = x$ and $\|y\| \le m\|x\|$, hence

$$\begin{aligned} \|R_\lambda x\| &= \|R_\lambda Ay\| = \|AR_\lambda y\| \le m\|AR_\lambda\|\,\|x\| \\ &\le \lambda[\|C\| + \sup_{\lambda>0}\|\lambda(\lambda - A)^{-1}C\|]\,\|x\| \\ &= \lambda M\|x\|, \end{aligned}$$

where $M = \|C\| + \sup\limits_{\lambda>0}\|\lambda(\lambda - A)^{-1}C\|$, which implies

$$\|R_\lambda|_{R(A)}\| \le \lambda M. \tag{3.2}$$

On the other hand, by (3.1),

$$R_\lambda|_{\ker A} = C|_{\ker A}. \tag{3.3}$$

And by [Ta, p.247], the projection Q of X onto $\ker A$ along $R(A)$ is continuous. Let $P = CQ$. For any $x \in X$, $x = Qx + (1-Q)x$, note that $R(Q) = \ker A$, $\ker Q = R(A)$, together with (3.2) and (3.3), we have

$$\begin{aligned} \|R_\lambda x - Px\| &= \|R_\lambda Qx - CQx + R_\lambda(1-Q)x\| \\ &= \|R_\lambda(1-Q)x\| \le \lambda M\|(1-Q)x\| \end{aligned}$$

so that $\|\lambda(\lambda - A)^{-1}C - P\| \to 0$ as $\lambda \to 0$, proving our result. □

Corollary 3.4.2 *If $(T(t))_{t\ge0}$ is a C-regularized semigroup generated by A and $0 \in \rho(A)$, then $(T(t))_{t\ge0}$ is uniformly A-ergodic.*

This is an immediate consequence of Theorem 3.4.1; one can also derive it directly from $\frac{1}{t}\int_0^t T(s)ds = \frac{1}{t}(A^{-1}T(t) - A^{-1}C) \to 0$.

It is well known that the characterization of a uniformly A-ergodic strongly continuous semigroup is that $R(A)$ is closed (cf.[Zhe]). However, it is not true for C-regularized semigroups. This can be illustrated by Example 1.3.3. And we will see the C-regularized semigroup we give here is orbitally A-ergodic.

Example 3.4.3 Suppose X is a complex Banach space and $C_0 \in B(X)$ is injective with $\overline{R(C_0)} \ne X$. Let $A = \begin{pmatrix} i & C_0^{-1} \\ 0 & -i \end{pmatrix}$ with $D(A) = X \times R(C_0)$, and $C = \begin{pmatrix} C_0 & 0 \\ 0 & C_0 \end{pmatrix}$.

It is not hard to show that A generates a bounded C-regularized semigroup

$$T(t) = \begin{pmatrix} e^{it}C_0 & \frac{1}{2i}(e^{it} - e^{-it}) \\ 0 & e^{-it}C_0 \end{pmatrix}$$

and

$$(\lambda - A)^{-1}C \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} (\lambda - i)^{-1}C_0x_1 + (\lambda - i)^{-1}(\lambda + i)^{-1}x_2 \\ (\lambda + i)^{-1}C_0x_2 \end{pmatrix}.$$

Obviously $\lambda(\lambda - A)^{-1}C \to 0$ as $\lambda \to 0$. But $D(A) = X \times R(C_0)$, $R(A) = X \times R(C_0)$. So $R(A)$ may not be closed even if $R(C_0)$ is dense in X.

Next we show $(T(t))_{t\geq 0}$ is orbitally A-ergodic. In fact,

$$\begin{aligned} & \frac{1}{s}\int_0^s T(t+r)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} dr \\ = \; & \frac{1}{s}\int_0^s \begin{pmatrix} e^{i(t+r)}C_0x_1 + \frac{1}{2i}(e^{i(t+r)} - e^{-i(t+r)})x_2 \\ e^{-i(t+r)}C_0x_2 \end{pmatrix} dr \\ = \; & \frac{1}{2is}\begin{pmatrix} 2(e^{is} - 1)e^{it}C_0x_1 - i[(e^{is} - 1)e^{it} - (e^{-is} - 1)e^{-it}]x_2 \\ 2(1 - e^{-is})e^{-it}C_0x_2 \end{pmatrix} \end{aligned}$$

converges to 0 as $s \to \infty$ uniformly for $t \geq 0$.

At last, we discuss the relations between strong C-ergodicity and uniform C-ergodicity by using the poles of the C-resolvent on the imaginary axis.

Theorem 3.4.4 *Assume that $\|T(t)\| = O(t)(t \to \infty)$ and $\sigma_C(A)\bigcap i\mathbf{R}$ consists of poles of the C-resolvent only. Then the following are equivalent.*

(a) $(T(t))_{t\geq 0}$ is strongly C-ergodic;

(b) every point in $\sigma_C(A)\bigcap i\mathbf{R}$ is a pole of order 1;

(c) $(T(t))_{t\geq 0}$ is uniformly C-ergodic; i.e., $\frac{1}{t}\int_0^t T(s)ds$ converges in the operator norm.

Proof. If λ_0 is a pole of $(\lambda - A)^{-1}C$, then $\lambda \mapsto (\lambda - A)^{-1}C$ can be expanded as a Laurent series $(\lambda - A)^{-1}C = \sum\limits_{n=-k}^{+\infty}(\lambda - \lambda_0)^n B_n$ for $0 < |\lambda - \lambda_0| < \delta$ and some $\delta > 0$. The coefficients B_n are bounded linear operators, given by

$$B_n = \frac{1}{2\pi i}\int_\Gamma (z - \lambda_0)^{-(n+1)}(z - A)^{-1}Cdz, \; n \in \mathbf{Z} \tag{3.4}$$

where $\Gamma = \{z \in \mathbf{C} : |z - \lambda_0| = \frac{\delta}{2}\}$. Write $B_{-1} = P$, from (3.4) one deduces that $B_{-(n+1)} = (\lambda - A)^n P$ for $n \geq 0$.

$(a) \Rightarrow (b)$. If $i\eta$ is a pole of order $k \geq 2$, then $B_{-k} \neq 0$ while $B_{-(k+1)} = 0$. So that there exists $x \in D(A^k)$ such that $(A - i\eta)^k x \neq 0$ but $(A - i\eta)^{k+1} x = 0$. Hence

$$e^{-i\eta t}T(t)x = Cx + t(A - i\eta)Cx + \frac{t^2}{2!}(A - i\eta)^2 Cx + \cdots + \frac{t^{k-1}}{(k-1)!}(A - i\eta)^{k-1}Cx.$$

This implies that $T(t)x$ is not strongly C-ergodic.

$(b) \Rightarrow (c)$. Since 0 is a pole of order 1 or regular point of the resolvent, so

$$(\lambda - A)^{-1}C = \frac{P}{\lambda} + \sum_{n=1}^{\infty} \lambda^n B_n,$$

for $|\lambda| < \delta$ and some $\delta > 0$, it follows that $\lambda(\lambda - A)^{-1}C \to P$ as $\lambda \to 0$. $\square$

Chapter 4

Norm continuity of regularized semigroups

Starting with the Hille-Yosida theorem it is a fundamental problem in the theory of strongly continuous semigroups to characterize properties of the semigroup in terms of its generator and its resolvent. In 1992, P. You [Yo] proved that for an exponentially stable semigroup $(T(t))_{t\geq 0}$ with generator A on a Hilbert space the norm continuity for $t > 0$ is equivalent to the condition

$$\lim_{\mu \to \pm\infty} \|R(i\mu, A)\| = 0. \tag{4.1}$$

Later, O. El-Mennaoui and K.-J. Engel [EE1] gave a much simpler and more straight forward proof of this result. More general cases concerning norm continuity of semigroups on Banach spaces appeared in [EE2, BM].

In this chapter, we consider the norm continuity of C-regularized semigroups. Section 4.1 study the norm continuity on Banach spaces. In Section 4.2, we show that if a loss of regularity is accepted, then the decay to zero or square-integrability of the C-resolvent of the generator of a regularized semigroup along the imaginary axis also gives a norm continuous regularized semigroup.

4.1 Norm continuity on Banach spaces

The following results are generalizations of those on strongly continuous semigroups, see [EE1, EE2, BM]; and some but not all of the proofs are also similar. For completeness we write all out.

Theorem 4.1.1 *Let $(T(t))_{t\geq 0}$ be a C-regularized semigroup generated by A with $\|T(t)\| \leq Me^{-\omega t}$. Consider the following statements:*

(a) $\quad (T(t))_{t\geq 0}$ *is norm continuous for $t > 0$.*

(b) $\lim_{\lambda\to\infty} \|(\lambda R(\lambda, A) - I)\, T(t)\| = 0$ *and the convergence is uniform for* t *on compact intervals.*

(c) $\{\langle R^n(i\cdot, A)Cx, \phi\rangle : \|x\| \leq 1, \|\phi\| \leq 1\}$ *is a relatively compact subset of* $L^1(\mathbf{R})$ *for some* $n \in \mathbf{N}$.

(d) $\{\langle T(\cdot)x, \phi\rangle : \|x\| \leq 1, \|\phi\| \leq 1\}$ *is a relatively compact set of* $L^p(\mathbf{R}^+)$ *for some* $1 \leq p < \infty$.

(d)′ $\{\langle T(\cdot)x, \phi\rangle : \|x\| \leq 1, \|\phi\| \leq 1\}$ *is a relatively compact set of* $L^p(\mathbf{R}^+)$ *for all* $1 \leq p < \infty$.

(e) $\{\langle R(i\cdot, A)Cx, \phi\rangle : \|x\| \leq 1, \|\phi\| \leq 1\}$ *is a relatively compact subset of* $L^2(\mathbf{R})$.

(f) $\lim_{M\to\infty} \int_M^\infty |\langle R(is, A)Cx, \phi\rangle|^2 ds = 0$ *uniformly for* $\|x\| \leq 1$, $\|\phi\| \leq 1$.

(g) $\lim_{|r|\to\infty} \|R(ir, A)C\|dr = 0$.

(h) $\lim_{M\to\infty} \int_M^\infty |\langle R(is, A)Cx, \phi\rangle|^3 ds = 0$ *uniformly for* $\|x\| \leq 1$, $\|\phi\| \leq 1$.

Then the following implications hold:

$$(c) \Rightarrow (a) \Leftrightarrow (b) \Rightarrow (d) \Leftrightarrow (d)' \Leftrightarrow (e) \Leftrightarrow (f) \Rightarrow (g) \Rightarrow (h)$$

Proof. (a) $\Rightarrow$ (b) Since $(T(t))_{t\geq 0}$ is exponentially stable, we have

$$R(\lambda, A)T(t) = \int_0^\infty e^{-\lambda s}T(t+s)ds, \qquad \forall \lambda > 0.$$

Thus

$$\begin{aligned}(\lambda R(\lambda, A) - I)T(t) &= \lambda \int_0^\infty e^{-\lambda s}T(t+s)ds - T(t) \\ &= \lambda \int_0^\infty e^{-\lambda s}[T(t+s) - T(t)]ds \\ &= \int_0^\infty e^{-s}[T(t+\frac{s}{\lambda}) - T(t)]ds.\end{aligned}$$

Now choose for given $\epsilon > 0$ a constant $N > 0$ such that

$$\| \int_N^\infty e^{-s}(T(\frac{s}{\lambda}+t) - T(t))ds\| < \frac{\epsilon}{2}.$$

Using the norm continuity of $(T(t))_{t\geq 0}$ there exists $\lambda_0 > 0$ such that

$$\|T(t+\frac{s}{\lambda}) - T(t)\| < \frac{\epsilon}{2N}$$

for all $s \in [0, N]$ and $\lambda > \lambda_0$, and hence

$$\| \int_0^N e^{-s}(T(t + \frac{s}{\lambda}) - T(t))ds\| < \frac{\epsilon}{2}$$

for all $\lambda > \lambda_0$; also, such λ_0 can be chosen uniformly for t on closed interval. This proves (b).
$(b) \Rightarrow (a)$. Let $\epsilon > 0$, $t > 0$, there exist $\lambda_0 > 0$ and $\delta > 0$ such that

$$\|(\lambda_0 R(\lambda_0, A) - I)T(t + \Delta t)\| < \epsilon, \qquad |\Delta t| < \delta.$$

Using this we obtain for $|h| < \delta$ and $x \in X$,

$$\begin{aligned}
\|T(t+h)x - T(t)x\| &\leq \|T(t+h)x - \lambda_0 R(\lambda_0, A)T(t+h)x\| \\
&\quad + \|T(t)x - \lambda_0 R(\lambda_0, A)T(t)x\| \\
&\quad\quad + \|\lambda_0 R(\lambda_0, A)(T(t+h) - T(t))x\| \\
&\leq 2\epsilon\|x\| + \|\lambda_0 R(\lambda_0, A)A \int_t^{t+h} T(s)xds\| \\
&= 2\epsilon\|x\| + \|\lambda_0 \int_t^{t+h} AR(\lambda_0, A)T(s)xds\| \\
&= 2\epsilon\|x\| + \|\lambda_0 \int_t^{t+h} (\lambda_0 R(\lambda_0, A) - I)T(s)xds\| \\
&\leq 2\epsilon\|x\| + \lambda_0 \cdot \epsilon h\|x\|,
\end{aligned}$$

which gives (a).
$(c) \Rightarrow (a)$. Let $T_n(t) := t^{n-1}T(t)$. It suffices to show that $T_n(t) : (0, \infty) \to B(X)$ is continuous, or in other words, the family

$$\{\langle T_n(\cdot)x, \phi\rangle : \|x\| \leq 1,\ \|\phi\| \leq 1\}$$

is equicontinuous. As

$$\lim_{t\to\infty} \|t^{n-1}T(t)\| = 0,$$

Arzela-Ascoli's theorem asserts that it is enough to show that

$$\{\langle T_n(\cdot)x, \phi\rangle : \|x\| \leq 1,\ \|\phi\| \leq 1\}$$

is relatively compact in $C((0, \infty))$.

Given any $x \in X$, $\phi \in X^*$, we have that $\langle (n-1)!\, R^n(i\cdot\,, A)Cx, \phi\rangle$ is the Fourier transform of $\langle T_n(\cdot)x, \phi\rangle$. Now observe that

$$\mathcal{U} = \{\langle (n-1)!\, R^n(i\cdot\,, A)Cx, \phi\rangle : \|x\| \leq 1,\ \|\phi\| \leq 1\} \subset L^1(\mathbf{R})$$

is mapped continuously by the Fourier antitransform $\mathcal{F} : L^1(\mathbf{R}) \to C(\mathbf{R}^+)$ onto:

$$\Upsilon = \{\langle T_n(\cdot)x, \phi\rangle : \|x\| \leq 1,\ \|\phi\| \leq 1\} \subset C((0, \infty)),$$

which yields the desired conclusion.
$(d) \Leftrightarrow (d)'$ Suppose (d) holds for p_0. Given $1 \le p < \infty$ and a sequence of pairs $\{(x_n, \phi_n)\} \subset X \times X^*$ with $\|x_n\|, \|\phi_n\| \le 1$, we have to find a subsequence n_k such that

$$f_k(t) = \langle T(t)x_{n_k}, \phi_{n_k}\rangle \text{ converges in } L^p((0,\infty)).$$

From the assumption, there exist a function $f \in L^{p_0}((0,\infty))$ and a subsequence n_k such that

$$g_k(t) = \langle T(t)x_{n_k}, \phi_{n_k}\rangle - f(t) \to 0 \quad a.e.$$

Using the fact that $|g_k(t)|^p \le 2^p M^p e^{-p\omega t}$ and the Lebesgue dominated convergence theorem the desired result is achieved.
$(a) \Rightarrow (d)$ To show $\{\langle T(\cdot)x, \phi\rangle : \|x\| \le 1, \|\phi\| \le 1\}$ is relatively compact in $L^p(\mathbf{R}^+)$, it suffices to show that

$$\lim_{h\to 0}\int_0^\infty |\langle T(t+h)x - T(t)x, \phi\rangle|^p dt = 0$$

and

$$\lim_{M\to\infty}\int_M^\infty |\langle T(t)x, \phi\rangle|^p dt = 0$$

hold uniformly for x and ϕ. It is obvious that the latter holds. For the former, we split the integral as follows

$$\begin{aligned}
&\int_0^\infty |\langle T(t+h)x, \phi\rangle - \langle T(t)x, \phi\rangle|^p dt \\
= &\left(\int_0^{\frac{1}{m}} + \int_{\frac{1}{m}}^m + \int_m^\infty\right) |\langle T(t+h)x, \phi\rangle - \langle T(t)x, \phi\rangle|^p dt \\
\le &\frac{2^p M^p}{m} + \int_{\frac{1}{m}}^m \|T(t) - T(t+h)\|^p dt + \frac{2^p M^p e^{-\omega p m}}{\omega p}.
\end{aligned}$$

Now, given $\epsilon > 0$, we first take $m > 1$ such that $\max\{\frac{2^p M^p}{m}, \frac{2^p M^p}{\omega p} e^{-\omega p m}\} < \frac{\epsilon}{3}$ and then, using the uniform continuity on $[\frac{1}{m}, m]$, find $\delta > 0$ such that if, $0 < h < \delta$, then

$$\|T(t) - T(t+h)\|^p < \frac{\epsilon}{3(m - \frac{1}{m})}, \qquad t \in [\frac{1}{m}, m].$$

Hence, for $0 < h < \delta$, we have $\int_0^\infty |\langle T(t)x, \phi\rangle - \langle T(t+h)x, \phi\rangle|^p dt < \epsilon$ uniformly in $\|x\| \le 1$, $\|\phi\| \le 1$.
$(d) \Rightarrow (g)$ For $x \in X$, $\phi \in X^*$ with $\|x\| \le 1$, $\|\phi\| \le 1$, if $s > 0$, then

$$\begin{aligned}
\langle R(is, A)Cx, \phi\rangle &= \int_0^\infty e^{-ist}\langle T(t)x, \phi\rangle dt \\
&= \frac{1}{2}\int_0^\infty (e^{-ist} - e^{-i(t-\frac{\pi}{s})s})\langle T(t)x, \phi\rangle dt \\
&= -\frac{1}{2}\int_0^{\frac{\pi}{s}} e^{-i(t-\frac{\pi}{s})}\langle T(t)x, \phi\rangle dt \\
&\qquad + \frac{1}{2}\int_0^\infty e^{-ist}\langle T(t)x - T(t+\frac{\pi}{s})x, \phi\rangle dt.
\end{aligned}$$

Thus

$$|\langle R(is\,,A)Cx,\phi\rangle|\leq\frac{1}{2}\int_0^{\frac{\pi}{s}}|\langle T(t)x,\phi\rangle|dt+\frac{1}{2}\int_0^{\infty}|\langle(T(t)-T(t+\frac{\pi}{s}))x,\phi\rangle|dt,$$

which implies $\|R(is,A)Cx\|\to 0$ as $s\to+\infty$. The same holds for $s<0$.
$(e)\Leftrightarrow(d)$ It follows from the fact that the Fourier transform is an isomorphism in $L^2(\mathbf{R})$.
$(e)\Leftrightarrow(f)$ From the proof of $(a)\Rightarrow(d)$, we only need to show

$$\lim_{h\to 0}\int_{\mathbf{R}}|\langle R(i(s+h),A)Cx,\phi\rangle-\langle R(is,A)Cx,\phi\rangle|^2ds=0$$

uniformly for x and ϕ. Given $\epsilon>0$, since

$$\begin{aligned}
&\int_{\mathbf{R}}|\langle R(i(s+h),A)Cx,\phi\rangle-\langle R(is,A)Cx,\phi\rangle|^2ds\\
=\;&\int_{\mathbf{R}}\langle R(i(s+h),A)Cx,\phi\rangle^2ds+\int_{\mathbf{R}}\langle R(is,A)Cx,\phi\rangle^2ds\\
&\qquad-2\int_{\mathbf{R}}\langle R(i(s+h),A)Cx,\phi\rangle\cdot\langle R(is,A)Cx,\phi\rangle ds\\
=\;&2\int_{\mathbf{R}}\langle R(is,A)Cx,\phi\rangle(\langle R(i(s+h),A)Cx,\phi\rangle-\langle R(is,A)Cx,\phi\rangle)ds\\
=\;&2(\int_{|s|\leq N}+\int_{|s|\geq N})\langle R(is,A)Cx,\phi\rangle\\
&\qquad\cdot(\langle R(i(s+h),A)Cx,\phi\rangle-\langle R(is,A)Cx,\phi\rangle)ds\\
:=\;&2(I_1+I_2),
\end{aligned}$$

by Hölder inequality,

$$|I_2|\leq(\int_{\mathbf{R}}|2\langle R(is,A)Cx,\phi\rangle|^2ds)^{\frac{1}{2}}\cdot(\int_{|s|\geq N}|\langle R(is,A)Cx,\phi\rangle|^2ds)^{\frac{1}{2}}<\epsilon$$

if N is chose large enough; while by the continuity of $\langle R(i\cdot\,,A)Cx,\phi\rangle$, if h is small enough, then $|I_1|<\epsilon$.
$(g)\Rightarrow(h)$ Since

$$\begin{aligned}
&\int_M^{\infty}|\langle R(is,A)Cx,\phi\rangle|^3ds\\
\leq\;&\sup_{|s|\geq M}\|R(is,A)C\|\cdot\int_{-\infty}^{\infty}|\langle R(is,A)Cx,\phi\rangle|^2ds\\
=\;&\sup_{|s|\geq M}\|R(is,A)C\|\cdot\int_{-\infty}^{\infty}|\langle\int_0^{\infty}e^{its}T(t)x\,dt,\phi\rangle|^2ds\\
=\;&2\pi\sup_{|s|\geq M}\|R(is,A)C\|\cdot\int_0^{\infty}|\langle T(t)x,\phi\rangle|^2dt\\
\to\;&0\qquad\text{as } M\to\infty
\end{aligned}$$

where in the above we used the Plancherel theorem for the Fourier transform. □

4.2 Norm continuity on Hilbert spaces

In this section, we consider the norm continuity of regularized semigroup on Hilbert spaces. First we show that a condition similar to (4.1) on C-resolvent of the generator A of a C-regularized semigroup also guarantees that A generates a norm continuous C^3-regularized semigroup.

Theorem 4.2.1 *Let A be a densely defined closed operator on a Hilbert space H. Then A generates an exponentially stable norm continuous regularized semigroup if and only if there is an injective operator C on H such that*

(a) $C^{-1}AC = A$;

(b) *There exist constants $M, \omega > 0$ such that for every $r > -\omega$ and $n \in \mathbf{N}$, $D((r-A)^{-n}) \supset R(C)$ and*

$$\|(r+\omega)^n (r-A)^{-n} C\| \leq M;$$

(c) $\lim_{s\to\infty} \|R(is,\, A)C\| = 0.$

Proof. The necessity follows from the properties of the generator of a regularized semigroup and Theorem 4.1.1.

For the converse, by the assumptions (a) and (b), we know that A generates an exponentially stable C-regularized semigroup $(T(t))_{t\geq 0}$. Now we show that $(T(t)C^2)_{t\geq 0}$ is a norm continuous C^3-regularized semigroup. In order to prove it, by Theorem 4.1.1.$(c) \Rightarrow (a)$, we only need to show that

$$\{\langle R^3(i\cdot, A)C^3x, y\rangle : \|x\|, \|y\| \leq 1\}$$

is a relatively compact subset of $L^1(\mathbf{R})$. So what we need to verify are

$$\lim_{h\to 0} \int_0^\infty |\langle R^3(i(s+h), A)C^3x - R^3(is, A)C^3x, y\rangle| ds = 0$$

and

$$\lim_{\eta\to\infty} \int_\eta^\infty |\langle R^3(is, A)C^3x, y\rangle| ds = 0.$$

Once the latter is proved, the former can be showed similarly as that of $(a) \Rightarrow (d)$ in Theorem 4.1.1. By the Plancherel Theorem in a Hilbert space, we know that $R(i\cdot, A)Cx \in L^2(\mathbf{R}, H)$ for every $x \in H$ with $\|R(i\cdot, A)Cx\|_{L^2} = \sqrt{2\pi}\|T(\cdot)x\|_{L^2}$ and also $(R(i\cdot, A)C)^*y \in L^2(\mathbf{R}, H)$ for every $y \in H$ with $\|(R(i\cdot, A)C)^*y\|_{L^2} = \sqrt{2\pi}\|T(\cdot)^*y\|_{L^2}$, the assumption (c), together with the Hölder inequality, implies that

$$\int_\eta^\infty |\langle R^3(is, A)C^3x, y\rangle| ds \quad \leq \quad \sup_{s\geq\eta}\{\|R(is, A)C\|\} \cdot (\int_0^\infty \|R(is, A)Cx\|^2 ds)^{\frac{1}{2}}$$

$$
\begin{aligned}
&\quad \cdot\left(\int_0^\infty \|(R(is,A)C)^*y\|^2ds\right)^{\frac{1}{2}}\\
\leq&\quad 2\pi \sup_{s\geq\eta}\{\|R(is,A)C\|\}\cdot\left(\int_0^\infty \|T(t)x\|^2dt\right)^{\frac{1}{2}}\\
&\quad \cdot\left(\int_0^\infty \|T(t)^*y\|^2dt\right)^{\frac{1}{2}}\\
\to&\quad 0 \text{ as } \eta\to\infty
\end{aligned}
$$

this completes our proof. □

In our next result, the regularity of the norm continuous regularized semigroup is better than the previous one.

Theorem 4.2.2 *Let A be a densely defined closed operator on a Hilbert space H. Then A generates an exponentially stable norm continuous regularized semigroup if and only if there is an injective operator C on H such that*

(a) $C^{-1}AC = A$;

(b) *There exist constants $M, \omega > 0$ such that for every $r > -\omega$ and $n \in \mathbf{N}$, $D((r-A)^{-n}) \supset R(C)$ and*

$$\|(r+\omega)^n(r-A)^{-n}C\| \leq M;$$

(c) *If either*

$$\lim_{\eta\to\infty}\int_\eta^\infty \|R(\pm is,\, A)Cx\|^2ds = 0$$

or

$$\lim_{\eta\to\infty}\int_\eta^\infty \|(R(\pm is,\, A)C)^*y\|^2ds = 0$$

holds uniformly for $\|x\| \leq 1$ or $\|y\| \leq 1$, where $x \in H, y \in H^*$, respectively.

Proof. Necessity. Let $(T(t))_{t\geq 0}$ be the norm continuous C-regularized semigroup generated by A. The proof of (c) is similar to that of [Ya, Theorem 1.2].

Sufficiency. By (a) and (b), let $(T(t))_{t\geq 0}$ be the C-regularized semigroup generated by A. We show that $(T(t)C)_{t\geq 0}$ is a norm continuous C^2-regularized semigroup generated by A. By $(c) \Rightarrow (a)$ of Theorem 4.1.1, we only need to show that

$$\{\langle R^2(i\cdot\,, A)C^2x, y\rangle : \|x\|, \|y\| \leq 1\}$$

is a relatively compact subset of $L^1(\mathbf{R})$. Again, we only need to verify that

$$\lim_{\eta\to\infty}\int_\eta^\infty |\langle R^2(is,A)C^2x,y\rangle|ds = 0;$$

this follows from the inequlity

$$\int_\eta^\infty |\langle R^2(is,A)C^2x,y\rangle|ds \leq \left(\int_\eta^\infty \|R(is,A)Cx\|^2ds\right)^{\frac{1}{2}}\cdot\left(\int_\eta^\infty \|(R(is,A)C)^*y\|^2ds\right)^{\frac{1}{2}}$$

and the assumption (c). □

Chapter 5

Exponential growth and stability

We have seen a lot of work on stability of strongly continuous semigroups. The first result was due to F.L. Huang ([Hu1], [Hu2]) in 1983.

Theorem A. *If $(T(t))_{t\geq 0}$ is a bounded strongly continuous semigroup with generator A, and suppose that the spectrum of A is contained in the left half plane $\{\lambda \in \mathbf{C} : \mathrm{Re}\lambda < 0\}$, then $\|T(t)x\| \to 0$ as $t \to \infty$ for all $x \in X$.*

In 1988, Arendt and Batty [AB], and Lyubich and Vũ [LV] proved the following result independently.

Theorem B. *Let $(T(t))_{t\geq 0}$ be a bounded strongly continuous semigroup on a Banach space X with generator A. If*

(i) $\sigma(A) \cap i\mathbf{R}$ is countable,

(ii) $\sigma_p(A^*) \cap i\mathbf{R} = \emptyset$,

then $(T(t))_{t\geq 0}$ is stable.

There are examples show that none of the assumptions above can be relaxed, so this result is optimal in some sense. Many variants of it have subsequently been obtained, see [Ba], [Hu3], [Hu4], [dVW] for example. More recently, Batty, van Neerven and Räbiger [BNR1] gave the following more general version of Theorem B.

Theorem C. *Let $(T(t))_{t\geq 0}$ be a bounded strongly continuous semigroup on a Banach space X with generator A. Suppose that there is a dense subspace Y of X such that for all $x \in Y$, $\sigma_u(A, x)$ is at most countable, where $\sigma_u(A, x)$ is the set of points $\lambda \in i\mathbf{R}$ to which $R(\lambda, A)x$ cannot be extended holomorphically, and $\sigma_r(A) \cap i\mathbf{R} = \emptyset$. Then $(T(t))_{t\geq 0}$ is stable.*

On the other hand, to the best of our knowledge, there are no such results for C-regularized semigroups. However, it is of basically importance since the stability of a C-regularized semigroup means the stability of the solution of the corresponding abstract Cauchy problem with initial value Cx for every $x \in X$. And we also note that the stability of C-regularized semigroups is different from that of strongly continuous semigroups in essence. It is well known that if $(T(t))_{t\geq 0}$ is a strongly continuous semigroup then the condition "there is $t_0 > 0$ such that

$\|T(t_0)\| < 1$ " implies that $(T(t))_{t\geq 0}$ is stable (even exponentially stable). But in the case of $(T(t))_{t\geq 0}$ being a C-regularized semigroup it is not true. A simple example is $T(t) = e^{\overline{it}}C$, where $\|C\| < 1$, so $\|T(t)\| = \|C\| < 1$, $\forall t \geq 0$; but $(T(t))_{t\geq 0}$ is not stable.

In this chapter, we try to study the stability of C-regularized semigroups. We will not make use of the results on stability of strongly continuous semigroups directly, but a Tauberian theorem on asymptotic behavior of vector-valued functions which was developed in [BNR2]. And the Hille-Yosida space techniques will be employed. This chapter is organized as follows.

In Section 5.1, we first give two characterizations on the resolvent of the generator of an exponentially bounded C-regularized semigroup. The first is a Hille-Yosida type characterization concerning the growth of $\|R(\lambda, A)^n C\|$; but it is not easy to verify, so a second one is given which involves an improper integral of the C-resolvent along a vertical line; this idea has appeared in [PA]. Next we investigate the growth bounds of the strong solution of the corresponding abstract Cauchy problem when the C-resolvent of the generator is uniformly bounded on the right half plane. Such work for strongly continuous semigroup was due to van Neerven [Ne], and later was exploited in [XL] for higher order abstract Cauchy problems.

Section 5.2 is devoted to the stability of C-regularized semigroups. We start with a necessary and sufficient condition for exponential stability (Theorem 5.2.1). Next, we show that a C-regularized semigroup $(T(t))_{t\geq 0}$ is exponentially stable if and only if there is $\omega > 0$ such that the Hille-Yosida space for $A + \omega I$ contains $R(C)$, and $(T(t))_{t\geq 0}$ is stable if and only if there is an interpolated space W such that $R(C) \subset W \hookrightarrow X$ such that $A|_W$ generates a stable strongly continuous semigroup (Theorem 5.2.2). Our main results in this section are Theorem 5.2.4 and Theorem 5.2.7. Theorem 5.2.4 says that if there is a dense space Y of X such that for every $x \in Y$, $\sigma_u(A, Cx)$ is at most countable and there is a closed subspace W of Z_0, the Hille-Yosida space for A, such that $A|_W$ generates a strongly continuous semigroup on W and $\sigma_r(A|_W) \cap i\mathbf{R} = \emptyset$, then $(T(t))_{t\geq 0}$ is stable. Theorem 5.2.7 shows that in the case of $R(C)$ being dense in X, the latter condition can be replaced by $\sigma_r(A) \cap i\mathbf{R} = \emptyset$. Moreover, it is proved in Proposition 5.2.8 that the condition $\sigma_r(A) \cap i\mathbf{R} = \emptyset$ is also necessary; and if X is reflexive, this condition can be replaced by $\sigma_p(A) \cap i\mathbf{R} = \emptyset$ (Corollary 5.2.11). Proposition 5.2.12 clarifies the relations between the spectrum of A and its restriction on Z_0.

5.1 Exponential growth

For an operator A on a Banach space X, if $\lambda - A$ is injective, we will denote by $R(\lambda, A)$ its inverse. And the point spectrum, continuous spectrum and residue spectrum of A are denoted by $\sigma_p(A)$, $\sigma_c(A)$ and $\sigma_r(A)$, respectively.

A C-regularized semigroup $(T(t))_{t\geq 0}$ is called *exponentially bounded* if there are constants $\omega \in \mathbf{R}$ and $M > 0$ such that $\|T(t)\| \leq Me^{\omega t}$ for all $t \geq 0$. First we give two equivalent conditions on the C-resolvent of the generator which guarantee the exponential boundedness.

Theorem 5.1.1 *Suppose that X is a Banach space, and A generates a C-regularized semigroup $(T(t))_{t\geq 0}$. Then the following statements are equivalent:*

(1) *$(T(t))_{t\geq 0}$ is exponentially bounded.*

(2) *There exists $\omega \in \mathbf{R}$ such that $\{\lambda : \mathrm{Re}\lambda > \omega\} \subset \rho_C(A)$ and*

$$\|(\lambda - \omega)^n R(\lambda, A)^n C\| \leq M, \qquad \forall \lambda > \omega.$$

(3) *There exists $\omega \in \mathbf{R}$ such that $\{\lambda : \mathrm{Re}\lambda > \omega\} \subset \rho_C(A)$ and*

$$\|\lambda R(\lambda, A)C\| \leq M, \qquad \forall \lambda > \omega$$

$$\sup_{\alpha > \omega} \int_{-\infty}^{+\infty} |\langle R(\alpha + ir, A)Cx, y\rangle|^2 dr \leq M\|x\|^2\|y\|^2, \ \forall x \in X, y \in X^*.$$

Proof. (1) $\Rightarrow$ (2) is obvious.
(2) $\Rightarrow$ (1). By the Arendt-Widder theorem, there is a family $(F(t))_{t\geq 0} \subset B(X)$ such that

$$R(\lambda, A)C = \lambda \int_0^\infty e^{-\lambda t} F(t)dt, \ \forall \lambda > \omega$$

and $\|F(t+h) - F(t)\| \leq Mhe^{\omega t}\max(e^{\omega h}, 1)$. Similarly as the proof of [Ar, Theorem 3.1], it is not hard to verify that $(F(t))_{t\geq 0}$ is an integrated C-regularized semigroup (see [LS]), that is

$$F(t)F(s)x = (\int_0^{t+s} - \int_0^t - \int_0^s)F(r)Cxdr.$$

Next, we show that A is the generator. If $x \in D(A)$, then

$$\begin{aligned}
\lambda \int_0^\infty e^{-\lambda t} F(t)Axdt &= R(\lambda, A)CAx \\
&= -Cx + \lambda R(\lambda, A)Cx \\
&= -Cx + \lambda^2 \int_0^\infty e^{-\lambda t} F(t)xdt \\
&= -\lambda \int_0^\infty e^{-\lambda t} Cxdt + \lambda^2 \int_0^\infty e^{-\lambda t} F(t)xdt,
\end{aligned}$$

so $F(t)x$ is differentiable, by the uniqueness of Laplace transform, and

$$F'(t)x = F(t)Ax + Cx$$

that is,

$$F(t)x - tCx = \int_0^t F(r)Axdr.$$

Therefore, $x \in D(B)$ with $Bx = Ax$, where B is the generator of $(F(t))_{t\geq 0}$; which is saying, B is an extension of A.

Conversely, let $x \in D(B)$, then $F'(t)x = F(t)Bx + Cx$. So we have

$$R(\lambda, A)Cx = \int_0^\infty e^{-\lambda t}(F(t)Bx + Cx)dt = \frac{1}{\lambda}[R(\lambda, A)CBx + Cx],$$

which means that $Cx \in D(A)$ and $ACx = BCx = CBx \in R(C)$, together with the fact that $A = C^{-1}AC$, we know $x \in D(A)$.

Now we have proved that A is the generator of $(F(t))_{t\geq 0}$; on the other hand, since A generates C-regularized semigroup $(T(t))_{t\geq 0}$, $(\int_0^t T(r)dr)_{t\geq 0}$ is an integrated C-regularized semigroup generated by A. By the uniqueness of the generator, $T(t) = F'(t)$, hence $\|T(t)\| \leq Me^{\omega t}$.
(1) $\Rightarrow$ (3). Since

$$\langle R(\lambda, A)Cx, y\rangle = \int_0^\infty e^{-\lambda t}\langle T(t)x, y\rangle dt,$$

the second inequality of (3) follows from the Plancherel theorem.
(3) $\Rightarrow$ (1). We can assume that $\omega = 0$. Let $x \in X, y \in X^*$, by the second inequality of (3) and the Paley-Wiener theorem, there exists a function $g(\cdot, x, y) \in L^2(0, \infty)$ such that

$$\langle R(\lambda, A)Cx, y\rangle = \int_0^\infty e^{-\lambda t}g(t, x, y)dt, \ \forall \ \mathrm{Re}\lambda > 0$$

and $\|g(\cdot, x, y)\|_{L^2}^2 \leq M\|x\|^2\|y\|^2$. For $\lambda > 0$, define

$$G(e^{-\lambda \cdot}) = \int_0^\infty e^{-\lambda t}g(t, x, y)dt,$$

then

$$\begin{aligned}
|G(e^{-\lambda \cdot})| &= |\int_0^\infty e^{-\lambda t}g(t, x, y)dt| \\
&= |\langle R(\lambda, A)Cx, y\rangle| \cdot |\lambda \int_0^\infty e^{-\lambda t}dt| \\
&\leq \|\lambda R(\lambda, A)C\| \, \|x\| \, \|y\| \, |\int_0^\infty e^{-\lambda t}dt| \\
&\leq M \, \|x\| \, \|y\| \, \|e^{-\lambda \cdot}\|_{L^1(0,\infty)} \qquad (*)
\end{aligned}$$

Thus, G defines a bounded linear functional on $\mathrm{span}\{e^{-\lambda \cdot} : \lambda > 0\} \subset L^1(0, \infty)$. Since $\{e^{-\lambda \cdot} : \lambda > 0\}$ is total in $L^1(0, \infty)$, G can be extended to $L^1(0, \infty)$; and by $(*)$, $g(\cdot, x, y) \in (L^1(0, \infty))^* = L^\infty(0, \infty)$ with $\|g(\cdot, x, y)\|_\infty \leq M\|x\| \, \|y\|$. Next we define $F(t, x, y) = \int_0^t g(s, x, y)ds$. Obviously, $F(t, x, y)$ is linear with respect to x

and y, and it is routine to show that there exists a family $(F(t))_{t\geq 0} \subset B(X)$ such that $\langle F(t)x, y\rangle = F(t, x, y)$ for all $t \geq 0$ and $F(\cdot)$ is strongly continuous. Hence, we reach at

$$R(\lambda, A)Cx = \lambda \int_0^\infty e^{-\lambda t} F(t)xdt, \forall \lambda > 0.$$

From the proof of (2) $\Rightarrow$ (1), we know $F(t)x = \int_0^t T(r)xdr$, so $g(t, x, y) = \langle T(t)x, y\rangle$ a.e. for all $x \in X, y \in X^*$, which implies that $|\langle T(t)x, y\rangle| \leq M \|x\| \|y\|$, consequently, we get $\|T(t)\| \leq M$. □

It is well known that when A generates a C-regularized semigroup $(T(t))_{t\geq 0}$, then for all $x \in D(A)$, the corresponding Cauchy problem

$$\frac{d}{dt}u(t) = Au(t), u(0) = Cx$$

has a unique strong solution given by $T(t)x$. Now we try to study the asymptotic property of such solutions under the condition that the C-resolvent of A is bounded on the right half plane. This idea goes back to [Ne].

Theorem 5.1.2 *Suppose that $(T(t))_{t\geq 0}$ is a C-regularized semigroup generated by A satisfying $\|T(t)\| \leq Me^{\omega t}$ for some constants M and ω. If for some $x_0 \in X$, the map $z \mapsto R(z, A)Cx_0$ admits a bounded analytic continuation to the half plane* $\{\mathrm{Re} z > 0\}$*, then for each λ with* $\mathrm{Re}\lambda > \omega$*, there exists a constant K such that*

$$\|R(\lambda, A)T(t)x_0\| \leq K(1 + t), \ \forall t \geq 0.$$

Proof. Since $(T(t))_{t\geq 0}$ is exponentially bounded, for $\mathrm{Re} z > \omega$, $T(t)x_0 \in R(z - A)$ and

$$R(z, A)T(t)x_0 = \int_0^\infty e^{-zs} T(t + s)x_0 ds.$$

In particular,

$$R(z, A)Cx_0 = \int_0^\infty e^{-zt} T(t)x_0 dt.$$

Moreover, for $\mathrm{Re}\lambda > \omega$, we have

$$R(\lambda, A)R(z, A)Cx_0 = \int_0^\infty e^{-zt} R(\lambda, A)T(t)x_0 dt,$$

hence

$$\langle R(\lambda, A)R(z, A)Cx_0, x^*\rangle = \int_0^\infty e^{-zt} \langle R(\lambda, A)T(t)x_0, x^*\rangle dt.$$

Let $z = \xi + i\eta$, $\mathrm{Re}\lambda, \xi > \omega$, by the resolvent identity we have

$$\begin{aligned} & \langle R(\lambda, A)R(\xi + i\cdot, A)Cx_0, x^*\rangle \\ = \ & \langle \frac{1}{\lambda - \xi - i\cdot}(R(\lambda, A)Cx_0 - R(\xi + i\cdot, A)Cx_0), x^*\rangle, \end{aligned}$$

then from $R(\lambda, A)T(t)x_0 - R(\lambda, A)Cx_0 = \int_0^t AR(\lambda, A)T(s)x_0 ds$ and by the inverse transform, the following

$$\begin{aligned}
& \langle R(\lambda, A)T(t)x_0, x^* \rangle \\
= & \frac{1}{2\pi i} \int_{\xi + i\mathbf{R}} \langle e^{zt} R(\lambda, A)R(z, A)Cx_0, x^* \rangle dz \\
= & \frac{1}{2\pi i} \int_{\xi + i\mathbf{R}} e^{zt} \frac{1}{\lambda - z} [\langle R(\lambda, A)Cx_0, x^* \rangle - \langle R(z, A)Cx_0, x^* \rangle] dz
\end{aligned}$$

is valid. The rest is analogous to the proof of [Ne, Theorem 2], so is omitted. □

Corollary 5.1.3 *Suppose that $(T(t))_{t \geq 0}$ is an exponentially bounded C-regularized semigroup generated by A. If $R(\lambda, A)C$ exists and is uniformly bounded in the right half plane, then there is a constant M such that*

$$\|T(t)x\| \leq M(1+t)\|x\|_{[D(A)]}, \ \forall x \in D(A).$$

Proof. Pick λ large enough, since $T(t)x = R(\lambda, A)T(t)(\lambda - A)x$, from the proof of the above theorem we know there is a constant M' which is independent of x such that

$$\|R(\lambda, A)T(t)(\lambda - A)x\| \leq M'\|(\lambda - A)x\| \leq M\|x\|_{[D(A)]}.$$

□

Theorem 4 and Corollary 5 in [Ne] can also be generalized to the case of C-regularized semigroups, but we omit them here.

5.2 Stability

A C-regularized semigroup $(T(t))_{t \geq 0}$ is called *exponentially stable* if there exists $\omega > 0$ such that $\|T(t)\| \leq Me^{-\omega t}$ for all $t \geq 0$; $(T(t))_{t \geq 0}$ is *stable* if $T(t)x$ converges to 0 as $t \to \infty$ for every $x \in X$.

Theorem 5.2.1 *Suppose $(T(t))_{t \geq 0}$ is a C-regularized semigroup generated by A. Then $(T(t))_{t \geq 0}$ is exponentially stable if and only if there exist constants $\omega, M > 0$ such that $\{\lambda \in \mathbf{C} : \mathrm{Re}\lambda > -\omega\} \subset \rho_C(A)$ and*

$$\|(\lambda + \omega)^n R(\lambda, A)^n C\| \leq M, \ \ \forall \mathrm{Re}\lambda > -\omega.$$

Theorem 5.2.2 *Let $(T(t))_{t \geq 0}$ be a C-regularized semigroup with generator A. Then*

(1) $(T(t))_{t\geq 0}$ *is exponentially stable if and only if there is some* $\omega > 0$ *such that* $Z_{-\omega}$*, the Hille-Yosida space for* $A + \omega I$*, contains* $R(C)$*.*

(2) $(T(t))_{t\geq 0}$ *is stable if and only if there is a closed supspace* W *of* Z_0*, where* Z_0 *is the Hille-Yosida for* A*, such that*

$$R(C) \subset W \subset Z_0$$

and $A|_W$ *generates a stable strongly continuous semigroup on* W*.*

Proof. (1) Suppose $(T(t))_{t\geq 0}$ is exponentially stable, that is, $\|T(t)\| \leq Me^{-\alpha t}$ for some $\alpha > 0$. Choose $0 < \omega < \alpha$, then $e^{\omega t}T(t)$ is bounded and uniformly strongly continuous, so the Hille-Yosida space for its generator $A + \omega I$, $Z_{-\omega}$, contains $R(C)$.

Conversely, let $T(t) = e^{-\omega t}e^{t(A+\omega I)|_{Z_0}}C$, then $(T(t))_{t\geq 0}$ is an exponentially bounded C-regularized semigroup.

(2) If $(T(t))_{t\geq 0}$ is stable, let

$$\begin{aligned} W &= \{x \in X : C^{-1}T(t)x \text{ exists, continuous and } C^{-1}T(t)x \to 0, t \to \infty\} \\ &= \{x \in Z_0 : \|e^{tA|_{Z_0}}x\| \to 0, t \to \infty\} \end{aligned}$$

then W is a closed subspace of Z_0 and $R(C) \subset W \subset Z_0$. Suppose $(S(t))_{t\geq 0}$ is the strongly continuous semigroup generated by $A|_{Z_0}$ on Z_0, then W is $S(t)$-invariant. Denote by $S(t)_|$ the restriction of $S(t)$ on W, then $(S(t)_|)_{t\geq 0}$ is also a strongly continuous semigroup on W. Suppose B is the generator of $(S(t)_|)_{t\geq 0}$, we will show that $B= A|_W$. In fact, if $x \in D(B)$, then

$$\frac{S(t)_|x - x}{t} \to Bx \quad \text{in } W \subset Z_0,$$

since C is a bounded operator on Z_0, we have

$$\frac{S(t)Cx - Cx}{t} \to CBx \quad \text{in } Z_0,$$

which implies

$$\frac{T(t)x - Cx}{t} \to CBx \quad \text{in } X$$

since $T(t) = S(t)C$. Hence $x \in D(A)$ with $Ax = Bx \in W$. On the other hand, if $x \in D(A|_W)$, then $x, Ax \in W \subset Z_0$. By Lemma 1.1.5, we know that $T(t)x$, $T(t)Ax \in R(C)$ for all $t \geq 0$ and $C^{-1}T(t)x$, $C^{-1}T(t)Ax$ are continuous with respect to t. So acting C^{-1} on both sides of

$$T(t)x = Cx + \int_0^t T(s)Axds,$$

and by the fact that $S(t)_{|}x = C^{-1}T(t)x$, $S(t)_{|}Ax = C^{-1}T(t)Ax$, the following

$$S(t)_{|}x = x + \int_0^t S(s)_{|}Axds$$

holds in X. Moreover, since both $S(t)_{|}x$ and $S(t)_{|}Ax$ are continuous in W, so the above equality also holds in W. Thus it follows that $x \in D(B)$.

The converse is obvious. □

In the sequel, we try to derive the stability of a C-regularized semigroup from the spectral conditions on its generator. To begin with, we can deduce the following individual stability result directly from [BNR2, Theorem 4.1].

Proposition 5.2.3 *Suppose that $(T(t))_{t\geq 0}$ is a bounded strongly uniformly continuous C-regularized semigroup with generator A. Let $x \in X$. If*

(a) $\sigma_u(A, Cx)$ is at most countable, where $\sigma_u(A, Cx)$ is the set of all those points $\lambda_0 = e^{i\theta}$ to which $R(\lambda, A)Cx$ can not be extended holomorphically and

(b) $\lim_{\alpha\to 0} \alpha \int_0^\infty e^{-(\alpha+\lambda)t}T(t+s)xdt = 0$ uniformly on $s \geq 0$ for every $\lambda \in \sigma_u(A, Cx)$,

then $\lim_{t\to\infty} \|T(t)x\| = 0$.

Now we can prove the following result in the general cases.

Theorem 5.2.4 *Suppose that $(T(t))_{t\geq 0}$ is a bounded strongly uniformly continuous C-regularized semigroup with generator A. If*

(i) there is a dense space Y of X such that for every $x \in Y$, $\sigma_u(A, Cx)$ is at most countable and

(ii) there is a closed subspace W of Z_0 satisfying $R(C) \subset W \subset Z_0$ such that $A|_W$ generates a contraction strongly continuous semigroup on W and $\sigma_r(A|_W) \cap i\mathbf{R} = \emptyset$,

then $(T(t))_{t\geq 0}$ is stable.

Proof. Let $x \in Y$, then $\sigma_u(A, Cx)$ is at most countable. In order to show that $T(t)x$ converges to 0 as $t \to \infty$, by Proposition 5.2.3, we only need to verify that (b) therein is satisfied. We note that (b) in Proposition 5.2.3 is equivalent to

$$\alpha \int_0^\infty e^{-(\alpha+\lambda)t}S(t)Cxdt \to 0 \text{ in } Z_0(\text{also } W) \text{ as } \alpha \to 0,\ \forall \lambda \in \sigma_u(A, Cx), \tag{5.1}$$

where $(S(t))_{t\geq 0}$ is strongly continuous semigroup generated by $A|_W$, by the norm defined on Z_0.

We claim that

$$\sigma_u(A, Cx) \cap i\mathbf{R} \subset \sigma_u(A|_W, Cx) \cap i\mathbf{R}. \tag{5.2}$$

In fact, if $\lambda \in i\mathbf{R}\backslash\sigma_u(A|_W, Cx)$, then $R(\lambda, A|_W)Cx$ can be extended holomorphically to a neighbor of λ_0 in W. Suppose $F(\lambda)$ is its extension, then $F(\lambda)$ is differentiable at a neighbor of λ_0 in W; since $W \hookrightarrow X$, $F(\lambda)$ is also differentiable at the same neighbor of λ_0 in X. Hence $F(\lambda)$ is also an holomorphic extension of $R(\lambda, A)Cx$ in X, this means that (5.2) holds.

By (5.2), the following condition

$$\alpha \int_0^\infty e^{-(\alpha+\lambda)t} S(t)Cxdt \to 0 \text{ in } W, \text{ as } \alpha \to 0, \forall \lambda \in \sigma_u(A|_W, Cx)$$

is stronger than the condition (5.1); on the other hand, by [BNR2, Proposition 2.2] and Remark (2) after it, the above condition is also weaker than the condition that $\sigma_r(A|_W) \cap i\mathbf{R} = \emptyset$. So by our assumption (ii), we have that $\|T(t)x\| \to 0$ as $t \to \infty$.

Now we have proved that for $x \in Y$, $\lim_{t\to\infty} \|T(t)x\| = 0$. Our proof is complete since Y is dense in X and $(T(t))_{t\geq 0}$ is bounded. □

Compare the following immediate corollary of Theorem 5.2.4 with [Ba, Theorem 5]. The operator C behaves like an intertwining operator.

Corollary 5.2.5 *Suppose $(T(t))_{t\geq 0}$ is a bounded strongly uniformly continuous C-regularized semigroup generated by A. If*

(i) $\sigma_C(A) \cap i\mathbf{R}$ is at most countable and

(ii) there is a closed subspace W of Z_0 such that $R(C) \subset W \subset Z_0$ and a strongly continuous semigroup $(S(t))_{t\geq 0}$ on W generated by B with $\sigma_r(B) \cap i\mathbf{R} = \emptyset$ such that $T(t) = S(t)C$,
then $(T(t))_{t\geq 0}$ is stable.

Corollary 5.2.6 *Suppose $(T(t))_{t\geq 0}$ is a bounded strongly uniformly continuous C-regularized semigroup with generator A. Suppose that $\forall x \in X$, $\sigma_u(A, Cx) = \emptyset$, then $(T(t))_{t\geq 0}$ is stable. In particular, if $i\mathbf{R} \subseteq \rho_C(A)$, then $(T(t))_{t\geq 0}$ is stable.*

Now we restrict our attention to the case when $R(C)$ is dense in X.

Theorem 5.2.7 *Suppose $(T(t))_{t\geq 0}$ is a bounded C-regularized semigroup generated by A and $\overline{R(C)} = X$. If*

(i) there is a dense subspace Y contained in X such that $\sigma_u(A, Cx)$ is at most countable for all x in Y and

(ii) $\sigma_r(A) \cap i\mathbf{R} = \emptyset$,
then $(T(t))_{t\geq 0}$ is stable.

Proof. I. By Theorem 5.2.4, we only need to show that there is a subspace W satisfying $R(C) \subset W \subset Z_0$ such that $A|_W$ generates a strongly continuous

semigroup on W and $\sigma_r(A|_W) \cap i\mathbf{R} = \emptyset$.
II. Construct such W. Define

$$F = \bigcup_{x \in X} \{T(t)x : t \geq 0\},$$

then set $E = \operatorname{span} F$, and the closure of E in Z_0 is defined as W. Thus $R(C) \subset W \subset Z_0$. Suppose $(S(t))_{t\geq 0}$ is the strongly continuous semigroup generated by $A|_{Z_0}$, since W is $S(t)$-invariant, the restriction of $(S(t))_{t\geq 0}$ on W is also a strongly continuous semigroup; from the proof of Theorem 5.2.2.(2), we know that $A|_W$ is its generator.
III. $\sigma_r(A|_W) \cap i\mathbf{R} \subseteq \sigma_r(A) \cap i\mathbf{R}$.

Suppose that $\eta \in \sigma_r(A|_W) \cap i\mathbf{R}$. We can assume that $\eta = 0$. Then there is an $w^* \in W^*$, $w^* \neq 0$ such that $(A|_W)^* w^* = 0$. Hence,

$$\langle A|_W x, w^* \rangle = 0, \ \forall x \in D(A|_W)$$

which implies that

$$\langle S(t)x, w^* \rangle = \langle x, w^* \rangle, \ \forall x \in W.$$

Define a linear form on X by

$$\langle x, x^* \rangle := \langle Cx, w^* \rangle.$$

Since $|\langle x, x^* \rangle| = |\langle Cx, w^* \rangle| \leq \|Cx\|_{Z_0} \|w^*\| \leq M\|x\| \, \|w^*\|$, $x^* \in X^*$. Moreover, we have

$$\langle T(t)x, x^* \rangle = \langle CT(t)x, w^* \rangle = \langle S(t)C^2 x, w^* \rangle = \langle C^2 x, w^* \rangle = \langle Cx, x^* \rangle, \ \forall x \in X$$

which implies that

$$\langle CAx, x^* \rangle = 0, \ \forall x \in D(A). \tag{5.3}$$

On the other hand, since $w^* \neq 0$ and E, which is spanned by F, is dense in W, thus we can find $x_0 \in X$ and $t_0 \geq 0$ such that $\langle T(t_0)x_0, w^* \rangle \neq 0$, so

$$\langle x_0, x^* \rangle = \langle Cx_0, w^* \rangle = \langle S(t_0)Cx_0, w^* \rangle = \langle T(t_0)x_0, w^* \rangle \neq 0;$$

this means that $x^* \neq 0$. Hence $C^* x^* \neq 0$ by the denseness of $R(C)$; otherwise, $\langle x, C^* x^* \rangle = \langle Cx, x^* \rangle = 0$ for all $x \in X$, this is a contrary. But (5.3) says that $C^* x^* \in D(A^*)$ and $A^*(C^* x^*) = 0$, so we have $0 \in \sigma_p(A^*) = \sigma_r(A)$. $\square$

From Theorem 5.2.2.(2) and [AB, Proposition 2.1], we know that the condition (ii) in Theorem 5.2.4 is necessary; also, the condition (ii) in Theorem 5.2.7 is necessary. We can see it from the following Proposition. This result and the two lemmas after it, are generalizations of Proposition 2.1, Lemma 2.3 and Proposition 2.2 in [AB], respectively, the proofs are just a little modifications. We write them out for completeness.

Proposition 5.2.8 *Suppose $(T(t))_{t\geq 0}$ is a stable C-regularized semigroup generated by A and $R(C)$ is dense in X, then $\sigma_r(A)\cap i\mathbf{R}=\emptyset$.*

Proof. By the Hahn-Banach theorem, $\sigma_r(A)=\sigma_p(A^*)$. So if $is\in\sigma_r(A)\cap i\mathbf{R}\neq\emptyset$, then there is some $x^*\in X^*$, $x^*\neq 0$, such that $A^*x^*=isx^*$, that is, $\langle Ax,x^*\rangle=\langle -isx,x^*\rangle$, $\forall x\in D(A)$. Hence

$$\frac{d}{dt}\langle T(t)x,x^*\rangle=\langle T(t)Ax,x^*\rangle=-is\langle T(t)x,x^*\rangle,\ \forall x\in D(A),$$

since $D(A)$ is dense in X, we have

$$\langle T(t)x,x^*\rangle=e^{-ist}\langle Cx,x^*\rangle\ \forall x\in X.$$

But $R(C)$ is dense in X, so there is an x in X such that $\langle Cx,x^*\rangle\neq 0$, this is contrary to our assumption that $(T(t))_{t\geq 0}$ is stable. □

Lemma 5.2.9 *Suppose $(T(t))_{t\geq 0}$ is bounded and $\overline{R(C)}=X$, then $\sigma_p(A)\cap i\mathbf{R}\subset\sigma_r(A)$.*

Proof. Let $\eta\in\sigma_p(A)\cap i\mathbf{R}$. We can assume that $\eta=0$. Then we can choose $u\neq 0$ such that $T(t)u=Cu$, $t\geq 0$; next, pick $u^*\in X^*$ such that $\langle Cu,u^*\rangle=1$. Let ϕ be a translation-invariant linear form on $L^\infty([0,\infty))$ with $\phi(\mathbf{1})=1$, where $\mathbf{1}$ is the function with constant value 1. Define $x^*\in X^*$ by

$$\langle x,x^*\rangle=\phi\langle T(\cdot)x,u^*\rangle,\ \forall x\in X,$$

then $\langle u,x^*\rangle=\phi\langle T(\cdot)u,u^*\rangle=1$, while

$$\langle T(t)x,x^*\rangle=\phi\langle T(\cdot+t)Cx,u^*\rangle=\phi\langle T(\cdot)Cx,u^*\rangle=\langle Cx,x^*\rangle,\ \forall x\in X.$$

Similar to the proof of Theorem 5.2.7 (Step III), we can show that $0\in\sigma_r(A)$, this completes the proof. □

Lemma 5.2.10 *Suppose X is reflexive, $(T(t))_{t\geq 0}$ is a bounded C-regularized semigroup and $\overline{R(C)}=X$. Then $\sigma_r(A)\cap i\mathbf{R}=\sigma_p(A)\cap i\mathbf{R}$.*

Proof. Since X is reflexive, it is not hard to show that $(T(t)^*)_{t\geq 0}$ is a bounded C^*-regularized semigroup on X^* with generator A^*. By Lemma 5.2.9, $\sigma_p(A)\cap i\mathbf{R}\subset\sigma_r(A)$; on the other hand, $\sigma_r(A)\cap i\mathbf{R}=\sigma_p(A^*)\cap i\mathbf{R}\subset\sigma_r(A^*)=\sigma_p(A^{**})=\sigma_p(A)$.
□

Corollary 5.2.11 *Suppose X is a reflexive Banach space, $(T(t))_{t\geq 0}$ is a bounded C-regularized semigroup and $\overline{R(C)} = X$. If*

(1) *there is a dense subspace Y of X such that for every $x \in Y$, $\sigma_u(A, Cx)$ is at most countable and*

(2) $\sigma_p(A) \cap i\mathbf{R} = \emptyset$

then $(T(t))_{t\geq 0}$ is stable.

The following proposition establishes the relations between the spectrum of A and its restriction on Z_0, it may be of some independent interest.

Proposition 5.2.12 *Suppose A generates a bounded C-regularized semigroup $(T(t))_{t\geq 0}$. Denote the restriction of A on its Hille-Yosida space Z_0 by A_0. Then*

(1) $\sigma(A_0) \subseteq \sigma(A)$;

(2) *if $R(C)$ is dense in X, then*

- **(a)** $\sigma_p(A_0) = \sigma_p(A)$;
- **(b)** $\sigma_c(A_0) \subseteq \sigma_c(A)$;
- **(c)** $\sigma_r(A_0) \supseteq \sigma_r(A)$.

Proof. (1) comes from [dV].
(2) (a) is obvious.
(b) Let $\lambda \in \sigma_c(A_0)$, thus the image of $\lambda - A_0$ is dense in Z_0. So for every $y \in Z_0$, there is a sequence $(x_n) \subset Z_0$ such that $(\lambda - A_0)x_n \to y$ in Z_0. Since $Z_0 \hookrightarrow X$, we also have $(\lambda - A)x_n = (\lambda - A_0)x_n \to y$ in X. Moreover, since $R(C) \subset Z_0$ and $R(C)$ is dense in X, the image of $(\lambda - A)$ is dense in X; therefore, $\lambda \notin \sigma_r(A)$.
(c) Let $\lambda \in \sigma_r(A)$, then there is $x^* \neq 0$ such that

$$\langle (\lambda - A)x, x^* \rangle = 0, \ \forall x \in D(A)$$

but the action of x^* on Z_0 is also a functional on Z_0, we denote it by $x^*_|$, thus

$$\langle (\lambda - A_0)x, x^*_| \rangle = 0, \forall x \in D(A_0).$$

Since $x^* \neq 0$ and $\overline{R(C)} = X$, $x^*_| \neq 0$. So $R(\lambda - A_0)$ is not dense in Z_0, this implies that $\lambda \in \sigma_r(A_0)$. $\square$

We end this chpater with an open question.

Open question: If $R(C)$ is dense in X and A generates a bounded C-regularized semigroup, is $R(C)$ dense in Z_0, the Hille-Yosida space for A?

If it is true, then from the proof of Theorem 5.2.7, we have $\sigma_r(A) \cap i\mathbf{R} = \sigma_r(A|_{Z_0}) \cap i\mathbf{R}$.

Bibliography

[Ar] W. Arendt, *Vector-valued Laplace transforms and Cauchy problems*, Israel J. Math. **59** (1987), 327-352.

[AB] W. Arendt and C.J.K. Batty, *Tauberian theorems and stability of one-parameter semigroups*, Trans. Amer. Math. Soc. **306** (1988) , 837-852.

[BG] H. Bart and S. Goldberg, *Characterizations of almost periodic strongly continuous groups and semigroups*, Math. Ann. **236** (1978), 105-116.

[Ba] C.J.K. Batty, *Spectral conditions for stability of one-parameter semigroups*, J. Differential Equations **127** (1996), 87-96.

[BNR1] C.J.K. Batty, J.M.A.M. van Neerven and F. Räbiger, *Local spectra and individual stability of uniformly bounded C_0-semigroup*, Trans. Amer. Math. Soc. **350** (1998), 2071-2085.

[BNR2] C.J.K. Batty, J.M.A.M. van Neerven and F. Räbiger, *Tauberian theorems and stability of solutions of the Cauchy problem*, Trans. Amer. Math. Soc. **350** (1998), 2087-2103.

[BM] O. Blasco and J. Martínez, *Norm continuity and related notions for semigroups on Banach spaces*, Arch. Math. **66** (1996), 470-478.

[Ci1] I. Cioranescu, *On the second order Cauchy problem associated with a linear operator*, J. Math. Anal. Appl. **154** (1991), 238-242.

[Ci2] I. Cioranescu, *Characterizations of almost periodic strongly continuous cosine operator functions*, J. Math. Anal. Appl. **116** (1986), 222-229.

[DP] E.B. Davies and M.M. Pang, *The Cauchy problem and a generalization of the Hille-Yosida theorem*, Proc. London Math. Soc. **55**(1987), 181-208.

[dL1] R.deLaubenfels, *Holomorphic C-existence families*, Tokyo. J. Math., **15** (1992), 17-38.

[dL2] R. deLaubenfels, *C-semigroups and the Cauchy problem*, J. Funct. Anal. **111** (1993), 44-61.

[dL3] R. deLaubenfels, *C-semigroups and strongly continuous semigroups*, Isreal J. Math. **81** (1993), 227-255.

[dL4] R. deLaubenfels, "Existence families, functional calculi and evolution equations", Lecture Notes in Math. **1570**, Springer Verlag, 1994.

[dSW] R. deLaubenfels, G.Z. Sun and S.W. Wang, *Regularized semigroups, existence families and the abstract Cauchy problem*, Diff. and Int. Equ. **8** (1995), 1477-1496.

[dV] R. deLaubenfels and Vũ Quôc Phóng, *Stability and almost periodicity of solutions of ill-posed abstract Cauchy problems*, Proc. Amer. Math. Soc. **125** (1997), 235-241.

[dVW] R. deLaubenfels, Vũ Quôc Phóng and Shengwang Wang, *Stability of semigroups of operators and spectral subspaces*, Semigroup Forum, to appear.

[DG] K. Deleeuw and I. Glicksberg, *Applications of almost periodic compactifications*, Acta Math. **105** (1961), 63-97.

[Du] N. Dunford, *Spectral theory I, Convergence to projection*, Tran. Amer. Soc., **54** (1943), 185-217.

[EE1] O. El-Mennaoui and K.-J. Engel, *On the characterization of eventually norm continuous semigroups in Hilbert spaces*, Arch. Math. **63** (1994), 437-440.

[EE2] O. El-Mennaoui and K.-J. Engel, *Towards a characterization of eventually norm continuous semigroups on Banach space,* Quaestioes Mathematicae **19**(1996), 183-190.

[Fa] H.O. Fattorini, "Second order linear differential equations in Banach spaces", Elsevier Science Publishers B.V., Amsterdam, 1985.

[Fi] A.M. Fink, "Almost periodic differential equations", Lecture Notes in Math. **377**, Springer-Verlag, Berlin, 1974.

[Go] S. Goldberg, "Unbounded linear operators theory and applications", McGraw-Hill, New York, 1966.

[Gol] J.A. Goldstein, "Semigroups of linear operators and applications", Oxford, New York, 1985.

[GLH] X.H. Gu, M. Li and F.L. Huang, *Almost periodicity of C-semigroups, integrated semigroups and C-cosine functions*, submitted.

[HP] E.Hille and R.S.Phillips, "Functional analysis and semigroups", Amer. Math. Soc. Coll. Publ. **31**, Providence R.I., 1957.

[Hu1] F.L. Huang, *Characteristic property of semigroups of isometric linear operators in Banach spaces*, J. Sichuan Univ. **4** (1983), 1-8. (in Chinese)

[Hu2] F.L. Huang, *Asymptotic stability theory for linear dynamical systems in Banach spaces*, Kexue Tongbao **10** (1983), 583-586. (in Chinese)

[Hu3] F.L. Huang, *Strong asymptotic stability of linear dynamical systems in Banach spaces*, J. Differential Equations **104** (1993), 307-324.

[Hu4] F.L. Huang, *Spectral properties and stability of one-parameter semigroups*, J. Differential Equations **104** (1993), 182-195.

[Hu5] F.L. Huang, *Characteristic conditions for exponential stability of linear dynamical systems in Hilbert spaces*, Ann. Diff. Equ. **1** (1985), 43-55.

[Ka] S. Kantorovitz, *The Hille-Yosida space of an arbitrary operator*, J. Math. Anal. Appl. **136** (1988), 107-111.

[LH1] M. Li and F.L. Huang, *An extrapolating space for a bounded C-semigroup*, Appl. Funct. Anal. **3** (1997), 77-81.

[LH2] M. Li and F.L. Huang, *Characterizations of contraction C-semigroups*, Proc. Amer. Math. Soc. **126** (1998), 1063-1069.

[LH3] M. Li and F.L. Huang, *Norm continuity of regularized semigroups*, in preparation.

[LHC] M. Li, F.L. Huang and X.L. Chu, *Ergodic theory for C-semigroups,* J. Sichuan Univ. **36** (1999), 645-651.

[Li] M. Li, *Exponential growth and stability of C-regularized semigroups*, submitted.

[LS] Y.C. Li and S.Y. Shaw, *N-times integrated C-semigroups and the abstract Cauchy problem*, Taiwaneses J. of Math. **1** (1997), 75-102.

[Lin] M. Lin, *On the uniform ergodic theorem* II, Proc. Amer. Math. Soc., **46**(1974), 217-225.

[LV] Y.I. Lyubich and Vũ Quôc Phóng, *Asymptotic stability of linear differential equations on Banach spaces*, Studia Math. **88** (1988), 37-42.

[Mi] P. Milnes, *On vector-valued weakly almost periodic functions*, J. London Math. Soc. (2) **22** (1980), 467-472.

[Na] R. Nagel, "One-parameter semigroups of positive operators," (ed.) Lect. Notes in Math. **1184**, Springer-Verlag, 1986.

[Ne] J.M.A.M. van Neerven, *Individual stability of C_0-semigroups with uniformly bounded local resolvent*, Semigroup Forum **53** (1996), 155-161.

[Pa] A. Pazy, "Semigroups of linear operators and applications to partial differential equations", Springer-Verlag, New York, 1983.

[PA] J.G. Peng, L. B. Ai, *A characterization equivalent to the Hille-Yosida conditions,* J. Math. Res. Exp. **18** (1998), 135-140. (in Chinese)

[Pi1] S.I. Piskarev, *Periodic and almost periodic cosine functions*, Math. USSR-Sb. **46** (1982), 391-402.

[Pi2] S.I. Piskarev, *Almost periodic solutions of second-order differential equations*, Siberian Math. J. **25** (1984), 451-460.

[RS1] W.M. Ruess and W.H. Summers, *Asymptotic almost periodicity and motions of semigroups of operators*, Linear Algebra Appl. **84** (1986), 335-351.

[RS2] W.M. Ruess and W.H. Summers, *Minimal sets of almost periodic motions,* Math. Ann. **276** (1986), 145-158.

[RS3] W.M. Ruess and W.H. Summers, *Compactness in spaces of vector valued continuous functions and asymptotic almost periodicity*, Math. Nachr. **135** (1988), 7-33.

[RS4] W.M. Ruess and W.H. Summers, *Weakly almost periodic semigroups of operators*, Pacific J. Math. **143** (1990), 175-193.

[RS5] W.M. Ruess and W.H. Summers, *Ergodic theorems for semigroup of operators*, Proc. Amer. Math. Soc. **114** (1992), 423-432.

[RV] W.M. Ruess and Vũ Quôc Phóng, *Asymptotically almost periodic solutions of evolution equations in Banach spaces*, J. Differential Equations **122** (1995), 282-301.

[Sh1] S.Y. Shaw, *Uniform convergence of ergodic limits and approximate solutions*, Proc. Amer. Math. Soc. **114:2**(1992), 405-411.

[Sh2] S.Y. Shaw, *On Cesáro and Abel limits of C-semigroups*, Soochow J. Math. **20:4**(1994), 547-553.

[Ta] N. Tanaka, *On the exponentially bounded C-semigroups*, Tokyo J. Math. **10** (1987), 107-117.

[TM1] N. Tanaka and I. Miyadera, *Exponentially bounded C-semigroups and integrated semigroups*, Tokyo J. Math. **12**(1989), 99-115.

[TM2] N. Tanaka and I. Miyadera, *Exponentially bounded C-semigroups and generation of semigroups*, J. Math. Anal. Appl. **143** (1989), 358-378.

[TL] A.E. Taylor and D.C. Lay, "Introduction to functional analysis", 2nd ed., Wiley, New York, 1980.

[Vu] Vũ Quôc Phóng, *Asymptotic almost periodicity of solutions of differential equations on Banach spaces*, Semesterbericht Funktionalanalysis Tübingen, **17** (1989/90), 213-226.

[XL] T.J. Xiao and J. Liang, "The Cauchy problem for Higher-order abstract differential equations," Lect. Notes in Math. **1701**, Springer-Verlag, 1998.

[XLH] L.H. Xie, M. Li and F. L. Huang, *Asymptotic almost periodicity of C-semigroups*, submitted.

[Ya] P.F. Yao, *On the inversion of the Laplace transform of C_0-semigroups and its applications*, SIAM J. Math. Anal. **26** (1995), 1331-1341.

[Yo] P.H. You, *Characteristic conditions for a C_0-semigroup with continuity in the uniform topology for $t > 0$ in Hilbert space*, Proc. Amer. Math. Soc. **116** (1992), 991-997.

[Zha] J.Z. Zhang, Regularized operator families and applications to differential operators, Dissertation, Huazhong University of Science and Technology, 1996.

[Zhe] Q. Zheng, "Strongly Continuous Semigroups of Linear Operators", Huazhong University of Science and Technology, Wuhan, 1994. (in Chinese)

[ZL] Q. Zheng and L.P. Liu, *Almost periodic regularized groups, semigroups, and cosine functions*, J. Math. Anal. Appl. **197** (1996), 90-112.

Vita

Name: Miao Li

Date of Birth: April 30, 1973

Sex: Female

Academic Degrees:

- Bachelor's Degree of Science, Sichuan University, People's Republic of China, 1993

  Speciality: Mathematics

- Master's Degree of Science, Sichuan University, People's Republic of China, 1996

  Speciality: Applied Mathematics

Experience:

- Assistant Professor, Sichuan University, People's Republic of China, 1996-1998
- Lecturer, Sichuan University, People's Republic of China, 1998-Present

Tutor: Prefessor Fa-Lun Huang