A Stieltjes type convolution for integrated semigroups of strong bounded variation and L_p-solutions to the abstract Cauchy Problem

Horst R. Thieme *
Department of Mathematics
Arizona State University
Tempe, AZ 85287-1804, USA

Hauke Voßeler †
Mathematisches Seminar
Universität Kiel
D-24098 Kiel, Germany

July 14, 2000

Abstract. A convolution of Stieltjes type is introduced for operator families of strong bounded variation and vector valued L_1-functions. Using this tool, perturbation theorems for integrated semigroups of strong bounded variation are derived, and improved results on L_p-solutions to the inhomogeneous Cauchy problem are obtained.

Key words: Integrated semigroups, perturbation theory, functions of bounded p-variation, resolvent positive operators
AMS subject classifications : 47D06, 34G10, 45N05, 46G12, 47A55

Introduction

Integrated semigroups have turned out to be a useful tool to study the inhomogeneous Cauchy problem

$$u'(t) = Au(t) + f(t), \qquad u(0) = x \in X, \tag{1}$$

where A is a closed linear operator (possible not densely defined), X a Banach space and $f : [0, T] \to X$ a vector valued inhomogeneity (see [Are87b], [KH89] and [Hie91]). An integrated semigroup is a strongly continuous operator family $(S(s))_{s\geq 0}$ (S for short) with $S(0) = 0$ and

$$S(t)S(s) = \int_t^{t+s} S(r)dr - \int_0^s S(r)dr, \qquad t, s \geq 0.$$

*Research partially supported by NSF grant DMS-9706787
†Supported by a scholarship of "Studienstiftung des deutschen Volkes"

Since we consider a class of integrated semigroups which may not be exponentially bounded, the generator is to be understood in the following sense, namely that for $x, y \in X$

$$x \in D(A),\ y = Ax \quad \Longleftrightarrow \quad \forall t \geq 0 \quad S(t)x = tx + \int_0^t S(s)y\,ds.$$

If $(S(t))_{t\geq 0}$ is an integrated semigroup, the solution of (1) is formally given by the variation of parameter formula

$$u(t) - x = \frac{d}{dt}S(t)x + \int_0^t \frac{d}{ds}S(s)f(t-s)ds. \tag{2}$$

Since $S(\cdot)x$ is not differentiable for all $x \in X$ (even not in Sobolev's sense), we introduce a convolution

$$(S \star f)(t) := \lim_{h \searrow 0} \frac{1}{h} \int_0^t (S(u+h) - S(u))f(t-u)du$$

for $f \in L_1(\mathbf{R}_+, X)$ and an operator family $(S(t))_{t\geq 0}$ that is of strong bounded p-variation on $\mathbf{R}_+$, $1 \leq p < \infty$. If S is of bounded variation in the operator norm and f is continuous, then

$$(S \star f)(t) = \int_0^t S(du)f(t-u),$$

which is why we call this convolution to be of Stieltjes type. It turns out that under the above assumptions one has

$$S \star f \in L_p(\mathbf{R}_+, X)$$

and so we have found a class of integrated semigroups that is appropriate to study W_p^1-solutions of (1) resp. L_p-solutions of its integral version

$$u(t) - x = A\int_0^t u(s)ds + \int_0^t f(s)ds. \tag{3}$$

Here is a preview of one of our results (Thm. 5.5 (2)(a)):

If A is the generator of an integrated semigroup of strong bounded p-variation, $1 \leq p < \infty$, and $x \in D(A)$, $f \in W_1^1(I, X)$ with $Ax + f(0) \in \overline{D(A)}$, $I = [0, a)$, then there exists a solution $u \in W_p^1(I, X)$ of (1) in the sense that $u(t) \in D(A)$ and the differential equation is satisfied for almost all $t \in I$.

Recall that the case $p = \infty$ is covered by the well established theory of locally Lipschitz continuous integrated semigroups (see [dPS87], [KH89]) in which case u is continuously differentiable and (1) holds everywhere.

For A densely defined our approach is closely connected to the theory of L_p-flows introduced by Weis in [Wei91] and some of our results were announced in [Wei91]. For an operator A that generates an L_p-flow actually generates an integrated semigroup of strong bounded p-variation. We also remind of the result by Arendt [Are87a] that a resolvent positive operator with negative spectral bound on an abstract L-space generates an increasing integrated semigroup which automatically is of strong
bounded variation.

The Widder inversion theorem for vector-valued Laplace transforms in $L_p(X)$ spaces [Wei93] allows us to characterize operators A that generate integrated semigroups of strong bounded p-variation. Resolvent positive and resolvent majorized operators in abstract L spaces are generators of integrated semigroup of strong bounded variation as well.

We will give several perturbation results which all rely on the fact that $S \star f$ is of bounded p-variation if S is of bounded p-variation and f is of bounded variation. If the underlying Banach space is an abstract L space, we have the following result which is based on the vector-valued version of Bernstein's inversion theorem for completely monotonic functions [Are87a]:

If A is a resolvent positive operator on an abstract L space X and $B_+, B_- : D(A) \to X$ are positive and give rise to a resolvent positive perturbation $A + B_+ + B_-$, then the perturbation $A + B_+ - B_- - w$ generates an integrated semigroup of strong bounded variation on $[0, \infty)$ for some $w \in \mathbf{R}$.

The Widder inversion theorem [Wei93] will be used to derive a perturbation result for general Banach spaces which has as a special case that integrated semigroups of strong bounded p-variation are preserved under Miyadera-Voigt perturbations [Voi77] of densely defined generators A.

In forthcoming work we will use the Stieltjes type convolution to analyze semilinear perturbations $A + G$ of Hille-Yosida operators A which contain as a special case $G = F \circ B$ with F being a nonlinear Lipschitz continuous operator and B a linear operator of Miyadera-Voigt type.

Our paper is organized as follows. Section 1 deals with preliminaries on functions of essential bounded p-variation. In section 2 we introduce the Stieltjes type convolution $\star$ and a modification $\diamond$ which is associative and allows to treat jumps at zero. Section 3 is devoted to the study of exponentially weighed functions of bounded pseudo-p-variation which will be useful in perturbation theory. In section 4 we give a generation theorem for integrated semigroups that are of strong bounded p-variation and deal with examples. In section 5, we give a discussion of L_p-solutions of the inhomogeneous Cauchy problem. Finally, in section 6 we will deal with perturbation theory.

1 Functions of essential bounded p-variation

Let $I \subseteq \mathbf{R}$ be an interval which is bounded from below and let

$$l(I) := \inf I \qquad \text{and} \qquad r(I) := \sup(I)$$

denote its left and right end-points. Unless explicitly stated otherwise, we always assume that $l(I) \in I$ and $r(I) \notin I$. For a locally integrable $f : I \to X$ we have:

$$\lim_{h \to 0} \frac{1}{h} \int_t^{t+h} f(s)ds = f(t) \qquad \text{for almost all } t \in I. \tag{4}$$

Let $K(f)$ denote the set of those $t \in I$ for which (4) holds. We say $Z = (t_0, \ldots, t_k) \in I^{k+1}$ is an *f-essential subdivision of I*, if $l(I) \leq t_0 < t_1 < \ldots < t_k < r(I)$ and $t_0, \ldots, t_k \in K(f)$. We call $Z = (t_0, \ldots, t_k) \in I^{k+1}$ an *subdivision of I* if $l(I) = t_0 < t_1 < \ldots < t_k < r(I)$. Furthermore we define

$$\mathcal{Z}(f, I) := \{Z \mid Z \text{ is an } f\text{-essential subdivision of } I\}. \tag{5}$$

1.1 Definition. Let $1 \leq p < \infty$. A locally integrable function $f : I \to X$ is of *essential bounded p-variation* if

$$\mathbf{ev}_p(f; I) := \sup_{Z \in \mathcal{Z}(f;I)} \left(\sum_{j=0}^{k-1} \frac{\|f(t_{j+1}) - f(t_j)\|^p}{|t_{j+1} - t_j|^{p-1}} \right)^{\frac{1}{p}} < \infty.$$

If we replace $\mathcal{Z}(f;I)$ by the set of all subdivisions of I, we replace $\mathbf{ev}_p$ by $\mathbf{v}_p$ and say that f is of *bounded p-variation.* Notice that $v_p(f,I) = ev_p(f,I) + \|f(0+) - f(0)\|$ for functions of bounded p-variation.

In the following proposition we collect some known facts about functions of bounded p-variation which we will use in the following sections. For the general theory and proofs we refer to [Vie95] (for more literature on this topic see [DU77], [BT38] and the references given there).
Let $\phi : I \to X$ be a function that is of bounded variation on every bounded subinterval of I. Define

$$\mathrm{Var}_\phi : I \to \mathbf{R},\ t \mapsto \begin{cases} \left\|1_{[l(I),t]}\phi\right\|_{v_1} & t > l(I), \\ \\ 0 & t = l(I), \end{cases} \tag{6}$$

where $\|\cdot\|_{v_1} = \mathbf{v}_1(\cdot, I)$. Furthermore we call $f : I \to X$ normalized if $f(l(I)) = 0$ and if f is continuous from the right in the interior of I, and we define

$$V_p(I,X) := \{f : I \to X \mid \text{f is normalized and of bounded } p\text{-variation}\}.$$

1.2 Proposition. *1. Let $1 \le p \le \infty$. Then the vector space $V_p(I,X)$ equipped with $v_p(\cdot, I)$ is a Banach space.*

2. *If there is a locally integrable function $g : I \to X$ such that $\int_{l(I)}^{\cdot} g(s)ds = \Phi$ then $\Phi \in V_1(I,X)$ and*

$$\mathrm{Var}_\phi(t) = \int_{l(I)}^{t} \|g(u)\|\, du \quad \text{for all } t \in I.$$

3. *Let $\Phi \in V_p(I,X)$ for some $1 < p < \infty$. Then Φ is absolutely continuous in I and $\Phi \in V_1(K,X)$ for every compact interval of $K \subseteq I$. Furthermore we have $\|\mathrm{Var}_\Phi\|_p = \|\Phi\|_p$.*

4. *If $\Phi \in V_1(I,X)$ then $\|\mathrm{Var}_\Phi\|_{v_1} = \|\Phi\|_{v_1}$ and for all $f \in C_0(I)$ we have*

$$\left\| \int_I f(t) d\Phi(t) \right\| \le \int_I |f(t)|\, d\,\mathrm{Var}_\Phi(t).$$

5. *Let $1 < p \le \infty$ and $\Phi \in V_p(I,X)$. Let $f \in L_{p'}(I)$ for $\frac{1}{p} + \frac{1}{p'} = 1$. Then the Stieltjes integral $\int_I f(t)d\Phi(t)$ exists and we have*

$$\left\| \int_I f(t) d\Phi(t) \right\| \le \|f\|_{p'} \|\Phi\|_{v_p}.$$

We next show how the case $p = \infty$ relates to the definition of essential bounded p-variation.

1.3 Proposition. *Let $1 \le p_0 < \infty$ and $f : I \to X$ be of essential bounded p-variation for every $p > p_0$ and $\sup_{p>p_0} \mathbf{ev}_p(f;I) < \infty$. Then f is essentially Lipschitz continuous on I, i.e.*

$$\mathbf{ev}_\infty(f;I) := \sup \left\{ \frac{\|f(t) - f(s)\|}{|t-s|} \mid t,s \in K(f),\ t \ne s \right\} < \infty.$$

Conversely, if I is bounded and f essentially Lipschitz continuous on I, then f is of essential bounded p-variation for all $1 \le p < \infty$ with

$$\mathbf{ev}_p(f;I) \le \mathbf{ev}_\infty(f;I)\ \ |I|^{\frac{1}{p}}. \tag{7}$$

Proof. Let $s, t \in K(f)$ and $t \neq s$. Consider $p > p_0$ and $\frac{1}{q} := 1 - \frac{1}{p}$. Set $r := \frac{p-1}{p}$. Since $pr = p - 1$ and $qr = 1$ we obtain

$$\begin{aligned}\|f(t) - f(s)\| = \frac{\|f(t) - f(s)\| \, |t-s|^r}{|t-s|^r} &\leq \left(\frac{\|f(t) - f(s)\|^p}{|t-s|^{pr}} \right)^{\frac{1}{p}} \left(|t-s|^{rq} \right)^{\frac{1}{q}} \\ &\leq \mathbf{ev}_p(f; I) \, |t-s|^{\frac{1}{q}} \leq M \, |t-s|^{\frac{1}{q}} \underset{q \to 1}{\longrightarrow} M \, |t-s| \, .\end{aligned}$$

The proof of the second statement is obvious and so we are done. □

It is an easy consequence of Hölder's inequality that for every $f \in L_p(I, X)$ the function $\int_0^{\cdot} f(s) ds$ is of bounded p-variation. Furthermore, if $1 < p < \infty$, every scalar valued function of bounded p-variation has an L_p-derivative and we have

$$\|f\|_{v_p} = \|f'\|_p = \left(\int_{l(I)}^{r(I)} \|f'(s)\|^p \, ds \right)^{\frac{1}{p}} .$$

In order to obtain an integral representation of essential p-variation, we introduce the notation of pseudo-p-variation for locally integrable functions. This concept which will turn out to be equivalent will better fit the Stieltjes type convolution we are going to introduce in the next section.

1.4 Definition. Let $1 \leq p < \infty$. A locally integrable function $f : I \to X$ is of *bounded pseudo-p-variation*, if

$$\mathbf{pv}_p(f; I) := \overline{\lim_{h \searrow 0}} \, \frac{1}{h} \left(\int_{l(I)}^{r(I)-h} \|f(s+h) - f(s)\|^p \, ds \right)^{\frac{1}{p}} < \infty.$$

1.5 Remark. Let $f : I \to X$ be of bounded pseudo-p-variation for $1 \leq p < \infty$. Using Hölder's inequality we see that for a finite interval I and $1 \leq q \leq p$

$$\mathbf{pv}_q(f; I) \leq (r(I) - l(I))^{\frac{p-q}{pq}} \cdot \mathbf{pv}_p(f; I)$$

and

$$\overline{\lim_{h \searrow 0}} \, \frac{1}{h} \int_{l(I)}^{\infty} e^{-\varepsilon s} \, \|f(s+h) - f(s)\| \, ds \leq \left(\frac{e^{\varepsilon p' l(I)}}{\varepsilon p'} \right)^{\frac{1}{p'}} \mathbf{pv}_p(f; I)$$

for all $\varepsilon > 0$ if I is unbounded and $\frac{1}{p} + \frac{1}{p'} = 1$.

We obtain

1.6 Proposition. *Let $f : I \to X$ be locally integrable. Then f is of essential bounded p-variation if and only if f is of bounded pseudo-p-variation. Furthermore we have*

$$\mathbf{ev}_p(f; I) = \mathbf{pv}_p(f; I).$$

Proof. For simplicity we choose $l(I) = 0$. To show '$\leq$' we start by dealing with the case $r(I) < \infty$. Let $h > 0$ and n be the largest number such that $nh < r(I) - h$.

Then

$$\begin{aligned}
&\int_0^{r(I)-h}\left(\frac{\|f(s+h)-f(s)\|}{h}\right)^p ds \\
= \;& \sum_{j=0}^{n-1}\frac{1}{h^p}\int_{jh}^{(j+1)h}\|f(s+h)-f(s)\|^p\,ds+\frac{1}{h^p}\int_{nh}^{r(I)-h}\|f(s+h)-f(s)\|^p\,ds \\
& \hfill (8) \\
= \;& \sum_{j=0}^{n-1}\frac{1}{h^p}\int_0^{h}\|f(s+jh+h)-f(s+jh)\|^p\,ds \\
& +\frac{1}{h^p}\int_0^{r(I)-(n+1)h}\|f(s+nh+h)-f(s+nh)\|^p\,ds \\
= \;& \sum_{j=0}^{n-1}\frac{1}{h^p}\int_{r(I)-(n+1)h}^{h}\|f(s+jh+h)-f(s+jh)\|^p\,ds \\
& +\sum_{j=0}^{n}\frac{1}{h^p}\int_0^{r(I)-(n+1)h}\|f(s+jh+h)-f(s+jh)\|^p\,ds \\
= \;& \int_{\frac{r(I)}{h}-(n+1)}^{1}\sum_{j=0}^{n-1}\frac{1}{h^{p-1}}\|f(sh+jh+h)-f(sh+jh)\|^p\,ds \\
& +\int_0^{\frac{r(I)}{h}-(n+1)}\sum_{j=0}^{n}\frac{1}{h^{p-1}}\|f(sh+jh+h)-f(sh+jh)\|^p\,ds \\
= \;& (+). \qquad (9)
\end{aligned}$$

Since $I\setminus K(f)$ has measure zero we obtain

$$(+)\le\int_{\frac{r(I)}{h}-(n+1)}^{1}\mathbf{ev}_p(f;I)^p ds+\int_0^{\frac{r(I)}{h}-(n+1)}\mathbf{ev}_p(f;I)^p ds=\mathbf{ev}_p(f;I)^p. \qquad (10)$$

This implies $\mathbf{pv}_p(f;I)\le\mathbf{ev}_p(f;I)$. If $r(I)=\infty$ then the term

$$\frac{1}{h^p}\int_{nh}^{r(I)-h}\|f(s+h)-f(s)\|^p\,ds$$

is missing in (8). In this case we obtain by Lebesgues dominated convergence theorem

$$\begin{aligned}
\frac{1}{h^p}\int_0^{\infty}\|f(s+h)-f(s)\|^p\,ds &= \int_0^1\sum_{j=0}^{\infty}\frac{\|f(sh+jh+h)-f(sh+jh)\|^p}{h^{p-1}}ds \\
&\le \int_0^1\mathbf{ev}_p(f;I)^p ds \;=\; \mathbf{ev}_p(f;I)^p.
\end{aligned}$$

We now deal with the other inequality. Let $\varepsilon>0$, $I_\varepsilon:=[0,r(I)-\varepsilon)$ and $Z=(t_0,\dots,t_k)\in\mathcal{Z}(f,I_\varepsilon)$. Then Hölder's inequality yields:

$$\int_{t_j}^{t_{j+1}}\|f(s+h)-f(s)\|\,ds\le(t_{j+1}-t_j)^{1-\frac{1}{p}}\left(\int_{t_j}^{t_{j+1}}\|f(s+h)-f(s)\|^p\,ds\right)^{\frac{1}{p}}.$$

Thus we have for $0 < h < \varepsilon$

$$\left(\frac{1}{h^p}\int_0^{r(I)-h} \|f(s+h)-f(s)\|^p\, ds\right)^{\frac{1}{p}} \geq \left(\sum_{j=0}^{n-1}\frac{1}{h^p}\int_{t_j}^{t_{j+1}} \|f(s+h)-f(s)\|^p\, ds\right)^{\frac{1}{p}}$$

$$\geq \left(\sum_{j=0}^{n-1}\frac{1}{(t_{j+1}-t_j)^{p-1}}\left(\int_{t_j}^{t_{j+1}}\frac{1}{h}\|f(s+h)-f(s)\|\, ds\right)^p\right)^{\frac{1}{p}}$$

$$\geq \left(\sum_{j=0}^{n-1}\frac{1}{(t_{j+1}-t_j)^{p-1}}\left\|\int_{t_j}^{t_{j+1}}\frac{1}{h}(f(s+h)-f(s))ds\right\|^p\right)^{\frac{1}{p}}$$

$$= \left(\sum_{j=0}^{n-1}\frac{1}{(t_{j+1}-t_j)^{p-1}}\left\|\frac{1}{h}\int_{t_{j+1}}^{t_{j+1}+h} f(s)ds - \frac{1}{h}\int_{t_j}^{t_j+h} f(s)ds\right\|^p\right)^{\frac{1}{p}}.$$

Recalling that $Z \in \mathcal{Z}(f, I_\varepsilon)$ and taking the limes superior,

$$\mathbf{pv}_p(f;I) \geq \left(\sum_{j=0}^{n-1}\frac{1}{(t_{j+1}-t_j)^{p-1}}\|f(t_{j+1})-f(t_j)\|^p\right)^{\frac{1}{p}}.$$

Therefore $\mathbf{pv}_p(f;I) \geq \mathbf{ev}_p(f;I_\varepsilon)$. Since $r(I) \notin I$ we have

$$\mathbf{ev}_p(f;I) = \sup_{\varepsilon>0}\mathbf{ev}_p(f;I_\varepsilon) \leq \mathbf{pv}_p(f;I).$$

□

1.7 Corollary. *Let $f : I \to X$ be of bounded pseudo-p-variation.*

(a) *Then f is essentially bounded on every finite subinterval of I and for any such finite subinterval $\tilde{I}$ of I we have*

$$\left\|f(t) - \frac{1}{|\tilde{I}|}\int_{\tilde{I}} f(r)dr\right\| \leq \mathbf{ev}_1(f;[l(I),t]\cup\tilde{I}) \quad \text{for a.a. } t \in I,$$

and in particular

$$\|f(t)\| \leq \mathbf{ev}_1(f;[l(I),t]\cup\tilde{I}) + \frac{1}{|\tilde{I}|}\int_{\tilde{I}}\|f(r)\|dr, \quad \text{for a.a. } t \in I.$$

(b) *Furthermore*

$$f^{\oplus}(t) := \lim_{h\searrow 0}\frac{1}{h}\int_t^{t+h} f(s)ds \tag{11}$$

exists for all $t \in I$ and $f^{\oplus}$ is of bounded p-variation on $I\setminus\{r(I)\}$, $\mathbf{v}_p(f^{\oplus}, I) = \mathbf{pv}_p(f, I)$. Further $(f^{\oplus})^{\oplus} = f^{\oplus}$ is right continuous,

$$\|f(t) - f^{\oplus}(0)\| \leq \mathbf{ev}_1(f;[l(I),t]) \text{ for a.a. } t \in I.$$

If $p > 1$, $f^{\oplus}$ is absolutely continuous on I and Hölder continuous of class $1-(1/p)$.

Proof. (a) f is of essential bounded variation on every finite subinterval of I. Let $\tilde{I}$ be a finite subinterval of I. We have for $t, r \in K(f)$, $r \in \tilde{I}$,

$$\|f(t) - f(r)\| \leq \mathbf{ev}_1(f; [l(I), t] \cup \tilde{I}).$$

Then, for all $t \in K(f)$,

$$\begin{aligned}\left\| f(t) - \frac{1}{|\tilde{I}|}\int_{\tilde{I}} f(r)dr \right\| = \left\| \frac{1}{|\tilde{I}|}\int_{\tilde{I}} (f(t) - f(r))dr \right\| &\leq \frac{1}{|\tilde{I}|}\int_{\tilde{I}} \|f(t) - f(r)\| dr \\ &\leq \mathbf{ev}_1(f, [l(I), t] \cup \tilde{I}).\end{aligned}$$

The second inequality follows from the first.
(b) We assume that there is a $t \in I \setminus \{r(I)\}$ such that $f^{\oplus}(t)$ does not exists. Then there is a sequence $(t_n)_{n\in\mathbf{N}}$ such that $t_n \in K(f)$ and $t_n \searrow t$ and $(f(t_n)_{n\in\mathbf{N}})$ is not a Cauchy sequence. After choosing a subsequence we have that $(t_{n_j})_{j\in\mathbf{N}}$ is decreasing and

$$\underline{\lim}_{j\to\infty} \left\| f(t_{n_j}) - f(t_{n_{j-1}}) \right\| > 0.$$

That is a contradiction to the fact that f is of essential bounded variation on any finite interval including t.
Since $f^{\oplus} = f$ on $K(f, I)$, $\mathbf{ev}_p(f^{\oplus}, I) = \mathbf{ev}_p(f, I)$. As $K(f^{\oplus}, I) = I$, $f^{\oplus}$ is of bounded p-variation and the p-variation coincides with the pseudo p-variation. Since $f^{\oplus}$ is of bounded p-variation, the right limit $f^{\oplus}(t+)$ exists for every $t \in I$ and $f^{\oplus}(t+) = (f^{\oplus})^{\oplus}(t) = f^{\oplus}(t)$. In (a) take $\tilde{I} = [l(I), l(I) + \varepsilon]$. Since $f = f^{\oplus}$ a.e., we can replace f by $f^{\oplus}$ in the integral and the estimate follows by taking the limit $\varepsilon \to 0$.
The absolute continuity of $f^{\oplus}$ follows from Hölder's inequality and the Hölder continuity directly from the definition of p-variation. □

Since every function of essential bounded p-variation (or equivalently of bounded pseudo-p-variation) has a representative that is of bounded p-variation, the proof that the functions of essential bounded p-variation form a Banach space reduces to the standard proof for functions of bounded p-variation.

1.8 Proposition. *If $1 \leq p < \infty$, the vector space*

$$\mathcal{PV}_p(I; X) := \{f^{\oplus} | f : I \to X \textit{ is of bounded pseudo } p\textit{-variation}\}$$

is a Banach space under the norm

$$\mathbf{pv}_p^{\oplus}(f; I) = \mathbf{pv}_p(f; I) + \|f^{\oplus}(0)\|.$$

If $p = 1$ or I is a finite interval, $\mathcal{PV}_p(I; X) \subseteq L_\infty(I, X)$ and $\|f\|_\infty \leq \mathbf{pv}_1^{\oplus}(f; I)$.

To illustrate the usefulness of this space we remark that the subspace of $\mathcal{PV}_q(I; X^*)$ with $g^{\oplus}(0) = 0$ can be identified with the dual space of $L_p(I; X)$, $1 < p < \infty$, $(1/p) + (1/q) = 1$, via

$$\langle f, g \rangle := \lim_{h\downarrow 0} \frac{1}{h} \int_{l(I)}^{r(I)-h} \langle f(s), g(s + h) - g(s) \rangle ds, \quad f \in L_p(I; X), g \in \mathcal{PV}_q(I; X^*).$$

In the next section we want to define a convolution between a family of bounded operators and vector valued L_1-functions. The following class of operator families turns out to be appropriate for this purpose.

1.9 Definitions. Let Z be a closed subspace of the Banach space X. A family of bounded linear operators $(U(t))_{t\in I}$ is of *strong bounded pseudo-p-variation on Z* if for all $x \in Z$

$$\mathbf{pv}_p(U(\cdot)x; I) < \infty.$$

It is easy to check that $x \mapsto U(\cdot)x$ is a closed linear operator from Z into the Banach space $\mathcal{PV}_p(I, X)$ and so a bounded linear operator; this implies that

$$\mathbf{PV}_p(U; I; Z) := \sup\left\{\mathbf{pv}_p(U(\cdot)x; I) \mid x \in Z,\ \|x\| \le 1\right\} < \infty$$

and for all $x \in Z$

$$\mathbf{pv}_p(U(\cdot)x; I) \le \mathbf{PV}_p(U; I; Z)\,\|x\|\,. \tag{12}$$

Furthermore

$$\mathbf{PV}^{\oplus}(U; I, Z) := \sup\left\{\mathbf{pv}_1^{\oplus}(U(\cdot)x; I) \mid x \in Z,\ \|x\| \le 1\right\}$$

is finite and

$$\mathbf{pv}_1^{\oplus}(U(\cdot)x; I) \le \mathbf{PV}^{\oplus}(U; I, Z)\,\|x\| \qquad \forall x \in Z.$$

If $p = 1$ or I is a bounded interval, it follows from Corollary 1.7 (b), that

$$\sup_{\|x\|\le 1} \|U(\cdot)x\|_\infty \le \mathbf{PV}^{\oplus}(U; I, X).$$

If I is bounded, $U(\cdot)x$ is essentially bounded by Corollary 1.7 for every $x \in X$. It follows from the closed graph theorem that $U : X \to L_\infty(I, X),\ x \mapsto U(\cdot)x$ is bounded and

$$\sup_{\|x\|\le 1} \|U(\cdot)x\|_\infty < \infty.$$

Again by Corollary 1.7, the operators

$$U^{\oplus}(t)x = \lim_{h\searrow 0} \frac{1}{h}\int_t^{t+h} U(s)x ds$$

define linear operators which, by the previous remark, are bounded. The last statement also holds, of course, if I is unbounded. One easily checks that, for bounded I or $p = 1$,

$$\sup_{t\in I} \|U^{\oplus}(t)\| = \sup_{\|x\|\le 1} \|U(\cdot)x\|_\infty.$$

In turn, let W be a linear bounded operator from X to $\mathcal{PV}_p(I; X)$. Then

$$W^{\oplus}(t)x = (Wx)^{\oplus}(t)$$

defines a family of linear bounded operators $W^{\oplus}(t)$ which is of strong bounded p-variation and strongly right-continuous.

2 A convolution operator of Stieltjes type

Let $(U(t))_{t\in I}$ be a family of bounded operators that is of strong bounded pseudo-p-variation on an interval I of the form $[0, r(I))$ and $\mathcal{P}_I$ be the set of trigonometric polynomials on I, if I is bounded, and $\mathcal{P}_I := \bigcup_{r\in[0,\infty)} \mathcal{P}_{[0,r)}$ otherwise (we neglect the embeddings $i_r : \mathcal{P}_{[0,r)} \to L_1([0, r), X),\ f \mapsto f 1_{[0,r)}$). $\mathcal{P}_I$ is a dense subset of $L_p(I, X)$, as one can see by extending the proof of the scalar case to the vector-valued situation. Let $f \in \mathcal{P}_I$. Then there are $n \in \mathbf{N}$ and $x_{-n}, \dots, x_n \in X$ such that

$$f = \sum_{k=-n}^{n} e^{i\frac{2\pi}{r}\ k(\cdot)} x_k\ 1_{[0,r)}(\cdot),$$

where $r = r(I)$ if I bounded and $r \in [0, \infty)$ otherwise.
We want to define a Stieltjes type convolution between U and $f \in L_1(I, X)$:

$$(U \star f)(t) := \lim_{h \searrow 0} \frac{1}{h} \int_0^t (U(s+h) - U(s)) f(t-s) \, ds \tag{13}$$

and relate it to the standard convolution

$$(U * f)(t) = \int_0^t U(s) f(t-s) \, ds. \tag{14}$$

For this purpose we consider the right hand sides of (13) and (14) for $f \in \mathcal{P}_I$ or rather for trigonometric monomials f, because $\star$ is linear, whenever it is defined. Obviously $U * f$ is defined for $f \in \mathcal{P}_I$.
If $f(t) = g(t)x$ is a trigonometric monomial, we have

$$\begin{aligned}
& \frac{1}{h} \int_0^t (U(s+h) - U(s)) f(t-s) ds \\
= \quad & \frac{1}{h} \int_t^{t+h} U(s) x g(t-s+h) ds - \frac{1}{h} \int_0^h U(s) x g(t+h-s) ds \\
+ \quad & \frac{1}{h} \int_0^t U(s) x \left(g(t-s+h) - g(t-s) \right) ds \\
=: \quad & (+)_h \\
= \quad & \frac{1}{h} \int_0^{t+h} U(s) f(t-s+h) ds \\
- \quad & \frac{1}{h} \int_0^t U(s) f(t-s) ds - \frac{1}{h} \int_0^h U(s) x g(t-s+h) ds \\
= \quad & \frac{1}{h}((U * f)(t+h) - (U * f)(t)) - g(t+h) \frac{1}{h} \int_0^h U(s) x g(-s) ds.
\end{aligned}$$

So the convergence of one of these expressions implies the convergence of the others. But

$$\begin{aligned}
(+)_h \quad = \quad & g(h) \frac{1}{h} \int_t^{t+h} U(s) x g(t-s) ds + \frac{1}{h}(g(t+h) - g(t)) \int_0^t U(s) x g(-s) ds \\
& -g(t+h) \frac{1}{h} \int_0^h U(s) x g(-s) ds \\
\to \quad & U^{\oplus}(t) x + g'(t) \int_0^t U(s) x g(-s) ds - g(t) U^{\oplus}(0) x \\
= \quad & U^{\oplus}(t) f(0) + \int_0^t U(s) f'(t-s) ds - U^{\oplus}(0) f(t).
\end{aligned}$$

We define for $f \in \mathcal{P}(I)$,

$$(U \star f)(t) := \begin{cases} \lim_{h \searrow 0} \frac{1}{h} \int_0^t (U(s+h) - U(s)) f(t-s) ds, & t \in [0, r), \\ \\ 0, & \textit{otherwise}, \end{cases} \tag{15}$$

$$(U * f)(t) := \int_0^t U(s) f(t-s) ds, \quad t \in [0, r),$$

and obtain

$$(U \star f)(t) = U^{\oplus}(t) f(0) + (U * f')(t) - U^{\oplus}(0) f(t) = (U * f)'(t) - U^{\oplus}(0) f(t). \tag{16}$$

2.1 Theorem. *For $1 \le p < \infty$ the mapping*

$$U\star : \mathcal{P}_I \to L_p(I,X)$$

has a unique extension in $L(L_1(I,X), L_p(I,X))$. This extension we denote also by $U\star$ and obtain

$$\|U\star\| \;\le\; \mathbf{PV}_p(U;I). \tag{17}$$

Furthermore we have for all $f \in L_1(I,X)$ the representation

$$U \star f = \|\cdot\|_1 - \lim_{h\searrow 0} \frac{1}{h}\int_0^{\cdot} (U(s+h) - U(s)) f(\cdot - s) ds. \tag{18}$$

Proof. We first consider a trigonometric polynomial $f \in \mathcal{P}_I$. By (9), (10) and Proposition 1.6 we have for all $0 < h < r(I)$ and $h < s < r(I)$

$$\begin{aligned}\left(\int_0^{r(I)-s} \left\|\frac{1}{h}(U(t+h) - U(t))f(s)\right\|^p dt\right)^{\frac{1}{p}} &\le \mathbf{ev}_p(U(\cdot)f(s);I)\\ &= \mathbf{pv}_p(U(\cdot)f(s);I)\\ &\le \mathbf{PV}_p(U;I)\left\|f(s)\right\|. \end{aligned}\tag{19}$$

We remark that if $r(I) = \infty$ this estimate is true for all $s > h > 0$. We rewrite $U \star f$ as

$$(U\star f)(t) = \lim_{h\searrow 0}\frac{1}{h}\int_0^t (U(t-s+h) - U(t-s))f(s)ds.$$

By the continuous Minkowski inequality we have together with (19)

$$\begin{aligned}\|U\star f\|_p &\le \overline{\lim_{h\searrow 0}}\left(\int_0^{r(I)}\left\|\int_0^t \frac{1}{h}(U(t-s+h) - U(t-s))f(s)ds\right\|^p dt\right)^{\frac{1}{p}}\\ &\le \overline{\lim_{h\searrow 0}}\int_0^{r(I)}\left(\int_s^{r(I)}\frac{1}{h^p}\left\|(U(t-s+h) - U(t-s))f(s)\right\|^p dt\right)^{\frac{1}{p}} ds\\ &\underset{(i)}{=} \int_0^{r(I)}\overline{\lim_{h\searrow 0}}\left(\int_0^{r(I)-s}\frac{1}{h^p}\left\|(U(t+h) - U(t))f(s)\right\|^p dt\right)^{\frac{1}{p}} ds\\ &\le \int_0^{r(I)}\mathbf{PV}_p(U;I)\left\|f(s)\right\| ds \;=\; \mathbf{PV}_p(U;I)\left\|f\right\|_1, \end{aligned}\tag{20}$$

where we applied Fatou's lemma in (i). If $r(I) = \infty$ the assumptions of Fatou's lemma are verified by (19). If $r(I) < \infty$ we obtain (with the convention $\|U(t)\| = 0$ for $t > r(I)$) from (19)

$$\begin{aligned}&\int_0^{r(I)}\left(\int_0^{r(I)-s}\frac{1}{h^p}\left\|(U(t+h) - U(t))f(s)\right\|^p dt\right)^{\frac{1}{p}} ds\\ &\qquad\le \frac{1}{h}\int_0^h \left(\|U(t)f(s)\|^p\right)^{\frac{1}{p}} ds + \mathbf{PV}_p(U;I)\int_0^{r(I)}\|f(s)\| ds\\ &\qquad\le r(I)^{\frac{1}{p}}\sup_{\|x\|\le 1}\|U(\cdot)x\|_\infty\frac{1}{h}\int_0^h\|f(s)\| ds + \mathbf{PV}_p(U;I)\int_0^{r(I)}\|f(s)\| ds\end{aligned}$$

and therefore we can apply Fatou's lemma in (i).
Since $U\star$ is linear and $\mathcal{P}_I$ dense in $L_p(I,X)$ we see from (20) that (17) holds.

We now show (18). By

$$(A_h f)(t) = \frac{1}{h}\int_0^t (U(s+h) - U(s))f(t-s)ds, \qquad 0 < h < 1,$$

a family of bounded operators from $L_1(I,X)$ to $L_p(I,X)$ is defined and together with (20) we see that

$$\overline{\lim_{h\searrow 0}}\, \frac{1}{h}\, \|A_h f\|_p \le \mathbf{PV}_1(U;I)\, \|f\|_p\,, \qquad f \in L_1(I,X).$$

Further $\lim_{h\searrow 0} A_h f$ exists for f in the dense subset $\mathcal{P}_I$ of $L_p(I,X)$; see the derivation leading to (9). As a consequence of the Banach-Steinhaus theorem, the limit $\lim_{h\searrow 0} A_h x$ exists for all $f \in L_1(I,X)$ and defines a bounded linear operator ([DS88], Theorem II.1.18). Since it coincides with $U\star$ on a dense subset, it also does so on the whole space, and formula (13) follows. □

The same arguments and the remarks at the end of Section 1 provide the following result for the standard convolution.

2.2 Theorem. *For $1 \le p < \infty$ and bounded I, the mapping*

$$U* : \mathcal{P}_I \to L_\infty(I,X)$$

has a unique extension in $L(L_1(I,X), L_\infty(I,X))$. This extension is also denoted by $U*$ and satisfies the estimate

$$\|U*\| \;\le\; \sup_{\|x\|\le 1} \|U(\cdot)x\|_\infty\,.$$

We will use the following notation. For $1 \le p \le \infty$

$$W_p^1(I;X) := \left\{ f : I \to X \;\middle|\; \exists f' \in L_p(I;X)\ \forall a,b \in I:\ f(b) - f(a) = \int_a^b f'(t)dt \right\}.$$

The Stieltjes type convolution is related to the derivative of the standard convolution as follows:

2.3 Proposition. *Let $(U(t))_{t\in I}$ be a family of bounded operators that is of strong bounded pseudo-p-variation for some $1 \le p < \infty$. Then*

(a) $U * f \in W_1^1(I,X) \cap L_\infty(I,X)$ for all $f \in L_1(I,X)$ and we have almost everywhere

$$(U \star f)(t) = (U * f)'(t) - U^{\oplus}(0)f(t).$$

(b) For all $f \in W_1^1(I,X)$ we have almost everywhere

$$(U \star f)(t) = (U * f')(t) + U^{\oplus}(t)f(0) - U^{\oplus}(0)f(t).$$

If I is an unbounded interval, these results also hold with the caveat that the various function spaces are replaced by their local counterparts.

Proof. We look at the equation (16). Since $U\star$ and $U*$ are linear, $U^{\oplus}(t)$ is bounded and $\mathcal{P}_I$ is dense in $L_p(I,X)$ and in $W_p^1(I,X)$ ($\mathcal{P}_I$ is a core for $\frac{d}{dt}$) we see by the standard approximation argument using Theorem 2.1 and Theorem 2.2 that (a) and (b) hold. □

Since the $\star$-convolution has the flaw that it is not multiplicative under Laplace transforms, we make the following definition (see Lemma 3.4).

2.4 Definition. If $(U(t))_{t\in I}$ is a family of bounded linear operators which is of strong bounded variation and $f \in L_1(I,X)$, we define

$$U \diamond f := (U * f)' = U \star f + U^{\oplus}(0)f.$$

2.5 Corollary. *For any $f \in L_1(I,Z)$,*

$$\|U \diamond f\|_1 \le \mathbf{PV}^{\oplus}(U;I,Z)\|f\|_1.$$

Proof. Extend U by 0 outside of I. By Theorem 2.1 and Proposition 2.3,

$$\begin{aligned}
&\|(U * f)'\|_1 \\
= \;& \int_0^{r(I)} \left\| \|\cdot\|_1\text{-}\lim_{h\downarrow 0} \frac{1}{h}\int_0^t (U(t-s+h) - U(t-s))f(s)ds + U^{\oplus}(0)f(t)\right\| dt.
\end{aligned}$$

Thus, by the triangle inequality and Fubini's theorem,

$$\begin{aligned}
&\|U \diamond f\|_1 \\
\le \;& \lim_{h\searrow 0} \int_0^{r(I)} \frac{1}{h}\int_0^t \|(U(t-s+h) - U(t-s))f(s)\|dsdt + \int_0^{r(I)} \|U^{\oplus}(0)f(s)\|ds \\
\le \;& \lim_{h\searrow 0} \int_0^{r(I)} \left(\frac{1}{h}\int_s^{r(I)} \|(U((t-s+h) - U(t-s))f(s)\|dt + \|U^{\oplus}(0)f(s)\|\right) ds \\
\le \;& \lim_{h\searrow 0} \int_0^{r(I)} \left(\frac{1}{h}\int_0^{r(I)-s} \|(U((r+h) - U(r))f(s)\|dr + \|U^{\oplus}(0)f(s)\|\right) ds.
\end{aligned}$$

Recalling Definition 1.4, Proposition 1.8 and Definition 1.9,

$$\|U \diamond f\|_1 \;\le\; \int_0^{r(I)} \mathbf{pv}_1^{\oplus}(U(\cdot)f(s);I)ds \;\le\; \mathbf{PV}^{\oplus}(U;I,Z)\int_0^{r(I)} \|f(s)\|ds.$$

□

The following result will be useful for perturbation theory.

2.6 Theorem. *Let $1 \le p < \infty$ and $(V(t))_{t\in I}$ be a family of bounded operators that is of strong bounded pseudo p-variation on I and $f : I \to Z$ be a function of bounded pseudo-variation on I. Then*

(a) $V \star f$ is of bounded pseudo-p-variation on I, $(V \star f)^{\oplus}(0) = 0$, and

$$\mathbf{pv}_p(V \star f;I) \le \mathbf{PV}_p(V;I,Z)\ \mathbf{pv}_1^{\oplus}(f;I).$$

(b) If $p = 1$, $V \diamond f$ is of bounded pseudo-variation,

$$\begin{aligned}
\mathbf{pv}_1(V \diamond f;I) &\le \mathbf{PV}^{\oplus}(V;I,Z)\ \mathbf{pv}_1(f;I) + \mathbf{pv}_1(V(\cdot)f^{\oplus}(0);I), \\
\mathbf{pv}_1^{\oplus}(V \diamond f;I) &\le \mathbf{PV}^{\oplus}(V;I,Z)\ \mathbf{pv}_1^{\oplus}(f;I).
\end{aligned}$$

and

$$(V \diamond f)(t) = \lim_{h\searrow 0} \frac{1}{h}\int_0^t V(t-s)(f(s+h) - f(s))ds + V^{\oplus}(t)f^{\oplus}(0).$$

Proof. (a) Define $f_l(s) := f(s+l)$ for $l > 0$ and $s \in I$ such that $s + l \in I$ and $f_l(s) = 0$ if $s + l \notin I$. We have almost everywhere

$$\begin{aligned}
&(V \star f)(s+l) - (V \star f)(s) \\
&\qquad\qquad = \lim_{h \searrow 0} \frac{1}{h} \int_0^{s+l} (V(u+h) - V(u)) f(s+l-u) du \\
&\qquad\qquad\quad - \lim_{h \searrow 0} \frac{1}{h} \int_0^{s} (V(u+h) - V(u)) f(s-u) du \\
&\qquad = \lim_{h \searrow 0} \frac{1}{h} \int_0^{s} (V(u+h) - V(u))(f(s+l-u) - f(s-u)) du \\
&\qquad\quad + \lim_{h \searrow 0} \frac{1}{h} \int_s^{s+l} (V(u+h) - V(u)) f(s+l-u) du \\
&\qquad\qquad = (V \star (f_l - f))(s) + (V \star f \cdot 1_{[0,l]})(s+l),
\end{aligned}$$

and therefore by Theorem 2.1

$$\begin{aligned}
&\left(\int_0^{r-l} \|(V \star f)(s+l) - (V \star f)(s)\|^p \, ds \right)^{\frac{1}{p}} \\
&\quad \leq \left(\int_0^{r-l} \|(V \star (f_l - f))(s)\|^p \, ds \right)^{\frac{1}{p}} + \left(\int_l^{r} \left\|(V \star f \cdot 1_{[0,l]})(s)\right\|^p \, ds \right)^{\frac{1}{p}} \\
&\qquad \leq \mathbf{PV}_p(V; I) \int_0^{r-l} \|f(s+l) - f(s)\| \, ds + \mathbf{PV}_p(V; I) \int_0^{l} \|f(s)\| \, ds.
\end{aligned}$$

The claim is obtained by multiplying with $\frac{1}{l}$ and letting $l \to 0$, if we recall that

$$\|f(\cdot)\|^{\oplus} = \left\|f^{\oplus}(\cdot)\right\|^{\oplus} = \left\|f^{\oplus}(\cdot)\right\|.$$

We now show $(V \star f)^{\oplus}(0) = 0$. By Proposition 2.3 (a),

$$\int_0^t (V \star f)(s) ds = (V * f)(t) - V^{\oplus}(0) \int_0^t f(s) ds.$$

Corollary 1.7 (b) allows to replace V and f by $V^{\oplus}$ and $f^{\oplus}$ in the right hand side of this equation. Thus $(V \star f)^{\oplus}(0) = 0$ follows from the fact that $f^{\oplus}$ is right continuous and $V^{\oplus}$ strongly right continuous.
The representation formula for $(V \diamond f)(t)$ follows in the same way as the derivation of (15) and (16).
(b) Notice that

$$\begin{aligned}
& \lim_{h \searrow 0} \frac{1}{h} \int_s^{s+l} (V(u+h) - V(u)) f(s+l-u) du \\
= \quad & (V(s+l) - V(s)) f^{\oplus}(0) + \lim_{h \searrow 0} \frac{1}{h} \int_s^{s+l} (V(u+h) - V(u)) g(s+l-u) du
\end{aligned}$$

with $g(t) = f^{\oplus}(t) - f^{\oplus}(0)$. By Proposition 2.3 (a) and the derivation above in part (a),

$$\begin{aligned}
(V \diamond f)(s+l) - (V \diamond f)(s) \quad = \quad & (V \diamond (f_l - f))(s) + (V(s+l) - V(s)) f^{\oplus}(0) \\
& + V \star (g \cdot 1_{[0,l]})(s+l).
\end{aligned}$$

Thus by Corollary 2.5,

$$
\begin{aligned}
& \int_0^{r(I)-l} \|(V \diamond f)(s+l) - (V \diamond f)(s)\| ds \\
\leq \;& \mathbf{PV}^{\oplus}(V; I, Z) \int_0^{r(I)-l} \|f(s+l) - f(s)\| ds \\
& + \int_0^{r(I)-l} \|(V(s+l) - V(s)) f^{\oplus}(0)\| ds + \mathbf{PV}_1(V; I, Z) \int_0^l \|g(s)\| ds.
\end{aligned}
$$

Multiplying by $\frac{1}{l}$, letting $l \to 0$, and noticing that $g(s) \to 0$ as $s \to 0$ we obtain the first inequality of (b).
By Proposition 2.3 (a), $((V * f)')^{\oplus} = (V \star f)^{\oplus} + V^{\oplus}(0) f^{\oplus}$. As we have shown above that $(V \star f)^{\oplus}(0) = 0$, $(V \diamond f)^{\oplus}(0) = V^{\oplus}(0) f^{\oplus}(0)$. So, by the definition of $\mathbf{pv}_1^{\oplus}$ (Proposition 1.8),

$$
\begin{aligned}
& \mathbf{pv}_1^{\oplus}(V \diamond f; I) = \mathbf{pv}_1(V \diamond f; I) + \|(V \diamond f)^{\oplus}(0)\| \\
\leq \;& \mathbf{PV}^{\oplus}(V; I, Z) \mathbf{pv}_1(f, I) + \mathbf{pv}_1(V(\cdot) f^{\oplus}(0); I) + \|V^{\oplus}(0) f^{\oplus}(0)\| \\
= \;& \mathbf{PV}^{\oplus}(V; I, Z) \mathbf{pv}_1(f, I) + \mathbf{pv}^{\oplus}(V(\cdot) f^{\oplus}(0); I).
\end{aligned}
$$

By Definition 1.9 and the subsequent deliberations,

$$
\mathbf{pv}_1^{\oplus}(V \diamond f; I) \leq \mathbf{PV}^{\oplus}(V; I, Z)(\mathbf{pv}_1(f, I) + \|f^{\oplus}(0)\|) = \mathbf{PV}^{\oplus}(V; I, Z)\ \mathbf{pv}^{\oplus}(f, I).
$$

□

2.7 Definition. If I is an unbounded interval, then a locally integrable function $f : I \to X$ is called to be of *local bounded (essential, pseudo) p-variation* if f is of bounded (essential, pseudo) p-variation on every bounded subinterval of I. Analogously we define an operator family to be of *local strong bounded (essential, pseudo) variation on I.*

If f is of bounded pseudo-variation on $\mathbf{R}_+$ and $(V(t))_{t \geq 0}$ is of local strong bounded pseudo-variation on $\mathbf{R}_+$, then the convolutions $V * f, V \star f, V \diamond f$, due to their local nature, can be lifted from the bounded intervals in $\mathbf{R}_+$ to $\mathbf{R}_+$ in a natural way. The following result flows from the representation formula for $V \diamond f$ in Theorem 2.6.

2.8 Proposition. *Let $f \in \mathcal{PV}(\mathbf{R}_+; X)$ and $(V(t))_{t \geq 0}$ be a family of bounded linear operators that is of local bounded variation on $\mathbf{R}_+$. If $(V(t))_{t \geq 0}$ is exponentially bounded, so is $V \diamond f$. More precisely, if $M, \lambda \geq 0$ are such that $\|V(t)x\| \leq M e^{\lambda t} \|x\|$ for a.a. $t \geq 0$ and all $x \in X$, then*

$$
\|(V \diamond f)(t)\| \leq M e^{\lambda t}\ \mathbf{pv}_1^{\oplus}(f, \mathbf{R}_+).
$$

If $(U(t))_{t \geq 0}$ and $(V(t))_{t \geq 0}$ are families of bounded linear operators which are of strong bounded pseudo-variation, then, for each x, $(U \star V(\cdot)x)$ and $(U \diamond V(\cdot)x)$ are defined as functions of bounded pseudo-variation, but difficult to turn into a family of bounded linear operators $(W(t))_{t \geq 0}$, because it is only uniquely determined outside of a set of Lebesgue measure 0 which depends on x. This situation can be remedied by the following definition

2.9 Definition. Let $(U(t))_{t \in I}$ and $(V(t))_{t \in I}$ be families of bounded linear operators that are of strong bounded pseudo-variation on I. Then for all $t \in I$

$$
(U \star V)(t)x := (U \star V(\cdot)x)^{\oplus}, \quad (U \diamond V)(t)x := (U \diamond V(\cdot)x)^{\oplus}.
$$

By Theorem 2.6, Corollary 1.7 (b), and the remark at the end of Section 1, we obtain part (a) of the following proposition. Notice that $U \star V = U \diamond V$, if $U^{\oplus}(0) = 0$.

2.10 Proposition. *Let $T > 0$ and $(U(t))_{t\in[0,T)}$ and $(V(t))_{t\in[0,T)}$ be families of bounded linear operators that are of strong bounded pseudo-variation, Z a closed subspace of X.*

(a) Then $U \diamond V$ and $U \star V$ are families of bounded linear operators on X that are of strong bounded variation and strongly right-continuous. If $V(t)$ maps X into Z for a.a. $t \in I$,

$$\mathbf{PV}^{\oplus}(U \diamond V; I, X) \leq \mathbf{PV}^{\oplus}(U; I, Z)\ \mathbf{PV}^{\oplus}(V; I, X).$$

If $(U(t))_{t\geq 0}$ is of strong bounded pseudo-p-variation and $V(t)$ maps into Z for a.a. $t \geq 0$, $U \star V$ is also of strong bounded p-variation,

$$\mathbf{PV}_p(U \star V; I, X) \leq \mathbf{PV}_p(U; I, Z)\ \mathbf{PV}^{\oplus}(V; I, X).$$

(b) If $f : [0, T) \to X$ is of bounded pseudo-variation, then

$$(U \diamond V) \diamond f = U \diamond (V \diamond f)$$

and

$$(U \star V) \star f = U \star (V \star f) + U \star (V^{\oplus}(0)f).$$

Proof. (a) By Theorem 2.6 and the remarks in 1.9,

$$\begin{aligned}\mathbf{pv}_1^{\oplus}(U \diamond [V(\cdot)x]; I, X) &\leq \mathbf{PV}^{\oplus}(U; I, Z)\ \mathbf{pv}_1^{\oplus}(V(\cdot)x; I)\\ &\leq \mathbf{PV}^{\oplus}(U; I, Z)\ \mathbf{PV}^{\oplus}(V; I, X)\|x\|,\end{aligned}$$

so $x \mapsto U \diamond [V(\cdot)x]$ is a bounded linear operator from X to $\mathcal{PV}(I, X)$ and thus $U \diamond V$ a family of strong bounded variation satisfying the estimate in (a). Similarly we proceed for $U \star V$.

(b) $(U * V) * f = U * (V * f)$ is a well-known fact which follows by changing the order of integration. By Corollary 1.7, $V * f \in W_1^1(I, X)$ and $(U * V)(\cdot)x \in W_1^1(I, X)$ for all $x \in X$. Differentiating the equality above, we obtain,

$$(U * V)' * f + (U * V)(0)f(t) = U * (V * f)' + U(t)(V * f)(0).$$

Since $(U * V)'(t)x := \frac{d}{dt}(U * V)(t)x = (U \diamond V)(t)x$ for a.a. $t \geq 0$ and $(U * V)(0) = 0$ and $(V * f)(0) = 0$,

$$(U \diamond V) * f = U * (V \diamond f).$$

Our first claim follows by differentiating once more. Indeed, the derivatives exist by Corollary 1.7 and we also obtain that

$$\begin{aligned}(U \diamond V) \diamond f &= (U \star V + U^{\oplus}(0)V) \diamond f\\ &= (U \star V) \star f) + (U \star V)(0)f + U^{\oplus}(0)(V \star f) + U^{\oplus}(0)V^{\oplus}(0)f,\end{aligned}$$

and

$$\begin{aligned}U \diamond (V \diamond f) &= U \diamond (V \star f + V^{\oplus}(0)f)\\ &= U \star (V \star f) + U^{\oplus}(0)(V \star f) + U \star (V^{\oplus}(0)f) + U^{\oplus}(0)V^{\oplus}(0)f.\end{aligned}$$

The second equality of (b) now follows by comparing the last two equations and using that $(U \star V)(0) = 0$ by definition of $U \star V$ and Theorem 2.6 (a). □

We now define recursively

$$U^{\diamond 1}(t) := U^{\oplus}(t), \qquad U^{\diamond(n+1)} := U \diamond U^{\diamond n}.$$

and have the following results from Theorem 2.6 and Proposition 2.10 by induction.

2.11 Proposition. *Let $(U(t))_{t\in I}$, be a family of bounded linear operators that is of strong bounded pseudo-variation on I. Then we have $U^{\diamond(n+1)} = U^{\diamond n} \diamond U$ for all $n \in \mathbf{N}$. If $U(t)$ maps X into a closed subspace Z of X for a.a. $t \in I$, then*

$$\mathbf{PV}(U^{\diamond(n+1)}; I, X) \le \mathbf{PV}(U; I, Z)^n \, \mathbf{PV}(U; I, X) \qquad \forall n \in \mathbf{N}.$$

3 Functions of bounded exponentially pseudo-variation on $\mathbf{R}_+$ and their Laplace transforms

In the following chapters we will deal with integrated semigroups that are of local strong bounded p-variation and with shifted integrated semigroups (see Lemma 3.6)

$$S^\lambda(t) = e^{-\lambda t} S(t) + \lambda \int_0^t e^{-\lambda s} S(s)\, ds$$

that are of strong bounded p-variation on $\mathbf{R}_+$ for some $\lambda \in \mathbf{R}$. For this reason we make the following

3.1 Definition. Let $\lambda \in \mathbf{R}$ and $1 \le p < \infty$. A locally integrable function $f : \mathbf{R}_+ \to X$ is called to be *of bounded exponentially weighed pseudo-p-variation (with weight λ)* if

$$\mathbf{pv}_p^\lambda(f) := \overline{\lim_{h\searrow 0}} \frac{1}{h} \left(\int_0^\infty e^{-p\lambda t} \|f(t+h) - f(t)\|^p dt \right)^p < \infty.$$

The theory of functions of bounded exponentially weighed variation can be reduced to the one without exponential weights using the following transformations,

$$f^\lambda(t) = e^{-\lambda t} f(t) + \lambda \int_0^t e^{-\lambda s} f(s) ds.$$

This transformation has the following properties.

3.2 Proposition. *Let $f : \mathbf{R}_+ \to X$ be locally integrable.*

(a) $(f^\lambda)^\mu = f^{\lambda+\mu}$ for all $\lambda, \mu \in \mathbf{R}_+$. In particular $f = (f^\lambda)^{-\lambda}$.

(b) If $\lambda, \mu \in \mathbf{R}$ and the Laplace transform of f at $\lambda + \mu$, $\mathcal{L}(f)(\mu + \lambda)$, converges absolutely, then the Laplace transform of f^λ at μ converges absolutely and the Laplace transforms are related by $\mu \mathcal{L}(f^\lambda)(\mu) = (\mu + \lambda)\mathcal{L}(f)(\mu + \lambda)$.

(c) Assume that $\lambda \in \mathbf{R}$ and the Laplace transform $\mathcal{L}(f)(\lambda)$ converges absolutely. Then f^λ is of bounded pseudo-p-variation if and only if f is of bounded exponentially weighed pseudo-p-variation with weight λ.

(d) Let $\lambda \in \mathbf{R}$. If f^λ is of bounded pseudo-p-variation on $\mathbf{R}$, then the Laplace transform $\mathcal{L}(f)(\mu)$ converges absolutely for $\operatorname{Re} \mu > \lambda$. If $p = 1$, this also hold for $\operatorname{Re} \mu = \lambda$.

Proof. (a) and (b) follow by changing the order of integration.
(c) This follows from

$$\begin{aligned} f^\lambda(t+h) - f^\lambda(t) &= e^{-\lambda t}(f(t+h)-f(t)) + (e^{-\lambda(t+h)} - e^{-\lambda t})f(t+h) \\ &\quad + \lambda \int_t^{t+h} e^{-\lambda s} f(s)ds \\ &= e^{-\lambda t}(f(t+h)-f(t)) + \lambda \int_t^{t+h} e^{-\lambda s}(f(s) - f(t+h))ds \end{aligned}$$

and Lebesgue's theorem of dominated convergence.
(d) Let f^λ be of bounded pseudo-p-variation. We can assume that $\mu > \lambda$. Then

$$\begin{aligned} &\mu \int_0^t e^{-\mu s}\|f(s)\|ds = \lim_{h\searrow 0} \int_0^t \frac{1}{h}(e^{-\mu(s-h)} - e^{-\mu s})\|f(s)\|ds \\ \leq\ & \varliminf_{h\searrow 0} \frac{1}{h}\left(\int_0^t e^{-\mu s}(\|f(s+h)\| - \|f(s)\|)ds + \int_0^h e^{-\mu(s-h)}\|f(s)\|ds\right) \\ \leq\ & \varliminf_{h\searrow 0} \frac{1}{h}\int_0^t e^{-(\mu-\lambda)s}e^{-\lambda s}\|f(s+h) - f(s)\|ds + \|f^\oplus(0)\| \\ \leq\ & \left(\int_0^t e^{-(\mu-\lambda)sq}ds\right)^{1/q} \varliminf_{h\searrow 0} \frac{1}{h}\left(\int_0^t e^{-\lambda sp}\|f(s+h)-f(s)\|^p ds\right)^{1/p} + \|f^\oplus(0)\| \\ \leq\ & \frac{1}{[(\mu-\lambda)q]^{1/q}}\mathbf{pv}_p^\lambda(f) + \|f^\oplus(0)\|. \end{aligned}$$

□

We combine the statements in this lemma and use the Post-Widder inversion formula ([HP57], Theorem 3.3.5) to obtain

3.3 Theorem. *Let $f : \mathbf{R}_+ \to X$ be locally integrable, and either f be of bounded exponentially weighed pseudo-p-variation with weight λ or f^λ be of bounded p-variation. Let $\mu > \lambda$ ($\mu \geq \lambda$, if $p = 1$). Then f is of bounded exponentially weighed pseudo-p-variation with weight μ, and f^μ is of bounded p-variation. Further the Laplace transforms $\mathcal{L}(f)(\mu)$ converge absolutely and*

$$\frac{1}{2}(f^\oplus(t) + f^\oplus(t-)) = \lim_{k\to\infty} \int_{k/t}^\infty (-1)^k \frac{\lambda^{k-1}}{(k-1)!}\lambda\mathcal{L}(f)(\lambda)d\lambda.$$

If $V(t), t \geq 0$, is a strongly locally integrable family of bounded linear operators, we define $V^\lambda(\cdot)x = (V(\cdot)x)^\lambda$.

3.4 Lemma. *Let f be locally integrable and $(U(t))_{t\geq 0}$, $(V(t))_{t\geq 0}$ be families of local strong bounded pseudo-variation on $\mathbf{R}_+$. Assume that the Laplace transforms of f, U, V converge absolutely when taken at sufficiently large λ. Then their Laplace transforms satisfy*

$$\mathcal{L}(U \diamond f) = \lambda\mathcal{L}(U)(\lambda)\mathcal{L}(f)(\lambda), \qquad \mathcal{L}(U \diamond V) = \lambda\mathcal{L}(U)(\lambda)\mathcal{L}(V)(\lambda),$$

for sufficiently large $\lambda > 0$.

Proof. By the well-known fact that the Laplace transform converts convolutions into compositions, for $\lambda > 0$ sufficiently large,

$$\mathcal{L}(U \diamond f)(\lambda) = \int_0^\infty e^{-\lambda t}(U * f)'(t)dt = \lambda\mathcal{L}(U * f)(\lambda) = \lambda\mathcal{L}(U)(\lambda)\mathcal{L}(f)(\lambda).$$

The analogous statement for $U \diamond V$ follows from Definition 2.4. □

3.5 Proposition. *Let f be locally integrable and $(U(t))_{t\geq 0}$, $(V(t))_{t\geq 0}$ be families of local strong bounded pseudo-variation on $\mathbf{R}_+$. Then $(U\diamond f)^\lambda = U^\lambda \diamond f^\lambda$ and $(U\diamond V)^\lambda = U^\lambda \diamond V^\lambda$.*

Proof. Let $r>0$ and modify U and f on (r,∞) such that they are 0 there. This does not affect U, f, and $U\diamond f$, U^μ, f^μ and $U^\mu \diamond f^\mu$ on $[0,r)$, but makes them exponentially bounded on $[0,\infty)$ and we can take their Laplace transforms. Further it makes U and U^μ of bounded pseudo-variation on $\mathbf{R}_+$. By Proposition 3.2 (b) and Lemma 3.4,

$$\begin{aligned} \lambda\mathcal{L}((U\diamond f)^\mu)(\lambda) &= (\lambda+\mu)\mathcal{L}(U\diamond f)(\lambda+\mu) \\ &= (\lambda+\mu)\mathcal{L}(U)(\lambda+\mu)\ (\lambda+\mu)\mathcal{L}(f)(\lambda+\mu) \\ &= \lambda\mathcal{L}(U^\mu)(\lambda)\ \lambda\mathcal{L}(f^\mu)(\lambda) = \lambda\mathcal{L}(U^\mu\diamond f^\mu)(\lambda). \end{aligned}$$

Hence $(U\diamond f)^\mu = U^\mu \diamond f^\mu$ a.e. on $[0,r)$. Since $r>0$ was arbitrary, our assertion follows. □

It was noticed in [KH89], Remark 3.2, that the transformation U^λ preserves exponentially bonded integrated semigroups; actually this holds in general, as some lengthy, but straightforward calculations show which are omitted.

3.6 Lemma. *If $\lambda \in \mathbf{R}$ and A is the generator of a non-degenerate integrated semigroup S, then S^λ is also a non-degenerate integrated semigroup, with generator $A-\lambda$.*

4 Characterization of generators and examples

In [Are87b] and [Hie91] generation theorems for n-times (resp. α-times, $\alpha \notin \mathbb{N}$) integrated semigroups are obtained via Widder inversion theorems of the vector valued Laplace transform. So we look for a vector valued inversion theorem for the Laplace transforms of functions of bounded p-variation to derive a generation theorem for integrated semigroups that are of strong bounded-p-variation. The following theorem is due to L.Weis (Theorem 3.2 of [Wei93]) and is the vector valued version of a classical theorem due to Widder (see Theorem 15a, VII in [Wid46]). We reformulate it in our terms of pseudo-variation.

4.1 Theorem. *Let X be a Banach space and $g:(0,\infty)\to X$, $1\leq p<\infty$, $M\geq 0$, $x\in X$. The following assertions are equivalent:*

(a) There is an $f\in\mathcal{PV}_p(\mathbf{R}_+,X)$ with $\mathbf{pv}_p(f;\mathbf{R}_+)\leq M$, $\lim_{t\to\infty} f(t)=x$ and

$$g(\lambda)=\lambda\int_0^\infty e^{-\lambda t}f(t)dt, \quad \lambda>0.$$

(b) g is infinitely often differentiable, $\lim_{\lambda\to\infty} g(\lambda)=x$ and

$$\overline{\lim_{k\to\infty}}\left(\int_0^\infty\left(\frac{\lambda^{k-1}}{(k-1)!}\|g^{(k)}(\lambda)\|\right)^p d\lambda\right)^{\frac{1}{p}}\leq M.$$

If one and then both of (a) and (b) hold, then we have for all $t>0$

$$\frac{1}{2}(f^\oplus(t)+f^\oplus(t-)) = \lim_{k\to\infty}\int_{k/t}^\infty(-1)^k\frac{\lambda^{k-1}}{(k-1)!}g^{(k)}(\lambda)d\lambda.$$

4.1 Widder type characterizations of generators

Combining Theorem 4.1 with Theorem 3.3 yields

4.2 Theorem. *Let A be a closed operator, $n \in \mathbf{N}$, $w \in \mathbf{R}$ and $1 \leq p < \infty$. The following assertions are equivalent:*

(a) A is the generator of an integrated semigroup $(S(t))_{t \geq 0}$ that is of local strong bounded p-variation such that S^w is of strong bounded p-variation on $\mathbf{R}_+$.

(b) $(w, \infty) \subseteq \rho(A; X)$ and for all $x \in X$ we have

$$\sup_{k \in \mathbf{N}} \left(\int_0^\infty \left(\left\| k\lambda^{k-1} R(\lambda + w, A)^{k+1} x \right\| \right)^p d\lambda \right)^{\frac{1}{p}} < \infty. \tag{21}$$

If $(S(t))_{t \geq 0}$ is of strong bounded p-variation on $\mathbf{R}_+$ we have for all $x \in X$ and $t > 0$

$$S(t)x = \lim_{k \to \infty} \int_{k/t}^\infty k\lambda^{k-1} R(\lambda + w, A)^{k+1} x \, d\lambda, \qquad \forall x \in X, \ t > 0. \tag{22}$$

4.3 Remark. (1) For A densely defined Theorem 4.2 is due to L.Weis and was announced in [Wei91] in terms of L_p-flow-theory.

(2) The extension of this theorem to α-times integrated semigroups for $1 \leq \alpha \notin \mathbb{N}$ is obvious.

(3) The formula in (a) holds as soon as the integrated semigroup S has absolutely convergent Laplace transforms $\mathcal{L}(S(\cdot)x)(\mu)$ for sufficiently large μ and all $x \in X$ [HP57, Theorem 6.3.5].

Proof. (a) $\Rightarrow$ (b) follows from Theorem 4.1.
For simplicity we consider from now on the case $w = 0$. The general case is due to rescaling.
(b) $\Rightarrow$ (a) We want to apply Theorem 4.1 to $g_x(\lambda) = R(\lambda, A)x$ for all $x \in X$.
We first show that $\lim_{\lambda \to \infty} R(\lambda, A)x = 0$ for all $x \in X$. First we show that the Limit exists for all $x \in X$. Since

$$R(t, A)x - R(1, A) = \int_1^t R(s, A)'x \, ds, \qquad t > 1 \tag{23}$$

we have to show that $\int_1^\infty R(\lambda, A)'x d\lambda$ exists for all $x \in X$. If $p = 1$ this is a direct consequence of our assumption. For $1 < p < \infty$ we obtain by the integral transform $\xi(t) = \frac{1}{t}$ from our assumption for $k = 1$ and the Hölder inequality:

$$\int_0^1 \left\| \frac{1}{t^2} R(t^{-1}, A)'x \, dt \right\| \leq \left(\int_0^1 \left\| \frac{1}{t^2} R(t^{-1}, A)'x \right\|^p \, dt \right)^{\frac{1}{p}} < \infty,$$

what ensures that

$$\int_1^\infty R(\lambda, A)'x \, d\lambda = \int_0^1 \frac{1}{t^2} R(t^{-1}, A)'x dt$$

exists for all $x \in X$.
We obtain now from (23) that $U(\infty)x := \lim_{\lambda \to \infty} R(\lambda, A)x$ exists and the resolvent equation yields for $\mu \in \rho(A; X)$:

$$R(\mu, A)U(\infty)x = \lim_{\lambda \to \infty} \frac{R(\lambda, A)x - R(\mu, A)x}{(\mu - \lambda)} = \lim_{\lambda \to \infty} \frac{1}{(\mu - \lambda)} U(\infty)x = 0.$$

Since $R(\mu, A)$ is injective we have $U(\infty)x = 0$ for all $x \in X$.
By Theorem 4.1 applied to $g_x(\lambda) = R(\lambda, A)x$ we know that for all $x \in X$ there is a normalized $\Phi_x \in V_p(\mathbf{R}_+, X)$ such that

$$R(\lambda, A)x = \lambda \int_0^\infty e^{-\lambda t}\Phi_x(t)dt, \qquad \lambda > 0. \tag{24}$$

We will show that

$$S(t) : X \to X, \ x \mapsto \Phi_x$$

defines an integrated semigroup. By definition we have $S(0)x = 0$ for all $x \in X$.
By standard arguments using the uniqueness theorem for the Laplace-Stieltjes transform we can show that each operator $S(t)$ is linear and commutes with the resolvents of A and by the arguments at the end of section one $S(t)$ is bounded for each $t > 0$. The operator family $(S(t))_{t\geq 0}$ satisfies

$$R(\lambda, A)x = \lambda \int_0^\infty e^{-\lambda t} S(t)x \ dt, \qquad \lambda > w, \quad x \in X.$$

Therefore it is enough to show that $(S(t))_{t\geq 0}$ is strongly continuous (see Theorem 3.1 of [Are87b]). But this is done just as in the proof of Theorem 5.7 of [Are87a] and therefore omitted here.
We finally remark that (22) is a consequence of Theorem 3.3 and the fact that $\Phi_x = S(\cdot)x$ is continuous. □

Integrated semigroups on Banach lattices

The result in [Are87a] that resolvent positive operators in Banach lattices with regular cone generate increasing integrated semigroups motivates us to have a closer look at the interplay of bounded variation and positivity. We recall that a Banach lattice E is called an abstract L-space if in addition the norm is additive on the positive cone (in particular E may be a L_1-space or a space of measures). Following Arendt ([Are87a]) we call an operator A on E resolvent positive on (a, ∞), if $(a, \infty) \subseteq \rho(A; X)$ for some $a \in \mathbf{R}$ and $R(\lambda, A) \geq 0$ for all $\lambda > a$. We refer to [Sch74] for more details on ordered Banach spaces and positive operators.
We will say that a bounded operator B_1 is *absolutely majorized* by a bounded positive operator B_2 if

$$|B_1 x| \leq B_2 \, |x| \qquad \text{for all } x \in X.$$

If A_1 and A_2 are closed operators and A_2 is resolvent positive, we say A_1 is *absolutely resolvent majorized* by A_2 on (w, ∞) for some $w \in \mathbf{R}$ if $(w, \infty) \subseteq \rho(A_1, X) \cap \rho(A_2, X)$ and $R(\lambda, A_1)$ is absolutely majorized by $R(\lambda, A_2)$ for all $\lambda > w$. Furthermore we see by induction

$$|R(\lambda, A_1)^n x| \leq R(\lambda, A_2)^n \, |x| \, , \qquad n \in \mathbf{N}, \ x \in X$$

and this yields

$$\|R(\lambda, A_1)^n x\| \leq \|R(\lambda, A_2)^n x\| \, , \qquad \forall x \in X.$$

This situation occurs when A_1 generates an integrated semigroup $(S_1(t))_{t\geq 0}$ that is dominated by an exponentially bounded integrated semigroup $(S_2(t))_{t\geq 0}$ generated by A_2.
We say that an integrated semigroup S_2 *dominates* an integrated semigroup S_1, if $S_2(t)x$ is increasing in $t \geq 0$ for all $x \in X_+$ and

$$|S_1(t)x| \leq S_2(t)|x| \qquad \forall x \in X, t \geq 0.$$

4.4 Proposition. *Let S_1 and S_2 be integrated semigroups on a Banach lattice X with generators A_1 and A_2, S_2 increasing. Then the following statements are equivalent.*

(1) S_2 dominates S_1.

(2) A_1 is resolvent majorized by A_2.

(3) $|S_1(t) - S_1(r)x| \leq S_2(t)|x| - S_1(r)|x| \qquad \forall x \in X, t \geq r \geq 0.$

Proof. Obviously (3) implies (1).
(1) $\implies$ (2): Since every increasing integrated semigroup on a Banach lattice is exponentially bounded [Are87a],

$$R(\lambda, A_2) = \lambda \int_0^\infty e^{-\lambda t} S_2(t) dt.$$

As S_2 dominates S_1, S_1 is exponentially bounded as well and the analogous formula for $R(\lambda, A_1)$ shows that A_1 is resolvent majorized by A_2.
(2) $\implies$ (1) by the formula (22) and Remark 4.3 (3). □

Theorem 4.2 and Lemma 3.6 imply that being the generator of an integrated semigroup of bounded variation is inherited via resolvent majorization.

4.5 Theorem. *Let $w \in \mathbf{R}$, $1 \leq p < \infty$, and S_2 be an increasing integrated semigroup on a Banach lattice X such that S_2^w is of bounded p-variation on $\mathbf{R}_+$. Let A_2 be the generator of S_2 and A_1 a closed operator in X which is resolvent majorized by A_2. Then A_1 is the generator of an integrated semigroup S_1 which is of local bounded p-variation with S_1^w being of bounded p-variation on $\mathbf{R}_+$. S_1 is dominated by S_2.*

4.6 Corollary. *Let X be an abstract L space and B a closed operator in X which is resolvent majorized by a resolvent positive operator A. Then A and B are generators of integrated semigroups S and U such that S dominates U. Both S and U are of local bounded variation and S^w and U^w are of bounded variation on $\mathbf{R}_+$ for some $w > 0$.*

Proof. A generates an increasing integrated semigroup S which is exponentially bounded [Are87a]. By Lemma 3.6, there is some $w > 0$ such that

$$S^w(t) = \int_0^t e^{-ws} S(ds) = e^{-wt} + w \int_0^t e^{-ws} S(s) ds$$

an increasing bounded integrated semigroup, generated by $A - w$. Since the norm is additive on the positive cone of the abstract L space X, S^w is of strong bounded variation on $\mathbf{R}$. By Theorem 4.5, B generates an integrated semigroup U which is dominated by S such that U^w is of bounded variation on $\mathbf{R}_+$. □

4.7 Remark. Alternatively (and equivalently) we can assume that the generator of S is a resolvent positive operator.

Our corollary will turn out to be useful in perturbation theory.

4.2 Examples

(1) Semigroups of bounded operators $(T(t))_{t\geq 0}$ that are strongly continuous for $t>0$ but have a L_p-singularity in $t=0$ provide an integrated semigroup that is of strong bounded p-variation by

$$S(t)x := \int_0^t T(s)x ds, \qquad x\in X, \ \ t\geq 0.$$

For examples where A is densely defined see [HP57], [Sun70], [Oha71] and the references in [Fat83], Sec. 2.5. For the case that A is not densely defined we refer to [SS86], [Cer94] and [Bob97].

(2) All examples of L_p-flows given in [Wei91] are covered by our approach.

(3) In the section 6 we are going to show that every positive perturbation of an resolvent positive operator on an abstract L-space leads to an operator that generates an exponentially bounded semigroup that is of local strong bounded variation.

5 Application to the abstract Cauchy problem

In this section we show how one can use integrated semigroups of local bounded p-variation and the $\star$-convolution to proof results on L_p-solutions for the inhomogeneous Cauchy problem. We always consider $I=[0,T)$ with $T\in(0,\infty]$.

5.1 Definition. Let A be a closed operator, $1\leq p<\infty$ and $f\in L_{p,loc}(I,X)$. We say $v\in L_p(I,X)$ $(C(I,X))$ is a L_p-solution (C-solution) of (A,x,f) if for almost all $t\in I$

$$v(t)-x = A\int_0^t v(s)ds - \int_0^t f(s)ds. \tag{25}$$

We say that v is a W_p^1-solution (or Sobolev solution) of (A,x,f) if $v\in W^1_{p,loc}(I,X)$ and

$$v'(t) = Av(t)+f(t), \qquad v(0)=x, \qquad f.a.e. \ \ t\in I. \tag{26}$$

If A is the generator of an integrated semigroup, the integrated version of (A,x,f) has at most one continuous solution (see e.g. [Are87b], [Thi90]). Thus, by integrating the L_p-solutions of (A,x,f), we obtain that there is at most one L_p-solution of (A,x,f).
For a closed operator A we set $Y:=\overline{D(A)}$ and denote by A_Y the part of A in $D(A)$. Furthermore we set $A_Y^0 := \mathrm{Id}|_Y$.

5.2 Corollary. *Let A generate an integrated semigroup $(S(t))_{t\geq 0}$ on $[0,\infty)$ which is of local bounded p-variation for some $1\leq p<\infty$, then $(A,x,0)$ has for every $x\in Y$ the unique L_p-solution $S'(\cdot)x$. Furthermore if $x_n\to x$ then the solutions u_{x_n} of $(A,x_n,0)$ converges locally in the L_p-norm to the solution u_x of $(A,x,0)$.*

5.3 Remark. For $D(A)$ dense this result is due to L.Weis (see Theorem 2.2 in [Wei91]) and was stated in terms of the L_p-flow theory.

Proof. Again we assume $w=0$. For $x\in D(A)$ we have

$$S(t)x = tx+\int_0^t S(s)Axds. \tag{27}$$

Therefore $S(\cdot)x\in W^1_{p,loc}(\mathbf{R}_+,X)$. Since S commutes with the resolvent of A we have

$$\int_0^t S'(s)xds = S(t)x\in D(A) \tag{28}$$

and

$$A\int_0^t S'(s)xds = AS(t)x = S(t)Ax = S'(t)x - x. \tag{29}$$

Thus $(A, x, 0)$ has a L_p-solution for $x \in D(A)$.
Now, let $x \in Y$ and $x_n \in D(A)$ such that $x_n \to x$. By Fatou's lemma we have for $t \geq 0$

$$\begin{aligned}\left(\int_0^t \|S'(s)(x_n - x_m)\|^p \, ds\right)^{\frac{1}{p}} &\leq \varliminf_{h\to 0} \left(\int_0^t \frac{\|(S(s+h) - S(s))(x_n - x_m)\|^p}{h^p} ds\right)^{\frac{1}{p}} \\ &\leq \mathbf{PV}_p(S; [0,t)) \, \|x_n - x_m\| \underset{n,m\to\infty}{\longrightarrow} 0. \end{aligned}\tag{30}$$

Thus we have $S(\cdot)x \in W_p^1([0,t]; X)$ and

$$S(\cdot)'x = \|\cdot\|_p - \lim_{n\to\infty} S'(\cdot)x_n$$

on every bounded interval. Since A is closed we obtain by approximation from (29) that $S'(\cdot)x$ is a L_p-solution for $(A, x, 0)$ and we are done. □

5.4 Lemma. *Let $(S(t))_{t\geq 0}$ be an integrated semigroup on $[0,\infty)$ of strong bounded p-variation generated by A and $f \in L_1([0,\infty), X)$. Then $(S * f)(t) \in D(A)$ for almost all $t \in [0,\infty)$ with*

$$A(S * f)(t) = (S \star f)(t) - \int_0^t f(s)ds.$$

Proof. Consider $f \in C^1(I, X)$ with $f(0) = 0$. Then we obtain by partial integration for all $t \in I$:

$$\int_0^t S(s)f(t-s)xds = -\int_0^t \int_0^s S(r)f'(t-s)drds. \tag{31}$$

Together with

$$A\int_0^t S(s)xds = S(t)x - tx, \qquad t \geq 0$$

we conclude by Proposition 2.3 (b) and the fact that A is closed

$$\begin{aligned} A\int_0^t S(s)f(t-s)xds &= \int_0^t (S(s) - s)\, f'(t-s)ds \\ &= (S \star f)(t) - \int_0^t f(s)ds. \end{aligned}$$

Let $f \in L_1(I, X)$. Since the C^1-functions that a take value zero in zero are dense in $L_p(I, X)$ we find by Theorem 2.1 and Theorem 2.2 a sequence of C^1-functions $(f_n)_{n\in\mathbf{N}}$ with $f_n(0) = 0$ such that in $L_1(I, X)$

$$f = \lim_{n\to\infty} f_n, \quad (S * f) = \lim_{n\to\infty} (S * f_n), \quad (S \star f) = \lim_{n\to\infty} (S \star f_n).$$

Since A is closed we see that the claim holds. □

For a closed operator A we denote by X_A the Banach space that is obtained if one equips $D(A)$ with the graph norm.

5.5 Theorem. *Let $1 \leq p < \infty$. Suppose A generates an integrated semigroup $(S(t))_{t\geq 0}$ that is of strong bounded p-variation on I, then we have*

(1) *For all* $x \in \overline{D(A)} = Y$ *and* $f \in L_1(I, X)$ *there exists a unique* L_p*-solution of* (A, x, f) *which is given by*

$$v_x := S'(\cdot)x + S \star f.$$

(2) (A, x, f) *has the unique* W_p^1*-solution* v_x' *if one of the following conditions is satisfied*

 (a) $x \in D(A)$ *and* $f \in W_1^1(I, X)$ *with* $Ax + f(0) \in Y$,

 (b) $x \in D(A_Y)$ *and* $f \in L_1(I, X_A) \cap L_p(I, X)$.

5.6 Remark. (1) Theorem 5.5 was partially announced by L.Weis in [Wei91].

(2) Since every Hille-Yosida operator A generates an integrated semigroup S that is of strong bounded p-variation on I for every $1 \leq p < \infty$, we have the following slight improvement of a part of Theorem 8.3 in [dPS87]:

If I is finite, $x \in D(A_Y)$ for some $1 \leq k \leq n$ and $f \in L_1(I, X_A) \cap L_p(I, X)$ it is enough to assume that A satisfies an integrated semigroup S that is of strong bounded p-variation on I for some $1 \leq p < \infty$ instead of assuming that A is a Hille-Yosida operator to make sure that (A, x, f) has a unique Sobolev solution.

(3) For an abstract L-space our Theorem refines Theorem 1.4 [Are] (Recall that every resolvent positive operator in an abstract L-space generates an integrated semigroup of strong bounded variation):

Theorem ([Are]). *Let* A *be a resolvent positive operator on a Banach lattice* X, $f \in W_1^1((0, r), X), u_0 \in D(A)$. *Suppose that* $Au_0 + f(0)$ *in the closure of* $D(A)$. *Then there exists a continuous solution* u *of*

$$u(t) = u_0 + A \int_0^t u(s)ds + \int_0^t f(s)ds, \qquad t \geq 0.$$

Proof of Theorem 5.5. As we remarked at the begin of the section the uniqueness statements follow easily from the known uniqueness results for integrated semigroups.
(1) Let $x \in Y = \overline{D(A)}$ and $f \in L_1(I, X)$. Define

$$v_x := S'(\cdot)x + S \star f.$$

By Lemma 5.4 and Proposition 2.3 we have

$$A \int_0^t (S \star f)(s)ds = A(S * f)(t) = (S \star f)(t) - \int_0^t f(s)ds, \tag{32}$$

and from Corollary 5.2 we know that $S'(\cdot)x$ is the L_p-solution of $(A, x, 0)_{n-1}$. So we obtain almost everywhere

$$S'(t)x - x = A \int_0^t S'(s)xds.$$

Combined with (32) we have

$$v_x(t) - x = A \int_0^t v_x(s)ds + \int_0^t f(s)ds.$$

Before we prove the second part of this theorem we state a useful fact. The proof is a consequence of (1) and the fact that A is a closed operator (see proof of Theorem 5.2 in [Are87b]).

5.7 Proposition. *Let $(S(t))_{t\geq 0}$ be an integrated semigroup that is of strong bounded p-variation and $f \in L_1(I,X)$. Then (A,x,f) has a W^1_p-solution if and only if v_x is in $W^1_{p,loc}(I,X)$.*

We turn to the proof of the second part of Theorem 5.5.
(2) (a) Let $x \in D(A)$. By the above proposition it remains to show that v_x is differentiable. We have

$$S'(t)x - x = S(t)Ax$$

and together with (1) and Proposition 2.3 we obtain

$$\begin{aligned} v_x(t) &= S'(t)x + (S \star f)(t) = S(t)Ax + x + S(t)f(0) + (S * f')(t) \\ &= S(t)(Ax + f(0)) + (S * f')(t). \end{aligned}$$

Since $Ax + f(0) \in Y$ we have that $S(\cdot)(Ax + f(0))$ is in $W^1_{p,loc}(I,X)$ by Corollary 5.2 and by Corollary 2.3 we see that $S * f' \in W^1_{p,loc}(I,X)$. So the claim is proved.
(2) (b) By Corollary 5.2 we see that $S(\cdot)x \in W^{k+1}_{p,loc}(\mathbf{R}_+,X)$. Since A commutes with S we obtain by Theorem 2.1

$$v_x = S'(\cdot)x + S \star f \in L_{p,loc}(\mathbf{R}_+, X_A).$$

By (1) we see that v_x satisfies equation (19). This implies $v_x \in W^1_{p,loc}(\mathbf{R}_+,X)$ and we are done by Proposition 5.7. □

We close with a result for Banach spaces that have the Radon-Nikodym property.

5.8 Corollary. *Let $1 < p < \infty$ and X be a Banach space that has the Radon-Nikodym property. Suppose A generates an integrated semigroup $(S(t))_{t\geq 0}$ that is of strong bounded p-variation on I, then we have*

(1) For all $x \in X$ and $f \in L_1(I,X)$ there exists a unique L_p-solution of (A,x,f) which is given by

$$v_x := S'(\cdot)x + S \star f.$$

(2) (A,x,f) has the unique W^1_p-solution v'_x if one of the following conditions is satisfied

(a) $x \in D(A)$ and f is of strong bounded variation.

(b) $x \in D(A)$ and $f \in L_1(I,X_A) \cap L_p(I,X)$.

Proof. In Banach spaces that have the Radon-Nikodym property every function of bounded p-variation is W^1_p for all $1 < p < \infty$ (see e.g. [Wei93], Corollary 3.3). Therefore $S(\cdot)x \in W^1_{p,loc}(I,X)$ for all $x \in X$ and $S(\cdot)x \in W^2_{p,loc}(I,X)$ for all $x \in D(A)$. From the proof of Theorem 5.5 we obtain that the assertions (1) and (2)(b) hold.
It remains to show that

$$v_x := S'(\cdot)x + S \star f$$

is a W^1_p-solution for (A,x,f) if $x \in D(A)$ and f is of bounded variation. By Proposition 2.6, $(S\star f)^{\oplus}$ is of bounded p-variation and therefore $S\star f \in W^1_{p,loc}(I,X)$. Since $S'(\cdot)x \in W^1_{p,loc}(I,X)$ we see that $v_x \in W^1_{p,loc}(I,X)$ and from the first part of our theorem we know that v_x is a solution of (A,x,f). Finally, A is closed and thus we obtain almost everywhere

$$\begin{aligned} v'_x(t) = \lim_{h\searrow 0} \frac{1}{h}(v_x(t+h) - v_x(t)) &= \lim_{h\searrow 0} \frac{1}{h}\left(A\int_t^{t+h} v_x(u)du - \int_t^{t+h} f(u)du\right) \\ &= Av_x(t) - f(t). \end{aligned}$$

□

6 Perturbation theory

We give two types of perturbation theorems for integrated semigroups that are of strong bounded p-variation. The first relies on Bernstein's inversion theorem for completely monotonic functions and the fact that increasing functions with values in an abstract L space are of bounded variation. The other is valid in general Banach spaces and uses Widder's inversion theorem. But before, we prove some unifying, though somewhat technical results.

General theory

We make the following

Assumption 1. Let $1 \leq p \leq \infty$ and A be the generator of an integrated semigroup $(S(t))_{t\geq 0}$ that is of local bounded p-variation. Furthermore let B be an A-bounded operator such that for every $x \in X$,

$$B \int_0^{\cdot} S(s)x ds \in W^1_{1,loc}([0,\infty), X)$$

and

$$V(t)x = \frac{d}{dt} B \int_0^t S(s)x ds, \quad t \geq 0$$

is a function of local bounded pseudo-variation.

6.1 Remark. Replacing V by $V^{\oplus}$ we see that $B \int_0^t S(s)dx$ is right differentiable with derivative $V^{\oplus}(t)x$. So without restriction, we assume this from the very beginning, with $V(t)$ being strongly right continuous.

For $x \in D(A)$ and $t \geq 0$,

$$V(t)x = tBx + \frac{d}{dt} B \int_0^t \left(\int_0^r S(s)Axds \right) dr.$$

Since B is A-bounded,

$$V(t)x = tBx + \frac{d}{dt} \int_0^t \left(\int_0^r V(s)Axds \right) dr = tBx + \int_0^t V(s)Axds.$$

We see that $V(t)x$ is right differentiable and

$$V'(t)x = Bx + V(t)Ax, \quad x \in D(A). \tag{33}$$

We make the following additional

Assumption 2. There exists a family of bounded linear operators $(W(t))_{t\geq 0}$, that is of local strong bound pseudo-variation and satisfies

$$W = V + V \diamond W = V + W \diamond V \quad \text{on } \mathbf{R}_+.$$

6.2 Remark and Definition. W is automatically strongly right continuous and of local strong bounded variation. Further W is uniquely determined by these algebraic properties ([GLO90], Section 9.3, Lemma 3.5) and called the *$\diamond$-resolvent of V*.

6.3 Proposition. *Under Assumptions 1 and 2, $A + B$ is the generator of an integrated semigroup $(U(t))_{t\geq 0}$ of local strongly bounded p-variation and $(U(t))_{t\geq 0}$ is given by*

$$U = S + S \diamond W = S + U \diamond V.$$

Proof. We set
$$U := S + S \diamond W.$$
By construction this is a strongly right continuous family of local bounded p-variation (see Proposition 2.10). To show that U is strongly continuous we make the following consideration. Notice that a.e.
$$S(r)(S \diamond [W(\cdot)x])(t) = ([S(r)S(\cdot)] \diamond [W(\cdot)x])(t) = ([S(r)S(\cdot)] * [W(\cdot)x])'(t).$$
Since S is an integrated semigroup, $S(r)S(\cdot)$ is differentiable and $[S(r)S(\cdot)]' = S(r+\cdot) - S(r)$. So, for all $x \in X$
$$\begin{aligned} S(r)(S \diamond [W(\cdot)x])(t) &= ([S(r)S(\cdot)]' * [W(\cdot)x])(t) \\ &= (S(r+\cdot) * [W(\cdot)x])(t) - S(r)\int_0^t W(s)xds. \end{aligned}$$
This shows that
$$S(r)(S \diamond W)(t) = (S(r+\cdot) * W)(t) - S(r)\int_0^t W(s)ds = S(r)(S \diamond W)(t-).$$
Since S is a non-degenerate family, $(S \diamond W)(t-) = (S \diamond W)(t)$.
By Assumption 2 and Remark 6.2, W is the $\diamond$-resolvent of V and thus we have
$$U = S + S \diamond (V + W \diamond V) = S + S \diamond V + (S \diamond W) \diamond V = S + U \diamond V.$$
In the following we derive a couple of relations between U and $A + B$ which will imply that U is a non-degenerate integrated semigroup and $A+B$ its generator (see [Thi90], Theorem 3.1). For this purpose let $x \in X$. By Lemma 5.4,
$$\int_0^t U(s)xds = \int_0^t S(s)xds + (S * W)(t)x \in D(A)$$
and
$$A\int_0^t U(s)xds = S(t)x - tx + (S \diamond W)(t)x - \int_0^t W(s)xds.$$
Further
$$B\int_0^t U(s)xds = \int_0^t V(s)xds + (V * W)(t)x = \int_0^t W(s)xds.$$
Combining these two equations,
$$(A+B)\int_0^t U(s)xds = U(t)x - tx.$$
This calculation also shows that U is a non-degenerate operator family; indeed if $U(t)x = 0$ for all $t \geq 0$, then $W(t)x = 0$ for all $t \geq 0$ and so $S(t)x = 0$ for all $t \geq 0$ which implies $x = 0$ because S is non-degenerate.
Finally assume that $x, y \in X$ and
$$U(t)x = tx + \int_0^t U(s)yds.$$

Integrating this equation we find that $x \in D(A)$. Therefore we get

$$\begin{aligned} U(t)x &= tx + \int_0^t S(s)Ax\ ds + (U * V)'(t)x \\ &\underset{(33)}{=} tx + \int_0^t S(s)Ax\ ds + \int_0^t U(t-s)(Bx + V(s)Ax)ds \\ &= tx + \int_0^t S(s)Ax\ ds + \int_0^t U(s)Bxds + (U * V)(t)Ax \\ &= tx + \int_0^t (S(s) + (U \diamond V)(s))Ax\ ds + \int_0^t U(s)Bx\ ds \\ &= tx + \int_0^t U(s)(A+B)x\ ds \end{aligned}$$

and this implies

$$\int_0^t U(s)y\ ds = \int_0^t U(s)(A+B)x\ ds.$$

Finally, the non-degeneracy of U provides $(A+B)x = y$.
This shows that U is an integrated semigroup and $A + B$ its generator (Theorem 3.1 in [Thi90]). □

Assumption 3. There exists a closed subspace Z of X and some $T > 0$ such that B maps $D(A)$ into Z and

$$\mathbf{PV}^{\oplus}(V; [0,T), Z) < 1.$$

6.4 Theorem. *Under Assumptions 1 and 3, $A+B$ is the generator of an integrated semigroup $(U(t))_{t\geq 0}$ that is of local strong bounded p-variation, $U = S + U \diamond V$.*

Proof. It follows from Proposition 2.11 that

$$W = \sum_{n=1}^{\infty} V^{\diamond n}$$

exists in $\mathcal{PV}([0,T); X)$ for some $T > 0$ and

$$W = V + V \diamond W = V + W \diamond V.$$

Let $t > 0$ be arbitrary. Choosing $\lambda > 0$ large enough, we have for all $x \in Z$

$$\mathbf{pv}_1^{\lambda}(V(\cdot)x; [0,t), Z) + \|V^{\oplus}(0)x\| \leq \eta \|x\|$$

with some $\eta < 1$. Similar arguments as in Section 3 imply that

$$\mathbf{PV}^{\oplus}(V^{\lambda}; [0,t), Z) \leq \eta < 1,$$

and as before we find some W^{λ} of strong bounded pseudo-variation such that

$$W^{\lambda} = V^{\lambda} + V^{\lambda} \diamond W^{\lambda} = V^{\lambda} + W^{\lambda} \diamond V^{\lambda}.$$

Setting $W = (W^{\lambda})^{-\lambda}$, by Proposition 3.2 we find a $\diamond$-resolvent W of V,

$$W = V + V \diamond W = V + W \diamond V \quad \text{ on } [0,t).$$

Since the restriction of W to any smaller interval still has these properties, it coincides with the $\diamond$-resolvent of V on that interval. Since t was arbitrary, we can define a $\diamond$-resolvent W of V on $[0, \infty)$ which is of local strong variation. Thus the claim follows by Proposition 6.3. □

Perturbations in abstract L-spaces

6.5 Theorem. *Let $w \in \mathbf{R}_+$ Suppose that A, $\bar{A}$ are closed and $B, \bar{B}$ are linear operators on an abstract L-space X such that B is A-bounded and $\bar{B}$ is $\bar{A}$-bounded. Let (w, ∞) be contained in $\rho(A; X) \cap \rho(\bar{A}; X)$ and let $\bar{A}$ be resolvent positive and A be absolutely resolvent majorized by $\bar{A}$ on (w, ∞). Further let $B(\lambda - \bar{A})^{-1}$ absolutely majorize the operators $\bar{B}(\lambda - A)^{-1}$ for all $\lambda \in (w, \infty)$. If $\nu > w$ and*

$$\mathrm{spr}(\bar{\mathrm{B}}\mathrm{R}(\nu, \bar{\mathrm{A}})) < 1,$$

$A + B$ and $\bar{A} + \bar{B}$ generate exponentially bounded integrated semigroups $(U(t))_{t \geq 0}$ and $(\bar{U}(t))_{t \geq 0}$ of local bounded pseudo-variation such that $(\bar{U}(t))_{t \geq 0}$ dominates $(U(t))_{t \geq 0}$.
Let $\mu \geq \nu$. If the resolvent positive operator $\bar{A}$ generates an integrated semigroup $(\bar{S}(t))_{t \geq 0}$ such that $(\bar{S}^{\mu}(t))_{t \geq 0}$ is of strong bounded-p-variation on $\mathbf{R}_+$, $1 \leq p < \infty$, then $(U^{\mu}(t))_{t \geq 0}$ and $(\bar{U}^{\mu}(t))_{t \geq 0}$ are of strong bounded p-variation on $\mathbf{R}_+$.

This result can be generalized at the expense of some technicalities.

6.6 Remark. Suppose that X is a Banach lattice and $A, \bar{A}$ are closed linear operators such that A is resolvent-majorized by $\bar{A}$ and $\bar{A}$ generates an integrated semigroup $\bar{S}$. Let $\nu > s(\bar{A})$ and $\bar{S}^{\nu}$ be of strong bounded p-variation on $\mathbf{R}_+$. Further let B and $\bar{B}$ be A-bounded and $\bar{A}$-bounded linear operators respectively such that $B(\lambda - A)^{-1}$ is absolutely majorized by $\bar{B}(\lambda - \bar{A})^{-1}$ for sufficiently large λ. Assume that spr $\bar{B}(\nu - \bar{A})^{-1} < 1$. Finally assume $\bar{B}$ maps $D(\bar{A})$ into a closed subspace Z of X which can equivalently be renormalized such that the norm becomes additive on $X_+ \cap Z$ (e.g., let $\bar{B}$ be a finite rank operator). Then $A + B$ and $\bar{A} + \bar{B}$ generate integrated semigroups U and $\bar{U}$ such that U^{ν} and $\bar{U}^{\nu}$ are of strong bounded p-variation and $\bar{U}$ dominates U.

Proof of Theorem 6.5 . Since $\bar{A}$ is resolvent positive it follows from [Are87a] that $\bar{A}$ generates an increasing integrated semigroup $\bar{S}$. Since L is an abstract L-space every increasing family of bounded operators is of local strong bounded variation. Thus $\bar{A}$ satisfies the first part of Assumption 1.
We now show that $\bar{A} + \bar{B} - \mu$ generates an integrated semigroup that is of strong bounded p-variation on $\mathbf{R}_+$ if $\mu \geq \nu$. We consider for $\lambda > w$

$$\bar{F}(\lambda) := \bar{B}R(\lambda, \bar{A}).$$

Since $\bar{A}$ is resolvent positive for $\lambda > w$ and $\bar{B}$ is positive, we have

$$(-1)^n \frac{d^n}{d\lambda^n} \bar{F}(\lambda) = n! \bar{B}R(\lambda, \bar{A})^{n+1} \geq 0, \qquad \lambda > w,$$

i.e. $(\bar{F}(\lambda))_{\lambda > w}$ is completely monotonic. By the vector valued version of Bernstein's theorem (see [Are87a], section 5) there exists an increasing family of bounded operators $(\bar{V}(t))_{t \geq 0}$ with

$$\lambda \int_0^{\infty} e^{\lambda t} \bar{V}(t)\, dt = \bar{B}R(\lambda, \bar{A}), \quad \lambda > w. \tag{34}$$

Using the uniqueness of the Laplace-Stieltjes transform and partial integration, it is easy to see that $\bar{V}$ and $\bar{B}$ satisfy Assumption 1. Since $\mathrm{spr}(\bar{\mathrm{F}}(\cdot))$ is a decreasing function, we have $\mathrm{spr}(\bar{\mathrm{F}}(\lambda)) < 1$ for all $\lambda \geq \nu$. Moreover, the semi-continuous dependence of the spectral radius on the operator and the norm-continuity of $BR(\lambda, A)$ allows us to choose some $\tilde{\nu} \in (w, \nu)$ such that the spectral radius of $BR(\tilde{\nu}, A)$ is still strictly smaller than 1. Define

$$\bar{G}(\lambda) = \sum_{j=1}^{\infty} \bar{F}(\lambda)^j, \qquad \lambda \geq \tilde{\nu}. \tag{35}$$

It follows from the binomial formula for the differentiation of products that the composition of two operator families that are completely monotonic is again completely monotonic. Therefore $(\bar{G}(\lambda))_{\lambda>\nu}$ is completely monotonic and by the vector valued version of Bernstein's theorem we find an exponentially bounded increasing family $(\bar{W}(t))_{t\geq 0}$ such that

$$\bar{G}(\lambda) = \lambda \int_0^\infty e^{-\lambda t}\bar{W}(t)\, dt, \qquad \lambda > \tilde{\nu}. \tag{36}$$

From (35) we obtain

$$\bar{G}(\lambda) = \bar{F}(\lambda) + \bar{F}(\lambda)\bar{G}(\lambda) = \bar{F}(\lambda) + \bar{G}(\lambda)\bar{F}(\lambda)$$

and the uniqueness of the Laplace-Stieltjes transform yields

$$\bar{W} = \bar{V} + \bar{V} \diamond \bar{W} = \bar{V} + \bar{W} \diamond \bar{V}.$$

Since every bounded increasing family on an abstract L-space is of strong bounded variation we see that the Assumption 2 holds. Thus, by Proposition 6.3, $\bar{A}+\bar{B}$ generates an integrated semigroup $\bar{U}$ of local strong bounded variation which is given by $\bar{U} = \bar{S} + \bar{S} \diamond \bar{W}$. Now suppose S^μ is of strong bounded p-variation on $\mathbf{R}_+$ for some $\mu \geq \nu$. Since $\mu \geq \nu > \tilde{\nu}$, $\bar{W}^\mu$ is a bounded increasing family in an abstract L-space and therefore of strong bounded variation. Since $\bar{U}^\mu = S^\mu + S^\mu \diamond \bar{W}^\mu$ by Proposition 3.5, we conclude from Proposition 2.10 that $\bar{U}$ is an integrated semigroup of local strong bounded p-variation and that $\bar{U}^\mu$ is of strong bounded p-variation on $\mathbf{R}_+$. Furthermore $\bar{U}^\mu$ is generated by $\bar{A}+\bar{B}-\mu$. We now show that $A+B$ is absolutely resolvent majorized by $\bar{A}+\bar{B}$ and this will finish the proof by Theorem 4.5. Since $F(\lambda) := BR(\lambda, A) \leq \bar{B}R(\lambda,\bar{A})$ for $\lambda \geq \nu$ we have that $\mathrm{spr}(\mathrm{F}(\lambda)) < 1$ and $\lambda \in \rho(A+B;X)$ with

$$\begin{aligned}
|R(\lambda, A+B)x| &= \left|R(\lambda,A)(\mathrm{Id}-F(\lambda))^{-1}x\right| \leq R(\lambda,\bar{A})\left|(\mathrm{Id}-F(\lambda))^{-1}x\right| \\
&\leq R(\lambda,\bar{A})\sum_{j=1}^{\infty}\left|(F(\lambda))^j x\right| \\
&\leq R(\lambda,\bar{A})\sum_{j=1}^{\infty}\left|(\bar{F}(\lambda))^j\right||x| = R(\lambda,\bar{A})(\mathrm{Id}-\bar{F}(\lambda))^{-1}|x| \\
&= R(\lambda,\bar{A}+\bar{B})\,|x|
\end{aligned}$$

for all $x \in X$. □

We state some consequences of this result. We recall that every increasing integrated semigroup on an abstract L-space is generated by an resolvent positive operator.

6.7 Corollary. *Suppose A generates an increasing integrated semigroup $(S(t))_{t\geq 0}$ of positive operators on an abstract L-space such that $(S^w(t))_{t\geq 0}$ is of strong bounded p-variation on $\mathbf{R}_+$ for some $1 \leq p \leq \infty$ and $w \geq w(S)$. Let $B = B_+ - B_-$ be the difference of two A-bounded operators such that $B_+R(\lambda,A)$ and $B_-R(\lambda,A)$ are positive for all $\lambda > w$. If*

$$\mathrm{spr}((\mathrm{B}_+ + \mathrm{B}_-)\mathrm{R}(\nu,\mathrm{A})) < 1$$

for some $\nu > w$ we have that $A+B$ generates an integrated semigroup $(U(t))_{t\geq 0}$ that is of local strong bounded p-variation and $(U^\mu(t))_{t\geq 0}$ is of strong bounded p-variation on $\mathbf{R}_+$ for all $\mu \geq \nu$.

6.8 Corollary. *If A is a resolvent positive operator on an abstract L space X and $B_+, B_- : D(A) \to X$ are positive and give rise to a resolvent positive perturbation $A + B_+ + B_-$, then for some $w \in \mathbf{R}$ the perturbation $A + B_+ - B_- - w$ generates an integrated semigroup of strong bounded variation.*

Proof. Since $A + B_- + B_+$ is resolvent positive we find $w \in \mathbf{R}$ such that $A + B_- + B_+ - w$ generates an increasing bounded integrated semigroup S (see Theorem 5.7 of [Are87a]). Since X is an abstract L-space we know that S is of strong bounded variation on $\mathbf{R}_+$.
Theorem 1.1 of [Voi89] ensures that $\operatorname{spr}((B_+ + B_-)R(\lambda, A)) < 1$ if $\lambda > w$ and the result follows from Corollary 6.7. □

6.9 Remark. In the situation that $p = \infty$ and A densely defined, Corollary 6.7 was proved in [Des]. The special case $B_- = 0$ was also shown in [Voi89], and, without assuming the density of $D(A)$, in [Thi96].

Miyadera perturbations on Banach spaces

We will use the Widder characterization of functions of bounded p-variation to obtain perturbation results on Banach spaces for integrated semigroups that are of strong bounded p-variation. As a consequence we obtain a generalization of the Miyadera-Voigt theorem for the case that A is densely defined and $1 \leq p \leq \infty$.

6.10 Theorem. *Let A be the generator of an exponentially bounded integrated semigroup $(S(t))_{t\geq 0}$ that is of local strong bounded p-variation for some $1 \leq p \leq \infty$. Suppose B is an A-bounded linear operator on X and there is a $w \in \mathbf{R}$ with $(w, \infty) \subseteq \rho(A; X)$ and*

$$\sup_{k\in\mathbf{N},\ \|x\|\leq 1} \int_0^\infty k\lambda^{k-1} \left\|BR(\lambda + w, A)^{k+1}x\right\| d\lambda < 1. \tag{37}$$

Then $A + B$ generates a exponentially bounded integrated semigroup $(U(t))_{t\geq 0}$ that is of local strong bounded p-variation and we have

$$U = S + (U \diamond V), \tag{38}$$

where V is the Laplace kernel of $BR(\cdot, A)$, i.e.

$$BR(\lambda, A)x = \int_0^\infty e^{-\lambda t} dV(t)x, \qquad \lambda > w, \qquad x \in X.$$

Furthermore if $(S^w(t))_{t\geq 0}$ is of bounded p-variation on $\mathbf{R}_+$ (or, equivalently, the condition in Theorem 4.3 (b) holds), then U^w is of bounded p-variation on $\mathbf{R}_+$ as well.

Proof. By Proposition 3.2 and Lemma 3.6, we can assume $w = 0$. It follows from (37) and

$$BR(t, A)x - BR(1, A)x = \int_1^t (BR(s, A))'x\, ds, \qquad x \in X,\ t > 1 \tag{39}$$

that $\lim_{\lambda\to\infty} BR(\lambda, A)x$ exists for all $x \in X$. By (37) and Theorem 4.2 (b), we find some $\gamma < 1$ and for every and for every $x \in X$ a function $V_x \in V_p(\mathbf{R}_+, X)$ such that $\|V_x\|_{v_p} \leq \gamma \|x\|$ and

$$BR(\lambda, A)x = \int_0^\infty e^{-\lambda t} dV_x(t), \qquad \lambda > 0, \qquad x \in X.$$

Therefore $\|V_x(t)\| = \left\|\int_0^t dV_x(s)\right\| \le \gamma \|x\|$ and by $V(t)x := V_x(t)$ we define a bounded operator on X for every $t \ge 0$. A short calculation using the uniqueness of the Laplace-Stieltjes transform shows that $(V(t))_{t\ge 0}$ satisfies Assumption 1. Furthermore we have

$$\mathbf{PV}^{\oplus}(V;[0,\infty),X) < 1. \tag{40}$$

Thus by Theorem 6.4, $A+B$ generates an integrated semigroup U that is of local strong variation and we have

$$U = S + U \diamond V = S + S \diamond \sum_{j=1}^{\infty} V^{\diamond j}.$$

By (40) and Proposition 2.10, $\mathbf{PV}^{\oplus}(\sum_{j=1}^{\infty} V^{\diamond j};[0,\infty),X) < \infty$. If S^w is of strong bounded p-variation on $\mathbf{R}_+$ for some $w \in \mathbf{R}_+$ then by

$$U^w = S^w + U^w \diamond V^w = S^w + S^w \diamond \sum_{j=1}^{\infty} (V^w)^{\diamond j}$$

U^w of strong bounded p-variation on $\mathbf{R}_+$ (see Proposition 2.10).
If S is locally Lipschitz continuous ($p=\infty$) we have by Proposition 2.10 for all $1 \le p < \infty$ (recall $S \diamond W = S \star W$ since $S(0)=0$)

$$\begin{aligned}\mathbf{PV}_p(S \diamond \sum_{j=1}^{\infty} V^{\diamond j}x;[0,T)) &\le \mathbf{PV}_p(S;[0,T),X)\ \mathbf{PV}^{\oplus}(\sum_{j=1}^{\infty} V^{\diamond j};[0,T),X) \\ &\underset{(7)}{\le} T^{\frac{1}{p}}\ \mathbf{PV}_\infty(S;[0,T),X)\ \mathbf{PV}^{\oplus}(\sum_{j=1}^{\infty} V^{\diamond j};[0,\infty),X).\end{aligned}$$

Thus by Proposition 1.3, U is locally Lipschitz continuous. □

6.11 Remark. If B maps $D(A)$ into a closed subspace Z of X, it is possible to assume the estimate (31) for $x \in Z$ only and to assume just the existence of the supremum for $x \in X$.

The next results deal with the situation that B is bounded on $\overline{D(A)}$.

6.12 Lemma. *Let A be the generator of an integrated semigroup that is of strong bounded p-variation on $\mathbf{R}_+$ for some $1 < p < \infty$. Then, for any $\varepsilon > 0$, there is some $w > 0$ such that*

$$\sup_{k\in\mathbf{N},\ \|x\|\le 1} \int_0^\infty k\lambda^{k-1} \left\|R(\lambda+w,A)^{k+1}x\right\| d\lambda \ < \ \varepsilon.$$

Proof. Recall the notation $S^r(t) = \int_0^t e^{-rs} dS(s)$. By Proposition 3.2 (c) we obtain for $x \in X$ and $r > 0$

$$\begin{aligned}\|S^r(\cdot)x\|_{v_1} &= \overline{\lim_{h\searrow 0}} \frac{1}{h} \int_0^\infty e^{-ru} \|(S(u+h)-S(u))x\|\, du \\ &\le \left(\int_0^\infty e^{-rqt} dt\right)^{\frac{1}{q}} \overline{\lim_{h\searrow 0}} \frac{1}{h} \left(\int_0^\infty \|(S(u+h)-S(u))x\|^p\, du\right)^{\frac{1}{p}} \\ &= \left(\frac{1}{rq}\right)^{\frac{1}{q}} \mathbf{PV}_p(S;\mathbf{R}_+)\, \|x\|.\end{aligned} \tag{41}$$

Let $\varepsilon > 0$. Choose $w > 0$ such that $\left(\frac{1}{wq}\right)^{\frac{1}{q}}$ $\mathbf{PV}_p(S;\mathbf{R}_+) < \varepsilon$. By Theorem 4.1 we obtain

$$\sup_{k\in\mathbf{N},\ \|x\|\le 1} \int_0^\infty \left\|k\lambda^{k-1}R(\lambda+w,A)^{k+1}x\right\| d\lambda < \varepsilon.$$

□

As an immediate consequence of Theorem 6.10 and Lemma 6.12 we obtain

6.13 Corollary. *Let A be the generator of an integrated semigroup that is of strong bounded p-variation on $\mathbf{R}_+$ for some $1 < p < \infty$ and $B \in L(\overline{D(A)}, X)$. Then there is a $w > 0$ such that $A + B - w$ generates an integrated semigroup that is of strong bounded variation.*

For the case that $B \in L(\overline{D(A)})$ commutes with A we have also a result for $p = 1$.

6.14 Proposition (cf. Prop. 3.1 in [KH89]). *Suppose A is the generator of an integrated semigroup $(S(t))_{t\ge 0}$ that is of strong bounded variation on $\mathbf{R}_+$ and $B \in L(\overline{D(A)})$ commutes with the resolvents of A on $\overline{D(A)}$. Then $A + B - \|B\| - \varepsilon$ generates for all $\varepsilon > 0$ an integrated semigroup $(\Psi(t))_{t\ge 0}$ that is of strong bounded variation and we have for $x \in X$ and $t > 0$:*

$$\Psi(t)x = e^{t(B-\|B\|-\varepsilon)}S(t)x - (B - \|B\| - \varepsilon)\int_0^t e^{s(B-\|B\|-\varepsilon)}S(s)xds.$$

Proof. In Proposition 3.1 of [KH89] this assertion was shown except the fact that Ψ is of strong bounded variation on $\mathbf{R}_+$. Since $e^{-\varepsilon\cdot}S(\cdot)x \in L_1(\mathbf{R}_+, X)$ we have that

$$t \mapsto \int_0^t e^{s(B-\|B\|-\varepsilon)}S(s)xds$$

is of bounded variation on $\mathbf{R}_+$ and so is the second term in the representation formula. Since $S(u)$ maps X into $\overline{D(A)}$ we obtain that

$$\begin{aligned}
&\overline{\lim_{h\searrow 0}}\ \frac{1}{h}\int_0^\infty \left\|e^{(u+h)(B-\|B\|-\varepsilon)}S(u+h)x - e^{u(B-\|B\|-\varepsilon)}S(u)x\right\| du\\
\le\ &\overline{\lim_{h\searrow 0}}\ \frac{1}{h}\int_0^\infty e^{-u(\|B\|+\varepsilon)}e^{u\|B\|}\left\|e^{h(B-\|B\|-\varepsilon)}(S(u+h)-S(u))x\right\| du\\
&+\overline{\lim_{h\searrow 0}}\ \frac{1}{h}\int_0^\infty e^{-u(\|B\|+\varepsilon)}e^{u\|B\|}\left\|e^{h(B-\|B\|-\varepsilon)}S(u)x - S(u)x\right\| du\\
\le\ &\left(\mathbf{PV}_1(S;\mathbf{R}_+) + \|(B-\|B\|-\varepsilon)\|\ \sup_{r>0}\|S(r)\|\int_0^\infty e^{-u\varepsilon}du\right)\|x\| < \infty.
\end{aligned}$$

Therefore is $\Psi = \Psi^{\oplus}$ of strong bounded variation on $\mathbf{R}_+$. □

We now turn to an extension of the Miyadera-Voigt perturbation theory to integrated semigroups of strong bounded p-variation with densely defined operator.

6.15 Theorem (Thm. 1 of [Voi77] for $p = \infty$). *Let $1 \le p \le \infty$ and A be densely defined and the generator of an integrated semigroup $(S(t))_{t\ge 0}$ that is of strong bounded p-variation on $\mathbf{R}_+$. Suppose there is a linear operator B and a dense subspace D such that the following conditions are satisfied:*

(1) $D \subseteq D(A) \cap D(B)$, $S'(t)D \subseteq D$, and for all $x \in D$ the function $BS'(\cdot)x$ is continuous.

(2) There are $0 < \alpha \leq \infty$ and $\gamma \in (0,1)$ such that

$$\int_0^\alpha \|BS'(t)x\|\,dt \leq \gamma\ \|x\| \qquad \textit{for all } x \in D.$$

Then there is an unique A-bounded extension $\hat{B}$ of $B|_D$ to $D(A)$ and $A+\hat{B}$ generates a once integrated semigroup $(\Psi(t))_{t\geq 0}$ that is of strong bounded p-variation on $\mathbf{R}_+$. We have

$$\Psi'(t)x = S'(t)x + (\Psi \diamond BS'x)(t) \qquad \textit{for } x \in D \textit{ a.e..}$$

6.16 Remark. (a) If B maps D into a closed subspace Z of X such that $D \cap Z$ is dense in Z, in (2) it is sufficient to assume $\gamma < 1$ only for $x \in Z$, while for $x \in X$ an arbitrary constant $\gamma < \infty$ will do.

(b) One can define a Lebesgue convolution for Ψ' with L_1-functions and denote the variation of constant formula in the classical form

$$\Psi'(t)x = S'(t)x + (\Psi' * BS'x)(t) \qquad \text{for } x \in D \text{ a.e..}$$

Proof. Recall that every locally Lipschitz continuous integrated semigroup is an integrated C_0-semigroup if $D(A)$ is dense (combine Theorem 2.4 of [KH89] and the Hille-Yosida theorem). For $D(A)$ dense we obtain from Corollary 5.2 that $S(\cdot)x \in W^1_{loc}(\mathbf{R}_+, X)$ for all $x \in X$. We first show that for every $0 < \varepsilon < \frac{\alpha}{2}$ there is a $\mu > 0$ such that for all $x \in D$

$$\int_0^\infty e^{-\mu t}\,\|BS'(t)x\|\,dt \leq \frac{1}{1-\alpha^{-1}2\varepsilon}\gamma\,\|x\|\,. \tag{42}$$

Let $0 < \varepsilon < \frac{\alpha}{2}$. Choose a $\mu > 0$ such that

$$\int_{\frac{\alpha}{2}}^\alpha e^{-\mu s}\,\|S'(s)x\|\,ds \leq e^{-\mu\frac{\alpha}{2}}\mathbf{PV}_1(S;[0,\alpha))\,\|x\| \leq \varepsilon\,\|x\|\,.$$

Let $\xi = 2\alpha^{-1}\varepsilon$. For every $x \in D$ we find a $t_x \in (\frac{\alpha}{2}, \alpha)$ such that

$$e^{-\mu t_x}\,\|S'(t_x)x\| \leq \xi\,\|x\|\,. \tag{43}$$

We show by induction that for all $n \in \mathbf{N}$ and $x \in D$ we have

$$\int_0^{\alpha+nt_x} \left\|e^{-\mu t}BS'(t)x\right\|dt \leq \left(\sum_{j=0}^{n-1}\gamma\xi^j + \xi^n\right)\|x\|\,. \tag{44}$$

For $n = 0$ the assertion is true by our assumption. Suppose the assertion is true for $n \geq 0$. Since $S'(t+t_x)x = S'(t)S'(t_x)x$ and $S'(t_x)x \in D$, we have by (43)

$$\begin{aligned}\int_0^{\alpha+(n+1)t_x} e^{-\mu t}\,\|BS'(t)x\|\,dt &= \int_0^{t_x} e^{-\mu t}\,\|BS'(t)x\|\,dt \\ &\quad + \int_0^{\alpha+nt_x} e^{-\mu t}\,\left\|BS'(t)(e^{-\mu t_x}S'(t_x)x)\right\|dt \\ &\leq \gamma\,\|x\| + \left(\sum_{j=0}^{n-1}\gamma\xi^j + \xi^n\right)\left\|e^{-\mu t_x}S'(t_x)x\right\| \\ &\leq \left(\sum_{j=0}^{n}\gamma\xi^j + \xi^{n+1}\right)\|x\|\,. \end{aligned} \tag{45}$$

Thus the assertion is true for $n+1$, and from (44) we obtain (42) by letting n tend to ∞.
We choose $0 < \varepsilon < \frac{\alpha}{2}$ such that $c := \frac{1}{1-\alpha^{-1}2\varepsilon}\gamma < 1$ and $\mu > 0$ according to ε such that (42) is true. By (42), for $\lambda > \mu$

$$D \to L_1(\mathbf{R}_+, X),\ x \mapsto e^{-\lambda(\cdot)}BS'(\cdot)x$$

has an unique bounded extension K to X. Furthermore define for all $\lambda > \mu$

$$F(\lambda)x = \int_0^\infty e^{-\lambda t}(Kx)(t)dt, \qquad x \in X. \tag{46}$$

We first remark that by (42) we have (see Theorem 4.1)

$$\sup_{k\in\mathbf{N},\ \|x\|\le 1} \int_0^\infty \left\|(F(\lambda)x)^{(k)}\right\| \frac{\lambda^{k-1}}{(k-1)!}d\lambda < 1 \tag{47}$$

and $\lim_{\lambda\to\infty} F(\lambda)x = 0$ for all $x \in X$. We now verify that the so called resolvent output identity

$$F(\lambda)R(z,A) = \frac{1}{\lambda - z}(F(z) - F(\lambda)) \tag{48}$$

holds for $\lambda, \eta > \mu$. For $x \in D$ the resolvent output identity can be derived from (46) by using the boundedness of K and the semigroup property of S'. Since D is dense (48) is established on X. It is now easy to see that the definition

$$\hat{B} := F(\lambda)(\lambda - A)$$

is independent of $\lambda > \mu$ and provides a linear A-bounded operator $\hat{B}$ (see [Thi96], section 4).
Since (47) holds, our Theorem 6.10 applied to $\hat{B}R(\lambda, A) = F(\lambda)$ yields that $A + \hat{B}$ generates an integrated semigroup Ψ that is of strong bounded p-variation.
We show that $\hat{B}|_D = B|_D$. Since $s \mapsto BS'(s)x$ is continuous for $x \in D$ we have

$$Bx = \lim_{\lambda\to\infty} \int_0^\infty e^{-t}BS'\left(\frac{t}{\lambda}\right)xdt = \lim_{\lambda\to\infty} \lambda \int_0^\infty e^{-\lambda t}BS'(t)xdt \tag{49}$$

and together with (48) we obtain

$$\begin{aligned} Bx = \lim_{\lambda\to\infty} \lambda \int_0^\infty e^{-\lambda t}BS'(t)xdt &= \lim_{\lambda\to\infty} \lambda F(\lambda)x \\ &= \lim_{\lambda\to\infty} \lambda F(\lambda)R(z,A)(z-A)x \\ &= \lim_{\lambda\to\infty} \frac{\lambda}{\lambda - z}(F(z)(z-A)x - F(\lambda)(z-A)x) \\ &= F(z)(z-A)x = \hat{B}x. \end{aligned}$$

Now, suppose $\tilde{B}$ is an A-bounded extension of $B|_D$ to $D(A)$. Let $\nu \in \rho(A;X)$. We have for $\lambda > \mu$ and $x \in D$

$$\begin{aligned} F(\lambda)x &= \int_0^\infty e^{-\lambda t}\tilde{B}S'(t)xdt = \int_0^\infty e^{-\lambda t}\tilde{B}S'(t)R(\nu,A)(\nu - A)xdt \\ &= \int_0^\infty e^{-\lambda t}\tilde{B}R(\nu,A)S'(t)(\nu - A)xdt = \tilde{B}R(\nu,A)\int_0^\infty e^{-\lambda t}S'(t)(\nu - A)xdt \\ &= \tilde{B}R(\nu,A)R(\lambda,A)(\nu - A)x = \tilde{B}R(\lambda,A)x. \end{aligned}$$

Since D is dense we conclude $\hat{B}R(\lambda, A) = F(\lambda) = \tilde{B}R(\lambda, A)$ as desired.

It only remains to show the variation of parameter formula. By (38) of Theorem 6.10 we have together with Corollary 5.2

$$\Psi'(t)x = S'(t)x + (\Psi \star V(\cdot)x)'(t) \qquad a.e.. \tag{50}$$

where

$$\int_0^\infty e^{-\lambda t} dV(t)x = \hat{B}R(\lambda, A)x = F(\lambda)x = \int_0^\infty e^{-\lambda t} BS'(t)x dt, \quad x \in D.$$

The uniqueness theorem of the Laplace transform yields

$$V(t)x = \int_0^t BS'(u)x du$$

and since $V(0)x = 0 = \Psi(0)$ for all $x \in D$, we obtain from Proposition 2.3 (b)

$$(\Psi \star V(\cdot)x)'(t) = (\Psi * BS'x)'(t) = (\Psi \diamond BS'x)(t) \qquad \text{for } x \in D \text{ a.e..}$$

By (50) we are done. □

Acknowledgement. H.R. T. thanks S. Oharu for useful discussions. The foundations of this paper were laid, when H. V. visited Arizona State University from January to August 1999, and H. V. is grateful to Arizona State University for their hospitality.

Literatur

[Are] W. Arendt. Resolvent positive operators and inhomogeneous boundary conditions. Ulmer Seminare zur Funktionalanlysis 1999.

[Are87a] W. Arendt. Resolvent positive operators. *Proc. London Math. Soc. 54*, pages 321–359, 1987.

[Are87b] W. Arendt. Vector valued Laplace transforms and Cauchy problems. *Israel J. Math. 59*, pages 327–352, 1987.

[Bob97] A. Bobrowski. On the generation of non-continuous semigroups. *Semigroup Forum 54*, pages 237–252, 1997.

[BT38] S. Bochner and A. E. Taylor. Linear functionals on certain spaces of abstractly-valued functions. *Ann. of Math. 2*, pages 163–168, 1938.

[Cer94] S. Cerrai. A Hille-Yosida theorem for weakly continuous semigroups. *Semigroup Forum 49*, pages 349–367, 1994.

[Des] W. Desch. Perturbations of positive operators on AL-spaces. 1985. unpublished.

[dPS87] G. da Prato and E. Sinestrari. Differential operators with nondense domain. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. 14 (2)*, pages 285–344, 1987.

[DS88] N. Dunford and J.T. Schwartz. *Linear Operators PartI: General Theory.* Wiley, London, 1988.

[DU77] J. Diestel and J.J. Uhl. *Vector Measures.* AMS, Providence, 1977.

[Fat83] H.O. Fattorini. *The Cauchy Problem.* Addison-Wesley, Reading, 1983.

[GLO90] G. Gripenberg, S. O. Londen, and O.Staffans. *Volterra Integral and Functional Equations.* Cambridge Univerity Press, Cambridge, 1990.

[Hie91] M. Hieber. Laplace transforms and α-times integrated semigroups. *Forum Math. 3*, pages 595–612, 1991.

[HP57] E. Hille and R. Phillips. *Functional analysis and semigroups.* AMS, 1957.

[KH89] H. Kellermann and M. Hieber. Integrated semigroups. *J. Func. Anal. 84*, pages 160–180, 1989.

[Oha71] S. Oharu. Semigroups of linear operators in a Banach space. *Publ. RIMS, Kyoto Univ. 7*, pages 205–260, 1971.

[Sch74] H.H. Schaefer. *Banach Lattices and Positive Operators.* Springer Verlag, Berlin-Heidelberg-New York, 1974.

[SS86] J. I. Silchenko and P.E. Sobolevskii. Solvability of the Cauchy problem for an evolution equation in a Banach space with a non-densely given operator coefficient which generates a semigroup with a singularity. *Sibirsk. Mat. Zh. 27*, pages 93–104, 1986.

[Sun70] H. Sunouchi. Convergence of semi-discrete difference schemes of Cauchy problems. *Tohoku Math. 22*, pages 394–408, 1970.

[Thi90] H. R. Thieme. Integrated semigroups and integral solutions to abstract Cauchy problems. *J. Math. Analysis Appl.*, pages 416–447, 1990.

[Thi96] H. R. Thieme. Positive perturbations of dual and integrated semigroups. *Advances in Mathematical sciences and Applications*, pages 445–507, 1996.

[Vie95] P. Vieten. *Laplace Transformation und Holomorphie banachraumwertiger Funktionen.* Shaker, Aachen, 1995.

[Voi77] J. Voigt. On the perturbation theory for strongly continuous semigroups. *Math. Annalen 229*, pages 163–171, 1977.

[Voi89] J. Voigt. On resolvent positive operators and positive C_0-semigroups on AL-spaces. *Semigroup Forum 38*, pages 263–266, 1989.

[Wei91] L. Weis. L_p-flows. Seminar notes. Louisiana State University, 1991.

[Wei93] L. Weis. Inversion of the vector valued Laplace transform in $L_p(X)$-spaces. *Proceedings of the conference on Differential Equations in Banach spaces, Bologna 1991, Marcel Dekker*, pages 235–253, 1993.

[Wid46] D. Widder. *The Laplace Transform.* Princeton University Press, Princeton, 1946.