Linear delay equations in the L_p-context

Lahcen Maniar and Jürgen Voigt

Abstract

We show well-posedness of the Cauchy problem for the delay equation $u'(t) = Au(t) + \Phi u_t$, with initial values in $X \times L_p(-h, 0;\ X)$; X a Banach space, A the generator of a C_0-semigroup on X, $h \in \{1, \infty\}$, $1 \leqslant p < \infty$. In the first result, Φ is defined by a Stieltjes integral, and $p = 1$. In the second result, Φ is a continuous linear mapping from $W_p^1(-h, 0;\ X)$ to the Favard class of A.

MSC 2000: 34K06
Keywords: linear delay equation, functional differential equation, perturbation

Introduction

We study linear delay equations of the form

$$\text{(DE)} \qquad \begin{cases} u'(t) = Au(t) + \Phi u_t \quad (t \geqslant 0), \\ u(0) = x, \quad u_0 = f \end{cases}$$

on a Banach space X, in the L_p-context, for $1 \leqslant p < \infty$. Here, A is the generator of a C_0-semigroup on X, and Φ is a linear operator from $W_p^1(-h, 0; X)$ to X, where $h = 1$ or $h = \infty$. The aim is to show that equation (DE) is solvable for initial values

$$(x, f) \in D(A) \times W_p^1(-h, 0; X)$$

satisfying $x = f(0)$, and that the solution depends continuously on (x, f) with respect to the topology of $X \times L_p(-h, 0; X)$.

It was shown in [1; Sec.2] that (DE) is equivalent to an abstract Cauchy problem

$$\begin{cases} \mathcal{U}'(t) = \mathcal{A}_\Phi \mathcal{U}(t) \quad (t \geqslant 0), \\ \mathcal{U}(0) = \binom{x}{f} \end{cases}$$

on the space $\mathcal{E} := X \times L_p(-h, 0; X)$, where $\mathcal{A}_\Phi$ is given by

$$\mathcal{A}_\Phi := \begin{pmatrix} A & \Phi \\ 0 & \frac{d}{d\tau} \end{pmatrix},$$

with domain

$$D(\mathcal{A}_\Phi) := \left\{(x,f) \in D(A) \times W_p^1(-h,0;X);\ f(0)=x\right\}.$$

Formulated in this context, the aim of the present paper is to show that $\mathcal{A}_\Phi$ is a generator.

We treat two types of mappings Φ, and accordingly, the paper is divided into two parts.

In the first part we treat continuous linear mappings $\Phi\colon C([-h,0];X) \to X$ (of a special form; see Section 1). Here we consider only the case $p=1$, the case $1<p<\infty$ having been treated in [1]. (This is not strictly true since in [1] only the case $h=1$ of finite delay is considered. The proofs, however, cover also the case $h=\infty$ of infinite delay.) In this sense, this part should be considered as a supplement of [1; Example 3.4(b)].

In the second part we show that for continuous linear mappings $\Phi\colon W_p^1(-h,0;X) \to D(A)$ the operator $\mathcal{A}_\Phi$ defined above is a generator. (In fact, the result is slightly more general.) This generalizes results of [3], [4].

In Section 1 we will provide some notation, state the first result and recall the Miyadera perturbation theorem.

Section 2 contains the proof of the first result.

In Section 3 we state and prove our second result. In the proof we use a theorem of Desch and Schappacher [4].

In the Appendix, the last mentioned result is stated and provided with a new proof.

Acknowledgement. This paper was started during a stay of J. Voigt at Marrakesh, Morocco, which was supported by the DAAD as well as the Cadi Ayyad University, Marrakesh.

1 Notation and first result

Let X be a Banach space, and let A be the generator of a C_0-semigroup $(S(t);\ t \geqslant 0)$ on X. Let $1 \leqslant p < \infty$. We define a C_0-semigroup on the Banach space

$$\mathcal{E} := X \times L_p(-h,0;X)$$

(norm given by $\|(x,f)\| := \|x\| + \|f\|_p$) by

$$\mathcal{T}(t) := \begin{pmatrix} S(t) & 0 \\ S_t & T_0(t) \end{pmatrix},$$

where $(T_0(t);\ t \geqslant 0)$, given by

$$T_0(t)f := f(\cdot + t),$$

is the C_0-semigroup of left translation on $L_p(-h,0;X)$, and

$$S_t x(\tau) := \begin{cases} S(t+\tau)x & \text{for } -t \leqslant \tau < 0, \\ 0 & \text{for } -h < \tau < -t; \end{cases}$$

cf. [1]. The generator $\mathcal{A}$ of $\mathcal{T}(\cdot)$ is then given by

$$\mathcal{A} = \begin{pmatrix} A & 0 \\ 0 & \frac{d}{d\tau} \end{pmatrix},$$

with domain

$$D(\mathcal{A}) = \left\{(x,f) \in D(A) \times W_p^1(-h,0;X);\ f(0) = x\right\}.$$

(We will write vectors of a product space as row vectors; only if an operator matrix is applied to such a vector we will write it as a column vector.)

The operator Φ which we will treat here is described in the following way. Let

$$\eta\colon [-h,0] \to L(X)$$

be of bounded variation. Let

$$\Phi\colon C([-h,0];X) \to X$$

be the bounded linear operator given by the Stieltjes integral

$$\Phi(f) := \int_{[-h,0]} d\eta(\tau) f(\tau). \tag{1.1}$$

(Here we have used the notation

$$C([-\infty,0];\, X) := \{f \in C((-\infty,0];\, X);\ f(-\infty) := \lim_{\tau\to-\infty} f(\tau) \text{ exists}\}.)$$

Note that, by Sobolev's embedding theorem, $W_p^1(-h,0;\,X) \subseteq C([-h,0];\,X)$ with continuous embedding. (In fact, for $h = \infty$ one has

$$W_p^1(-\infty,0;\,X) \subseteq \{f \in C([-\infty,0];\,X);\ f(-\infty) = 0\}$$

with continuous embedding.) Thus, the operator

$$\mathcal{B} := \begin{pmatrix} 0 & \Phi \\ 0 & 0 \end{pmatrix}$$

is defined on $D(\mathcal{A})$, and is $\mathcal{A}$-bounded.

In order to explain the terminology in (DE) completely we recall that, for a function $u\colon (-h,\infty) \to X$, the function $u_t\colon (-h,0) \to X$ is defined by

$$u_t(\tau) := u(t+\tau),$$

for $t \geqslant 0$, $-h < \tau < 0$.

It was shown in [1; Theorem 2.8] that (DE) is well-posed if and only if

$$\mathcal{A}_\Phi := \begin{pmatrix} A & \Phi \\ 0 & \frac{d}{d\tau} \end{pmatrix},$$

with $D(\mathcal{A}_\Phi) = D(\mathcal{A})$, is a generator. In this way, the problem of solvability of (DE) is transformed into the question to show that $\mathcal{A}_\Phi$ is a generator.

It was further shown in [1; Example 3.4 (b)] that, for $1 < p < \infty$ and $h = 1$, $\mathcal{A}_\Phi$ is a generator. This was shown by proving that $\mathcal{B}$ defined above is an infinitesimally small Miyadera perturbation of $\mathcal{A}$ (see the terminology below). The same result, with the same proof, holds for $p = \infty$. Also, for $p = 1$ (and $h = 1$), it was shown that $\mathcal{B}$ is a small Miyadera perturbation of $\mathcal{A}$ if the total variation $|\eta|([-h, 0])$ of η is small enough, and therefore $\mathcal{A}_\Phi = \mathcal{A} + \mathcal{B}$ is a generator.

The first result of this paper is the following.

1.1 Theorem. *Let A, Φ be as above, and let $p = 1$. Then $\mathcal{A}_\Phi = \mathcal{A} + \mathcal{B}$ is a generator in $\mathcal{E}$. As a consequence,* (DE) *has a unique solution for all $(x, f) \in D(\mathcal{A}_\Phi) = D(\mathcal{A})$, and the solution depends continuously on (x, f).*

1.2 Remarks. (a) For the case that A is bounded Theorem 1.1 has been proved in [3] (for $\dim X < \infty$) and [4] (see also [6]), even for more general Φ. In this case, the operator A can be included in Φ. (This, in a way, is in strong contrast to our method since we will assume that η has no contribution at zero; see Section 2.) The method of proof in [3], [4], [6] is different from our method.

(b) For the case that $h = 1$ and η is given by a density $\phi \in L_1(-1, 0; L(X))$, it is shown in [2; Lemma 4.1] that Φ is an infinitesimally small Miyadera perturbation of $\mathcal{A}$ (and thus $\mathcal{A} + \mathcal{B}$ is a generator).

(c) For a variety of existence results for differential equations with delay, in Hilbert space, we refer to [12].

An essential tool in the proof of Theorem 1.1 will be the Miyadera perturbation theorem. Let $(T(t);\, t \geqslant 0)$ be a C_0-semigroup on the Banach space X, and let A be its generator. Let B be an A-bounded operator. Then B is called a Miyadera perturbation of A if there exist $\alpha > 0$, $\gamma \geqslant 0$ such that

$$\int_0^\alpha \|BT(t)x\| dt \leqslant \gamma \|x\|$$

for all $x \in D(A)$. The operator B is called a *small* Miyadera perturbation if α, γ can be chosen such that $\gamma < 1$, and *infinitesimally small* if the infimum of all possible γ is zero.

The Miyadera perturbation theorem states that, for a small Miyadera perturbation B, the operator $A + B$ is a generator; cf. [8], [9], [11], [7; III, Theorem 3.14].

2 Proof of the first result.

Let A, η, Φ, $\mathcal{A}$, $\mathcal{A}_\Phi$ be as in Section 1, with $p = 1$.

We will assume that η has no mass at zero, i.e.

$$\eta(0) = \lim_{\tau \to 0} \eta(\tau). \tag{2.1}$$

This can be done without restriction because otherwise one can add $\eta_0 := \eta(0) - \lim_{\tau\to 0-} \eta(\tau)$ to A; note that the operators $\mathcal{A}$ and $\begin{pmatrix} A+\eta_0 & \Phi - \eta_0\delta_0 \\ 0 & \frac{d}{d\tau} \end{pmatrix}$ coincide on $D(\mathcal{A})$. For convenience, we will henceforth assume that η is continuous from the left. This can be done without restriction since η has right and left limits at every point in $[-h,0]$, and replacing η by its left-continuous version does not affect the definition of Φ.

As a matter of notation, we will denote by $|\eta|$ the variation of η,

$$|\eta|(\tau) := \sup\left\{\sum_{j=1}^{n} \|\eta(\tau_j) - \eta(\tau_{j-1})\|; -h = \tau_0 < \ldots < \tau_n = \tau, n \in \mathbb{N}\right\} \ (-h \leqslant \tau \leqslant 0),$$

but also the measure associated with that monotone function.

The main idea of our proof is to add $\mathcal{B}$ to $\mathcal{A}$ in two steps. For $0 < \alpha \leqslant 1$ we define

$$\eta_\alpha(\tau) := \begin{cases} \eta(\tau) & \text{for} \quad -h \leqslant \tau < -\alpha, \\ \eta(-\alpha) & \text{for} \quad -\alpha \leqslant \tau \leqslant 0, \end{cases}$$

and $\tilde{\eta}_\alpha := \eta - \eta_\alpha$. Then

$$|\eta_\alpha|([-h,0]) = |\eta|([-h,-\alpha)) \leqslant |\eta|([-h,0]) \quad (-h \leqslant \alpha < 0),$$

and $|\tilde{\eta}_\alpha|([-h,0]) = |\eta|([-\alpha,0]) \to 0$ as $\alpha \to 0$, according to the requirement (2.1). As in (1.1), we define Φ_α, $\tilde{\Phi}_\alpha$ corresponding to η_α, $\tilde{\eta}_\alpha$, and $\mathcal{B}_\alpha = \begin{pmatrix} 0 & \Phi_\alpha \\ 0 & 0 \end{pmatrix}$, $\tilde{\mathcal{B}}_\alpha = \begin{pmatrix} 0 & \tilde{\Phi}_\alpha \\ 0 & 0 \end{pmatrix}$, respectively. Then we first show that $\mathcal{A} + \mathcal{B}_\alpha$ is a generator, and then that $\tilde{\mathcal{B}}_\alpha$ is a small Miyadera perturbation of $\mathcal{A} + \mathcal{B}_\alpha$, if α is small.

2.1 Lemma. *For $0 < \alpha \leqslant 1$, $\mathcal{A} + \mathcal{B}_\alpha$ is the generator of a C_0-semigroup $\mathcal{T}_\alpha(\cdot)$, and*

$$c := \sup\{\|\pi_2 \mathcal{T}_\alpha(t)\|; 0 < t \leqslant \alpha \leqslant 1\} < \infty.$$

(Here, π_2 denotes the canonical projection onto the second component of $X \times L_1(-h,0;X)$.)

Proof. (i) First we show that $\mathcal{A} + \mathcal{B}_\alpha$ is a generator.

For $(x,f) \in D(\mathcal{A})$, the second component of $\mathcal{T}(t)\binom{x}{f}$ is given by u_t, with a function $u\colon (-h,\infty) \to X$, $u|_{(-h,0)} = f$. Therefore we obtain

$$\begin{aligned} \int_0^\alpha \left\|\mathcal{B}_\alpha \mathcal{T}(t)\begin{pmatrix} x \\ f \end{pmatrix}\right\| dt &= \int_0^\alpha \|\Phi_\alpha u_t\| dt = \int_0^\alpha \left\|\int_{[-h,-\alpha)} d\eta(\tau) u(t+\tau)\right\| dt \\ &\leqslant \int_{[-h,-\alpha)} \int_0^\alpha \|f(t+\tau)\| dt\, d|\eta|(\tau) \leqslant |\eta|([-h,-\alpha))\, \|f\|_1 . \end{aligned} \tag{2.2}$$

Choose $n \in \mathbb{N}$ such that $|\eta|([-h,-\alpha)) < n$. Then (2.2) shows that $\frac{1}{n}\mathcal{B}_\alpha$ is a small Miyadera perturbation of $\mathcal{A}$, and therefore $\mathcal{A} + \frac{1}{n}\mathcal{B}_\alpha$ is the generator of a C_0-semigroup

$\mathcal{T}_{\alpha,1}(\cdot)$. It is clear that $\pi_2\mathcal{T}_{\alpha,1}(t)\binom{x}{f}$ and $\pi_2\mathcal{T}(t)\binom{x}{f}$ coincide on $(-h,-\alpha)$, for $0\leqslant t\leqslant\alpha$. From the fact that η_α has no mass in $[-\alpha,0]$ it follows that $\mathcal{B}_\alpha\mathcal{T}_{\alpha,1}(t)\binom{x}{f}=\mathcal{B}_\alpha\mathcal{T}(t)\binom{x}{f}$ for $0\leqslant t\leqslant\alpha$, and therefore one obtains the same estimate as (2.2) for $\mathcal{T}_{\alpha,1}$,

$$\int_0^\alpha \left\|\mathcal{B}_\alpha\mathcal{T}_{\alpha,1}(t)\begin{pmatrix}x\\f\end{pmatrix}\right\|dt\leqslant|\eta|([-h,-\alpha))\,\|f\|_1\,,$$

for all $(x,f)\in D(\mathcal{A})=D(\mathcal{A}+\frac{1}{n}\mathcal{B}_\alpha)$. As before we conclude that $(\mathcal{A}+\frac{1}{n}\mathcal{B}_\alpha)+\frac{1}{n}\mathcal{B}_\alpha=\mathcal{A}+\frac{2}{n}\mathcal{B}_\alpha$ is a generator. Repeating this argument n times we obtain that $\mathcal{A}+\mathcal{B}_\alpha$ is a generator.

(ii) Since η_α has no mass in $[-\alpha,0]$ the delay equation

$$u'(t)=Au(t)+\Phi_\alpha(u_t),$$

with initial values $u(0)=x\in D(A)$, $u_0=f\in W_1^1((-h,0);X)$, $f(0)=x$, is solved for $0\leqslant t\leqslant\alpha$ by

$$u(t)=S(t)x+\int_0^t S(t-s)\int_{[-h,-\alpha)}d\eta(\tau)f(s+\tau)ds.$$

With this solution we have

$$\mathcal{T}_\alpha(t)\begin{pmatrix}x\\f\end{pmatrix}=\begin{pmatrix}u(t)\\u_t\end{pmatrix}$$

(where $u|_{(-h,0)}=f$), and therefore, with $c_S:=\sup\limits_{0\leqslant t\leqslant 1}\|S(t)\|$,

$$\begin{aligned}
&\left\|\pi_2\mathcal{T}_\alpha(t)\begin{pmatrix}x\\f\end{pmatrix}\right\|=\|u_t\|_1\\
&\leqslant\|f\|_1+\alpha c_S\left(\|x\|+\int_0^\alpha\int_{[-h,-\alpha)}\|f(s+\tau)\|d|\eta|(\tau)ds\right)\\
&\leqslant\|f\|_1+\alpha c_S\left(\|x\|+\int_{[-h,-\alpha)}d|\eta|(\tau)\,\|f\|_1\right)\\
&=\alpha c_S\|x\|+\big(1+\alpha c_S|\eta|([-h,-\alpha))\big)\,\|f\|_1\,.
\end{aligned}$$

■

2.2 Lemma. *With the constant c from Lemma 2.1 one has*

$$\int_0^\alpha\left\|\tilde{\mathcal{B}}_\alpha\mathcal{T}_\alpha(t)\begin{pmatrix}x\\f\end{pmatrix}\right\|dt\leqslant c\,|\eta|([-\alpha,0])\ \|(x,f)\|$$

for all $0<\alpha\leqslant 1$, $(x,f)\in D(\mathcal{A})$.

Proof. With the estimate from Lemma 2.1 we obtain (denoting $u_t=\pi_2\mathcal{T}_\alpha(t)\binom{x}{f}$ as before)

$$\begin{aligned}
&\int_0^\alpha\left\|\tilde{\mathcal{B}}_\alpha\mathcal{T}_\alpha(t)\begin{pmatrix}x\\f\end{pmatrix}\right\|dt=\int_0^\alpha\left\|\int_{[-\alpha,0]}d\eta(\tau)u_t(\tau)\right\|dt\\
&\leqslant\int_0^\alpha\int_{[-\alpha,0]}\|u(t+\tau)\|d|\eta|(\tau)\,dt=\int_{[-\alpha,0]}\int_0^\alpha\|u(t+\tau)\|dt\,d|\eta|(\tau)\\
&\leqslant\int_{[-\alpha,0]}\|u_{\tau+\alpha}\|_1\ d|\eta|(\tau)\leqslant c\,|\eta|([-\alpha,0])\,\|(x,f)\|\,.
\end{aligned}$$

■

Proof of Theorem 1.1. For the constant c from Lemma 2.1, we choose $\alpha > 0$ small enough to obtain $c\,|\eta|([-\alpha, 0]) < 1$. From Lemma 2.1 we know that $\mathcal{A} + \mathcal{B}_\alpha$ is a generator. Then Lemma 2.2 states that $\tilde{\mathcal{B}}_\alpha$ is a small Miyadera perturbation of $\mathcal{A} + \mathcal{B}_\alpha$, and this implies that $\mathcal{A} + \mathcal{B}_\alpha + \tilde{\mathcal{B}}_\alpha = \mathcal{A} + \mathcal{B} = \mathcal{A}_\Phi$ is a generator. ■

3 The second result

Let $1 \leqslant p < \infty$. In this section we treat the case where

$$\Phi\colon \{f \in W_p^1(-h, 0; X);\ f(0) \in D(A)\} \to F_A \tag{3.1}$$

is linear and continuous. Here, $\{f \in W_p^1(-h, 0; X);\ f(0) \in D(A)\}$ is endowed with the norm coming from the graph norm of $D(\mathcal{A})$, $\|f\|_{\mathcal{A}} := \|f\|_{W_p^1} + \|f(0)\| + \|Af(0)\|$, and F_A is the *Favard class* of A defined by

$$F_A := \{x \in X;\ \|x\|_{F_A} := \sup_{0<t\leqslant 1} \frac{1}{t}\|T(t)x - x\| < \infty\}.$$

Recall that D_A is continuously embedded in F_A, where $D_A = (D(A), \|\cdot\|_A)$, with the graph norm $\|x\|_A = \|x\| + \|Ax\|$.

3.1 Theorem. *The operator*

$$\mathcal{A}_\Phi := \begin{pmatrix} A & \Phi \\ 0 & \frac{d}{d\tau} \end{pmatrix},$$

with $D(\mathcal{A}_\Phi) = \{(x, f) \in D(A) \times W_p^1(-h, 0; X);\ f(0) = x\}$ *is a generator.*

3.2 Remarks. (a) If $A \in L(X)$, then A can be included in Φ, and therefore this case enters the case $A = 0$ treated in [4], [6]; see also Remark 1.2 (a).

(b) Our method of proof is essentially the proof in [4]; we only have to choose the auxiliary space Z differently.

Proof of Theorem 3.1. We will apply [4; Theorem on p. 330], which we recall and prove in the Appendix.

We are going to show that Theorem A.1 is applicable with $Z := F_A \times \{0\}$, for the C_0-semigroup $\mathcal{T}(\cdot)$ generated by $\mathcal{A} := \begin{pmatrix} A & 0 \\ 0 & \frac{d}{dt} \end{pmatrix}$ with $D(\mathcal{A}) = D(\mathcal{A}_\Phi)$ as above, and the perturbation $\mathcal{B} := \begin{pmatrix} 0 & \Phi \\ 0 & 0 \end{pmatrix}$. Recall that $\mathcal{T}(\cdot)$ is given as follows: For $(x, f) \in \mathcal{E}$ let $u\colon (-h, \infty) \to X$,

$$u(t) := \begin{cases} f(t) & \text{for} \quad -h < t < 0, \\ S(t)x & \text{for} \qquad 0 \leqslant t. \end{cases}$$

Then $\mathcal{T}(t)(x, f) = (u(t), u_t)$ $(t \geqslant 0)$.

Now we show that Z satisfies (Z1), (Z2) of Theorem A.1. (Z1) is evident. In order to show (Z2) we take φ: $[0,1] \to Z$ continuous, $\varphi(t) = (\varphi_1(t), 0)$. Then

$$\mathcal{T}(t-s)\varphi(s) = (S(t-s)\varphi_1(s), \tau \mapsto S(t-s+\tau)\varphi_1(s))$$

for $0 \leqslant s \leqslant t$ (where $S(t) := 0$ for $t < 0$),

$$\int_0^t \mathcal{T}(t-s)\varphi(s)ds = \left(\int_0^t S(t-s)\varphi_1(s)ds, \tau \mapsto \int_0^{t+\tau} S(t+\tau-s)\varphi_1(s)\, ds\right). \tag{3.2}$$

We recall that F_A satisfies properties (Z1) and (Z2) of Theorem A.1 (for $Z = F_A$); cf. [4; p.336]. In fact, there exists $c_A \geqslant 0$ such that

$$\|A \int_0^t S(t-s)\varphi_1(s)\, ds\| \leqslant c_A t \sup_{0 \leqslant s \leqslant t} \|\varphi_1(s)\|_{F_A}$$

for $0 \leqslant t \leqslant 1$. From (Z1) for F_A we obtain $\int_0^t S(t-s)\varphi_1(s)ds \in D(A)$, and this is also the value of the second component in (3.2) for $\tau = 0$. Using standard methods and properties (Z1), (Z2) (for $Z = F_A$, as above) one shows that the second component

$$\psi_1(\tau) := \int_0^{t+\tau} S(t+\tau-s)\varphi_1(s)\, ds$$

is right differentiable on $[-t, 0)$,

$$\left(\frac{d}{d\tau}\right)_{\text{right}} \psi_1(\tau) = \varphi_1(t+\tau) + A \int_0^{t+\tau} S(t+\tau-s)\varphi_1(s)ds.$$

Using again (Z2) one shows that the last expression depends continuously on τ, and therefore ψ_1 is continuously differentiable on $[-t, 0]$.

Noting $\psi_1(-t) = 0$ we obtain $\int_0^t \mathcal{T}(t-s)\varphi(s)ds \in D(\mathcal{A})$. Moreover

$$\begin{aligned}
&\left\|\mathcal{A}\int_0^t \mathcal{T}(t-s)\varphi(s)ds\right\| \\
= \ & \left\|\left(A\int_0^t S(t-s)\varphi_1(s)ds, \tau \mapsto (\varphi_1(t+\tau) + A\int_0^{t+\tau} S(t+\tau-s)\varphi_1(s)ds)\right)\right\| \\
\leqslant \ & \left\|A\int_0^t S(t-s)\varphi_1(s)ds\right\| + \|\varphi_1(t+\cdot)\|_{L_p} + \left\|\tau \mapsto A\int_0^{t+\tau} S(t+\tau-s)\varphi_1(s)ds\right\|_{L_p} \\
\leqslant \ & c_A t \sup_{0\leqslant s\leqslant t} \|\varphi_1(s)\|_{F_A} + \|\varphi_1(t+\cdot)\|_{L_p} + c_A\left(\int_{-t}^0 (t+\tau)^p d\tau\right)^{1/p} \sup_{0\leqslant s\leqslant t}\|\varphi_1(s)\|_{F_A} \\
\leqslant \ & c_A t \sup_{0\leqslant s\leqslant t} \|\varphi_1(s)\|_{F_A} + t^{1/p}\sup_{0\leqslant s\leqslant t}\|\varphi_1(s)\| + c_A\left(\frac{1}{p+1}t^{p+1}\right)^{1/p} \sup_{0\leqslant s\leqslant t}\|\varphi_1(s)\|_{F_A}.
\end{aligned}$$

This shows the inequality needed in (Z2).

Assumption (3.1) just means that $\mathcal{B}$: $D_{\mathcal{A}} \to Z$ is continuous. ■

3.3 Remark. Note that the proof given above also implies that the space Z satisfies condition (Z_p) of [5; see p.134].

Appendix

In this appendix we recall the result of Desch and Schappacher used in Section 3. We also provide a new proof of this result.

A.1 Theorem. *(Desch, Schappacher [4; p. 330]) Let A be the generator of a C_0-semigroup $T(\cdot)$ on a Banach space E. Let $(Z, \|\cdot\|_Z)$ be a Banach space satisfying*

(Z1) Z is continuously embedded in X;

(Z2) there exist $t_0 > 0$, γ: $(0, t_0] \to [0, \infty)$, $\lim_{t\to 0} \gamma(t) = 0$, such that for all continuous functions φ: $[0, t_0] \to Z$ one has

$$\int_0^t T(t-s)\varphi(s)ds \in D(A),$$

$$\left\| A \int_0^t T(t-s)\varphi(s)ds \right\| \leqslant \gamma(t) \sup_{0\leqslant s\leqslant t} \|\varphi(s)\|_Z \,.$$

Let B: $D_A \to Z$ be linear and continuous, where $D_A := (D(A), \|\cdot\|_A)$.
Then $A + B$ is a generator.

Proof. We are going to show that (i) there exists $\hat{t} > 0$ such that for all $x \in D(A+B) = D(A)$ the Cauchy problem

$$u'(t) = (A+B)u(t), \quad u(0) = x \tag{A.1}$$

has a unique solution on $[0, \hat{t}]$, (ii) $\rho(A+B) \neq \emptyset$. These properties imply that $A+B$ is a generator; cf. [10; Ch. 4, Theorem 1.3].

(i) There exists $\hat{t} > 0$ such that

$$c := \left(\hat{t}\, c_T \, \|id\|_{L(Z,X)} + \sup_{0<t\leqslant\hat{t}} \gamma(t) \right) \|B\|_{L(D_A, Z)} < 1,$$

where $c_T := \sup_{0\leqslant t\leqslant 1} \|T(t)\|$. Let $x \in D(A)$. We want to show that there exists a unique solution $f \in C([0, \hat{t}]; D_A)$ of the integral equation

$$f(t) = T(t)x + \int_0^t T(t-s)Bf(s)ds. \tag{A.2}$$

In order to show this we note first that for $f \in C([0,\hat{t}]; D_A)$ the function Ψf given by

$$\Psi f(t) := T(t)x + \int_0^t T(t-s)Bf(s)ds$$

is again in $C([0,\hat{t}]; D_A)$; cf. [4; p. 331]. Moreover, it follows from the choice of $\hat{t}$ that Ψ is a contraction with contraction constant $c < 1$. Therefore the contraction mapping principle implies the existence of a unique fixed point f of Ψ, i.e., of a solution f of (A.2).

Next we show that $t \mapsto \int_0^t T(t-s)Bf(s)ds$ is differentiable as an X-valued function. In

$$\begin{aligned} &\frac{1}{h}\left(\int_0^{t+h} T(t+h-s)Bf(s)ds - \int_0^t T(t-s)Bf(s)ds\right) \\ &= \frac{1}{h}\int_t^{t+h} T(t+h-s)Bf(s)ds + \frac{1}{h}(T(t+h)-I)\int_0^t T(t-s)Bf(s)ds \end{aligned}$$

we let $h \to 0+$ and obtain

$$Bf(t) + A\int_0^t T(t-s)Bf(s)ds.$$

Since this function is continuous we obtain the differentiability, and as a consequence, f is a solution of (A.1).

Since every solution of (A.1) is also a solution of (A.2) the uniqueness of solutions of (A.2) implies the same for (A.1).

(ii) In [4; p. 332] it is shown that there exists $\lambda \in \rho(A)$ such that $(\lambda - A)^{-1}B$ is contractive as a mapping in D_A. This implies that $I_{D_A} - (\lambda - A)^{-1}B$ is invertible in $L(D_A)$; therefore

$$\lambda - A - B = (\lambda - A)(I_{D_A} - (\lambda - A)^{-1}B)$$

has the inverse

$$\left(I_{D_A} - (\lambda - A)^{-1}B\right)^{-1}(\lambda - A)^{-1} \in L(X, D_A) \subseteq L(X),$$

i.e., $\lambda \in \rho(A+B)$. ■

References

[1] A. Bátkai and S. Piazzera: *Semigroups and linear partial differential equations with delay.* J. Math. Anal. Appl., to appear.

[2] S. Brendle: *On regularity properties of the Dyson-Phillips series.* Preprint.

[3] M. C. Delfour: *The largest class of hereditary systems defining a C_0 semigroup on the product space.* Can. J. Math. **32**, 969–978 (1980).

[4] W. Desch and W. Schappacher: *On relatively bounded perturbations of linear C_0-semigroups.* Ann. Sc. Norm. Super. Pisa, Cl. Sci., Ser. IV - Vol. **XI**, 327–341 (1984).

[5] ______: *Some generation results for perturbed semigroups.* In: Semigroup Theory and Applications (Clément, Invernizzi, Mitidieri, Vrabie eds.) Lect. Notes Pure Appl. Math. **116**, 125–152 (1989).

[6] K.-J. Engel: *Spectral theory and generator property of one-sided coupled operator matrices.* Semigroup Forum **58**, 267–295 (1999).

[7] K.-J. Engel and R. Nagel: *One-parameter semigroups for linear evolution equations.* Springer, New York, 1999.

[8] I. Miyadera: *On perturbation theory for semi-groups of operators.* Tôhoku Math. J. **18**, 299–310 (1966).

[9] ______: *On perturbation for semigroups of linear operators* (Japanese). Scientific Researches, School of Education, Waseda Univ. **21**, 21–24 (1972).

[10] A. Pazy: *Semigroups of linear operators and applications to partial differential equations.* Springer, New York, 1983.

[11] J. Voigt: *On the perturbation theory for strongly continuous semigroups.* Math. Ann. **229**, 163–171 (1977).

[12] G. F. Webb: *Functional differential equations and nonlinear semigroups in L^p-spaces.* J. Differ. Equations **20**, 71–89 (1976).

Lahcen Maniar
Cadi Ayyad University
Faculty of Sciences Semlalia, B.P. 2390
Marrakesh, Morocco
e-mail: `maniar@ucam.ac.ma`

Jürgen Voigt
Technische Universität Dresden
Fachrichtung Mathematik
01062 Dresden, Germany
e-mail: `voigt@math.tu-dresden.de`