Evolution Equations with Non-Linear Constitutive Laws and Memory Effects

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Abstract

Abstract: Evolution equations are considered as operator equations involving a sum of the time-derivative operator ∂_0 regarded as a normal operator in a suitable Hilbert space setting and another fairly arbitrary spatial operator A acting in a Hilbert space H. As an extension of the linear theory the case of a class of non-linear constitutive laws with memory effects is considered.

Classification: 47D06, 44A05

1 Introduction.

1.1. We consider initial value problems of the form

$$\begin{cases} \partial_0 W = 2\pi\mathrm{i}(AU + F_0) \quad \text{ on } \mathbb{R}^+, \\ W(0+) = W_0, \\ W = M(U), \end{cases} \tag{1.1}$$

as operator equations in a suitable Hilbert space frame-work. In (1.1), ∂_0 denotes differentiation with respect to the time parameter, i denotes the imaginary unit, A is a densely defined, closed linear operator on a separable (complex) Hilbert space H with inner product $\langle \cdot \,|\, \cdot \rangle$, assumed to be linear in the second factor, and induced norm $|| \cdot ||$. The initial value W_0 and the source term F_0 are given; the coefficient $2\pi\mathrm{i}$ is introduced to conveniently adapt the operator to the Fourier-Laplace transform we shall use later.

Our goal is to develop a unified approach to this type of equations, in the sense that the results we present apply to quite general situations, which include, but are not limited to, problems which are classically referred to as "parabolic" or "hyperbolic". For example, our approach would cover Schrödinger type equations, which in the classical sense are of neither kind.

To interpret (1.1) as an operator equation, we consider the initial data W_0 as an additional source term at time $t = 0$, by means of a δ distribution; more precisely, we define $\delta \otimes W_0$ as a functional acting on $C_0^\infty(\mathbb{R}, \mathbb{C})$ (i.e., on complex-valued C^∞-functions with compact support in $\mathbb{R}$), by

$$(\delta \otimes U_0)(\varphi) := \varphi(0)\, U_0 \tag{1.2}$$

for all $\varphi \in C_0^\infty(\mathbb{R}, H)$. Writing formally $D_0 U$ for $\frac{1}{2\pi\mathrm{i}}\partial_0 U$ and extending F_0 by zero on $]-\infty, 0[$, problem (1.1) can be formulated as

$$\begin{cases} D_0\, W - A\, U = F_0 + \frac{1}{2\pi\mathrm{i}}\delta \otimes W_0, \\ W = M(U). \end{cases} \tag{1.3}$$

This problem becomes completely determined when we specify the map M, which we interpret as a description of some given material relation between the fields W and U. Here, we consider a particular relation modelling memory effects; that is, $M(U)(t) := M(t, U)$ is defined by

$$M(U)(t) := L\, U + \int_{-\infty}^{t} K(t-s)\, \zeta(U)(s)\, ds, \tag{1.4}$$

where $L : H \to H$ is positive definite and bounded, $\zeta : H \to H$ is a non-linear term and K a convolution kernel. In this context, we believe that our results provide a sensible generalization to the existing theory, which, as far as we can determine, is quite well developed

for models with linear memory terms, such as in Fabrizio, [3], [4], [5], for Maxwell's equations, or for specific models in one dimension of space, such as in Kurbanov, [7], again for Maxwell's equations, or for perturbations of conservations laws, such as in Sampalmieri, [15] and [16]. For other physical systems characterized by memory type material relations, see also Mattioli-Salvatori, [9], who consider a model in nonlinear optics; Brandon-Hrusa, [2], for a model for heat flows; and Bloom, [1], for Maxwell's equations. An extensive overview is provided in Prüss' monograph [13], and a wealth of results, as well as references, can be found in Ruess, [14]. We refer to all these references, and to the references quoted therein, for further background material and specific results.

1.2. In evolution problems with memory terms, it is usually assumed that the prehistory of U prior to the time $t = 0$ is known, as a given function $U_{-\infty}(t)$ for $t < 0$. This means that the solution to (1.3) has the form

$$U(t) := \begin{cases} U_{-\infty}(t) & \text{for} \quad t < 0, \\ V(t) & \text{for} \quad t > 0. \end{cases} \tag{1.5}$$

We also assume that $U_{-\infty}(0-)$, $V(0+)$ are well defined in H, and that the "jump"

$$V(0+) - U_{-\infty}(0-) =: V_0, \tag{1.6}$$

is also given, with

$$W(0+) = M(V(0+)) = W_0 = M(U_{-\infty}(0-) + V_0). \tag{1.7}$$

Formally differentiating (1.4) with respect to "time", we obtain

$$D_0\, M(U) = L\, D_0\, U + K(0)\, \zeta(U) + \int_{-\infty}^{(\cdot)} K'(\cdot - s)\, \zeta(U)(s)\, ds; \tag{1.8}$$

thus, denoting the extension of V by zero on $]-\infty, 0[$ again by V, we obtain

$$\begin{aligned} (D_0 - L^{-1} A)\, V = {} & F + \frac{1}{2\pi \mathrm{i}} \delta \otimes L^{-1}\, W_0 \; - L^{-1}\, K(0)\, \zeta(V) + \\ & - \frac{1}{2\pi \mathrm{i}}\, L^{-1}\, \chi_+ \int_0^{(\cdot)} K'(\cdot - s)\, \zeta(V(s))\, ds, \end{aligned} \tag{1.9}$$

where χ_+ denotes the characteristic function of $[0, +\infty[$, and

$$F \; := L^{-1}\, F_0 - \frac{1}{2\pi \mathrm{i}}\, L^{-1}\, \chi_+ \int_{-\infty}^{0} K'(\cdot - s)\, \zeta(U_{-\infty}(s))\, ds. \tag{1.10}$$

In this way, the prehistory $U_{-\infty}$ can also be made part of the source terms.

To preserve the spectral properties of A in the transition to $L^{-1} A$ we need to modify the inner product of H into

$$\langle\, \cdot | \cdot\, \rangle_0 := \langle L^{1/2}\, \cdot \,|\, L^{1/2} \cdot \rangle = \langle\, \cdot \,|\, L\, \cdot \rangle\,; \tag{1.11}$$

we denote by H_0 the corresponding Hilbert space. Finally, setting $\mathcal{A} := L^{-1}A$, $\mathcal{K} := L^{-1}K'$, $\Gamma(V) := L^{-1}K(0)\zeta(V)$, and assuming $\mathcal{K} \equiv 0$ for $t < 0$, we are led to consider an evolution problem of the form

$$(D_0 - \mathcal{A})\, V = F + \frac{1}{2\pi \mathrm{i}}\delta \otimes V_0 - \Gamma(V) \, - \int_0^{(\cdot)} \mathcal{K}(\cdot - s)\, \zeta(V(s))\, ds \tag{1.12}$$

on the whole line $-\infty < t < +\infty$.

1.3. In the next section, we consider general equations of the form (1.12), and interpret them in a suitable function space framework. We state our main result, which we prove in §3. In §4 we give some remarks on how our theory can be applied to more general nonlinearities, i.e. not necessarily of convolution type. Finally, in §5 we present an application of our results to a model of nonlinear Maxwell's equations for materials with fading memory.

2 Formulation of the Problem.

2.1. We shall consider (1.12) in a Sobolev lattice setting, as developed in [12] (compare also [11]). Let $A : D(A) \subseteq H_0 \to H_0$ be a densely defined, closed, linear operator with non-empty resolvent set. We fix $\lambda_0 \in \varrho(A)$, and set

$$H_\infty := \bigcap_{k\in\mathbb{Z}} D\left((\lambda_0 - A)^k)\right), \tag{2.1}$$

$$C_0^{\infty,+}(\mathbb{R}) := C_0^\infty(\mathbb{R}) \, + \, \alpha_0 \mathrm{Lin}_{\mathbb{C}}\left\{x \mapsto x^n \mid n \in \mathbb{N}\right\}, \tag{2.2}$$

where $\alpha_0 \in C^\infty(\mathbb{R})$ is a cut-off function with support in $[0, +\infty[$ and $\mathrm{supp}(1-\alpha_0) \subset]-\infty, 1]$. Finally, if X is a space, we denote by 1_X the identity in X.

Equation (1.12) can be appropriately discussed in the lattice of Hilbert spaces

$$(H_{\nu,j,k})_{j,k\,\in\mathbb{Z}}\,, \qquad \nu \in \mathbb{R}, \tag{2.3}$$

where $H_{\nu,j,k}$ denotes the Hilbert space obtained by completion of

$$C_0^{\infty,+}(\mathbb{R}) \underset{a}{\otimes} H_\infty \, := \mathrm{Lin}_{\mathbb{C}}\left\{t \mapsto \varphi(t)w \;\middle|\; \varphi \in C_0^{\infty,+}(\mathbb{R}),\; w \in H_\infty\right\}$$

with respect to the norm $|\cdot|_{\nu,j,k}$ induced by the inner product $\{\varphi, \psi\} \mapsto \langle\varphi|\psi\rangle_{\nu,j,k}$ defined for $j, k \in \mathbb{Z}$ by

$$\left\langle (D_0 + (\nu+1)\mathrm{i})^j(\lambda_0 - A)^k\varphi \,|\, (D_0 + (\nu+1)\mathrm{i})^j(\lambda_0 - A)^k\psi\right\rangle_{\nu,0,0}. \tag{2.4}$$

In particular, the inner product $\langle\cdot\,|\,\cdot\rangle_{\nu,0,0}$ is the inner product of the tensor product Hilbert space $H_{\nu,0} \otimes H_0$, where $H_{\nu,0}$ is the weighted L^2-space with inner product $\langle\cdot\,|\,\cdot\rangle_{\nu,0}$ defined, for $\varphi,\, \psi \in C_0^{\infty,+}(\mathbb{R})$, by

$$(\varphi, \psi) \mapsto \int_{\mathbb{R}} \varphi(t)^*\, \psi(t)\, \exp(-4\pi\nu t)\, dt \tag{2.5}$$

(the asterix denotes complex conjugation). The lattice structure is given by the continuous and dense imbeddings

$$H_{\nu,j,k} \;\hookrightarrow\; H_{\nu,u,w}, \qquad j,k,u,w \in \mathbb{Z}, \qquad j \geq u,\; k \geq w. \tag{2.6}$$

We recall that, identifying as usual $1_{H_{\nu,0}} \otimes \mathcal{A}$ with $\mathcal{A}$ and $D_0 \otimes 1_{H_0}$ with D_0, D_0 and $\mathcal{A}$ have continuous extensions to the space

$$H_{\nu,-\infty,-\infty} := \bigcup_{j,k\,\in\mathbb{Z}} H_{\nu,j,k}\,, \tag{2.7}$$

where continuity is understood in the sense that for every j, $k \in \mathbb{Z}$ the operators D_0 and $\mathcal{A}$ have (unique) continuous extensions, considered as mappings from $H_{\nu,j,k} \to H_{\nu,j_0,k_0}$ at first for some, and then in fact for all, j_0, $k_0 \in \mathbb{Z}$ sufficiently large and negative. We shall keep the notations D_0 and $\mathcal{A}$ to denote these extensions. Next, we define the time support of a function $f \in H_{\nu,-\infty,-\infty}$. At first, we say that $f = 0$ in an open (time) interval $I \subset \mathbb{R}$ if the map

$$\varphi \mapsto \langle f \,|\, \varphi \otimes u \rangle_{\nu,0,0}$$

vanishes for all $\varphi \in C_0^\infty(\mathbb{R})$ such that $\operatorname{supp}\varphi \subseteq I$, and all $u \in H_\infty$. The time support of $f \in H_{\nu,-\infty,-\infty}$ is then defined as

$$\operatorname{supp}_0 f \;:=\; \mathbb{R} \setminus \big(\cup\{I \subseteq \mathbb{R} \,|\, I \text{ open interval},\; f = 0 \text{ in } I\}\big)\,. \tag{2.8}$$

Finally, if $I \subseteq \mathbb{R}$ is a closed interval, we set

$$H_{\nu,0,0}(I) := \{\varphi \in H_{\nu,0,0} \;\mid\; \operatorname{supp}_0\varphi \subseteq I\}\,, \tag{2.9}$$

$$H_{\infty,0,0}(I) := \bigcap_{\nu_0\in\mathbb{R}} \bigcup_{\nu>\nu_0} H_{\nu,0,0}(I). \tag{2.10}$$

2.2. Before specifying the assumptions on the data and the nonlinear terms $\mathcal{K}$, ζ, we recall from [12] the following result for the linear equation

$$(D_0 - \mathcal{A})\; V = F, \tag{2.11}$$

which corresponds to the principal part of (1.12):

Theorem 2.1. *Assume there is $\nu_0 > 0$ such that the imaginary part of the spectrum of $\mathcal{A}$, $\mathfrak{Im}\,\sigma(\mathcal{A})$, is bounded below by $-\nu_0$, and that for $\nu > \nu_0$ the operators*

$$(\lambda - \mathrm{i}\,\nu - \mathcal{A})^{-1} : H_0 \to H_0$$

are uniformly bounded for all $\lambda \in \mathbb{C}$ with $\mathfrak{Im}\,\lambda < 0$, that is,

$$\varrho(\mathcal{A},\nu) := \sup\,\big\{\,\big\|(\lambda - \mathrm{i}\,\nu - \mathcal{A})^{-1}\big\| \;\big|\; \lambda \in \mathbb{C},\; \mathfrak{Im}\,\lambda < 0\big\} = O(1) \tag{2.12}$$

as $\nu \to +\infty$. Then $(D_0 - \mathcal{A})$ is continuously invertible in $H_{\nu,j,k}$ for every $\nu > \nu_0$ and j, $k \in \mathbb{Z}$. Moreover, each operator

$$(D_0 - \mathcal{A})^{-1} : H_{\nu,j,k} \to H_{\nu,j,k}, \qquad \nu > \nu_0,\ j,\ k \in \mathbb{Z},$$

is also (forward) causal, in the sense that if $f \in H_{\nu,j,k}$ has time support bounded below by some $\kappa \in \mathbb{R}$, then $(D_0 - \mathcal{A})^{-1} f$ has also time support with the same lower bound.

We remark that, as shown in [12], we can here also allow for some "regularity loss" of the solutions, in the sense that we can consider operators $\mathcal{A}$ which, instead of (2.12), admit polynomial growth in $\mathfrak{Im}\,\lambda$ of the resolvent; that is, (2.12) can be replaced by

$$\sup \left\{ (\lambda - \mathrm{i})^{-m} \left\| (\lambda - \mathrm{i}\,\nu - \mathcal{A})^{-1} \right\| \;\middle|\; \lambda \in \mathbb{C},\ \mathfrak{Im}\,\lambda < 0 \right\} = O(1) \tag{2.13}$$

as $\nu \to +\infty$, for some positive integer m. In this sense, the case $m = 0$ of (2.12) corresponds to the situation when solutions experience no regularity loss in their time derivatives. Note also that Theorem 2.1 is quite general, in that, as we have mentioned, it can be applied to a wider class of problems than just "parabolic" or "hyperbolic" ones.

For the applications to the nonlinear theory that we want to consider, we need more restrictive assumptions on $\mathcal{A}$: instead of (2.12), we require that

$$\varrho(\mathcal{A}, \nu) = o(1) \tag{2.14}$$

as $\nu \to +\infty$. In particular, this condition is satisfied if

$$\varrho(\mathcal{A}, \nu) = O\left(\frac{1}{\nu - \nu_0} \right), \tag{2.15}$$

which is the case if $\mathcal{A}$ is the generator of a C_0-semigroup. When (2.14) holds, by a suitable choice of the parameter $\nu > \nu_0$ we can guarantee that the operator norm

$$\left\| (D_0 - \mathcal{A})^{-1} : H_{\nu,j,k} \to H_{\nu,j,k} \right\|$$

is arbitrarily small for all $j, k \in \mathbb{Z}$. In particular, this is true for $j = k = 0$, which is the case we consider later. Although this result is a simple corollary to the solution theory presented in [12], we give a proof here for sake of self-containedness.

Lemma 2.2. *Assume that (2.14) holds. Then, as $\nu \to +\infty$,*

$$\left\| (D_0 - \mathcal{A})^{-1} : H_{\nu,0,0} \to H_{\nu,0,0} \right\| = o(1). \tag{2.16}$$

Proof. We use the Fourier-Laplace transform $\mathcal{L}_\nu$ in the continuous extension of

$$\mathcal{L}_\nu : C_0^{\infty,+}(\mathbb{R}) \underset{a}{\otimes} H_0 \subset H_{\nu,0,0} \to H_{0,0,0},$$

which is defined for $\varphi \in C_0^{\infty,+}(\mathbb{R}) \underset{a}{\otimes} H_0$ by

$$(\mathcal{L}_\nu \varphi)(x) := \int_{\mathbb{R}} \exp(-2\pi \mathrm{i}\,(x - \mathrm{i}\,\nu)\,t)\,\varphi(t)\,dt.$$

Here $H_{0,0,0} := L^2(\mathbb{R}) \otimes H_0 \equiv L^2(\mathbb{R}, H)$ denotes the Hilbert space of $H-$valued, generalized L^2-"functions". The transforms $\mathcal{L}_\nu$ extend to unitary mappings, which we still denote by $\mathcal{L}_\nu$. Then

$$\mathcal{L}_\nu \, (D_0 - \mathcal{A})^{-1} = (m - \mathrm{i}\,\nu - \mathcal{A})^{-1} \, \mathcal{L}_\nu, \tag{2.17}$$

where m denotes the operator "multiplication by the argument", that is,

$$\left((m - \mathrm{i}\,\nu - \mathcal{A})^{-1} \, \varphi\right)(\lambda) = (\lambda - \mathrm{i}\,\nu - \mathcal{A})^{-1} \, \varphi(\lambda), \tag{2.18}$$

for $\lambda \in \mathbb{R}$, $\nu > \nu_0$ and, e.g., $\varphi \in C_0^\infty(\mathbb{R}) \underset{a}{\otimes} H_0$. We compute then

$$\begin{aligned}
\left|(D_0 - \mathcal{A})^{-1} \, \varphi\right|^2_{\nu,0,0} &= \left|\mathcal{L}_\nu \, (D_0 - \mathcal{A})^{-1} \, \varphi\right|^2_{0,0,0} = \\
&= \left|(m - \mathrm{i}\,\nu - \mathcal{A})^{-1} \, \mathcal{L}_\nu \, \varphi\right|^2_{0,0,0} = \\
&= \int_{\mathbb{R}} \left|(\lambda - \mathrm{i}\,\nu - \mathcal{A})^{-1} \, (\mathcal{L}_\nu \, \varphi)(\lambda)\right|^2_0 \, d\lambda \leq \\
&\leq \varrho(\mathcal{A}, \nu) \int_{\mathbb{R}} |(\mathcal{L}_\nu \varphi)(\lambda)|^2_0 \, d\lambda \leq \varrho(\mathcal{A}, \nu) \, |\varphi|^2_{\nu,0,0} \, .
\end{aligned}$$

Consequently,

$$\left\|(D_0 - \mathcal{A})^{-1} : \, H_{\nu,0,0} \to H_{\nu,0,0}\right\| \leq \varrho(\mathcal{A}, \nu), \tag{2.19}$$

which clearly implies the claim. □

2.3. We now go back to the general equation (1.12). We assume that the operator $\mathcal{A}$ satisfies the assumptions of Theorem 2.1, and that the other terms in (1.12) satisfy the following conditions:

1. The source term F and the "initial value" V_0 satisfy:

$$F \in H_{\nu,0,0} \;\text{ for some }\; \nu \in [0, +\infty[, \quad \mathrm{supp}_0 F \subseteq [0, +\infty[; \tag{2.20}$$

$$V_0 \in H_1. \tag{2.21}$$

2. The operators $\Gamma : H_0 \to H_0$ and $\zeta : H_0 \to H_0$ are Lipschitz continuous, with (best) Lipschitz constants $|\Gamma|_{\mathrm{Lip}}$ and $|\zeta|_{\mathrm{Lip}}$, respectively. These operators induce corresponding Lipschitz continuous mappings in each $H_{\nu,0,0}$, which we denote by the same name:

$$\Gamma(\varphi)(t) := \Gamma(\varphi(t)), \qquad \zeta(\varphi)(t) := \zeta(\varphi(t)), \tag{2.22}$$

for $t \in \mathbb{R}$ and, e.g., $\varphi \in C_0^{\infty,+}(\mathbb{R}) \underset{a}{\otimes} H_0$. The best Lipschitz constants of these induced mappings remain the same.

3. The integral kernel $\mathcal{K}(t) : H_0 \to H_0$ is a bounded, linear operator for every $t \in \mathbb{R}$, and $\mathcal{K}(t) = 0$ for $t < 0$. Thus, $\mathcal{K}$ is a $1-$parameter family of bounded linear operators. In addition, the function $t \mapsto \|\mathcal{K}(t)\|_{\mathcal{L}(H_0 \mapsto H_0)} =: \kappa(t)$ is measurable, and there is $\nu_K \in \mathbb{R}$ such that

$$L_K := \int_0^{+\infty} \kappa(s)\, e^{-2\pi\nu_K s}\, ds < +\infty. \tag{2.23}$$

In particular, if $\nu_K = 0$ assumption (2.23) describes the so-called materials with "fading memory" (see e.g. Hrusa, [6], and the references therein), characterized by convolution kernels with integrable operator norm (that is, $\kappa \in L^1(0, +\infty)$).

3 The Solution Theory.

3.1. As a consequence of the above assumptions on $\mathcal{K}$ and ζ, we deduce that if $\nu \geq \nu_0$, the map

$$V \mapsto \int_0^{+\infty} \mathcal{K}(\cdot - s)\, \zeta(V(s))\, ds =: Z(V) \tag{3.1}$$

is also Lipschitz continuous in $H_{\nu,0,0}$, with (best) Lipschitz constant $L_K\ |\zeta|_{\text{Lip}}$. Indeed, we estimate

$$\begin{aligned}
&\left| \int_0^{+\infty} \mathcal{K}(\cdot - s)\zeta(\varphi(s))\, ds - \int_0^{+\infty} \mathcal{K}(\cdot - s)\zeta(\psi(s))\, ds \right|^2_{\nu,0,0} \leq \\
&\quad \leq |\zeta|^2_{\text{Lip}} \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \kappa(t-s)\ |\varphi(s) - \psi(s)|_0\ ds \right)^2 e^{-4\pi\nu t}\, dt \leq \\
&\quad \leq |\zeta|^2_{\text{Lip}}\ L_K \int_{\mathbb{R}} \int_{\mathbb{R}} \kappa(t-s)\ |\varphi(s) - \psi(s)|^2_0\ \cdot \\
&\qquad\qquad \cdot e^{2\pi\nu_K (t-s)}\, ds\, e^{2\pi\nu_K t}\, e^{-4\pi\nu t}\, dt\ \leq \\
&\quad \leq |\zeta|^2_{\text{Lip}}\ L_K \int_{\mathbb{R}} \int_{\mathbb{R}} \kappa(t-s)\ |\varphi(s) - \psi(s)|^2_0\ \cdot \\
&\qquad\qquad \cdot e^{-2\pi\nu_K (t-s)}\, e^{-4\pi\nu s}\, e^{-4\pi(\nu-\nu_K)(t-s)}\, dt\, ds\ \leq \\
&\quad \leq |\zeta|^2_{\text{Lip}}\ L_K^2 \int_{\mathbb{R}} |\varphi(s) - \psi(s)|^2_0\ e^{-4\pi\nu s}\, ds\ = \\
&\quad = |\zeta|^2_{\text{Lip}}\ L_K^2\ |\varphi - \psi|^2_{\nu,0,0}\,.
\end{aligned}$$

That is, we find that for all $\varphi,\ \psi \in H_{\nu,0,0}$,

$$|Z(\varphi) - Z(\psi)|_{\nu,0,0} \leq\ L_K\, |\zeta|_{\text{Lip}}\, |\varphi - \psi|^2_{\nu,0,0}\,, \tag{3.2}$$

which shows the Lipschitz continuity of Z, uniformly in $\nu \geq \nu_0$.

3.2. This result allows us to obtain the well-posedness of equation (1.12) in the spaces $H_{\nu,0,0}$:

Theorem 3.1. *Under the same assumptions on $\mathcal{A}$ of Theorem 2.1, and those listed in §2.3 on the other terms, equation (1.12) has, for all sufficiently large $\nu > 0$, a unique solution $V \in H_{\nu,0,0}$. This solution does not depend on ν as long as ν is sufficiently large, and depends continuously on the data F and V_0, in the sense that there is a positive constant C such that for all pairs of source terms F_1, F_2, and initial values $V_{0,1}$, $V_{0,2}$,*

$$|V_1 - V_2|_{\nu,0,0} \leq C \left(|F_1 - F_2|_{\nu,0,0} + |V_{0,1} - V_{0,2}|_1 \right), \tag{3.3}$$

where V_i denotes the solution corresponding to the data F_i, $V_{0,i}$, $i = 1, 2$.

Proof. The result follows by a standard contraction argument. Indeed, for sufficiently large ν the mapping $T_\nu : H_{\nu,0,0} \to H_{\nu,0,0}$ defined by

$$\begin{aligned} T_\nu \varphi := (D_0 - \mathcal{A})^{-1} \Big(& F + \frac{1}{2\pi \mathrm{i}} \delta \otimes V_0 + \\ & - \Gamma(\varphi) - \int_0^{+\infty} \mathcal{K}(\cdot - s)\, \zeta(\varphi(s))\, ds \Big) \end{aligned} \tag{3.4}$$

is Lipschitz continuous, with Lipschitz constant

$$|T_\nu|_{\mathrm{Lip}} \leq \varrho(\mathcal{A}, \nu) \left(|\Gamma|_{\mathrm{Lip}} + |\zeta|_{\mathrm{Lip}}\, L_K \right). \tag{3.5}$$

Because of (2.14), the right side of (3.5) can be made smaller than 1 for all sufficiently large $\nu > 0$: the mapping T_ν becomes then a contraction, which implies the existence of a unique fixed point $V \in H_{\nu,0,0}$, together with the continuous dependence estimate. The fixed point V satisfies

$$\begin{aligned} V &= T_\nu V = \\ &= (D_0 - \mathcal{A})^{-1} \left(F + \frac{1}{2\pi \mathrm{i}} \delta \otimes V_0 - \Gamma(V) - \int_0^{+\infty} \mathcal{K}(\cdot - s) \zeta(V(s))\, ds \right); \end{aligned} \tag{3.6}$$

applying the operator $(D_0 - \mathcal{A}) : H_{\nu,0,0} \subseteq H_{\nu,-1,-1} \to H_{\nu,-1,-1}$ (in the sense of the Sobolev lattice constructed above) to both sides of (3.6) we conclude that V solves (1.12). Finally, to show the uniqueness of solutions to (1.12) in $H_{\nu,0,0}$, we remark that if

$$(D_0 - \mathcal{A})\, V = F + \frac{1}{2\pi \mathrm{i}} \delta \otimes V_0 - \Gamma(V) - \int_0^{+\infty} \mathcal{K}(\cdot - s)\, \zeta(V(s))\, ds, \tag{3.7}$$

applying $(D_0 - \mathcal{A})^{-1} : H_{\nu,-1,-1} \to H_{\nu,-1,-1}$ to both sides of (3.7) we obtain that

$$V = (D_0 - \mathcal{A})^{-1} \left(F + \frac{1}{2\pi \mathrm{i}} \delta \otimes V_0 - \Gamma(V) - \int_0^{+\infty} \mathcal{K}(\cdot - s)\, \zeta(V(s))\, ds \right). \tag{3.8}$$

Since $V \in H_{\nu,0,0}$, the right side of (3.8) is just $T_\nu(V)$, so by the uniqueness of the fixed point we deduce that, if $\nu > 0$ is sufficiently large, solutions of (1.12) are unique in $H_{\nu,0,0}$ as claimed. □

4 Generalizations.

4.1. In some cases it is possible to relax assumption (2.21) on the regularity of the initial value V_0, and consider arbitrary $V_0 \in H_0$. Indeed, the proof of Theorem 3.1 shows that this is true when $\mathcal{A}$ is such that for all $V_0 \in H_0$

$$(D_0 - \mathcal{A})^{-1} \ \frac{1}{2\pi \mathrm{i}} \delta \otimes V_0 \ \in H_{\nu,0,0}. \tag{4.1}$$

For example, this is the case if $\mathcal{A}$ is the generator of a continuous 1-parameter semigroup.

We can also show that the continuous dependence of solutions of (1.12) on the force term F can be expressed in terms of the continuous dependence on the "actual" source term F_0 and the prehistory $U_{-\infty}$, which appears in the term

$$\int_{-\infty}^{0} \mathcal{K}(\cdot - s)\, \zeta(U_{-\infty}(s))\, ds, \tag{4.2}$$

provided we take $U_{-\infty} \in H_{\nu_K,0,0}(]-\infty, 0])$. Recalling (2.23), this requirement means, roughly speaking, that if $\kappa(t)$ can grow as $t \to +\infty$ (case $\nu_K > 0$), then $U_{-\infty}(s)$ must decay as $s \to -\infty$ and, conversely, if $\kappa(t)$ decays at $+\infty$ (case $\nu_K < 0$), then $U_{-\infty}$ can grow at $-\infty$.

The stated continuous dependence on the past history is described by

Theorem 4.1. *Under the same assumptions of Theorem 3.1, for $\nu \geq \nu_K$ the solution $V \in H_{\nu,0,0}$ of (1.12) depends continuously on the source term F_0, the prehistory $U_{-\infty}$ and the initial data V_0, in the sense that there is a constant C such that (compare with (3.3))*

$$\begin{aligned} |V_1 - V_2|_{\nu,0,0} \leq C \Big(& |F_{0,1} - F_{0,2}|_{\nu,0,0} + \\ & + |U_{-\infty,1} - U_{-\infty,2}|_{\nu_K,0,0} + |V_{0,1} - V_{0,2}|_1 \Big), \end{aligned} \tag{4.3}$$

where for $i = 1, 2$, V_i is the solution of (1.12) corresponding to the initial value $V_{0,i}$ and the source term

$$F_i := F_{0,i} - \int_{-\infty}^{0} \mathcal{K}(\cdot - s)\zeta(U_{-\infty,i}(s))\, ds.$$

Proof. This result is a consequence of the estimates of Theorem 3.1, and of analogous Lipschitz estimates of the prehistory term (4.2). Recalling that $\nu \geq \nu_K$ and $U_{-\infty,i} \in H_{\nu_K,0,0}(]-\infty,0])$, we compute:

$$\begin{aligned}
&\left| \int_0^{+\infty} \mathcal{K}(\cdot - s)\zeta(U_{-\infty,1}(s))\,ds - \int_0^{+\infty} \mathcal{K}(\cdot - s)\zeta(U_{-\infty,2}(s))\,ds \right|^2_{\nu,0,0} \leq \\
&\quad \leq |\zeta|^2_{\mathrm{Lip}} \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \kappa(t-s)\ |U_{-\infty,1}(s) - U_{-\infty,2}(s)|_0\ ds \right)^2 e^{-4\pi\nu t)}\,dt \leq \\
&\quad \leq |\zeta|^2_{\mathrm{Lip}}\ L_K \int_{\mathbb{R}} \int_{\mathbb{R}} \kappa(t-s)\ |U_{-\infty,1}(s) - U_{-\infty,2}(s)|^2_0 \cdot \\
&\qquad\qquad \cdot e^{-2\pi\nu_K s}\,ds\, e^{2\pi\nu_K t}\, e^{-4\pi\nu t}\,dt \leq \\
&\quad \leq |\zeta|^2_{\mathrm{Lip}}\ L_K \int_{\mathbb{R}} \int_{\mathbb{R}} \kappa(t-s)\ |U_{-\infty,1}(s) - U_{-\infty,2}(s)|^2_0 \cdot \\
&\qquad\qquad \cdot e^{-2\pi\nu_K (t-s)}\, e^{-4\pi\nu_K s}\, e^{-4\pi(\nu-\nu_K)t}\,dt\,ds \leq \\
&\quad \leq |\zeta|^2_{\mathrm{Lip}}\ L_K^2 \int_{\mathbb{R}} |U_{-\infty,1}(s) - U_{-\infty,2}(s)|^2_0\ e^{-4\pi\nu_K s}\,ds\ = \\
&\quad = L_K^2\ |\zeta|^2_{\mathrm{Lip}}\ |U_{-\infty,1} - U_{-\infty,2}|^2_{\nu_K,0,0}\ .
\end{aligned}$$

Putting this estimate together with (3.2), it is then easy to obtain (4.3). □

4.2. We also remark that the same methods can be applied to more general nonlinearities than those appearing in the convolution term of (1.12), which can in fact be seen as a special case of a more general situation. More precisely, we can consider equations of the form

$$(D_0 - \mathcal{A})\,V = F + \frac{1}{2\pi \mathrm{i}}\delta \otimes V_0 + \Phi(V), \tag{4.4}$$

where the nonlinearity Φ is constructed, on each space $H_{\nu,0,0}$, as the continuous extension $\Phi_\nu : H_{\nu,0,0} \to H_{\nu,0,0}$ of a map Ψ initially defined as

$$\Psi : C_0^{\infty,+}(\mathbb{R}) \underset{a}{\otimes} H_0 \to H_{\infty,0,0}([0,+\infty[). \tag{4.5}$$

That is, Ψ is independent of ν, and Φ_ν is its extension on $H_{\nu,0,0}$; this construction makes sense, in particular, if

$$\limsup_{\nu \to +\infty} |\Phi_\nu|_{\mathrm{Lip}} < \infty. \tag{4.6}$$

In this case, we have

Theorem 4.2. *Assume that $\mathcal{A}$ is as in Theorem 2.1, and that (2.20), (2.21) and (4.6) hold. Equation (4.4) has then, for all sufficiently large $\nu > 0$, a unique solution $V \in$*

$H_{\nu,0,0}([0,+\infty[)$. *This solution depends continuously on the data* F *and* V_0, *in the sense that there is a constant* $C > 0$ *such that*

$$|V_1 - V_2|_{\nu,0,0} \leq C \left(|F_1 - F_2|_{\nu,0,0} + |V_{0,1} - V_{0,2}|_1 \right), \tag{4.7}$$

where V_i *denotes the solution corresponding to the forcing term* F_i *and the initial value* $V_{0,i}$, $i = 1, 2$. *As in Theorem 3.1,* V *does not depend on* ν *as long as* ν *is sufficiently large, and if (4.1) holds, estimate (4.7) can again be refined to*

$$|V_1 - V_2|_{\nu,0,0} \leq C \left(|F_1 - F_2|_{\nu,0,0} + |V_{0,1} - V_{0,2}|_0 \right). \tag{4.8}$$

We mention that, as expected, we can also obtain stronger regularity of solutions in the "parabolic" case, together with corresponding well-posedness results. As a specific example, the memory term considered in the previous section clearly satisfies the conditions required for Φ; in particular, (4.6) follows because the Lipschitz constant for the map Z introduced in (3.1) is independent of ν if $\nu \geq \nu_0$.

4.3. We conclude with one example, which also shows that our techniques apply to some nonlinearities of the type considered by Ruess, [14], in the context of "parabolic" functional evolution equations with "infinite delay"; compare also to the cases considered in the extensive references cited therein.

We first introduce the translations τ_t by

$$(\tau_t \psi)(x) = \psi(x + t),$$

for $t, x \in \mathbb{R}$ and, e.g., $\psi \in C_0^{\infty,+}(\mathbb{R}) \underset{a}{\otimes} H_0$; then, given $f : H_{\nu_0,0,0}(]-\infty, 0]) \to H_0$ for some $\nu_0 \in \mathbb{R}$, we consider the composition $f \circ \tau_{(\cdot)}$. We claim that if f is Lipschitz continuous, the map $\Phi : H_{\nu,0,0}([0,\infty[) \to H_{\nu,0,0}([0,+\infty[)$ defined by

$$\Phi(V) := f(\tau_{(\cdot)}(V + U_{-\infty})), \tag{4.9}$$

for a fixed prehistory $U_{-\infty}$, satisfies condition (4.6): that is, Lipschitz continuity holds uniformly with respect to ν if ν is sufficiently large. Indeed, letting $|f|_{\mathrm{Lip}}$ denote the (best) Lipschitz constant of f, for $\nu \geq \nu_0$ we estimate

$$\begin{aligned}
&\int_0^{+\infty} |f(\tau_t \varphi_1) - f(\tau_t \varphi_2)|_0^2 \, e^{-4\pi\nu t} \, dt \leq \\
&\quad \leq |f|_{\mathrm{Lip}}^2 \int_0^{+\infty} \int_{\mathbb{R}} |\varphi_1(t+s) - \varphi_2(t+s)|_0^2 \, e^{-4\pi\nu_0(t+s)} ds \, e^{-4\pi(\nu-\nu_0)t} \, dt \leq \\
&\quad \leq |f|_{\mathrm{Lip}}^2 \int_0^{+\infty} \int_{-\infty}^{t} |\varphi_1(u) - \varphi_2(u)|_0^2 \, e^{-4\pi\nu_0 u} \, du \, e^{-4\pi(\nu-\nu_0)t} \, dt \leq \\
&\quad \leq |f|_{\mathrm{Lip}}^2 \int_0^{+\infty} \int_{-\infty}^{0} |\varphi_1(u) - \varphi_2(u)|_0^2 \, e^{-4\pi\nu_0 u} \, du \, e^{-4\pi(\nu-\nu_0)t} \, dt \, +
\end{aligned}$$

$$+|f|^2_{\mathrm{Lip}}\int_0^{+\infty}\int_0^t |\varphi_1(u)-\varphi_2(u)|_0^2\; e^{-4\pi\nu_0 u}\,du\,e^{-4\pi(\nu-\nu_0)t}\,dt \le$$

$$\le \frac{|f|^2_{\mathrm{Lip}}}{4\pi(\nu-\nu_0)}\int_{-\infty}^0 |\varphi_1(u)-\varphi_2(u)|_0^2\; e^{-4\pi\nu_0 u}\,du +$$

$$+|f|^2_{\mathrm{Lip}}\int_0^{+\infty}\int_u^{+\infty} |\varphi_1(u)-\varphi_2(u)|_0^2\; e^{-4\pi\nu_0 u}\,e^{-4\pi(\nu-\nu_0)t}\,dt\,du \le$$

$$\le \frac{|f|^2_{\mathrm{Lip}}}{4\pi(\nu-\nu_0)}\int_{-\infty}^0 |\varphi_1(u)-\varphi_2(u)|_0^2\; e^{-4\pi\nu_0 u}\,du +$$

$$+\frac{|f|^2_{\mathrm{Lip}}}{4\pi(\nu-\nu_0)}\int_0^{+\infty} |\varphi_1(u)-\varphi_2(u)|_0^2\; e^{-4\pi\nu u}\,du.$$

Thus, we obtain

$$\begin{aligned}&\big|f(\tau_{(\cdot)}\varphi_1)-f(\tau_{(\cdot)}\varphi_2)\big|_{\nu,0,0}\le \\ &\quad\le \frac{|f|_{\mathrm{Lip}}}{2\sqrt{\pi(\nu-\nu_0)}}\left(\big|(\chi_-)(\varphi_1-\varphi_2)\big|_{\nu_0,0,0}+\big|(\chi_+)(\varphi_1-\varphi_2)\big|_{\nu,0,0}\right),\end{aligned}\tag{4.10}$$

where χ_- and χ_+ denote, respectively, the characteristic functions of the intervals $]-\infty,0]$ and $[0,+\infty[$. From (4.10) and (4.9) we conclude that for V_1, $V_2\in H_{\nu,0,0}([0,+\infty[)$,

$$|\Phi(V_1)-\Phi(V_2)|_{\nu,0,0}\le \frac{|f|_{\mathrm{Lip}}}{2\sqrt{\pi(\nu_1-\nu_0)}}\,|V_1-V_2|_{\nu,0,0}\,,\tag{4.11}$$

uniformly for $\nu\ge\nu_1>\nu_0$. This shows the desired uniform Lipschitz continuity of Φ, that is, (4.6); it is worth to point out that, in this case, the decay of the Lipschitz constant of Φ in ν_1 obtained in (4.11) allows us to relax the assumptions on $\mathcal{A}$ to (2.12).

Going back to nonlinearities in memory terms, we see that the map

$$V\mapsto\int_0^{+\infty}\mathcal{K}(\cdot-s)\zeta(V(s))\,ds+\int_{-\infty}^0\mathcal{K}(\cdot-s)\zeta(U_{-\infty}(s))\,ds$$

considered in §3 is of the desired form. Indeed, in this case the maps

$$V\mapsto\int_{-\infty}^{-t}\mathcal{K}(-u)\zeta(U_{-\infty}(u+t))\,du+\int_{-t}^0\mathcal{K}(-u)\zeta(V(u+t))\,du$$

are, for all $t\in[0,\infty[$, of the type (4.9), with

$$f(\varphi)\;:=\int_{-\infty}^0\mathcal{K}(-u)\zeta(\varphi(u))\,du.$$

In particular, $f:\,H_{\nu_0,0,0}\to H_0$ is Lipschitz continuous if, in the notations of (2.23),

$$\kappa(\cdot)\in H_{\nu_0,0}([0,+\infty[)\;:=\;\{\,\varphi\in H_{\nu_0,0}\mid \mathrm{supp}\varphi\subseteq[0,+\infty[\,\}\tag{4.12}$$

for some $\nu_0 \in \mathbb{R}$. In fact, we estimate

$$|f(\varphi_1) - f(\varphi_2)|_0 \leq |\zeta|_{\rm Lip} \int_{-\infty}^{0} \kappa(-u)\ |\varphi_1(u) - \varphi_2(u)|_0\ du \leq$$
$$\leq |\zeta|_{\rm Lip} \left(\int_0^{+\infty} (\kappa(u))^2 e^{-4\pi\nu_0 u}\, du \right)^{1/2} |\varphi_1 - \varphi_2|_{\nu_0,0,0}\,,$$

which shows the asserted Lipschitz continuity of f. Note that condition (4.12) on $\mathcal{K}$ is locally "stronger" than (2.23), in that κ is required to be locally of class L^2 rather than L^1.

5 Application: Maxwell's Equations.

5.1: The Equations. In this section we consider the system of the linear Maxwell's equations

$$\begin{cases} D_t - {\rm curl} H = G_1 \\ B_t + {\rm curl} E = G_2 \end{cases} \tag{5.1}$$

in materials with memory. This means that the constituent relations, which relate the electric and magnetic field E, H to the corresponding "inductions" D, B, and which are necessary to transform (5.1) into a determined system, contain nonlinearities of nonlocal type, described by convolution integrals of the kind we have considered in the previous sections. Our goal is to show how this model system (i.e., equations (5.1) together with the constituent relations) can be formulated as an abstract evolution equation of the form (1.12), to which the results of the previous sections can be applied. We refer to the monograph by Prüss, [13], for general background information on electromagnetic models in materials with memory, as well as to the papers [1], [3], [4], [5], [7], quoted in the references, for analogous models.

5.2: The Spaces. Given an arbitrary open set $\Omega \subset \mathbb{R}^3$, we define

$$\mathbb{L}^2(\Omega) := (L^2(\Omega))^3, \tag{5.2}$$
$$\mathbb{H} := \mathbb{L}^2(\Omega) \times \mathbb{L}^2(\Omega), \tag{5.3}$$

each equipped with the usual Euclidean norm from $L^2(\Omega)$; in particular, we denote by $\|\cdot\|$ and $(\cdot\,,\cdot)$ respectively the norm and the scalar product in $\mathbb{L}^2(\Omega)$. We set then

$$H({\rm curl}, \Omega) := \left\{ u \in \mathbb{L}^2(\Omega) \ \mid\ {\rm curl} u \in \mathbb{L}^2(\Omega) \right\}, \tag{5.4}$$

with norm defined by

$$\|u\|_C^2 \ :=\ \|u\|^2 + \|{\rm curl} u\|^2,$$

and consider its closed subspace

$$H_0(\mathrm{curl},\Omega) := \big\{u \in H(\mathrm{curl},\Omega) \;\mid\; \forall\, \varphi \in H(\mathrm{curl},\Omega), \tag{5.5}$$

$$(\mathrm{curl}u,\varphi) = (u,\mathrm{curl}\varphi)\big\}. \tag{5.6}$$

Note that the condition (5.6) which appears in the definition of the space $H_0(\mathrm{curl},\Omega)$ is a generalization to arbitrary open sets of the classical boundary condition

$$n \times u = 0 \qquad \text{on } \partial\Omega, \tag{5.7}$$

where n denotes the (exterior) normal to the boundary $\partial\Omega$ of the domain Ω. Indeed, if Ω is bounded, u, $\varphi \in C^1(\overline{\Omega})$ and $\partial\Omega$ is smooth, (5.6) is just the well known identity

$$\int_\Omega \mathrm{curl}u \cdot \varphi\, dx = \int_\Omega u \cdot \mathrm{curl}\varphi\, dx,$$

which holds if u satisfies (5.7) in the classical sense. Finally, we denote by $\mathbb{M}_{3\times 3}$ the set of 3×3 matrices with complex entries.

5.3: The Operators. We introduce in $\mathbb{H}$ the (unbounded) operator

$$A \;:=\; \frac{1}{2\pi \mathrm{i}} \begin{pmatrix} 0 & \mathrm{curl} \\ -\mathrm{curl} & 0 \end{pmatrix} : \tag{5.8}$$

A is then densely defined, closed and self-adjoint in $\mathbb{H}$ with domain

$$\mathcal{D}(A) \;=\; H_0(\mathrm{curl},\Omega) \times H(\mathrm{curl},\Omega). \tag{5.9}$$

This fact is in general quite well-known, see e.g. [8] or [10]; however, most proofs, even those reported in very recent literature, are given under more restrictive assumptions on the domain Ω, which unnecessarily limit the applicability of the result. In particular, we wish to point out explicitly that no trace results are needed for $\partial\Omega$ in order to establish the self-adjointness of A.

The constituent relations between the fields E, H, and the corresponding inductions D, B can then be described very generally by means of a linear, positive operator $L : \mathbb{H} \to \mathbb{H}$, and by a nonlinear, Lipschitz continuous map $\zeta : \mathbb{H} \to \mathbb{H}$, to construct, via a convolution kernel K, the map M defined in (1.4). A possible example, corresponding to a situation which seems to be quite common in applications, is an operator L which is a positive definite 2×2 block matrix

$$L \;=\; \begin{pmatrix} l_{11} & l_{12} \\ l_{21} & l_{22} \end{pmatrix}, \tag{5.10}$$

with entries $l_{ij} \in L^\infty(\Omega;\mathbb{M}_{3\times 3})$; however, L could also contain nonlocal terms such as integral operators (e.g., by convolution in the space variables). In (5.10), a typical choice

is $l_{11} = \varepsilon$ (dielectricity; a matrix-valued function related to displacement currents), $l_{22} = \mu$ (permeability; a matrix-valued function related to magnetic induction), and $l_{12} = l_{21} \equiv 0$; ε and μ are assumed uniformly positive definite. Analogously, a common choice (but by no means the only possible one) for the convolution kernel K is again a 2×2 block matrix, of the form

$$K(t) \;=\; \begin{pmatrix} k_{11}(t) & k_{12}(t) \\ k_{21}(t) & k_{22}(t) \end{pmatrix}; \tag{5.11}$$

with this choice, the case of "fading memory" considered e.g. by Bloom, [1], corresponds to the requirement that ($\nu_K = 0$ or $\nu_0 = 0$ and)

$$k_{ij} \in W^{1,1}(\mathbb{R}; L^\infty(\Omega; \mathbb{M}_{3\times 3})). \tag{5.12}$$

In conclusion, letting ζ_i denote, for $i = 1, 2$, the components of ζ, the special examples (5.10) and (5.11) would yield the constituent relations

$$\begin{aligned} D(t) &= l_{11}E(t) + l_{12}H(t) + \\ &\quad + \int_{-\infty}^{t} (k_{11}(t-s)\zeta_1(E(s),H(s)) + k_{12}(t-s)\zeta_2(E(s),H(s)))\, ds, \\ B(t) &= l_{21}E(t) + l_{22}H(t) + \\ &\quad + \int_{-\infty}^{t} (k_{21}(t-s)\zeta_1(E(s),H(s)) + k_{22}(t-s)\zeta_2(E(s),H(s)))\, ds. \end{aligned}$$

5.4: Conclusion. To complete the formal transformation of Maxwell's equations into the form (1.12), we define the external source term $F_0 = \frac{1}{2\pi i}(G_1, G_2)^\top$, where the upper index $\top$ denotes transposition. Note that the generalized boundary condition $E \in H_{\nu,0} \otimes H_0(\mathrm{curl}, \Omega)$ built into the domain of A (or $\mathcal{A} := L^{-1}A$) describes the so-called "perfect conductivity" condition on the boundary of the region Ω filled by the material; that is, the complement of Ω is perfectly conducting.

With $V := (E, H)^\top$, $\Gamma := L^{-1}K(0)\zeta$, $\mathcal{A} := L^{-1}A$, $\mathcal{K} := L^{-1}K'$ and F defined as in (1.10), the initial boundary value problem for Maxwell's equations in a region Ω with perfectly conducting boundary $\partial\Omega$ assumes then the desired form (1.12). Thus, the problem can be treated with the techniques we have presented here. Indeed, $\mathcal{A}$ is a selfadjoint operator (see e.g. [10]) in the Hilbert space H_0 constructed from $\mathbb{H}$ by introducing the modified inner product (compare with (1.11))

$$\langle \cdot \,|\, \cdot \rangle_0 := (L^{1/2}\cdot\,,\, L^{1/2}\cdot) = (\,\cdot\,,\, L\,\cdot). \tag{5.13}$$

Consequently, following [11] we conclude that $\mathcal{A}$ satisfies the required assumptions (refer also to our earlier comment on the generation of continuous families of operator semi-groups $(S(t))_{t\geq 0}$). In particular, Theorems 3.1, 4.1 and 4.2 are applicable.

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References

[1] **F. Bloom,** *Asymptotic Bounds for Solutions to a System of Damped Integrodifferential Equations of Electromagnetic Theory.* J. Math. An. Appl., 73 (1980), 524-542.

[2] **D. Brandon - W. J. Hrusa**, *Construction of a Class of Integral Models for Heat Flow in Materials with memory.* J. Int. Eqs. Appl., 1/2 (1988), 175-201.

[3] **M. Fabrizio**, *Una Teoria Fenomenologica per la Termodinamica del Campo Elettromagnetico.* Boll. U.M.I.(4), 4 (1971), 708-719.

[4] **M. Fabrizio - E. Santi**, *Esistenza, Unicità e Stabilità Asintotica per un Sistema Iperbolico con Condizioni al Contorno Ereditarie.* Rend. Mat. (7), 13 (1993), 397-414.

[5] **M. Fabrizio - A. Morro**, *A Boundary Condition with Memory in Electromagnetism.* Arch. Rat. Mech. Anal., 136 (1996), 359-381.

[6] **W. J. Hrusa**, *Global Existence of Classical Solutions to the Equations of Motions for Materials with Fading Memory.* In: Lecture Notes in Pure and Applied Mathematics, # 102; pp. 97-110. Dekker, New York, 1985.

[7] **I. Kurbanov**, *Solvability of the Linear Maxwell Equations with Memory.* Ukr. Mat. J., 43/5 (1991), 619-635.

[8] **R. Leis**, *Initial Boundary Value Problems in Mathematical Physics.* Chapter 8, Teubner & Wiley, Stuttgart, Chichester et al., 1986.

[9] **N. Mattioli - M. C. Salvatori**, *Un Teorema di Esistenza e Unicità in Ottica Nonlineare Dispersiva.* Atti Sem. Mat. Fis. Univ. Modena, 28 (1979), 405-424.

[10] **R. Picard**, *On the Low Frequency Asymptotics in Electromagnetic Theory.* J. Reine Angew. Math., 354 (1984), 50-73.

[11] **R. Picard**, *Evolution Equations as Space-Time Operator Equations.* J. Math. Anal. Appl., 173/2 (1993), 436-458.

[12] **R. Picard**, *Evolution Equations as Operator Equations in Lattices of Hilbert Spaces.* Glasnik Mat., (2000); to appear.

[13] **J. Prüss**, *Evolutionary Integral Equations and Applications.* Birkäuser, Basel, 1993.

[14] **W. M. Ruess**, *Existence and Stability of Solutions to Partial Functional Differential Equations with Delay.* Adv. Diff. Eqs., 4 (1999), 843-876.

[15] **R. Sampalmieri**, *Weak Solutions of a Conservation Law with Memory.* Boll. U.M.I., (7) 11-B (1997), 393-414.

[16] **R. Sampalmieri**, *Weak Solutions of a Conservation Law with Memory in Several Space Variables.* NoDEA, 5 (1998), 99-116.