VISCOSITY SOLUTIONS AND THE HAMILTONIAN-JACOBI-BELLMANN EQUATION

J.A. VAN CASTEREN

Abstract. In this paper we prove that the Lagrangian action, which may be phrased in terms of a non-linear Feynman-Kac formula, coincides under rather generous hypotheses with the unique viscosity solution to the Hamilton-Jacobi-Bellmann equation. The method of proof is based on martingale theory and Jensen inequality.

1. Introduction

It is pointed out that Zambrini and coworkers [9, 1, 6, 7] have kind of a transition scheme to go from classical stochastic calculus (with non-reversible processes) to physical real time (reversibile) quantum mechanics and vice versa. An important tool in this connection is the so-called Noether theorem. In fact, in Zambrini's words, reference [9] contains the first concrete application of this theorem. In [9] the author formulates a theorem like Theorem 1.1 below, he also uses so-called "Bernstein diffusions" (see e.g. [3]) for the "Euclidean Born interpretation" of quantum mechanics. The Bernstein diffusions are related to solutions of $\left(\dfrac{\partial}{\partial t} - \left(K_0\dot{+}V\right)\right)\eta = 0$, and of $\left(\dfrac{\partial}{\partial t} + K_0\dot{+}V\right)\eta^* = 0$. It would be nice to formulate the stochastic Noether theorem (Theorem 2.4 in [9]) in terms of the carré du champ operator and ideas from stochastic control. The operator K_0 generates a *diffusion* in the following sense: for every C^∞-function $\Phi : \mathbb{R}^\nu \to \mathbb{R}$, with $\Phi(0, \dots, 0) = 0$, the following identity is valid:

$$\begin{aligned} &K_0\left(\Phi(f_1, \dots, f_n)\right) \\ &= \sum_{j=1}^n \frac{\partial \Phi}{\partial x_j}(f_1, \dots, f_n)\, K_0 f_j - \frac{1}{2}\sum_{j,k=1}^n \frac{\partial^2 \Phi}{\partial x_j \partial x_k}(f_1, \dots, f_n)\, \Gamma_1(f_j, f_k) \end{aligned} \tag{1.1}$$

1991 *Mathematics Subject Classification.* 47D06, 47D07, 60G46, 60J25, 35S10.

Key words and phrases. Markov processes, Feynman-Kac formula, martingales, Hamilton-Jacobi-Bellmann equation, viscosity solutions.

Thanks are due to the University of Antwerp (UIA) and the Flemish Fund for Scientific Research (FWO, Flanders) for their material support. The author is indebted to K.T. Sturm (Universität Bonn), Xue-Mei Li (University of Connecticut, Storrs), and E. Pardoux (Marseille) for interesting discussions on the subject and some references.

for all functions $f_1, \dots, f_n$ in a rich enough algebra of functions $\mathcal{A}$, contained in the domain of the generator K_0, as described in below. The algebra $\mathcal{A}$ has to be "large" enough. To be specific, it is supposed to possess the following properties. Its domain is dense in $L^p(E,m)$ for all $1 \le p < \infty$ and it is a core for K_0 considered as an operator in these spaces. The same is true for the space $C_0(E)$. In addition, it is assumed that $\mathcal{A}$ is stable under composition with C^∞-functions of several variables, that vanish at the origin. Moreover, in order to obtain some nice results a rather technical condition is required: whenever $(f_n : n \in \mathbb{N})$ is a sequence in $\mathcal{A}$ that converges to f with respect to the graph norm of K_0 (in $L^2(E,m)$) and whenever $\Phi : \mathbb{R} \to \mathbb{R}$ is a C^∞-function, vanishing at 0, with bounded derivatives of all orders (including the order 0), then one may extract a subsequence $(\Phi(f_{n_k}) : k \in \mathbb{N})$ that converges to $\Phi(f)$ in $L^1(E,m)$, whereas the sequence $(K_0\Phi(f_{n_k}) : k \in \mathbb{N})$ converges in $L^1(E,m)$ to $K_0\Phi(f)$. Notice that all functions of the form $e^{\psi}f$, ψ, $f \in \mathcal{A}$, belong to $\mathcal{A}$. Also notice that the required properties of $\mathcal{A}$ depend on the generator K_0. In fact we will assume that the algebra $\mathcal{A}$ is also large enough for all operators of the form $f \mapsto e^{-\psi}K_0\left(e^{\psi}f\right)$, where ψ belongs to $\mathcal{A}$. We have the following result for generators of diffusions. For the notion of the squared gradient operator (carré du champ operateur) see equality (1.4). The operator K_0 acts on the first variable and so does the squared gradient operator Γ_1. The symbol D_1 stands for the operator $D_1 = \dfrac{\partial}{\partial t}$. Fix $u > t_0 \ge 0$. We work in the space $C_0(E \times (t_0, u])$

Theorem 1.1. *Let $\chi : E \times (t_0, u] \to [0, \infty]$ be a function such that*

$$\mathbb{E}_{x,t}^{M_{v,t}}\left[|\log \chi(X(u), u)|\right], \quad v \in D(D_1 - K_0)$$

is finite for $t_0 < t \le u$. Here $u > t_0 \ge 0$ are fixed times and

$$\{(\Omega, \mathcal{F}, \mathbb{P}_x), (X(t) : t \ge 0), (\vartheta_t : t \ge 0), (E, \mathcal{E})\}$$

is the strong Markov process generated by $-K_0$. Let S_L be a solution to the following Riccati type equation. This equation is called the Hamilton-Jacobi-Bellman equation. For $t_0 < s \le u$ and $x \in E$ the following identity is true:

$$\begin{cases} -\dfrac{\partial S_L}{\partial s}(x,s) + \dfrac{1}{2}\Gamma_1(S_L, S_L)(x,s) + K_0 S_L(x,s) - V(x,s) = 0; \\ S_L(x,u) = -\log \chi(x,u), \quad x \in E. \end{cases} \tag{1.2}$$

Then for any real valued $v \in D(D_1 - K_0)$ the following inequality is valid:

$$S_L(x,t) \le \mathbb{E}_{x,t}^{M_{v,t}}\left[\int_t^u \left(\frac{1}{2}\Gamma_1(v,v) + V\right)(X(\tau),\tau)\,d\tau\right] - \mathbb{E}_{x,t}^{M_{v,t}}\left[\log \chi(X(u),u)\right], \tag{1.3}$$

and equality is attained for the "Lagrangian action" $v = S_L$.

By definition $E_{x,t}[Y]$ is the expectation, conditoned at $X(t) = x$, of the random variable Y which is measurable with respect to the information from the future: i.e. with respect to $\sigma\{X(s) : s \ge t\}$. The measure $\mathbb{P}_{x,t}^{M_{v,t}}$ is defined in equality (1.5) below. Put $\eta_\chi = \exp(-S_L)$, where S_L satisfies (1.2). From (1.1) it follows that $\left(\dfrac{\partial}{\partial t} - \left(K_0 \dot{+} V\right)\right)\eta_\chi = 0$, provided that $K_0 1$ is interpreted as 0, i.e. $\int K_0 f\,dm = 0$ for all functions $f \in D(K_0)$, $K_0 f \in L^1(E,m)$.

Fix a function $v : E \times (t_0, u] \to \mathbb{R}$ in $D(D_1 - K_0)$, where, as above, $D_1 = \dfrac{\partial}{\partial t}$ is differentiation with respect to t. Let the process

$$\left\{(\Omega, \mathcal{F}, \mathbb{P}_{x,t}), ((q_v(t), t) : t \geq 0), (\vartheta_t : t \geq 0), (E \times \mathbb{R}, \mathcal{E} \otimes \mathcal{B})\right\}$$

be the Markov process generated by the operator $-K_v + D_1$, where K_v is defined by $K_v(f)(x,t) = K_0 f(x,t) + \Gamma_1(v, f)(x,t)$. Here, $\mathcal{B}$ denotes the Borel field of $\mathbb{R}$, and by $\Gamma_1(v, f)(x,t)$ we mean

$$\Gamma_1(v, f)(x,t) = \lim_{s \downarrow 0} \frac{1}{s} \mathbb{E}_x \left[(v(X(s), t) - v(X(0), t)) (f(X(s), t) - f(X(0), t)) \right]. \quad (1.4)$$

We also believe that the following version of the Cameron-Martin formula is true. For all finite n-tuples $t_1, \ldots, t_n$ in $(0, \infty)$ the identity (1.6 is valid:

$$\mathbb{E}_{x,t}^{M_{v,t}} \left[\prod_{j=1}^{n} f_j \left(X(t_j + t), t_j + t \right) \right] \quad (1.5)$$

$$= \mathbb{E}_{x,t} \left[\exp\left(-\frac{1}{2} \int_t^u \Gamma_1(v, v)(X(\tau), \tau)\, d\tau - M_{v,t}(u) \right) \prod_{j=1}^{n} f_j \left(X(t_j + t), t_j + t \right) \right]$$

$$= \mathbb{E}_{x,t} \left[\prod_{j=1}^{n} f_j \left(q_v(t_j + t), t_j + t \right) \right] \quad (1.6)$$

where the $\mathbb{E}_{x,t}$-martingale $M_{v,t}(s)$, $s \geq t$, is given by

$$M_{v,t}(s) = v(X(s), s) - v(X(t), t) + \int_t^s \left(-\frac{\partial}{\partial \tau} + K_0 \right) v(X(\tau), \tau)\, d\tau. \quad (1.7)$$

Its quadratic variation part $\langle M_{v,t} \rangle (s) := \langle M_{v,t}, M_{v,t} \rangle (s)$ is given by

$$\langle M_{v,t} \rangle (s) = \int_t^s \Gamma_1(v, v)(X(\tau), \tau)\, d\tau. \quad (1.8)$$

The equality in (1.5) serves as a definition of the measure $\mathbb{P}_{x,t}^{M_{v,t}}(\cdot)$, and the equality in (1.6) is a statement. The proof of Theorem 1.1 can be found in [8]. Let $\chi : E \times [t, u] \to [0, \infty]$ be as in Theorem 1.1. What is the relationship between the following expressions?

$$\sup_{\Phi \in D(D_1 - K_0)} \left\{ \Phi(x,t) :\ -\dot{\Phi} + K_0 \Phi + \frac{1}{2} \Gamma_1(\Phi, \Phi) \leq V,\ \Phi(\cdot, u) \leq -\log \chi(\cdot, u) \right\}; \quad (1.9)$$

$$-\log \mathbb{E}_{x,t} \left[\exp\left(-\int_t^u V(X(\sigma), \sigma)\, d\sigma \right) \chi(X(u), u) \right]; \quad (1.10)$$

$$\inf_{v \in D(K_0)} \left\{ \mathbb{E}_{x,t}^{M_{v,t}} \left[\int_t^u \left(\frac{1}{2} \Gamma_1(v, v) + V \right) (X(\tau), \tau) d\tau \right] - \mathbb{E}_{x,t}^{M_{v,t}} \left[\log \chi(X(u), u) \right] \right\}; \quad (1.11)$$

$$\inf_{\Phi \in D(D_1 - K_0)} \left\{ \Phi(x,t) :\ -\dot{\Phi} + K_0 \Phi + \frac{1}{2} \Gamma_1(\Phi, \Phi) \geq V,\ \Phi(\cdot, u) \geq -\log \chi(\cdot, u) \right\}. \quad (1.12)$$

In what follows we write $D_1 = \dfrac{\partial}{\partial t}$. We also write $D_1\varphi = \dot{\phi}$. A somewhat more general theorem reads as follows. As above, we work in the space $C_0\left(E\times(t_0,u]\right)$, where $u>t_0$ is fixed.

Theorem 1.2. (i) *The following inequalities are valid:*

$$\begin{aligned}
&\sup_{\Phi\in D(D_1-K_0)}\left\{\Phi(x,t):\ -\dot{\Phi}+K_0\Phi+\frac{1}{2}\Gamma_1\left(\Phi,\Phi\right)\le V,\ \Phi(\cdot,u)\le-\log\chi\left(\cdot,u\right)\right\}\\
&\le-\log\mathbb{E}_{x,t}\left[\exp\left(-\int_t^u V\left(X(\sigma),\sigma\right)d\sigma\right)\chi\left(X(u),u\right)\right]\\
&\le\inf_{v\in D(D_1-K_0)}\left\{\mathbb{E}_{x,t}^{M_{v,t}}\left[\int_t^u\left(\frac{1}{2}\Gamma_1\left(v,v\right)+V\right)\left(X(\tau),\tau\right)d\tau\right]-\mathbb{E}_{x,t}^{M_{v,t}}\left[\log\chi\left(X(u),u\right)\right]\right\}\\
&\le\inf_{\Phi\in D(D_1-K_0)}\left\{\Phi(x,t):\ -\dot{\Phi}+K_0\Phi+\frac{1}{2}\Gamma_1\left(\Phi,\Phi\right)\ge V,\ \Phi(\cdot,u)\ge-\log\chi\left(\cdot,u\right)\right\}.
\end{aligned}$$

(ii) *If the function S_L defined by the non-linaear Feynman-Kac formula*

$$S_L(x,t)=-\log\mathbb{E}_{x,t}\left[\exp\left(-\int_t^u V\left(X(\sigma),\sigma\right)d\sigma\right)\chi\left(X(u),u\right)\right] \tag{1.13}$$

belongs to $D\left(K_0\right)$, then the above 4 quantities are equal. Moreover the function S_L satisfies the Hamilton-Jacobi-Bellmann equation (1.28). The same is true if the expressons in (1.9) and in (1.12) are equal.

(iii) *In general the function in (1.13) is a viscosity solution of the Hamilton-Jacobi-Bellmann equation (1.28). This means that if (x,t) belongs to $E\times(t_0,u]$ and if $\varphi\in D\left(D_1-K_0\right)$ has the property that*

$$\left[S_L-\varphi\right](x,t)=\sup\left\{\left[S_L-\varphi\right](y,s):(y,s)\in E\times[t,u]\right\},$$

then

$$\left[-\dot{\varphi}+K_0\varphi\right](x,t)+\frac{1}{2}\Gamma_1\left(\varphi,\varphi\right)(x,t)\le V(x,t). \tag{1.14}$$

It also means that if (x,t) belongs to $E\times(t_0,u]$ and if $\varphi\in D\left(D_1-K_0\right)$ has the property that

$$\left[S_L-\varphi\right](x,t)=\inf\left\{\left[S_L-\varphi\right](y,s):(y,s)\in E\times[t,u]\right\},$$

then

$$\left[-\dot{\varphi}+K_0\varphi\right](x,t)+\frac{1}{2}\Gamma_1\left(\varphi,\varphi\right)(x,t)\ge V(x,t). \tag{1.15}$$

(iv) *If for all $(x,t)\in E\times(t_0,u]$ the expression*

$$\mathbb{E}_{x,t}\left[\exp\left(-\int_t^u V\left(X(\sigma),\sigma\right)d\sigma\right)\chi\left(X(u),u\right)\right],$$

is strictly positive, then the following equality is valid:

$$\begin{aligned}
&-\log\mathbb{E}_{x,t}\left[\exp\left(-\int_t^u V\left(X(\sigma),\sigma\right)d\sigma\right)\chi\left(X(u),u\right)\right] \qquad (1.16)\\
&=\inf_{v\in D(D_1-K_0)}\left\{\mathbb{E}_{x,t}^{M_{v,t}}\left[\int_t^u\left(\frac{1}{2}\Gamma_1\left(v,v\right)+V\right)\left(X(\tau),\tau\right)d\tau\right]\right.
\end{aligned}$$

$$-\mathbb{E}_{x,t}^{M_{v,t}}\left[\log\chi\left(X(u),u\right)\right]\Big\}. \tag{1.17}$$

First we introduce the following proposition.

Proposition 1.3. (i) *The operator $D_1 - K_0 - V$ extends to a generator of a semigroup $\exp\left(s\left(D_1 - K_0 - V\right)\right)$, $s \geq 0$, given by*

$$\begin{aligned}&\exp\left(s\left(D_1 - K_0 - V\right)\right)\Phi(x,t)\\ &= \mathbb{E}_x\left[\exp\left(-\int_0^s V\left(X(\tau),\tau+t\right)d\tau\right)\Phi\left(X(s),s+t\right)\right] \quad (1.18)\\ &= \mathbb{E}_{x,t}\left[\exp\left(-\int_t^{s+t} V\left(X(\tau),\tau\right)d\tau\right)\Phi\left(X(s+t),s+t\right)\right]. \quad (1.19)\end{aligned}$$

(ii) *Let the function $S_L(x,t)$ be given by*

$$\begin{aligned}S_L(x,t) &= -\log\mathbb{E}_x\left[\exp\left(-\int_0^{u-t} V\left(X(\tau),\tau+t\right)d\tau\right)\chi\left(X(u-t),u\right)\right]\\ &= -\log\mathbb{E}_{x,t}\left[\exp\left(-\int_t^{u} V\left(X(\tau),\tau\right)d\tau\right)\chi\left(X(u),u\right)\right].\end{aligned}$$

Then the following identity is valid

$$\left[\exp\left(s\left(D_1 - K_0 - V\right)\right)\exp\left(-S_L\right)\right](x,t) = \exp\left(-S_L(x,t)\right). \tag{1.20}$$

Remark. Suppose that the semigroup $\exp\left(-tK_0\right)$, $t \geq 0$, has an integral kernel $p_0(t,x,y)$, which is continuous on $(0,\infty)\times E\times E$, and hence, for $f : E \to [0,\infty)$ Borel measurable we have $\exp\left(-tK_0\right)f(x) = \int_E p_0(t,x,y)f(y)dm(y)$, where m is a non-negative Radon measure on E. Define the measure $\mu_{x,t}^{y,u}$ on the σ-field generated by $X(\tau)$, $t \leq \tau < u$ by

$$\mu_{x,t}^{y,u}(A) = \mathbb{E}_{x,t}\left[p_0\left(u-s,X(s),y\right)1_A\right],$$

where A belongs to the σ-field generated by $X(\tau)$, $t \leq \tau \leq s$, with $t \leq s < u$. By the $\mathbb{E}_{x,t}$-martingale property of the process $s \mapsto p_0(u-s,X(s),y)$, $t \leq s < u$, the measure is well defined and can be extended to the σ-field generated by $X(\tau)$, $t \leq \tau < u$. The latter can be done via the classical Kolmogorov extension theorem. The integral kernel of the operator $\exp\left(s\left(D_1 - K_0 - V\right)\right)$ is given by the Feynman-Kac formula:

$$\exp\left(s\left(D_1 - K_0 - V\right)\right)(x,y) = \int\exp\left(-\int_t^{s+t} V(X(\tau),\tau)d\tau\right)d\mu_{x,t}^{y,t+s}.$$

Under appropriate conditions on V, the integral kernel of $\exp\left(s\left(D_1 - K_0 - V\right)\right)$ is again continuous on $(0,u]\times E\times E$. Details for time-independent functions V can be found in e.g. [4].

Proof. (Proof of Proposition 1.3.) (i) Let s_1 and s_2 be positive real numbers, and let Φ be a non-negative Borel measurable function defined on $E\times[0,\infty)$. Then we have:

$$\begin{aligned}&\left[\exp\left(s_1\left(D_1 - K_0 - V\right)\right)\exp\left(s_2\left(D_1 - K_0 - V\right)\right)\Phi\right](x,t)\\ &= \mathbb{E}_x\left[\exp\left(-\int_0^{s_1} V\left(X(\tau),\tau+t\right)d\tau\right)\left\{\exp\left(s_2\left(D_1 - K_0 - V\right)\right)\Phi\left(X(s_1),s_1+t\right)\right\}\right]\end{aligned}$$

$$
\begin{aligned}
&= \mathbb{E}_x \left[\exp\left(-\int_0^{s_1} V\left(X(\tau), \tau + t\right) d\tau \right) \right.\\
&\qquad \left. \mathbb{E}_{X(s_1)} \left\{ \exp\left(-\int_0^{s_2} V\left(X(\tau), \tau + s_1 + t\right) d\tau \right) \Phi\left(X(s_2), s_1 + s_2 + t\right) \right\} \right]
\end{aligned}
$$

(Markov property)

$$
\begin{aligned}
&= \mathbb{E}_x \left[\exp\left(-\int_0^{s_1} V\left(X(\tau), \tau + t\right) d\tau \right) \right.\\
&\qquad \left. \exp\left(-\int_0^{s_2} V\left(X(\tau + s_1), \tau + s_1 + t\right) d\tau \right) \Phi\left(X(s_1 + s_2), s_1 + s_2 + t\right) \right]\\
&= \mathbb{E}_x \left[\exp\left(-\int_0^{s_1+s_2} V\left(X(\tau), \tau + t\right) d\tau \right) \Phi\left(X(s_1 + s_2), s_1 + s_2 + t\right) \right]\\
&= \left[\left[\exp\left((s_1 + s_2)\left(D_1 - K_0 - V\right)\right) \Phi \right] (x, t). \right. \qquad (1.21)
\end{aligned}
$$

Next we prove assertion (ii).

$$
\begin{aligned}
&\left[\exp\left(s\left(D_1 - K_0 - V\right)\right) \exp\left(-S_L\right)\right](x, t)\\
&= \mathbb{E}_x \left[\exp\left(-\int_0^{s} V\left(X(\tau), \tau + t\right) d\tau \right) \exp\left(-S_L\left(X(s), s + t\right)\right) \right]\\
&= \mathbb{E}_x \left[\exp\left(-\int_0^{s} V\left(X(\tau), \tau + t\right) d\tau \right) \right.\\
&\qquad \left. \mathbb{E}_{X(s)} \left\{ \exp\left(-\int_0^{u-t-s} V\left(X(\tau), \tau + s + t\right) d\tau \right) \chi\left(X(u - s - t), u\right) \right\} \right]
\end{aligned}
$$

(Markov property)

$$
\begin{aligned}
&= \mathbb{E}_x \left[\exp\left(-\int_0^{s} V\left(X(\tau), \tau + t\right) d\tau \right) \right.\\
&\qquad \left. \exp\left(-\int_0^{u-t-s} V\left(X(\tau + s), \tau + s + t\right) d\tau \right) \chi\left(X(u - t), u\right) \right]\\
&= \mathbb{E}_x \left[\exp\left(-\int_0^{s} V\left(X(\tau), \tau + t\right) d\tau \right) \right.\\
&\qquad \left. \exp\left(-\int_s^{u-t} V\left(X(\tau), \tau + t\right) d\tau \right) \chi\left(X(u - t), u\right) \right]\\
&= \mathbb{E}_x \left[\exp\left(-\int_0^{u-t} V\left(X(\tau), \tau + t\right) d\tau \right) \chi\left(X(u - t), u\right) \right]\\
&= \exp\left(-S_L(x, t)\right). \qquad (1.22)
\end{aligned}
$$

This proves assertion (ii). □

Proof. (Proof of Theorem 1.2.) (i) The first and the final inequality in (i) follow from the non-linear Feynman-Kac formula. For $\Phi \in D\left(D_1 - K_0\right)$ we have:

$$
\Phi(x, t) = -\log \mathbb{E}_{x,t} \left[\exp\left(-\int_t^{u} \left[-\dot{\Phi} + K_0 \Phi + \frac{1}{2} \Gamma_1\left(\Phi, \Phi\right) \right] \left(X(\tau), \tau\right) d\tau \right) \right].
$$

The second inequality of (i) is a consequence of Jensen inequality, and should be compared with the arguments in Zambrini [9], who used ideas from Fleming and Soner: see Chapter VI in [5]. The reader is also referred to [8]. The inequality we have in mind is the following one:

$$-\log \mathbb{E}_{x,t}^{M_{v,t}}\left[\exp\left(-\varphi\right)\right] \le \mathbb{E}_{x,t}^{M_{v,t}}\left[\varphi\right], \tag{1.23}$$

with equality only if φ is constant $\mathbb{P}_{x,t}$-almost surely. We apply 1.23 for the stochastic variable $\varphi = \varphi_v$, given by

$$\varphi_v = -\int_t^u \left[\frac{1}{2}\Gamma_1\left(v,v\right) + V\right]\left(X(\tau),\tau\right)d\tau - M_{v,t}(u) - \log\chi\left(X(u)\right). \tag{1.24}$$

We also notice that the following processes are $\mathbb{P}_{x,t}$ martingales on the interval $[t,u]$:

$$\exp\left(-\frac{1}{2}\left\langle M_{v,t}\right\rangle(s) - M_{v,t}(s)\right) \quad \text{and} \tag{1.25}$$

$$\exp\left(-\frac{1}{2}\left\langle M_{v,t}\right\rangle(s) - M_{v,t}(s)\right)\left(\left\langle M_{v,t}\right\rangle(s) + M_{v,t}(s)\right). \tag{1.26}$$

By the Jensen inequality we have

$$\mathbb{E}_{x,t}^{M_{v,t}}\left[\frac{1}{2}\left\langle M_{v,t}\right\rangle(u) + \int_t^u V\left(X(\tau),\tau\right)d\tau - \log\chi\left(X(u),u\right)\right]$$

(the process in (1.26) is a $\mathbb{P}_{x,t}^{M_{v,t}}$-martingale)

$$\mathbb{E}_{x,t}^{M_{v,t}}\left[-\frac{1}{2}\left\langle M_{v,t}\right\rangle(u) - M_{v,t}(u) + \int_t^u V\left(X(\tau),\tau\right)d\tau - \log\chi\left(X(u),u\right)\right] \tag{1.27}$$

(here we apply Jensen inequality)

$$\ge -\log \mathbb{E}_{x,t}^{M_{v,t}}\left[\exp\left(\frac{1}{2}\left\langle M_{v,t}\right\rangle(u) + M_{v,t}(u) - \int_t^u V\left(X(\tau),\tau\right)d\tau + \log\chi\left(X(u),u\right)\right)\right] \tag{1.28}$$

(definition of the probability measure $\mathbb{E}_{x,t}^{M_{v,t}}$)

$$= -\log \mathbb{E}_{x,t}\left[\exp\left(-\int_t^u V\left(X(\tau),\tau\right)d\tau + \log\chi\left(X(u),u\right)\right)\right].$$

(ii) The assertion in (ii) immediately follows from (i).
(iii) Let (x,t) belong to $E \times (t_0,u]$, and let φ be as in (1.14). Then we have

$$\begin{aligned}
&\left[-\dot{\varphi} + K_0\varphi - V\varphi\right](x,t) + \frac{1}{2}\Gamma_1\left(\varphi,\varphi\right)(x,t)\\
&= \exp\left(\varphi(x,t)\right)\left[\left(D_1 - K_0 - V\right)\exp\left(-\varphi\right)\right](x,t)\\
&= \exp\left(\varphi(x,t)\right)\lim_{s\downarrow 0}\frac{1}{s}\left[\left(\exp\left(sD_1 - sK_0 - sV\right) - I\right)\exp\left(-\varphi\right)\right](x,t)\\
&= \exp\left(\varphi(x,t)\right)\liminf_{s\downarrow 0}\frac{1}{s}\Bigg(\left[\exp\left(sD_1 - sK_0 - sV\right)\left\{\exp\left(S_L - \varphi\right)\right\}\exp\left(-S_L\right)\right](x,t)
\end{aligned}$$

$$
\begin{aligned}
&\qquad - \exp\left(S_L(x,t) - \varphi(x,t)\right)\exp\left(-S_L(x,t)\right)\Big)\\
&= \exp\left(\varphi(x,t)\right)\liminf_{s\downarrow 0}\frac{1}{s}\Big(\left[\exp\left(sD_1 - sK_0 - sV\right)\left\{\exp\left(S_L-\varphi\right)\right\}\exp\left(-S_L\right)\right](x,t)\\
&\qquad - \left\{\sup_{(y,\sigma)\in E\times[t,u]}\exp\left(S_L(y,s)-\varphi(y,s)\right)\right\}\exp\left(-S_L(x,t)\right)\Big)\\
&\le \exp\left(\varphi(x,t)\right)\liminf_{s\downarrow 0}\frac{1}{s}\\
&\quad\Bigg(\left[\exp\left(sD_1 - sK_0 - sV\right)\left\{\sup_{(y,\sigma)\in E\times[t,u]}\left\{\exp\left(S_L-\varphi\right)\right\}(y,\sigma)\right\}\exp\left(-S_L\right)\right](x,t)\\
&\qquad - \left\{\sup_{(y,\sigma)\in E\times[t,u]}\exp\left(S_L(y,\sigma)-\varphi(y,\sigma)\right)\right\}\exp\left(-S_L(x,t)\right)\Bigg)\\
&\le \sup\left\{\left\{\exp\left(S_L-\varphi\right)\right\}(y,\sigma) : (y,\sigma)\in E\times[t,u]\right\}\exp\left(\varphi(x,t)\right)\\
&\qquad \liminf_{s\downarrow 0}\frac{1}{s}\left(\left[\exp\left(sD_1 - sK_0 - sV\right)\exp\left(-S_L\right)\right](x,t) - \exp\left(-S_L(x,t)\right)\right)\\
&= \exp\left(S_L(x,t)-\varphi(x,t)\right)\exp\left(\varphi(x,t)\right)\cdot 0 = 0. \qquad (1.29)
\end{aligned}
$$

The latter equality follows because, for $s > 0$ the equality

$$\left[\exp\left(sD_1 - sK_0 - sV\right)\exp\left(-S_L\right)\right](x,t) = \exp\left(-S_L(x,t)\right)$$

is valid: see Proposition 1.3, assertion (ii).

The reverse inequality (1.15) follows in a similar manner.

(iv) In view of assertion we only have to prove that the expression in (1.17) is less than equal to the one in (1.16). To this end we consider ($v \in D\left(D_1 - K_0\right)$)

$$
\begin{aligned}
&\mathbb{E}_{x,t}^{M_{v,t}}\left[\frac{1}{2}\int_t^u \Gamma_1\left(v,v\right)\left(X(\tau),\tau\right)d\tau + \int_t^u V\left(X(\tau),\tau\right)d\tau - \log\chi\left(X(u),u\right)\right]\\
&= \mathbb{E}_{x,t}^{M_{v,t}}\left[\frac{1}{2}\left\langle M_{v,t}\right\rangle(u) + \int_t^u V\left(X(\tau),\tau\right)d\tau - \log\chi\left(X(u),u\right)\right]
\end{aligned}
$$

(the process in (1.26) is a martingale)

$$
= \mathbb{E}_{x,t}^{M_{v,t}}\left[-\frac{1}{2}\left\langle M_{v,t}\right\rangle(u) - M_{v,t}(u) + \int_t^u V\left(X(\tau),\tau\right)d\tau - \log\chi\left(X(u),u\right)\right]
$$

(definition of the martingale $M_{v,t}()$)

$$
\begin{aligned}
&= \mathbb{E}_{x,t}^{M_{v,t}}\Bigg[-\frac{1}{2}\int_t^u \Gamma_1\left(v,v\right)\left(X(\tau),\tau\right)d\tau - v\left(X(u),u\right) + v\left(X(t),t\right)\\
&\qquad - \int_t^u\left(-D_1+K_0\right)v\left(X(\tau),\tau\right)d\tau + \int_t^u V\left(X(\tau),\tau\right)d\tau - \log\chi\left(X(u),u\right)\Bigg]\\
&= \mathbb{E}_{x,t}^{M_{v,t}}\Bigg[\int_t^u\left\{\left(D_1-K_0\right)v - \frac{1}{2}\Gamma_1\left(v,v\right)\left(X(\tau),\tau\right)\right\}d\tau + \int_t^u V\left(X(\tau),\tau\right)d\tau
\end{aligned}
$$

$$+ v(X(t),t) - v(X(u),u) - \log\chi(X(u),u)\Big]$$

$((X(t),t) = (x,t)$, $\mathbb{P}_{x,t}$-almost surely)

$$= v(x,t) + \mathbb{E}_{x,t}^{M_{v,t}}\left[\int_t^u \left\{(D_1 - K_0)v - \frac{1}{2}\Gamma_1(v,v)(X(\tau),\tau)\right\}d\tau\right.$$
$$\left. + \int_t^u V(X(\tau),\tau)\,d\tau - v(X(u),u) - \log\chi(X(u),u)\right]$$
$$= v(x,t) + \mathbb{E}_{x,t}^{M_{v,t}}\left[\int_t^u \exp(v(X(\tau),\tau))\left[(-D_1 + K_0 + V)\exp(-v)\right](X(\tau),\tau)\,d\tau\right.$$
$$\left. - v(X(u),u) - \log\chi(X(u),u)\right]$$
$$= v(x,t) + \mathbb{E}_{x,t}\left[\exp\left(-\frac{1}{2}\langle M_{v,t}\rangle(u) - M_{v,t}(u) + \int_t^u V(X(\tau),\tau)\,d\tau\right)\right.$$
$$\exp\left(-\int_t^u V(X(\tau),\tau)\,d\tau\right)$$
$$\int_t^u \exp(v(X(\tau),\tau))\left[(-D_1 + K_0 + V)\exp(-v)\right](X(\tau),\tau)\,d\tau$$
$$\left. - v(X(u),u) - \log\chi(X(u),u)\right]$$
$$= v(x,t) + \mathbb{E}_{x,t}\left[\exp\left(\int_t^u \exp(v(X(\tau),\tau))\left[(-D_1 + K_0 + V)\exp(-v)\right](X(\tau),\tau)\,d\tau\right)\right.$$
$$\exp(v(X(t),t) - v(X(u),u))\exp\left(-\int_t^u V(X(\tau),\tau)\,d\tau\right)$$
$$\int_t^u \exp(v(X(\tau),\tau))\left[(-D_1 + K_0 + V)\exp(-v)\right](X(\tau),\tau)\,d\tau$$
$$\left. - v(X(u),u) - \log\chi(X(u),u)\right]. \tag{1.30}$$

Choose $w \in D(D_1 - K_0)$ and define for $s > 0$ the function v_s by

$$\exp(-v_s) = \frac{1}{s}\int_0^s \exp(\sigma(D_1 - K_0 - V))\exp(-w)ds. \tag{1.31}$$

Then

$$\exp(v_s)(-D_1 + K_0 + V)\exp(-v_s) = \frac{(I - \exp(s(D_1 - K_0 - V)))\exp(-w)}{\int_0^s \exp(\sigma(D_1 - K_0 - V))\exp(-w)d\sigma}. \tag{1.32}$$

So from (1.30) we obtain for w in the domain of $D_1 - K_0$ and $s > 0$ the inequality

$$\mathbb{E}_{x,t}^{M_{v_s},t}\left[\frac{1}{2}\int_t^u \Gamma_1(X(\tau),\tau)\,d\tau + \int_t^u V(X(\tau),\tau)\,d\tau - \log\chi(X(u),u)\right]$$

$$
\begin{aligned}
&\leq v_s(x,t) + \mathbb{E}_{x,t}\Bigg[\exp\left(\int_t^u \frac{\left[\left(I-\exp\left(s\left(D_1-K_0-V\right)\right)\right)\exp(-w)\right]\left(X(\tau),\tau\right)}{\int_0^s \left[\exp\left(\sigma\left(D_1-K_0-V\right)\right)\exp(-w)\right]\left(X(\tau),\tau\right)d\sigma} d\tau\right)\\
&\qquad \exp\left(v_s\left(X(t),t\right)-v_s\left(X(u),u\right)\right)\exp\left(-\int_t^u V\left(X(\tau),\tau\right)d\tau\right)\\
&\qquad \int_t^u \frac{\left[\left(I-\exp\left(s\left(D_1-K_0-V\right)\right)\right)\exp(-w)\right]\left(X(\tau),\tau\right)}{\int_0^s \left[\exp\left(\sigma\left(D_1-K_0-V\right)\right)\exp(-w)\right]\left(X(\tau),\tau\right)d\sigma} d\tau\\
&\qquad - v_s\left(X(u),u\right) - \log\chi\left(X(u),u\right)\Bigg]. \qquad (1.33)
\end{aligned}
$$

Upon letting $w \in D\left(D_1 - K_0\right)$ tend to the function S_L in an appropriate manner, we obtain by invoking Proposition 1.3 the inequality

$$
\begin{aligned}
&\inf_{v\in D(D_1-K_0)} \mathbb{E}_{x,t}^{M_{v_s},t}\left[\frac{1}{2}\int_t^u \Gamma_1\left(X(\tau),\tau\right)d\tau + \int_t^u V\left(X(\tau),\tau\right)d\tau - \log\chi\left(X(u),u\right)\right]\\
&\leq S_L(x,t).
\end{aligned}
$$

This proves assertion (iv). □

Conclusion. The main result of the present paper is contained in Theorem 1.2. The method of proof is based on martingale methods. For more information on viscosity solutions the reader is referrred to [2].

References

[1] S. Albeverio, J. Rezende and J.C. Zambrini, Probability and quantum symmetries 2, 1999, (in preparation).

[2] M.G. Crandall and H. Ishii and P.L. Lions, *User's guide to viscosity solutions of second order partial differential equations*, Bull. Amer. Math. Soc. (New Series), vol. **27**, (1992), 1–67.

[3] A.B. Cruzeiro and J.C. Zambrini, *Malliavin Calculus and Euclidean quantum mechanics I; Functional Calculus*, J. Functional Analysis, volume 96, (1991), 62–95.

[4] M. Demuth and J.A. van Casteren, Stochastic spectral theory of selfadjoint Feller operators, (book in preparation, will appear in 2000, Birkhäuser)

[5] W.H. Fleming and H. M. Soner, Controlled Markov processes and viscosity solutions, Applications of Mathematics, **Vol.** 25, Springer Verlag, Berlin, 1993.

[6] M. Thieullen and J.C. Zambrini, *Probability and quantum symmetries 1.The theorem of Noether in Schrodinger's Euclidean quantum mechanics*, Ann. Inst. H., volume 67, no. 3, (1997), 297–338.

[7] M. Thieullen and J.C. Zambrini, *Symmetries in the stochastic calculus of variations*, Prob. Theory and related fields, volume 107, (1997), 401–427.

[8] J. A. Van Casteren, *Feynman-Kac semigroups, martingales and wave operators*, *J. Korean Math. Soc.*, to appear.

[9] J.-C. Zambrini, *A special time-dependent quantum invariant and a general theorem on quantum symmetries*, Proceedings of the second International Workshop Stochastic Analysis and Mathematical Physics: ANESTOC'96, editor R. Rebolledo, Workshop: Vina del Mar Chile 16-20 December 1996, World Scientific, Singapore, 1998, 197–210.

Department of Mathematics and Computer Sciece University of Antwerp (UIA), 2610 Wilrijk/Antwerp, Belgium

E-mail address: vcaster@uia.ua.ac.be