A NOTE ON THE EXISTENCE OF WAVE OPERATORS

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ABSTRACT. Let, for $j = 0,\ 1$, $H_j = H_j^* \geq -\omega_j I$, $\omega_j > -\infty$, be a self-adjoint operator in the Hilbert space $\mathcal{H}_j$. Let $J : \mathcal{H}_1 \to \mathcal{H}_0$ be a continuous linear operator with domain $\mathcal{H}_1$ and range in $\mathcal{H}_0$. Write $H_j = \int_{-\omega_j}^{\infty} \xi E_j(d\xi)$, $j = 0,\ 1$. Then the wave operators

$$\Omega_{\pm}\left(H_0, J, H_1\right) f := \text{s-}\lim_{t \to \pm\infty} \exp\left(itH_0\right) J \exp\left(-itH_1\right) f, \quad f \in P_{\text{ac}}\left(H_1\right) \mathcal{H}_1,$$

exist if and only if for every $f \in P_{\text{ac}}\left(H_1\right) \mathcal{H}_1$ (absolutely continuous subspace of H_1) both strong limits

$$\Omega_{\pm}\left(H_0, E_0\left(A_0\right) J E_1\left(A_1\right), H_1\right) f := \text{s-}\lim_{t \to \pm\infty} \exp\left(itH_0\right) E_0\left(A_0\right) J E_1\left(A_1\right) \exp\left(-itH_1\right) f,$$

exist for every pair of bounded intervals A_0 and A_1 in $\mathbb{R}$, and the function

$$t \mapsto \exp\left(itH_0\right) J \exp\left(-itH_1\right) f$$

is slowly oscillating at $\pm\infty$. Moreover the wave operators $\Omega_{\pm}\left(H_0, E_0\left(A_0\right) J E_1\left(A_1\right), H_1\right)$, where A_0 and A_1 are arbitrary bounded intervals, exist if and only if for every $\varepsilon > 0$ the wave operators $\Omega_{\pm}\left(H_0, J_{\varepsilon}, H_1\right)$ exist. Here $J_{\varepsilon} = \int_0^1 \exp\left(i\tau\varepsilon H_0\right) J \exp\left(-i\tau\varepsilon H_0\right) d\tau$.

1. INTRODUCTION, EXISTENCE OF LOCAL WAVE OPERATORS

Whereas in [7, 8, 9, 10] we mainly considered compactness properties of differences of self-adjoint semigroups, in this paper we concentrate on time-dependent methods in dealing with the existence of (local) wave operators. Uniform continuity and slow oscillation of the (finite) wave vectors are the central issues. Throughout the paper we take the following setup. For $j = 0,\ 1$, $H_j = H_j^* \geq -\omega_j I$, $\omega_j > -\infty$, denotes a self-adjoint operator in the Hilbert space $\mathcal{H}_j$, and $J : \mathcal{H}_1 \to \mathcal{H}_0$ is a linear operator with domain in $\mathcal{H}_1$ and range in $\mathcal{H}_0$. We write $H_j = \int_{-\omega_j}^{\infty} \xi E_j(d\xi)$, $j = 0,\ 1$. This means that $E_0(\cdot)$ and $E_1(\cdot)$ are the spectral decompositions corresponding to H_0 and H_1, respectively. We begin with a definition.

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1. Definition. Let the operators $J : \mathcal{H}_1 \to \mathcal{H}_0$, $H_0 = H_0^*$ (with domain and range in $\mathcal{H}_0$), and $H_1 = H_1^*$ (with domain and range in $\mathcal{H}_1$) be as above; i.e. H_0 and H_1 are bounded from below and J is continuous. Fix $f \in \mathcal{H}_1$. By definition the wave vectors $\Omega_\pm (H_0, J, H_1) f$ exist if the strong limits

$$\Omega_\pm (H_0, J, H_1) f := \text{s-}\lim_{t\to\infty} \exp(\pm itH_0) J \exp(\mp itH_0) f \tag{1}$$

exist. The wave operators $\Omega_\pm (H_0, J, H_1)$ are said to exist if for every f belonging to the absolutely continuous subspace of H_1 (notation $f \in P_{\mathrm{ac}}(H_1)\mathcal{H}_1$) the strong limits in (1) exist. The vector $f \in \mathcal{H}_1$ is said to be *absolutely continuous with respect to* H_1 if the measure $A \mapsto \langle E_1(A) f, f\rangle$ is absolutely continuous with respect to the Lebesgue measure. The local wave operators are said to exist if for every pair of bounded intervals A_0 and A_1 the following wave vectors exist ($f \in P_{\mathrm{ac}}(H_1)\mathcal{H}_1$):

$$\Omega_\pm (H_0, E_0(A_0) J E_1(A_1), H_1) f := \text{s-}\lim_{t\to\infty} \exp(\pm itH_0) E_0(A_0) J(A_1) \exp(\mp itH_0) f.$$

In Theorem 1 we give several conditions which ensure the existence of local wave operators. The proofs depend very much on formulae in [11]: some of the results are related to the KMS-condition in the theory of von Neumann-algebras (e.g. see Sunder [6, page 63 and following]).

1. Theorem. *Fix $f \in \mathcal{H}_1$. The following assertions are equivalent.*

(*i*) *For every pair of bounded intervals the wave vectors $\Omega_\pm (H_0, E_0(A_0) J E_1(A_1), H_1) f$ exist.*

(*ii*) *For some (every) $t_0 > 0$ the following wave vectors exist:*

$$\Omega_\pm \left(H_0, \exp\left(-\frac{1}{2}t_0 H_0\right) J \exp\left(-\frac{1}{2}t_0 H_1\right), H_1\right) f.$$

(*iii*) *For some (every) $t_0 > 0$ the following wave vectors exist:*

$$\Omega_\pm \left(H_0, \int_0^{t_0} \exp(-sH_0) J \exp(-(t_0 - s) H_1)\, ds, H_1\right) f.$$

(*iv*) *For every $\varphi \in L^1(\mathbb{R})$ the following wave vectors exist:*

$$\Omega_\pm \left(H_0, \int_{-\infty}^{\infty} \varphi(\tau) \exp(i\tau H_0) J \exp(-i\tau H_1)\, d\tau, H_1\right) f.$$

(*v*) *For every $\varepsilon > 0$ the following wave vectors exist:*

$$\Omega_\pm \left(H_0, \int_0^1 \exp(i\tau\varepsilon H_0) J \exp(-i\tau\varepsilon H_1)\, d\tau, H_1\right) f.$$

(*vi*) *For every $\varphi \in C_0(\mathbb{R})$ the wave vectors $\Omega_\pm (H_0, \varphi(H_0) J, H_1) f$ exist.*

(vii) *For every bounded Borel measurable function φ with the property that $\lim_{\xi\to\infty}|\varphi(\xi)|=0$ the wave vectors $\Omega_\pm\left(H_0,\varphi\left(H_0\right)J,H_1\right)f$ exist.*

Proof. (i) $\Rightarrow$ (ii). We write ($f\in\mathcal{H}_j$)

$$\exp\left(-\frac{1}{2}t_0H_j\right)f=\frac{1}{2}t_0\int_{-\infty}^{\infty}e^{-\frac{1}{2}t_0\xi}E_j\left(-\infty,\xi\right)fd\xi,\quad j=0,\ 1.$$

(ii) $\Rightarrow$ (i). It suffices to notice the identity

$$\begin{aligned}&\Omega_\pm\left(H_0,E_0\left(A_0\right)JE_1\left(A_1\right),H_1\right)f\\&=E_0\left(A_0\right)\exp\left(\frac{1}{2}t_0H_0\right)\Omega_\pm\left(H_0,\exp\left(-\frac{1}{2}t_0H_0\right)J\exp\left(-\frac{1}{2}t_0H_1\right),H_1\right)\\&\qquad E_1\left(A_1\right)\exp\left(\frac{1}{2}t_0H_1\right)f,\end{aligned}$$

and to observe that, by symbolic calculus, the operator $E_0\left(A_0\right)\exp\left(\frac{1}{2}t_0H_0\right)$, is continuous, and that the vector $E_1\left(A_1\right)\exp\left(\frac{1}{2}t_0H_1\right)f$ can be approximated in the strong topology of $\mathcal{H}_1$ by vectors of the form $\widehat{\varphi}\left(H_1\right)f$, $\varphi\in L^1\left(\mathbb{R}\right)$.
(ii) $\Rightarrow$ (iii). Set $F(\tau)=\exp\left(it_0\tau H_0\right)\left(J\exp\left(-t_0H_1\right)-\exp\left(-t_0H_1\right)J\right)\exp\left(-it_0\tau H_1\right)$. We also introduce the "super-operator" $\mathcal{Q}\left(t_0\right)$, defined by

$$\begin{aligned}\mathcal{Q}\left(t_0\right)T&=\mathbb{E}^{\text{logistic}}\left[\exp\left(iU_1t_0H_0\right)T\exp\left(-iU_1t_0H_1\right)\right]\\&=\frac{\pi}{2}\int_{-\infty}^{\infty}\frac{1}{\left(\cosh(\pi\tau)\right)^2}\exp\left(i\tau t_0H_0\right)T\exp\left(-i\tau t_0H_1\right)d\tau.\end{aligned}\tag{2}$$

Here T is a linear operator from $\mathcal{H}_1$ to $\mathcal{H}_0$, and U_1 is a *logistically distributed* random variable (see Evans [5, §24]):

$$\mathbb{P}^{\text{logistic}}\left[U_1\in B\right]=\frac{\pi}{2}\int_B\frac{1}{\left(\cosh\pi\tau\right)^2}d\tau=\pi^2\int_0^\infty\frac{\sinh\pi\tau}{\left(\cosh\pi\tau\right)^3}\int_{-\tau}^{\tau}1_B(\sigma)d\sigma\,d\tau.$$

In addition we write $\psi(\tau)=\dfrac{\pi}{2}\dfrac{1}{\left(\cosh\pi\tau\right)^2}$. Then its Fourier transform $\widehat{\psi}(\xi)$ is given by $\widehat{\psi}(\xi)=\dfrac{\sinh\frac{1}{2}\xi}{\frac{1}{2}\xi}$. Like in the proof of Theorem 12 in [11] we have

$$\begin{aligned}&\int_0^{t_0}\exp\left(-sH_0\right)J\exp\left(-\left(t_0-s\right)H_1\right)ds\\&\qquad-\sum_{j=0}^{n}\binom{n+1}{j+1}(-1)^j\mathcal{Q}\left(t_0\right)^j\left[t_0\exp\left(-\frac{1}{2}t_0H_0\right)J\exp\left(-\frac{1}{2}t_0H_1\right)\right]\\&=-\frac{t_0}{\pi}\int_0^\infty\left(\int_0^\infty C_n'(\xi)\int_{\tau\xi}^{\infty}\frac{\sin\eta}{\eta}d\eta\,d\xi\right)\frac{1}{1+\tau^2}\left(F(\tau)-F(-\tau)\right)d\tau,\end{aligned}\tag{3}$$

where, $C_n(\xi) = \xi\left(1-\left(\frac{\partial}{\partial\xi}\right)^2\right)\left(\left(1-\widehat{\psi}(\xi)\right)^{n+1}\frac{1}{\xi}\right)$. Then by Lemma 11 in [11] we obtain

$$\begin{aligned}
&\Bigg\|\int_0^{t_0}\exp\left(-sH_0\right)J\exp\left(-\left(t_0-s\right)H_1\right)ds \\
&\qquad -\sum_{j=0}^{n}\binom{n+1}{j+1}(-1)^j\mathcal{Q}\left(t_0\right)^j\left[t_0\exp\left(-\frac{1}{2}t_0H_0\right)J\exp\left(-\frac{1}{2}t_0H_1\right)\right]\Bigg\| \\
&\leq\frac{t_0}{\pi}\int_0^\infty\left|\int_0^\infty C_n'(\xi)\int_{\tau\xi}^\infty\frac{\sin\eta}{\eta}d\eta\,d\xi\right|\frac{1}{1+\tau^2}\left\|\left(F(\tau)-F(-\tau)\right)\right\|d\tau \\
&\leq\frac{t_0}{\pi}C_0\int_0^\infty\int_0^\infty\left(1-\widehat{\psi}(\xi)\right)^{\frac{1}{2}n}\frac{\tau}{(1+\tau\xi)^2}d\xi\frac{1}{1+\tau^2}\left\|\left(F(\tau)-F(-\tau)\right)\right\|d\tau, \qquad (4)
\end{aligned}$$

where $C_0 < 86$. By Lebesgue's dominated convergence theorem the expression in (4) converges to 0 as n tends to ∞. In fact it is dominated by

$$C\times t_0\left\|\exp\left(-t_0H_0\right)J-J\exp\left(-t_0H_1\right)\right\|\frac{\log\log n}{\log n},\quad n\geq 4, \qquad (5)$$

where C is an absolute constant. From (ii) together with Lebesgue's dominated convergence theorem it follows that, for every $n\in\mathbb{N}$, the wave vectors

$$\Omega_\pm\left(H_0,\sum_{j=0}^{n}\binom{n+1}{j+1}(-1)^j\mathcal{Q}\left(t_0\right)^j\left[t_0\exp\left(-\frac{1}{2}t_0H_0\right)J\exp\left(-\frac{1}{2}t_0H_1\right)\right],H_1\right)f$$

exist. From (3), (4) and (5) (or Lebesgue's dominated convergence theorem) the claim in assertion (iii) follows.

(iii) $\Rightarrow$ (ii). This implication is a consequence of the identity:

$$\begin{aligned}
&t_0\exp\left(-\frac{1}{2}t_0H_0\right)T\exp\left(-\frac{1}{2}t_0H_1\right) \\
&=\mathbb{E}^{\text{logistic}}\left[\exp\left(it_0U_1H_0\right)\int_0^{t_0}\exp\left(-uH_0\right)T\exp\left(-(t_0-u)H_1\right)du\,\exp\left(-it_0U_1H_1\right)\right] \\
&=\frac{\pi}{2}\int_{-\infty}^\infty\frac{1}{\left(\cosh\pi\tau\right)^2}\exp\left(i\tau t_0H_0\right)\int_0^{t_0}\exp\left(-uH_0\right)T\exp\left(-\left(t_0-u\right)H_1\right)du \\
&\qquad\exp\left(-i\tau t_0H_1\right)d\tau. \qquad (6)
\end{aligned}$$

Here T is a linear operator from $\mathcal{H}_1$ to $\mathcal{H}_0$, and as above U_1 is a *logistically distributed* random variable: see Proposition 4 in [11].

(iii) $\Rightarrow$ (iv). This implication will be a consequence of the identity (see Theorem 1 (b) in [11]):

$$\begin{aligned}
&\int_0^{t_0}\exp\left(-uH_0\right)J\exp\left(-\left(t_0-u\right)H_1\right)du \\
&=\frac{t_0}{2\pi}\int_{-\infty}^\infty\log\left(\frac{\cosh\pi\tau+1}{\cosh\pi\tau-1}\right)\exp\left(i\tau t_0H_0\right)\left\{\exp\left(-t_0H_0\right)J+J\exp\left(-t_0H_1\right)\right\} \\
&\qquad\exp\left(-i\tau t_0H_1\right)d\tau. \qquad (7)
\end{aligned}$$

By (ii), which is equivalent to (iii), we see that the wave vectors

$$\Omega_{\pm}\left(H_0, \exp\left(-t_0 H_0\right) J, H_1\right) f$$

exist. Here we use the fact that the vector f belongs to the closure of the set

$$\left\{\widehat{\varphi}\left(H_1\right) \exp\left(-\frac{1}{2} t_0 H_1\right) f : \varphi \in L^1(\mathbb{R})\right\}.$$

We also employ the semigroup property of the family $\exp\left(-tH_0\right)$ and continuity of its members. Hence assertion (iii) together with (7) implies the existence of the wave vectors

$$\Omega_{\pm}\left(H_0, t_0 \int_{-\infty}^{\infty} \varphi_0(\tau) \exp\left(i\tau t_0 H_0\right) J \exp\left(-t_0 H_1\right) \exp\left(-i\tau t_0 H_1\right) d\tau, H_1\right) f.$$

Here $\varphi_0(\tau) = \dfrac{1}{2\pi} \log\left(\dfrac{\cosh \pi\tau + 1}{\cosh \pi\tau - 1}\right)$. Again using the fact that the vector f belongs to the closure of the set

$$\left\{\widehat{\varphi}\left(H_1\right) \exp\left(-t_0 H_1\right) f : \varphi \in L^1(\mathbb{R})\right\},$$

we infer assertion (iv) for the function φ_0. Since its Fourier transform $\widehat{\varphi}_0(\xi)$, which is equal to $\dfrac{\tanh \frac{1}{2}\xi}{\xi}$, vanishes nowhere, the classical Wiener Tauberian theorem (see e.g. Yosida [13]) implies that the subspace $\left\{\varphi_0 * \varphi : \varphi \in L^1(\mathbb{R})\right\}$ is dense in $L^1(\mathbb{R})$. A standard approximation argument (and a change of integration variable) then yields (iv) in its full generality.

(iv) $\Rightarrow$ (iii). This implication follows from assertion (iv) together with (7) by the change of integration variable $\tau \mapsto t_0\tau$ and the choice $\varphi(\tau) = \varphi_0\left(\tau/t_0\right)$. Moreover, the continuity of the operator $\exp\left(-t_0 H_0\right)$ is used and the fact that the vector $\exp\left(-t_0 H_1\right) f$ belongs to the closure of $\left\{\widehat{\varphi}\left(H_1\right) f : \varphi \in L^1\left(\mathbb{R}\right)\right\}$.

(iv) $\Rightarrow$ (v). Put $\varphi(\sigma) = \dfrac{1}{\varepsilon} 1_{[0,\varepsilon]}(\sigma)$, and set $\sigma = \varepsilon\tau$.

(v) $\Rightarrow$ (iv). From (v) it follows that all wave vectors of the form

$$\Omega_{\pm}\left(H_0, \int_a^b \exp\left(i\tau H_0\right) J \exp\left(-i\tau H_1\right) d\tau, H_1\right) f, \quad a < b,$$

exist. Since the linear span of $\left\{1_{[a,b]} : -\infty < a < b < \infty\right\}$ is dense in $L^1(\mathbb{R})$, assertion (iv) follows via an approximation argument.

(ii) $\Rightarrow$ (vi). Fix $t_0 > 0$ and put $\varphi_0(\xi) = \exp\left(-\frac{1}{2} t_0 \xi\right)$. The linear span of the polynomials $\{\varphi_0^n : n \in \mathbb{N},\ n \geq 1\}$ is dense in the space of bounded continuous functions on the interval $\left[-\omega_0, \infty\right)$, which tend to zero at ∞ (theorem of Stone-Weierstrass). By symbolic calculus for self-adjoint operators for every continuous function $\varphi : \left[-\omega_0, \infty\right) \to$

$\mathbb{R}$ which vanishes at $+\infty$, the operator $\varphi(H_0)$ belongs to the closed linear span of $\{\varphi_0^n(H_0) : n \in \mathbb{N},\ n \geq 1\}$. The assertion in (vi) then follows.

(ii) $\Rightarrow$ (vi). This implication follows from the choice $\varphi = \varphi_0$.

(vi) $\Rightarrow$ (vii). A bounded Borel B subset of $\mathbb{R}$ is majorized by a function $\varphi \in C_0(\mathbb{R})$. From (vi) it follows that (vii) is true for $\varphi = 1_B$. A general bounded Borel measurable function, which vanishes at ∞ is a finite linear combination of such functions φ which satisfy $0 \leq \varphi \leq 1$. The representation $\varphi = \int_0^1 1_{\{\varphi \geq \xi\}} d\xi$ then yields the desired result.

Finally, the implication (vii) $\Rightarrow$ (vi) is trivial.

In the following theorem we treat the uniform continuity of the function

$$t \mapsto \exp(itH_0)\, J \exp(-itH_1)\, f, \quad f \in \mathcal{H}_1.$$

2. Theorem. *Fix f in the Hilbert space $\mathcal{H}_1$. The following assertions are equivalent:*

(i) The function $t \mapsto \exp(itH_0)\, J \exp(-itH_1)\, f$ is uniformly continuous.

(ii) $\lim_{\tau \downarrow 0} \sup_{t \in \mathbb{R}} \|(\exp(i\tau H_0) - I)\, J \exp(-itH_1)\, f\| = 0$.

(iii) $\lim_{\tau \downarrow 0} \sup_{t \in \mathbb{R}} \|(I - \exp(-\tau H_0))\, J \exp(-itH_1)\, f\| = 0$.

(iv) $\lim_{n \to \infty} \sup_{t \in \mathbb{R}} \|E_0(n, \infty) J \exp(-itH_1)\, f\| = 0$.

(v) $\lim_{n \to \infty} \sup_{t \in \mathbb{R}} \|E_0(n, \infty) J E_1(A_1) \exp(-itH_1)\, f\| = 0$, *where A_1 is any bounded interval in $\mathbb{R}$.*

(vi) The sequence $\{t \mapsto E_0(-\infty, n) \exp(itH_0)\, J \exp(-itH_1)\, f : n \in \mathbb{N}\}$ is a Cauchy sequence in the space $(C_b(\mathbb{R}, \mathcal{H}_0), \|\cdot\|_\infty)$, where $\|g\|_\infty = \sup_{t \in \mathbb{R}} \|g(t)\|$, $g \in C_b(\mathbb{R}, \mathcal{H}_0)$.

Proof. (i) $\Rightarrow$ (ii). We notice the following equality:

$$\begin{aligned}
&\exp(itH_0)(\exp(i\tau H_0) - I)\, J(-itH_1)\, f \\
&= \exp(i(\tau + t)H_0)\, J \exp(-i(\tau + t)H_1)\, f - \exp(itH_0)\, J \exp(-itH_1)\, f \\
&\quad + \exp(i(\tau + t)H_0)\, J \exp(-itH_1)(I - \exp(-i\tau H_1))\, f.
\end{aligned} \tag{8}$$

From (8) we infer

$$\begin{aligned}
&\|\exp(itH_0)(\exp(i\tau H_0) - I)\, J(-itH_1)\, f\| \\
&\leq \|\exp(i(\tau + t)H_0)\, J \exp(-i(\tau + t)H_1)\, f - \exp(itH_0)\, J \exp(-itH_1)\, f\| \\
&\quad + \|\exp(i(\tau + t)H_0)\, J \exp(-itH_1)(I - \exp(-i\tau H_1))\, f\|,
\end{aligned} \tag{9}$$

and hence

$$\begin{aligned}
&\sup_{t \in \mathbb{R}} \|(\exp(i\tau H_0) - I)\, J(-itH_1)\, f\| \\
&\leq \sup_{t \in \mathbb{R}} \|\exp(i(\tau + t)H_0)\, J \exp(-i(\tau + t)H_1)\, f - \exp(itH_0)\, J \exp(-itH_1)\, f\| \\
&\quad + \|J\|\, \|(I - \exp(-i\tau H_1))\, f\|.
\end{aligned} \tag{10}$$

The conclusion in assertion (ii) follows from (10) and the continuity of the function $\tau \mapsto \exp(-i\tau H_1)\, f$.

(ii) $\Rightarrow$ (iii). Fix $\tau > 0$. The mappings $z \mapsto \exp(-\tau z H_0)\, g$, $g \in \mathcal{H}_0$, are bounded harmonic on functions the right half plane. In fact they are holomorphic there. As a consequence we get

$$\exp(-\tau H_0) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{1}{1+s^2}\exp(-i\tau s H_0)\, ds.$$

Consequently,

$$\begin{aligned}&(I - \exp(-\tau H_0))\, J \exp(-itH_1)\, f\\ &= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{1}{1+s^2}\,(I - \exp(-i\tau s H_0))\, J\exp(-itH_1)\, f\, ds.\end{aligned}$$

The assertion in (iii) now follows easily.

(iii) $\Rightarrow$ (iv). Fix $\tau > 0$. From symbolic calculus we obtain

$$\begin{aligned}&\left\|(I - \exp(-\tau H_0))\, J \exp(-itH_1)\, f\right\|^2\\ &= \int_{-\omega_0}^{\infty} \left(1 - e^{-\tau\xi}\right)^2 \langle E_0(d\xi) J \exp(-itH_1)\, f, \exp(-itH_1)\, f\rangle\\ &\geq \int_{n}^{\infty} \left(1 - e^{-\tau\xi}\right)^2 \langle E_0(d\xi) J \exp(-itH_1)\, f, \exp(-itH_1)\, f\rangle\\ &\geq \left(1 - e^{-\tau n}\right)^2 \langle E_0(n,\infty) J \exp(-itH_1)\, f, \exp(-itH_1)\, f\rangle,\end{aligned}$$

and hence

$$\left\|E_0(n,\infty) J\exp(-itH_1)\, f\right\| \leq \frac{\left\|(I - \exp(-\tau H_0))\, J\exp(-itH_1)\, f\right\|}{1 - e^{-\tau n}}. \tag{11}$$

Assertion (iv) easily follows from (11).

(iv) $\Rightarrow$ (v). Let φ be a function in $L^1(\mathbb{R})$. From (iv) together with Fubini's theorem it follows that

$$\lim_{n\to\infty}\sup_{t\in\mathbb{R}} \left\|E_0(n,\infty) J\widehat{\varphi}(H_1)\exp(-itH_1)\, f\right\| = 0. \tag{12}$$

Since the space of Fourier transforms of functions in $L^1(\mathbb{R})$ is dense in $C_0(\mathbb{R})$, symbolic calculus implies that (12) is not only true for functions of the form $\widehat{\varphi}(H_1)$, $\varphi \in L^1(\mathbb{R})$, but for all continuous functions $\widehat{\varphi}$ which vanish at infinity. The measure $A \mapsto E_1(A) f$ is a Hilbert space valued (regular) Borel measure. Hence for a given compact subset K of $\mathbb{R}$, a given open subset O of $\mathbb{R}$, containing K, and a given strictly positive real number $\varepsilon > 0$, there exists a function $\varphi \in C_0(\mathbb{R})$, such that $1_K \leq \varphi \leq 1_O$, and such that $\|\varphi(H_1)\, f - E_1(K) f\| \leq \varepsilon$. It also follows that $E_1(A_1)\, f$ can be approximated by vectors of the form $\varphi(H_1)\, f$, where φ belongs to $C_0(\mathbb{R})$.

(v) $\Rightarrow$ (iv). First we approximate f by $E_1(A_1)\, f$, where A_1 is a bounded closed interval. This is possible, because $\lim_{m\to\infty}\|E_1(m,\infty)\, f\| = 0$. After that we use (v) to conclude (iv).

(iv) $\Rightarrow$ (vi). Suppose $n > m$. Then

$$
\begin{aligned}
&\left\| E_0 \left(-\infty, n\right) \exp\left(itH_0\right) J \exp\left(-itH_1\right) f - E_0\left(-\infty, m\right) \exp\left(itH_0\right) J \exp\left(-itH_1\right) f \right\| \\
&= \left\| E_0 \left[m, n\right) \exp\left(itH_0\right) J \exp\left(-itH_1\right) f \right\| \leq \left\| E_0 \left[m, \infty\right) \exp\left(itH_0\right) J \exp\left(-itH_1\right) f \right\| \\
&= \left\| E_0 \left[m, \infty\right) J \exp\left(-itH_1\right) f \right\|. \qquad (13)
\end{aligned}
$$

Hence (vi) follows from (iv) and (13).

(vi) $\Rightarrow$ (i). For $n \in \mathbb{N}$ fixed the function $t \mapsto E_0\left(-\infty, n\right) \exp\left(itH_0\right) J \exp\left(-itH_1\right) f$ is uniformly continuous. Hence, by (vi), the function $t \mapsto \exp\left(itH_0\right) J \exp\left(-itH_1\right) f$ is a uniform limit of ones which are themselves uniformly continuous, and hence it inherits this property. This proves assertion (i).

2. Wave operators and slowly oscillating functions

It is convenient to insert the following definition of *slowly oscillating function*. It is kind of a uniform continuity at infinity. This definition is taken from Benedetto [1] page 124.

3. Definition. A Hilbert space valued function Φ defined on $\mathbb{R}$ is said to be *slowly oscillating* (at $\pm\infty$), if for every $\varepsilon > 0$, there exists $r > 0$ (large) and $\delta > 0$ (small) such that for t and $s \in \mathbb{R}$, for which $|t| \geq r$, $|s| \geq r$ and for which $|t - s| < \delta$ the inequality $\|\Phi(t) - \Phi(s)\| \leq \varepsilon$ holds. The notion of *slowly oscillating* at ∞ is now self-explanatory.

A uniformly continuous function, defined on $\mathbb{R}$, is slowly oscillating. If $\lim_{t\to\infty} \Phi(t)$ exists, then $\Phi(t)$ is slowly oscillating at ∞. A bounded Borel measurable function $\Phi : \mathbb{R} \to \mathcal{H}$ is slowly oscillating at $\pm\infty$ if and only if

$$
\inf_{\varphi \in L^1(\mathbb{R}), m \in \mathbb{N}} \sup_{|t| \geq m} \|\varphi * \Phi(t) - \Phi(t)\| = 0.
$$

The following proposition is a vector-valued version of Theorem 2.3.2 in [1], page 125. In fact Benedetto states the result for just one function $\varphi \in L^1(\mathbb{R})$ with non-vanishing Fourier transform. But if that is the case the linear span of its translates is dense in $L^1(\mathbb{R})$: see e.g. Theorem 1.1.3 (page 25) of [1].

4. Proposition. *Let $\mathcal{H}$ be a Hilbert space, and let $\Phi : \mathbb{R} \to \mathcal{H}$ be a bounded Borel measurable function which slowly oscillates at ∞. Suppose that, for some vector $f_0 \in \mathcal{H}$, $\lim_{t\to\infty} \varphi * \Phi(t) = \int_{\infty}^{\infty} \varphi(\tau) d\tau\, f_0$ for every $\varphi \in L^1(\mathbb{R})$. Then $\lim_{t\to\infty} \Phi(t) = f_0$.*

Remark. The arguments in the proof of Theorem 2 show that a function of the form $t \mapsto \exp\left(itH_0\right) J \exp\left(-itH_1\right) f$ is slowly oscillating at ∞ if and only if for every bounded interval A_1 in $\mathbb{R}$ we have

$$
\inf_{n,\ m \in \mathbb{N}} \sup_{t \geq m} \|E_0(n, \infty) J E_1\left(A_1\right) \exp\left(-itH_1\right) f\| = 0.
$$

The latter is the case if e.g. f belongs to $P_{\mathrm{ac}}\left(H_1\right) \mathcal{H}_1$ and if, for some $t_0 > 0$, the operator $\left(\exp\left(-t_0 H_0\right) J - J \exp\left(-t_0 H_0\right)\right) E_1\left(A_1\right)$, or if for some $n \in \mathbb{N}$ the operator $E_0(n, \infty) J E_1\left(A_1\right)$ is compact: see condition (8.20), page 274 of [4] and [3] as well.

It is also true if for some $n \in \mathbb{N}$ the expression

$$2\pi \int_0^\infty \|E_0(n,\infty) J E_1(A_1) \exp(-itH_1) f\|^2 \, dt$$
$$= \sup_{\varepsilon>0} 2\pi \int_0^\infty e^{-2\varepsilon t} \|E_0(n,\infty) J E_1(A_1) \exp(-itH_1) f\|^2 \, dt$$

(Plancherel's theorem in Hilbert space: see e.g. Yafaev [Yafaev], page 91)

$$= \lim_{\varepsilon \downarrow 0} \int_0^\infty \left\| E_0(n,\infty) J E_1(A_1) ((\xi + i\varepsilon) I - H_1)^{-1} f \right\|^2 d\xi$$

is finite. We also need the fact that the function $t \mapsto E_0(n,\infty) J E_1(A_1) \exp(-itH_1) f$ has a bounded derivative.

The following theorem contains the main result of the present paper.

5. Theorem. *Let the notation and hypotheses be as in Definition 1. The following assertions are equivalent:*

(i) *For every pair of bounded intervals A_0 and A_1 the wave operators*

$$\Omega_\pm(H_0, E_0(A_0) J E_1(A_1), H_1)$$

exist, and for every f in a dense subset of $P_{\mathrm{ac}}(H_1)\mathcal{H}_1$ the function

$$t \mapsto \exp(itH_0) J \exp(-itH_1) f$$

is slowly oscillating at $\pm\infty$.

(ii) *For every pair of bounded intervals A_0 and A_1 the wave operators*

$$\Omega_\pm(H_0, E_0(A_0) J E_1(A_1), H_1)$$

exist, and for every $f \in P_{\mathrm{ac}}(H_1)\mathcal{H}_1$ the function $t \mapsto \exp(itH_0) J \exp(-itH_1) f$ is uniformly continuous.

(iii) *The wave operators $\Omega_\pm(H_0, J, H_1)$ exist.*

Proof. (i) $\Rightarrow$ (ii). Fix $f \in P_{\mathrm{ac}}(H_1)\mathcal{H}_1$ for which the function $\Phi(t)$, defined by $\Phi(t) = \exp(itH_0) J \exp(-itH_1) f$ is slowly oscillating at $\pm\infty$. From assertion (iv) in Theorem 1 it follows that $\lim_{t\to\infty} \varphi * \Phi(t)$ exists for all functions $\varphi \in L^1(\mathbb{R})$. Pick φ_1 and φ_2 in $L^1(\mathbb{R})$. Since

$$\int_{-\infty}^\infty \varphi_2(\tau) d\tau \lim_{t\to\infty} \varphi_1 * \Phi(t) = \lim_{t\to\infty} \varphi_2 * \varphi_1 * \Phi(t) = \lim_{t\to\infty} \varphi_1 * \varphi_2 * \Phi(t)$$
$$= \int_{-\infty}^\infty \varphi_1(\tau) d\tau \lim_{t\to\infty} \varphi_2 * \Phi(t).$$

we obtain that, for some vector $\Omega_+ f \in \mathcal{H}_0$, which is independent of $\varphi \in L^1(\mathbb{R})$, we have $\lim_{t\to\infty} \varphi * \Phi(t) = \int_{-\infty}^\infty \varphi(\tau) d\tau \Omega_+ f$, for all $\varphi \in L^1(\mathbb{R})$. Since by assumption the function $t \mapsto \Phi(t)$ slowly oscillates at ∞ we infer that $\lim_{t\to\infty} \Phi(t) = \Omega_+ f$. A similar reasoning shows that $\lim_{t\to\infty} \Phi(-t) = \Omega_- f$ exists as well. Here we use the slowly oscillating character of $\Phi(t)$ at $-\infty$. Since a continuous function $t \mapsto \Phi(t)$ for which the limits $\lim_{t\to\pm\infty} \Phi(t)$ exist is uniformly continuous, a density argument yields the claim in (ii).

(ii) $\Rightarrow$ (iii). The arguments in the proof of the implication (i) $\Rightarrow$ (ii) can be repeated to conclude (iii) from (ii).

(iii) $\Rightarrow$ (i). This implication is obvious.

3. Some related observations

The following proposition indicates that many functions of the form

$$t \mapsto \exp(itH_0)\, J_0 \exp(-itH_0)\, f$$

are uniformly continuous. The one with J_0 as in (17) is rather significant. If the function $\Phi : t \mapsto \exp(itH_0)\, J \exp(-itH_1)\, f$ is uniformly continuous, then it can be uniformly approximated by functions of the form ($\varphi \in L^1(\mathbb{R})$)

$$t \mapsto \varphi * \Phi(t) = \exp(itH_0) \int_{-\infty}^{\infty} \varphi(\tau) \exp(i\tau H_0)\, J \exp(-i\tau H_1)\, d\tau \exp(-itH_1)\, f.$$

6. Proposition. *Let J_0 be any one of the operators*

$$E_0(A_0)\, J E_1(A_1), \quad A_0 \text{ and } A_1 \text{ are bounded intervals}; \tag{14}$$

$$\exp\left(-\frac{1}{2}t_0 H_0\right) J \exp\left(-\frac{1}{2}t_0 H_1\right), \quad t_0 > 0; \tag{15}$$

$$\int_0^{t_0} \exp(-sH_0)\, J \exp(-(t_0 - s) H_1)\, ds, \quad t_0 > 0; \tag{16}$$

$$\int_{-\infty}^{\infty} \varphi(\tau) \exp(i\tau H_0)\, J \exp(-i\tau H_1)\, d\tau, \quad \varphi \in L^1(\mathbb{R}); \tag{17}$$

$$\varphi(H_0)\, J, \quad \lim_{\xi\to\infty} |\varphi(\xi)| = 0, \quad \varphi \text{ bounded Borel measurable.} \tag{18}$$

Fix $f \in \mathcal{H}_1$. Then the function $t \mapsto \exp(itH_0)\, J_0 \exp(-itH_0)\, f$ is uniformly continuous.

Proof. We only need to prove the claim in Proposition 6 for J_0 given by (16), (17), and (18). The others then follow easily. In order to prove the uniform continuity of the function $t \mapsto \exp(itH_0)\, J_0 \exp(-itH_0)\, f$, where J_0 is as in (18), we assume without loss of generality that $0 \le \varphi \le 1$. We set $\varphi_n = 2^{-n} \sum_{j=1}^{2^n} 1_{\{\varphi \ge j2^{-n}\}}$. Then $\varphi_n \le \varphi_{n+1} \le \varphi \le \varphi_n + 2^{-n}$. From symbolic calculus it follows that it suffices to prove that, for $A := \{\varphi \ge \alpha\}$, with $\alpha > 0$, the function

$$t \mapsto \exp(itH_0)\, E_0(A)\, J \exp(-itH_0)\, f \tag{19}$$

is uniformly continuous. Since the function φ vanishes at ∞, the set A is contained in a bounded interval in $\mathbb{R}$. Consequently the $\mathcal{L}(\mathcal{H}_0)$-valued function $t \mapsto \exp(itH_0)\, E_0(A)$

is uniformly continuous. In addition the function $t \mapsto \exp(-itH_1) f$ is uniformly continuous, and consequently the same is true for the function in (19). (The same argument was used in the proof of the implication (vi) $\Rightarrow$ (i) of Theorem 2.) Next let φ be a function in $L^1(\mathbb{R})$. Then, with $J_0 = \int_{-\infty}^{\infty} \varphi(\tau) \exp(i\tau H_0) J \exp(-i\tau H_1)\, d\tau$, we see

$$\begin{aligned} &\exp(itH_0) J_0 \exp(-itH_1) f - \exp(isH_0) J_0 \exp(-isH_1) f \\ &= \int_{-\infty}^{\infty} \big(\varphi(\tau - t) - \varphi(\tau - s)\big) \exp(i\tau H_0) J \exp(-i\tau H_1) f d\tau, \end{aligned}$$

and hence

$$\begin{aligned} &\|\exp(itH_0) J_0 \exp(-itH_1) f - \exp(isH_0) J_0 \exp(-isH_1) f\| \\ &\le \int_{-\infty}^{\infty} \big|\varphi(\tau - t) - \varphi(\tau - s)\big| d\tau \, \|J\| \ \|f\|. \end{aligned}$$

The translation group $t \mapsto \varphi_t$, where $\varphi_t(\tau) = \varphi(\tau - t)$, is continuous in $L^1(\mathbb{R})$, and thus Proposition 6 follows if J_0 is as in (17). The representation in (7), in conjunction with the results for J_0 given by (17) and (18) shows that the claim in Proposition 6 is also true for the operator J_0 defined by (16).

Of course, the way we proved the result in Proposition 6 for the operator J_0 of the form (16) contains a big detour; i.e. it uses representation (7). A neater and better result is obtained in the following proposition.

7. Proposition. *Fix $t_0 > 0$, and $f \in \mathcal{H}_1$. Define the function $g(t)$ by*

$$g(t) = \exp(itH_0) \int_0^{t_0} \exp(-sH_0) J \exp(-(t_0 - s) H_1)\, ds \exp(-itH_1) f.$$

Then the function g possesses a bounded derivative, given by

$$g'(t) = -i \exp(itH_0) (\exp(-t_0 H_0) J - J \exp(-t_0 H_1)) \exp(-itH_1) f.$$

Proof. The result is an easy consequence of Duhamel's formula:

$$\int_0^{t_0} \exp(-sH_0)(JH_1 - H_0 J) \exp(-(t_0 - s) H_1)\, ds = \exp(-t_0 H_0) J - J \exp(-t_0 H_1).$$

We conclude this paper by mentioning how our results relate to those in Yafaev's book.

Observation. Finally, let $\mathcal{H}$ be an auxiliary Hilbert space and let $G_1 : \mathcal{H}_1 \to \mathcal{H}$ and $G_0 : \mathcal{H}_0 \to \mathcal{H}$ be operators such that, in form sense, $H_0 J - J H_1 = G_0^* G_1$: see e.g. Yafaev [12], page 50. Suppose that G_0 is H_0-bounded and suppose that for every pair of bounded intervals A_0 and A_1 one of the following conditions is satisfied

(a) the operators $G_0 E_0(A_0)$ and $G_1 E_1(A_1)$ are Hilbert-Schmidt (and hence the operator $E_0(A_0)(H_0 J - J H_1) E_1(A_1)$ is trace class);

(b) the operators $G_0E_0(A_0)$ and $G_1E_1(A_1)$ are smooth with respect to H_0 and H_1 respectively.

Then the wave operators $\Omega_\pm(H_0, E_0(A_0)JE_1(A_1), H_1)$ exist, and for $f \in P_{\rm ac}(H_1)\mathcal{H}_1$ the function $t \mapsto \exp(itH_0)J\exp(-itH_1)f$ oscillates slowly at $\pm\infty$. For these results we refer to Yafaev [12] page 205 (Lemma 3), page 206 (Theorem 5) (for the trace class situation), and to page 140 (Theorem 6) (in the case of smooth operators). In fact the arguments in the proof of Theorem 6 (page 140 of [12]) can be used to prove this slowly oscillating behavior. Condition (a) should be compared with condition (8.19) in [4], page 274: see also [3].

We also mention that an operator G_0 is called H_0-*smooth* if the expression

$$\frac{1}{2\pi}\sup_{f\in D(H_0):\|f\|\le 1}\int_{-\infty}^{\infty}\|G_0\exp(-itH_0)f\|^2\,dt = \sup_{A_0\subset\mathbb{R}}\frac{1}{|A_0|}\|G_0E_0(A_0)\|^2$$

is finite: see e.g. Yafaev [Yafaev], page 129.

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