STABILITY OF ASYMPTOTIC PROPERTIES OF HILLE-YOSIDA OPERATORS UNDER PERTURBATIONS

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ABSTRACT. We give conditions on the strongly continuous semigroup $(T_0(t))_{t\geq 0}$, generated by the part of a Hille-Yosida operator A on $X_0 := \overline{D(A)}$, and a non-autonomous family of operators $(B(t))_{t\geq 0}$ such that the evolution family $(U(t,s))_{t\geq s\geq 0}$, generated by the part of $A + B(t)$ in X_0, inherits some asymptotic properties of $(T_0(t))_{t\geq 0}$ as boundedness, asymptotic almost periodicity, uniform ergodicity and total uniform ergodicity. The proof of our main result is based on an extension result of Batty-Chill [4], which is proved here. We illustrate our abstract hypotheses and results by an example from population dynamics. Also, an application of the abstract results to non-autonomous retarded differential equations is given.

1. INTRODUCTION

The question whether some regularity properties or certain asymptotic behaviour, as the norm continuity, norm compactness, the differentiability, almost periodicity,..., of a C_0-semigroup persist under autonomous perturbations, is treated by many authors (see e.g., [5], [15], [18], [19], [20] and the references therein).

For non-autonomous perturbations of generators, Latushkin et al. [11] showed the persistence of exponential dichotomy under small bounded perturbations. Also, in [23], R. Schnaubelt studied the stability of the same property under Miyadera perturbations. In [6], with V. Casarino and S. Piazzera, we extended the results of [5] to non-autonomous perturbations.

This paper is motivated by some dynamical population equations perturbed at the boundary (see Example 3.5), for which we can not use the perturbations of semigroups to study the well-posedness (see, e.g. [16, 25]) and then we can not use the results of [6], to study their asymptotic behaviour. To this purpose, we generalize, in this paper, the results of [6] to the case of Hille-Yosida operators.

To be more precise, we consider an operator $(A, D(A))$ on a Banach space X with not necessarily dense domain satisfying the resolvent conditions in the Hille–Yosida theorem. Let $(T_0(t))_{t\geq 0}$ be the C_0-semigroup generated by its part in $X_0 := \overline{D(A)}$, such that $t \longmapsto T_0(t)x$, $x \in X_0$, belongs to some closed translation biinvariant subspace $\mathcal{E}$ of the space $C_{ub}(\mathbb{R}_+, X_0)$ (see Section 2 for definitions). Consider also a family $(B(t))_{t\geq 0}$ of

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bounded operators from X_0 to X. Assume that $t \longmapsto B(t)x$ is continuous for all $x \in X_0$. We know, from [21], that there is a strongly continuous evolution family $(U(t,s))_{t\geq s\geq 0}$ on X_0 given by the Dayson-Phillips series

$$U(t,s) = \sum_{n=0}^{+\infty} U_n(t,s), \qquad t \geq s \tag{1.1}$$

where $U_0(t,s) := T_0(t-s)$ and $U_n(t,s) := \displaystyle\int_s^t T_{-1}(t-\sigma)B(\sigma)U_{n-1}(\sigma,s)d\sigma$.
The question now is under which conditions does the map $t \longmapsto U(t,0)x$ belongs to $\mathcal{E}$ for all $x \in X_0$?
As the terms of the series (1.1) are given by convolutions between the semigroup $(T_{-1}(t))_{t\geq 0}$ and X-valued functions, we show first a result concerning the asymptotic behaviour of this convolutions, which will be the key to answer the above question, and then obtain the aim of the paper. An analogous result is obtained by C. Batty and R. Chill [4] in the case of infinitesimal generators.

For autonomous perturbations of Hille-Yosida operators, G. Nickel and A. Rhandi [16] showed the stability of the essential spectrum, also with A. Batkái and A. Rhandi, we have recently [3] studied the permanence of the eventual norm continuity, the compactness and the differentiability. We also cite a work of R.H.Thieme [24], where integrated semigroups are used.

These abstract results are applied to the following non-autonomous retarded differential equation

$$(RDE) \quad \begin{cases} x'(t) = Bx(t) + L(t)x_t, \quad t \geq 0, \\ x_0 = \varphi \in C([-r,0], E_0 := \overline{D(B)}), \end{cases}$$

where $(B, D(B))$ is a Hille-Yosida operator on a Banach space E, $(L(t))_{t\geq 0}$ is a family of bounded linear operators from $C([-r,0],E_0)$ to E. We prove that the asymptotic behaviour of the solution of (RDE) can be provided by the one of the semigroup $(S_0(t))_{t\geq 0}$, generated by the part of B in E_0. The same study is done in [6] in L^p-spaces for B a generator. The well-posedness of the equation (RDE) is studied in [21], when B generates a C_0-semigroup.

2. Preliminaries

In this section we check the extrapolation spaces for Hille-Yosida operators and give some needed definitions and results. Let X be a Banach space and A a linear operator with domain $D(A) \subset X$. We say that A is a *Hille-Yosida* operator on X if there exists $\omega \in \mathbb{R}_+$ such that $(\omega, +\infty) \subset \rho(A)$, the resolvent set of A, and

$$M := \sup\{\|(\lambda-\omega)^n R(\lambda,A)^n\| : \lambda > \omega,\, n \geq 0\} < \infty. \tag{2.1}$$

From the Hille-Yosida Theorem (cf. [10], Theorem. 12.2.4), we know that the part A_0 of A in $X_0 := \overline{D(A)}$ given by

$$D(A_0) := \{x \in D(A) : \ Ax \in X_0\}, \quad A_0x := Ax \quad \text{ for } x \in D(A_0),$$

generates a C_0-semigroup $(T_0(t))_{t\geq 0}$. Moreover, $\rho(A) \subset \rho(A_0)$ and for each $\lambda \in \rho(A)$, $R(\lambda, A_0)$ is the restriction of $R(\lambda, A)$ in X_0. We define on X_0 the norm

$\|x\|_{-1} := \|R(\lambda, A_0)x\|$ for a fixed $\lambda \in \rho(A)$ (different $\lambda \in \rho(A)$ yield equivalent norms). The completion X_{-1} of X_0 with respect to $\|\cdot\|_{-1}$ is called the *extrapolation space* of X_0. The *extrapolated semigroup* $(T_{-1}(t))_{t\geq 0}$ is the unique continuous extension of $(T_0(t))_{t\geq 0}$ to X_{-1}. It is strongly continuous and its generator $A_{-1} \in \mathcal{L}(X_0, X_{-1})$ is the unique continuous extension of A_0. Moreover, X is continuously embedded in X_{-1} and $R(\lambda, A_{-1})$ is an extension of $R(\lambda, A)$ for $\lambda \in \rho(A_{-1}) = \rho(A)$. Finally, A_0 and A are the parts of A_{-1} in X_0 and X, respectively.

It follows from [14], Proposition 3.3 that we have

$$\int_0^t T_{-1}(t-\tau)f(\tau)\,d\tau \in X_0$$

and

$$\left\|\int_0^t T_{-1}(t-\tau)f(\tau)\,d\tau\right\| \leq M_1 \int_0^t e^{\omega(t-\tau)}\|f(\tau)\|\,d\tau \tag{2.2}$$

for $f \in L^1_{loc}(\mathbb{R}_+, X)$.
Abstract extrapolation spaces have been introduced by Da Prato-Grisvard [7] and R. Nagel [13] and used for various purposes (cf. [3], [9], [12], [14],...).

For a function $f : \mathbb{R}_+ \to X$, $H(f)$ is the set of all translates $\{f(\cdot + t) : t \in \mathbb{R}_+\}$. Let $C_b(\mathbb{R}_+, X)$ be the Banach space of all bounded continuous functions from $\mathbb{R}_+$ to X, endowed with the uniform norm.

A function $f \in C_b(\mathbb{R}_+, X)$ is called *asymptotically almost periodic* (abbreviated as a.a.p.) if $H(f)$ is relatively compact in $C_b(\mathbb{R}_+, X)$.

A strongly continuous semigroup is called *strongly asymptotically almost periodic* if the function $t \longmapsto T(t)x$ is a.a.p. for every $x \in X$.

Let $C_{ub}(\mathbb{R}_+, X)$ denote the Banach space of all bounded uniformly continuous functions from $\mathbb{R}_+$ to X. A closed subspace $\mathcal{E}$ of $C_{ub}(\mathbb{R}_+, X)$ is said to be *translation biinvariant* if

$$\mathcal{E} = \{f \in C_{ub}(\mathbb{R}_+, X) : \ f(\cdot + t) \in \mathcal{E}\ \} \text{ for all}\, t \geq 0.$$

A closed subspace $\mathcal{E}$ of $C_{ub}(\mathbb{R}_+, X)$ is said to be *operator invariant* if $M \circ f \in \mathcal{E}$ for every $f \in \mathcal{E}$ and $M \in \mathcal{L}(X)$, where $M \circ f$ is defined by $(M \circ f)(t) = M(f(t))$, $t \geq 0$ and *homogeneous* if it is translation biinvariant and operator invariant.

From [4], the following classes of X-valued functions are closed and homogeneous subspaces of $C_{ub}(\mathbb{R}_+, X)$:

- the space $C_0(\mathbb{R}_+, X)$ of all continuous functions vanishing at infinity,
- the class of all asymptotically almost periodic functions from $\mathbb{R}_+$ to X,
- the class of all weakly asymptotically almost periodic functions in the sense of Eberlein,
- the class of uniformly ergodic functions from $\mathbb{R}_+$ to X,
- the class of totally (uniformly) ergodic functions from $\mathbb{R}_+$ to X.

For the sake of completeness we recall the definitions:

A continuous bounded function $f : \mathbb{R}_+ \longrightarrow X$ is called *weakly asymptotically almost periodic in the sense of Eberlein* if $H(f)$ is weakly relatively compact in $C_b(\mathbb{R}_+, X)$.

A function $f \in C_{ub}(\mathbb{R}_+, X)$ is said to be *uniformly ergodic* if the limit

$$\lim_{\alpha \to 0^+} i\alpha \int_0^\infty e^{-\alpha s} f(\cdot + s)\,ds$$

exists and defines an element of $C_{ub}(\mathbb{R}_+, X)$.

A function $f \in C_{ub}(\mathbb{R}_+, X)$ is said to be *totally (uniformly) ergodic* if the function $e^{i\theta\cdot} f(\cdot)$ is uniformly ergodic for all $\theta \in \mathbb{R}$. Since f is uniformly bounded, this is also equivalent to the existence in $C_{ub}(\mathbb{R}_+, X)$ of the Cesáro limit, see [2]

$$\lim_{t\to+\infty} \frac{1}{t} \int_0^t e^{i\theta s} f(\cdot + s)\, ds.$$

3. Main abstract results and example

Let $(A,\, D(A))$ be a Hille-Yosida operator on a Banach space X, $(T_0(t))_{t\geq 0}$ the C_0-semigroup generated by its part on $X_0 := \overline{D(A)}$. Consider a family of linear and bounded operators $(B(t))_{t\geq 0}$ such that $B(\cdot) \in C([0, +\infty), \mathcal{L}_S(X_0, X))$, the set of all strongly continuous functions from $[0, +\infty)$ into $\mathcal{L}(X_0, X)$.

We known from (see [17] and [21], Thm. 2.3) that the family of operators $\big((A + B(t))_{/X_0}\big)_{t\geq 0}$ generates a strongly continuous evolution family $(U(t, s))_{t\geq s\geq 0}$ on X_0 given by the Dyson-Phillips series

$$U(t, s) = \sum_{n=0}^{\infty} U_n(t, s) \qquad \text{for all } t \geq s \geq 0, \tag{3.1}$$

where $U_0(t, s) := T_0(t - s)$ and $U_n(t, s) := \int_s^t T_{-1}(\sigma - s)B(\sigma)U_{n-1}(t, \sigma)d\sigma$.
Moreover, this series converges uniformly on $\{(t, s) : 0 \leq s \leq t \leq T\}$, for each $T > 0$.

If we assume now that there exists a constant $0 \leq q < 1$ such that

$$\int_0^{\infty} ||B(s + \sigma)T_0(\sigma)x||d\sigma \leq q\, ||x|| \tag{3.2}$$

for all $s \geq 0$ and $x \in X_0$, we can show the uniform convergence of the above series on the set $\{(t, s) : t \geq s \geq 0\}$. To do this, we need the following lemma.

Lemma 3.1. *Assume that (3.2) is satisfied and $\|T_0(t)\| \leq 1$ for all $t \geq 0$. Then, for all $x \in X_0$ and $s \geq 0$, the following estimates hold*

$$\int_s^{+\infty} \|B(\sigma)U_n(\sigma, s)x\|d\sigma \leq q^{n+1}\|x\| \quad \textit{for all} \ \ n \in \mathbb{N} \tag{3.3}$$

and

$$\|U_n(t, s)x\| \leq q^n\|x\| \quad \textit{for all } t \geq s \textit{ and } n \in \mathbb{N}. \tag{3.4}$$

Proof. For $n = 0$, the estimate (3.3) is only the condition (3.2). Suppose now that the estimate holds for $n - 1$. Let $x \in X_0$ and $t \geq s$, and denote by $R_\lambda(B) := \lambda R(\lambda, B)$, for $\lambda \in \rho(B)$, for any closed linear operator B. From the definition of $U_n(t, s)$ and by using

Fubini's Theorem, we have

$$\int_s^t ||B(\sigma)R_\lambda(A_0)U_n(\sigma,s)x||d\sigma = \int_s^t ||B(\sigma)R_\lambda(A_0)\int_s^\sigma T_{-1}(\sigma-\tau)B(\tau)U_{n-1}(\tau,s)xd\tau||d\sigma$$

$$\leq \int_s^t\int_s^\sigma ||B(\sigma)R_\lambda(A_{-1})T_{-1}(\sigma-\tau)B(\tau)U_{n-1}(\tau-s)x||d\tau d\sigma$$

$$= \int_s^t\int_0^{\sigma-\tau} ||B(\sigma+\tau)T_0(\sigma)R_\lambda(A)B(\tau)U_{n-1}(\tau,s)x||d\sigma d\tau$$

$$\leq q\int_s^t ||R_\lambda(A)B(\tau)U_{n-1}(\tau,s)x||d\tau$$

$$\leq q\int_s^t ||B(\tau)U_{n-1}(\tau,s)x||d\tau.$$

Hence, By the hypothesis of recurrence and if we let λ go to ∞, we get the estimate. Since $\|T_0(t)\| \leq 1$, we have from the inequality (2.2) and the first estimate that, for each $x \in X_0$ and $n \in \mathbb{N}$

$$\begin{aligned}\|U_n(t,s)x\| &:= \|\int_s^t T_{-1}(t-\sigma)B(\sigma)U_{n-1}(\sigma,s)xd\sigma\| \\ &\leq \int_s^t \|B(\sigma)U_{n-1}(\sigma,s)x\|d\sigma \\ &\leq q^n\|x\|\end{aligned}$$

for all $t \geq s$. □

The following proposition is an immediate consequence of the above lemma.

Proposition 3.2. *Under the hypotheses of Lemma 3.1, the following holds*

(i) for all $x \in X_0$, the series defined by (3.1) converges uniformly on the set $\{(t,s) : t \geq s \geq 0\}$.

(ii) The evolution family $(U(t,s))_{t\geq s\geq 0}$ is uniformly bounded and

$$||U(t,s)|| \leq \frac{1}{1-q} \quad \textit{for all} \;\; t \geq s.$$

To study the asymptotic behaviour of the evolution family $(U(t,s))_{t\geq s\geq 0}$, we need a result concerning the asymptotic behaviour of the convolution between T_{-1} and a function $g \in L^1(\mathbb{R}, X)$, which is defined as

$$(T_{-1} * g)(t) = \int_0^{+\infty} T_{-1}(s)g(t-s)ds = \int_{-\infty}^t T_{-1}(t-s)g(s)ds\,,\; t \geq 0\,.$$

Now we can state the needed result. For an analogous result, in the case of generators, see Batty-Chill [4].

Lemma 3.3. *Let A be a Hille-Yosida operator on X such that the semigroup $(T_0(t))_{t\geq 0}$ is bounded. Let $\mathcal{E}$ be a closed biinvariant subspace of $C_{ub}(\mathbb{R}_+, X_0)$, such that the function*

$\mathbb{R}_+ \ni t \longmapsto T_0(t)x$ belongs to $\mathcal{E}$, for every $x \in X_0$. Then, $(T_{-1} * g)_{/\mathbb{R}_+} \in \mathcal{E}$ for all $g \in L^1(\mathbb{R}, X)$.

Proof. Following the technique used for the proof of Proposition 3.3 in [14] (see also [1]), we can show that $(T_{-1} * g)(t) \in X_0$ for all $g \in L^1(\mathbb{R}, X)$, and

$$\|(T_{-1} * g)(t)\| \leq M_1 \|g\|_{L^1}.$$

Furthermore, by this estimate we can prove that for all $g \in L^1(\mathbb{R}, X)$, $(T_{-1} * g)_{/\mathbb{R}_+} \in C_{ub}(\mathbb{R}_+, X_0)$ and then the map $g \longmapsto (T_{-1} * g)_{/\mathbb{R}_+}$ is bounded from $L^1(\mathbb{R}, X)$ into $C_{ub}(\mathbb{R}_+, X_0)$. For a simple function $g := 1_{(a,b)} \bigotimes x$, with $a \leq b \in \mathbb{R}$ and $x \in X$, we have that

$$\begin{aligned}(T_{-1} * g)(t) &= \int_a^b T_{-1}(t-s)x\,ds \\ &= T_0(t-b) \int_0^{b-a} T_{-1}(s)x\,ds\end{aligned}$$

for all $t \geq b$. Hence, since $\mathcal{E}$ is biinvariant and $t \longmapsto T_0(t)x$ belongs to $\mathcal{E}$ for every $x \in X_0$, we can conclude that $(T_{-1} * g)(\cdot) \in \mathcal{E}$.
Consequently, by linearity and density we get the result. □

Using Proposition 3.2, the estimate (3.3) and Lemma 3.3, we can now show the aim of this paper.

Theorem 3.4. *Let $(A, D(A))$ be a Hille-Yosida operator on X such that $\|T_0(t)\| \leq 1$, $t \geq 0$ and the condition (3.2) holds. If $\mathcal{E}$ is a homogeneous subspace of $C_{ub}(\mathbb{R}_+, X_0)$ and $t \longmapsto T_0(t)x$ belongs to $\mathcal{E}$ for all $x \in X_0$, then the function $t \longmapsto U(t,0)x$ belongs to $\mathcal{E}$ for all $x \in X_0$.*

Proof. For each $x \in X_0$, one has

$$U(t,0)x = \sum_{n=0}^{+\infty} U_n(t,0)x, \qquad t \geq 0$$

and from Proposition 3.2, this series converges uniformly on $\mathbb{R}_+$. Hence, it is sufficient to show that, for each $n \in \mathbb{N}$, $t \longmapsto U_n(t,0)x$ is in $\mathcal{E}$. By hypothesis this holds for $n = 0$. Assume that this is true for $n-1$. For n, we have

$$\begin{aligned}U_n(t,0)x &:= \int_0^t T_{-1}(t-\sigma)B(\sigma)U_{n-1}(\sigma,0)x d\sigma \\ &= (T_{-1} * g)(t),\end{aligned}$$

where g is the function defined on $\mathbb{R}$ by

$$g(t) := \begin{cases} B(t)U_{n-1}(t,0)x, & t \geq 0 \\ 0, & t < 0. \end{cases}$$

In view of Lemma 3.3, it suffices to show that this function g belongs to $L^1(\mathbb{R}, X)$. The claim now follows from the estimate (3.3). □

Example 3.5. We end this section by an application of Proposition 3.2 to an example of population equations. Moreover, by this example we illustrate our main hypothesis (3.2). Let the semigroup $(T_0(t))_{t\geq 0}$ defined on the Banach space $L^1(0,\infty)$ by

$$(T_0(t)f)\,(a) := \begin{cases} f(a-t), & a-t \geq 0, \\ 0, & otherwise. \end{cases}$$

It is clear that $(T_0(t))_{t\geq 0}$ is a contraction semigroup on $L^1(0,\infty)$.
Consider also the operator L from $L^1(0,\infty)$ to $\mathbb{R}$

$$Lf := \int_0^\infty \beta(a) f(a) da,$$

with $\beta \in L^\infty(0,\infty) \cap L^1(0,\infty)$ such that $\|\beta\|_{L^1} < 1$.
Let now the following C_0-semigroup on $\{0\} \times L^1(0,\infty)$

$$\mathcal{T}_0(t) := \begin{pmatrix} 0 & 0 \\ 0 & T_0(t) \end{pmatrix}, \quad t \geq 0$$

and the unbounded perturbation operator

$$\mathcal{B} := \begin{pmatrix} 0 & L \\ 0 & 0 \end{pmatrix}$$

from $\{0\} \times L^1(0,\infty)$ into the space $\mathbb{R} \times L^1(0,\infty)$.
By the above definitions and the Fubini's theorem, we have for all $f \in L^1(0,\infty)$

$$\begin{aligned} \int_0^\infty \|\mathcal{B}\mathcal{T}_0(t) \left(\begin{smallmatrix} 0 \\ f \end{smallmatrix}\right)\| dt &= \int_0^\infty \|LT_0(t)f\|_{L^1} dt \\ &= \int_0^\infty \int_0^\infty |\beta(a+t)||f(a)| da\, dt \\ &\leq ||\beta||_{L^1} ||f||_{L^1}. \end{aligned}$$

Hence, we obtain

$$\int_0^\infty \|\mathcal{B}\mathcal{T}_0(t) \left(\begin{smallmatrix} 0 \\ f \end{smallmatrix}\right)\| dt \leq |\beta|_{L^1} \| \left(\begin{smallmatrix} 0 \\ f \end{smallmatrix}\right) \|.$$

Consequently the condition (3.2) is satisfied for this example with $q := ||\beta||_{L^1} < 1$.
Now all hypotheses of Proposition 3.2 are satisfied. Hence, the perturbed semigroup on $\{0\} \times L^1(0,\infty)$, which can be identified with the solution semigroup $(S_0(t))_{t\geq 0}$, on $L^1(0,\infty)$, of the following dynamical population equation

$$\begin{cases} u_t(t,a) = -\dfrac{\partial}{\partial a} u(t,a), \quad t,\, a \geq 0, \\ u(t,0) = \displaystyle\int_0^\infty \beta(a) u(t,a) da, \quad t \geq 0, \end{cases}$$

is also bounded and one has

$$\|S_0(t)\| \leq \frac{1}{1 - ||\beta||_{L^1}} \text{ for all } t \geq 0.$$

For the well-posedness of this equation and especially for the above matrix operators, we refer to [16], [22] and [25]. Note that we can consider more general models and also non-autonomous births $\beta(t,\cdot)$, $t \geq 0$.

4. Application to non-autonomous retarded differential equations

Let $(B, D(B))$ be a Hille-Yosida operator on a Banach space E and the strongly continuous semigroup $(S_0(t))_{t\geq 0}$, generated by its part in $E_0 := \overline{D(B)}$ such that

$$\|S_0(t)\| \leq 1 \quad \text{for all } t \geq 0.$$

Consider the following non-autonomous retarded differential equation

$$(RDE) \qquad \begin{cases} x'(t) = Bx(t) + L(t)x_t, & t \geq s, \\ \quad x_s = \varphi \in C([-r,0], E_0), \end{cases}$$

where $(L(t))_{t\geq 0}$ is a family of bounded linear operators from $C([-r,0], E_0)$ to E.

It is known (see [9], [12], [21] and [24]) that the study of the well-posedness of (RDE) is equivalent to the one of the following abstract non-autonomous Cauchy problem on the product Banach space $X := E \times C([-r,0], E_0)$

$$\begin{cases} U'(t) = \mathcal{A}U(t) + \mathcal{B}(t)U(t), & t \geq s, \\ U(s) = \binom{0}{\varphi}, \end{cases} \tag{4.1}$$

where $\mathcal{A}$ is the operator defined on X by

$$\mathcal{A} := \begin{pmatrix} 0 & A\delta_0 - \delta' \\ 0 & \frac{d}{d\tau} \end{pmatrix}$$

with domain

$$D(\mathcal{A}) := \{0\} \times \left\{\phi \in C^1([-r,0], E_0) : \phi(0) \in D(A)\right\},$$

and for each $t \geq 0$

$$\mathcal{B}(t) := \begin{pmatrix} 0 & L(t) \\ 0 & 0 \end{pmatrix}.$$

We know, from [21] (see also [12]), that $\mathcal{A}$ is a Hille-Yosida operator on X. Its part on $X_0 := \overline{D(\mathcal{A})} = \{0\} \times C([-r,0], E_0)$ generates then the contraction C_0-semigroup $(\mathcal{T}_0(t))_{t\geq 0}$

$$\mathcal{T}_0(t) := \begin{pmatrix} 0 & 0 \\ 0 & T_0(t) \end{pmatrix},$$

where $(T_0(t))_{t\geq 0}$ is the C_0-semigroup given by

$$(T_0(t)\varphi)(\theta) := \begin{cases} \varphi(t+\theta), & t+\theta \leq 0, \\ S_0(t+\theta)\varphi(0), & t+\theta > 0. \end{cases} \tag{4.2}$$

It is clear that $(\mathcal{B}(t))_{t\geq 0}$ is a family of bounded operators from X_0 to X. Hence, by [21, Theorem 2.3], the family of operators $\left((\mathcal{A} + \mathcal{B}(t))_{/X_0}\right)_{t\geq 0}$ generates a strongly continuous evolution family $(\mathcal{U}(t,s))_{t\geq s\geq 0}$ on the space $\{0\} \times C([-r,0], E_0)$. We have, for each $t \geq 0$

$$(\mathcal{A} + \mathcal{B}(t))_{/X_0} = \begin{pmatrix} 0 & 0 \\ 0 & \frac{d}{d\tau} \end{pmatrix}$$

with domain

$$D((\mathcal{A} + \mathcal{B}(t))_{/X_0}) = \{0\} \times \left\{\phi \in C^1([-r,0], E_0) : \phi(0) \in D(A); \varphi'(0) = A\varphi(0) + L(t)\varphi\right\}.$$

Therefore the family $(A(t), D(A(t)))_{t\geq 0}$ of operators defined by

$$A(t) := \frac{d}{d\tau}, \quad D(A(t)) := \left\{\phi \in C^1([-r,0],E_0) : \phi(0) \in D(A); \varphi'(0) = A\varphi(0) + L(t)\varphi\right\}$$

generates a strongly continuous evolution family $(U(t,s))_{t\geq s}$ on the space $C([-r,0],E_0)$. We have also for all $t \geq s$, that

$$\mathcal{U}(t,s)\begin{pmatrix} 0 \\ \varphi \end{pmatrix} = \begin{pmatrix} 0 \\ U(t,s)\varphi \end{pmatrix} = \begin{pmatrix} 0 \\ T_0(t-s)\varphi \end{pmatrix} + \int_s^t \mathcal{T}_{-1}(t-\sigma)\begin{pmatrix} L(\sigma)U(\sigma,s)\varphi \\ 0 \end{pmatrix} d\sigma.$$

We can also show as in [21] that the solution of (RDE) is the function given by

$$x(t,s,\varphi) := \begin{cases} \varphi(t-s), & s-r \leq t \leq s \\ (U(t,s)\varphi)(0), & t \geq s. \end{cases} \tag{4.3}$$

From the definition, the semigroup $(\mathcal{T}_0(t))_{t\geq 0}$ has the same asymptotic behaviour as $(T_0(t))_{t\geq 0}$, and since this last semigroup is essentially given by $(S_0(t))_{t\geq 0}$, we can hope that they have the same asymptotic behaviour. In the following lemma, we give some common asymptotic behaviours.

Lemma 4.1. *Let $\varphi \in C([-r,0],E_0)$. Assume that the map $\mathbb{R}_+ \ni t \longmapsto S_0(t)\varphi(0)$ is*
(1) continuous and vanishing at infinity, or
(2) Asymptotically almost periodic, or
(3) ergodic, or
(4) (totally) uniformly ergodic.
Then, the function $\mathbb{R}_+ \ni t \longmapsto T_0(t)\varphi$ has the same property.

Remark 4.2. The proof of Lemma 4.1 is similar to the one given in [5] with small differences due to the fact that in [5], the authors considered the equation (RDE) in L^p-spaces.

We can now state the main result of this section.

Theorem 4.3. *Assume that there is a constant $0 \leq q < 1$ such that*

$$\int_0^\infty ||L(s+t)T_0(t)\varphi||dt \leq q||\varphi|| \text{ for all } \varphi \in C([-r,0],E_0) \text{ and } s \geq 0. \tag{4.4}$$

Assume moreover that the function $\mathbb{R}_+ \ni t \longmapsto S_0(t)\varphi(0)$ belongs to one of the classes (1)-(4) of Lemma 4.1 for all $\varphi \in C([-r,0],E_0)$. Then, for all $\varphi \in C([-r,0],E_0)$, the solution of (RDE) $\mathbb{R}_+ \ni t \longmapsto x(t,0,\varphi)$ belongs to the same class as the map $\mathbb{R}_+ \ni t \longmapsto S_0(t)\varphi(0)$.

Proof. From Lemma 4.1, the function $t \longmapsto \mathcal{T}_0(t)\begin{pmatrix} 0 \\ \varphi \end{pmatrix}$ is in the same class as $t \longmapsto S_0(t)\varphi(0)$. Also, by the definition of the family $(\mathcal{B}(t))_{t\geq 0}$, the assumption (4.4) is exactly the condition (3.2) of the previous section for $(\mathcal{T}_0(t))_{t\geq 0}$ and $(\mathcal{B}(t))_{t\geq 0}$. Hence, since the semigroup $(\mathcal{T}_0(t))_{t\geq 0}$ is of contractions, by Theorem 3.4, the map $t \longmapsto \mathcal{U}(t,0)\begin{pmatrix} 0 \\ \varphi \end{pmatrix} = \begin{pmatrix} 0 \\ U(t,0)\varphi \end{pmatrix}$ is in the same class as $t \longmapsto \mathcal{T}_0(t)\begin{pmatrix} 0 \\ \varphi \end{pmatrix}$. Then $t \longmapsto U(t,0)\varphi$ is in the same class as $t \longmapsto S_0(t)\varphi(0)$. Consequently, by (4.3) and since the classes (1)-(4) are operator-invariant, we get the result. $\square$

In the above theorem, we have considered particular asymptotic behaviours. Now, if one can give sufficient conditions on the map $t \longmapsto S_0(t)\varphi(0)$ such that the function

$t \longmapsto T_0(t)\varphi$ is in some homogeneous subspace $\mathcal{E}$ of $BUC(\mathbb{R}_+, C([-r,0],E_0))$, then by Theorem 3.4 we obtain this general theorem.

Theorem 4.4. *If $t \longmapsto T_0(t)\varphi$ belongs to $\mathcal{E}$ for all $\varphi \in C([-r,0],E_0)$, and the condition (4.4) is satisfied then, for all $\varphi \in C([-r,0],E_0)$, the function $\mathbb{R}_+ \ni t \longmapsto x_t(0,\varphi)$ is also in $\mathcal{E}$, where $x(t,0,\varphi)$ is the solution of (RDE).*

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