On a non-uniqueness in fragmentation models

J. Banasiak *

School of Mathematical and Statistical Sciences,
University of Natal, Durban 4041
South Africa

Abstract

Equations of fragmentation describing e.g. the polymer degradation, despite their apparent simplicity, have numerous non- typical features. One of them is the existence of multiple solutions. In this paper, applying classical results of Hille and Phillips, we explain their occurrence and prove that nevertheless the uniqueness holds for a large class of physically reasonable solutions.

Key words: conservative semigroups, fragmentation model, ill-posed problems
AMS subject classification: 47D06, 34G10.

1 Introduction

The process of cluster fragmentation has recently received a considerable interest in the field of polymer science since the degradation of bonds, or depolymerisation, results in fragmentation (see e.g. [1, 12, 11]). In this note we consider continuous models in which the breakup rate depends upon the size, and any polymer can break up only into two parts. If $u(t,x)$ gives the concentration of x-mers as a function of time, and $F(x,y)$ denotes the intrinsic rate that $(x+y)$-mer breaks up into x-mer and y-mer, then u obeys the equation

$$\partial_t u(t,x) = -u(t,x)\int\limits_0^x F(y,x-y)dy + 2\int\limits_x^\infty F(x,y-x)u(y)dy,. \tag{1.1}$$

Moreover, we consider only $F(x,y)=(x+y)^\alpha$, $\alpha\in\mathbb{R}$, that describes the cases in which the break-up depends upon the size of the object breaking up $(x+y)$, but not upon the individual sizes of

*This work was supported by the University of Natal Research Fund

the two resulting pieces. The equation reads then

$$\partial_t u(t,x) = -x^\alpha u(t,x) + 2\int\limits_x^\infty y^{\alpha-1}u(t,y)dy. \tag{1.2}$$

Equation (1.2) was investigated from analytical point of view in [1], a number of closed form solutions were found in [12], and a generalization of (1.1) (or (1.2) with $\alpha > 0$) to the multiple fragmentation case was analyzed in [11]. Some definite results including the identification of the generator in both $\alpha > 0$ and $\alpha < 0$ cases were obtained in [4]. We shall summarize results of [4] below.

All the considerations are carried out in the space

$$X_1 = L_1([0,\infty[; xdx)$$

where the norm

$$\|u\| = \int\limits_0^\infty |u(x)|xdx,$$

for $u \geq 0$ represents the total mass of the ensemble.

It was shown that there exists an extension T_α of the operator $(K_\alpha, D(K_\alpha))$ where

$$[K_\alpha u](x) = -x^\alpha u(x) + 2\int\limits_x^\infty y^{\alpha-1}u(y)dy, \tag{1.3}$$

and $D(K_\alpha) = \{f \in X_1;\ x^\alpha f \in X_1\}$, that generates a positive C_0-semigroup of contractions in X_1, denoted hereafter by $(G_\alpha(t))_{t\geq 0}$. Thus, for any $\mathring{u} \in D(T_\alpha)$ the function $t \to u(t,\mathring{u}) = G_\alpha(t)\mathring{u}$ is a unique classical solution to the problem

$$\begin{aligned} \partial_t u &= T_\alpha u, \qquad t > 0, \\ \lim_{t\to 0^+} u(t,\mathring{u}) &= \mathring{u}, \quad \text{in } X_1. \end{aligned} \tag{1.4}$$

The resolvent of T_α is given by

$$[R(\lambda, T_\alpha)f](x) = \frac{f}{\lambda + x^\alpha} + \frac{2}{(\lambda + x^\alpha)^{1+2/\alpha}}\int\limits_x^\infty f(s)(\lambda + s^\alpha)^{-1+2/\alpha}s^{\alpha-1}ds, \quad \lambda > 0. \tag{1.5}$$

It turns out that $K_\alpha \neq T_\alpha$ for all $\alpha \neq 0$ but for $\alpha > 0$ we have $T_\alpha = \overline{K_\alpha}$. On the other hand, for $\alpha < 0$, T_α is a proper extension of $\overline{K_\alpha}$.

Thus for $\alpha > 0$ the semigroup $(G_\alpha(t))_{t\geq 0}$ is conservative for non-negative initial data, however, for $\alpha < 0$, the solutions to Eq. (1.2) do not conserve mass (despite the equation being formally conservative!). This gives the analytic side to the the phenomenon of the loss of mass observed for explicit solutions in [13, 10] and explained as the "shattering" transition, that is, the creation of "zero" size particles with non-zero mass due to very fast fragmentation of very small particles (the fragmentation rate becomes infinite at 0).

This phenomenon is currently under intensive investigation and falls outside the scope of this paper as the main objective of it is another "pathology" of the fragmentation equation: the existence of multiple solutions, observed for $\alpha > 0$ (in fact, as we shall see later, there are no multiple solutions for $\alpha < 0$ – it seems that one pathology per case is enough).

Possibly the best known pair of multiple solutions to (1.2) (with $\alpha = 1$) is e.g. [5, 12]

$$u_1(t,x) = \frac{\exp t}{(1+x)^3}, \tag{1.6}$$

and

$$u_2(t,x) = e^{-xt}\left(\frac{1}{(1+x)^3} + \int\limits_x^\infty \frac{1}{(1+x)^3}[2t + t^2(y-x)]dy\right). \tag{1.7}$$

Their existence shows that (1.2) offers an incomplete specification of dynamics of the system. An immediate remedy seems to be offered by an observation that the latter solution is mass-conserving, whereas the former clearly is not, and should be ruled out from the considerations.

It turns out that requiring the mass conservation may be also not sufficient. In [1] the authors prove that the uniqueness holds among all solutions satisfying for $u(t) \in D(K_\alpha)$ for all $t \geq 0$ (such solutions are automatically mass-conserving). However, since we know that $D(K_\alpha) \neq D(T_\alpha)$, it could happen that no solution satisfies this requirement.

A step further was done in [11] where it was proved (for a more general multiple-fragmentation equation) that weak solutions satisfying $u(t) \in D(T_\alpha)$ for all $t \geq 0$ (thus given by the formula $u(t) = G_\alpha(t) \overset{\circ}{u}$, $\overset{\circ}{u} \in X_1$) with some additional restriction on the initial data, are unique. Since $T_\alpha = \overline{K_\alpha}$, such solutions are also mass-conserving.

However, the requirement that the solution is in $D(T_\alpha)$ is hardly physical and often rather difficult to check. In this paper we offer an explanation of the existence of multiple solutions and show that the uniqueness holds in what we believe to be the largest physically reasonable class of solutions. By this, roughly speaking, we mean continuous in time X_1-valued functions which are solutions to the integral version of (1.2), non-negative for non-negative initial data, and with non-increasing mass. The precise formulation is provided in Definition 4.1.

Remark 1.1 The reason why we assume that the mass is non-increasing rather than conserved is first, that mathematically this gives a larger class of solutions, and second, that since the loss of mass is physically acceptable for $\alpha < 0$, there could be a mechanism explaining the same phenomenon, if observed, for $\alpha > 0$.

The main tools used in our considerations are two old theorems by Hille and Phillips, [8], pp. 619–620, which we recall for convenience and slightly modify in Section 3.

2 Origin of multiple solutions

A quick calculation shows that the solutions of the form (1.6) can be obtained by the separation of variables method in (1.2) (with $\alpha = 1$): putting $u(t,x) = T(t)X(x)$ we obtain that $T' = \lambda T$ and

$$-xX(x) + 2\int_x^\infty X(y)dy = \lambda X(x),$$

which gives (1.6) for $\lambda = 1$.

Thus in general, for Eq. (1.2), with $\alpha > 0$ one can produce solutions in the form

$$u_{\lambda,\alpha}(t,x) = \frac{\exp \lambda t}{(\lambda + x^\alpha)^{1+2/\alpha}}, \tag{2.1}$$

where the x-dependent factor $X_{\lambda,\alpha}$ is the solution to

$$-x^\alpha X_{\lambda,\alpha}(x) + 2\int_x^\infty y^{\alpha-1} X_{\lambda,\alpha}(y)dy = \lambda X_{\lambda,\alpha}(x). \tag{2.2}$$

It is clear that $u_{\lambda,\alpha} \in C^1([0,\infty[, X_1)$. However, we have

Proposition 2.1 *For any $\alpha > 0$ and $t \geq 0$, $u_{\lambda,\alpha} \notin D(T_\alpha)$. Thus $u_{\lambda,\alpha}$ is neither a solution, nor a mild solution, nor a weak solution to (1.4).*

Proof. Otherwise we would have by (1.5)

$$\frac{1}{(\lambda + x)^{1+2/\alpha}} = \frac{f(x)}{\lambda + x^\alpha} + \frac{2}{(\lambda + x^\alpha)^{1+2/\alpha}} \int_x^\infty f(s)(\lambda + s^\alpha)^{-1+2/\alpha} s^{\alpha-1} ds,$$

for some $f \in X_1$. Multiplying by $(\lambda + x^\alpha)^{1+2/\alpha}$, we obtain

$$1 = \frac{f(x)(\lambda + x^\alpha)^{1+2/\alpha}}{\lambda + x^\alpha} + 2\int_x^\infty f(s)(\lambda + s^\alpha)^{-1+2/\alpha} s^{\alpha-1} ds.$$

Since $(\lambda + x^\alpha)^{-1+2/\alpha} x^{\alpha-2} = O(1)$ as $x \to \infty$ and $f \in X_1$, we see that

$$\frac{f(x)(\lambda + x^\alpha)^{1+2/\alpha}}{\lambda + x^\alpha} = O(1), \quad x \to \infty.$$

On the other hand $(\lambda + x^\alpha)^{1+2/\alpha}/(\lambda + x^\alpha) = O(x^2)$ as $x \to \infty$ thus

$$\lim_{x\to\infty} x^2 f(x) = 1.$$

However, no f satisfying this condition can belong to X_1.

Hence $u_{\lambda,\alpha}$ is not a strict solution to (1.4). It is also clear that $\int\limits_0^t u_{\lambda,\alpha}(s,x)ds \notin D(T_\alpha)$ and therefore $u_{\lambda,\alpha}$ is not a mild solution to (1.4). Moreover, by e.g. Theorem 2.4.6 of [7] any weak solution is also given by $u(t) = G_\alpha(t)\, \overset{\circ}{u}$, which by [6], p. 50, would yield $\int\limits_0^t u(s)ds \in D(T_\alpha)$. Hence $u_{\lambda,\alpha}$ is not a weak solution to (1.4) either. ■

Since $X_{\lambda,\alpha} \in X_1 \setminus D(T_\alpha)$, we see by Eq. (2.2) that T_α, which is the generator of the semigroup "solving" (1.2) is not maximal, that is, there exist functions in X_1 for which the right-hand side of (1.2) is also in X_1, but which do not belong to $D(T_\alpha)$. To make this statement precise, we introduce the operator

$$[T_{\alpha,max}u](x) = -x^\alpha u(x) + 2\int\limits_x^\infty y^{\alpha-1}u(y)dy$$

defined on the domain

$$D(T_{\alpha,max}) = \{u \in X_1, x \to -x^\alpha u(x) + 2\int\limits_x^\infty y^{\alpha-1}u(y)dy \in X_1\}. \tag{2.3}$$

According to the discussion above and Proposition A.1 we have (at least for $\alpha > 0$)

$$K_\alpha \underset{\neq}{\subseteq} T_\alpha \underset{\neq}{\subseteq} T_{\alpha,max},$$

Remark 2.1 Let us note that the definition above is not precise as we have not specified the way of evaluating the integral in (2.3). This can be the Lebesgue integral (which would be in agreement with the general setting) but also it can be an improper integral. It is easy to see that for $0 < \alpha \leq 2$ both are the same, due to the definition of X_1. However, a simple example that these two are different in general is furnished, e.g. for $\alpha = 3$, by $f(x) = \frac{\sin x}{x^3}$. Fortunately, as we shall see later, this distinction is not relevant in our case.

It should be also noted that the notation $T_{\alpha,max}$ is a little bit misleading as it is not really a maximal operator – using Zorn's lemma one can always find an extension defined on the whole space. What we have in mind here is finding the maximal operator in X_1 that is still a restriction of a "point-wise operator" defined by the expression

$$u(x) \to -x^{\alpha}u(x) + 2\int\limits_{x}^{\infty} y^{\alpha-1}u(y)dy.$$

Before we discuss the fragmentation model in detail, let us consider an abstract situation.

3 Extensions of generators

Let us consider a Banach space X and an operator $(A, D(A))$ that is a generator of a C_0 semigroup $(G(t))_{t\geq 0}$ in X. To simplify the notation we shall assume that $(G(t))_{t\geq 0}$ is contractive, which yields $\{\lambda;\ Re\lambda > 0\} \subset \rho(A)$.

Let us assume that there exists an extension A_e of A defined on the domain $D(A_e)$. We have the following basic result.

Lemma 3.1 *For any λ with $Re\lambda > 0$,*

$$D(A_e) = D(A) \oplus N(\lambda I - A_e). \tag{3.1}$$

If we equip $D(A_e)$ with the graph norm, then $D(A)$ is a closed subspace of $D(A_e)$ and the projection of $D(A_e)$ onto $D(A)$ along $N(\lambda I - A_e)$ is given by

$$x = Px' = R(\lambda, A)(\lambda I - A_e)x', \quad x' \in D(A_e). \tag{3.2}$$

Proof. Let us fix λ with $Re\,\lambda > 0$. Since $A \subset A_e$, then also

$$\lambda I - A \subset \lambda I - A_e, \tag{3.3}$$

therefore $\lambda I - A_e$ is a surjection for $Re\lambda > 0$. Since A is the generator of a contraction semigroup, for any $x' \in D(A_e)$ there exists a unique $x \in D(A)$ such that

$$(\lambda I - A)x = (\lambda I - A_e)x'.$$

Denote $P = R(\lambda, A)(\lambda I - A_e)$. By (3.3) it is a linear surjection onto $D(A)$, bounded as an operator from $D(A_e)$ into $D(A_e)$. Moreover, again by (3.3)

$$P^2 = R(\lambda, A)(\lambda I - A_e)R(\lambda, A)(\lambda I - A_e) = R(\lambda, A)(\lambda I - A)R(\lambda, A)(\lambda I - A_e) = R(\lambda, A)(\lambda I - A_e) = P,$$

thus it is a projection. Clearly, for $e_\lambda \in N(\lambda I - A_e)$ we have $Pe_\lambda = 0$, hence this is a projection parallel to $N(\lambda I - A_e)$. By [9], p. 155, $D(A)$ is a closed subspace of $D(A_e)$ and the decomposition (3.1) holds. ∎

The importance of the following corollary stems from Theorem 3.1 below.

Corollary 3.1 *If $D(A_e) \setminus D(A) \neq \emptyset$, then $\sigma_p(A_e) \supseteq \{\lambda \in \mathbb{C};\ Re\lambda > 0\}$. Moreover, there exists a holomorphic (in the norm of X) function $\{\lambda \in \mathbb{C};\ Re\lambda > 0\} \ni \lambda \to e_\lambda$ such that for any λ with $Re\,\lambda > 0$, $e_\lambda \in N(\lambda I - A_e)$, and which is bounded in any closed half-plane $\{\lambda \in \mathbb{C};\ Re\lambda \geq \gamma > 0\}$.*

Proof. Let $u \in D(A_e) \setminus D(A)$ and $A_e u = f$. For any λ with $Re\,\lambda > 0$, denote $g_\lambda = \lambda u - A_e u$ and $v = R(\lambda, A)g_\lambda$; by (3.3) $e'_\lambda = u - v \in N(\lambda I - A_e)$.

A quick calculation gives

$$\begin{aligned} e'_\lambda &= u - v = u - R(\lambda, A)(-f + \lambda u) = u - \lambda R(\lambda, A)u + R(\lambda, A)f \\ &= -AR(\lambda, A)u + R(\lambda, A)f \end{aligned}$$

Taking the representation $e'_\lambda = u - \lambda R(\lambda, A)u + R(\lambda, A)f$ we see that as $\lambda \to R(\lambda, A)$ is holomorphic for $Re\,\lambda > 0$, $\lambda \to e_\lambda$ is also holomorphic there. From the Hille-Yosida theorem we have the estimate $\|R(\lambda, A)\| \leq 1/Re\,\lambda$ for $Re\,\lambda > 0$. For any scalar function $C(\lambda)$, the element $e_\lambda = C(\lambda)e'_\lambda \in N(\lambda I - A_e)$ for each $Re\lambda > 0$. Thus taking e.g. $C(\lambda) = \lambda^{-1}$, we obtain e_λ that satisfies the required conditions. ∎

Proposition 3.1 *If for some $\lambda > 0$ the null-space $N(\lambda I - A_e)$ is closed in X, then A_e is closed.*

Proof. We know that A_e is closed if and only if $\lambda I - A_e$ is closed so that we shall prove the closedness of $\lambda I - A_e$. Let $x'_n \to x'$ and $(\lambda I - A_e)x'_n \to y$ in X. Operating on x'_n with the projector (3.2) we obtain that $x_n = R(\lambda, A)(\lambda I - A_e)x'_n$ converges to some $x \in D(A)$ (both in X and in $D(A)$). Thus $e_{\lambda,n} \in N(\lambda I - A_e)$ also converges in X, and by assumption,

$$e_\lambda = \lim_{n\to\infty} e_{\lambda,n} \in N(\lambda I - A_e).$$

Thus

$$x' = x + e_\lambda$$

and since both $D(A)$ and $N(\lambda I - A_e)$ are subspaces of $D(A_e)$, we have $x' \in D(A_e)$. Moreover, since $(\lambda I - A_e)x'_n \to y$ in X, we have $x_n = R(\lambda, A)(\lambda I - A_e)x'_n \to R(\lambda, A)y$, thus $R(\lambda, A)y = x$

and $(\lambda I - A_e)x = (\lambda I - A_e)R(\lambda, A)y = (\lambda I - A)R(\lambda, A)y = y$. This finally yields

$$(\lambda I - A_e)x' = (\lambda I - A_e)x + (\lambda I - A_e)e_\lambda = y$$

and A_e is closed. ■

If A_e is closed, for the solvability of the Cauchy problem

$$\begin{aligned} \partial_t u &= A_e u, \quad t > 0, \\ \lim_{t \to 0^+} u(t) &= \overset{\circ}{u}, \end{aligned} \tag{3.4}$$

we can use the theory presented in [8], pp. 619-621. By a solution of this problem we understand any function which is continuous on $[0, \infty[$, continuously differentiable on $]0, \infty[$, $u(t) \in D(A_e)$ for each $t > 0$, and such that (3.4) is satisfied. Also, following [8] we say that a solution is of normal type ω if

$$\lim \sup_{t \to +\infty} t^{-1} \log \|u(t)\| = \omega < +\infty. \tag{3.5}$$

We also call *a nul solution* of (3.4) any non-zero solution of (3.4) with $\overset{\circ}{u} = 0$.

We have the following result concerning the existence of nul-solutions of normal type ω.

Theorem 3.1 *If A_e is a closed operator, a necessary and sufficient condition that (3.4) have a nul solution of type $\leq \omega$ is that for each λ with $Re\,\lambda > \omega$ the characteristic equation*

$$A_e x_\lambda = \lambda x_\lambda \tag{3.6}$$

has a solution $x_\lambda \neq 0$, which is a bounded and holomorphic function of λ in each half-plane $Re\lambda \geq \omega + \epsilon$, $\epsilon > 0$.

Due to their importance in the main part of the paper, we briefly sketch the main constructions of the proof.

a) If $u(t)$ is a nul solution of type ω, then it follows that the Laplace transform

$$x_\lambda = \int_0^\infty e^{-\lambda t} u(t) dt$$

is a bounded holomorphic function in each half-plane $Re\lambda \geq \omega + \epsilon$, $\epsilon > 0$, which satisfies (3.6).

b) Conversely, let x_λ be a bounded holomorphic function in each half-plane $Re\lambda \geq \omega + \epsilon$, $\epsilon > 0$. It is possible that the inverse Laplace transform of x_λ is not a nul solution (inverse Laplace transforms of bounded functions not necessarily are continuous at 0). However, x_λ multiplied by any scalar function of λ is still a solution to (3.6). Then, e.g. defining $X_\lambda = (\lambda - \omega + 1)^{-3} x_\lambda$ and

$$u(t) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} e^{\lambda t} X_\lambda d\lambda,$$

one can check that u is a nul solution to (3.4) of type not exceeding ω.

To cover as large class of solutions as possible we introduce a concept of mild or integral solution to (3.4), in the same way as for semigroups. That is, we say that u is a mild solution of (3.4) if $u \in C([0, \infty[, X)$, $\int_0^t u(s)ds \in D(A_e)$ for any $t > 0$, and

$$u(t) = \overset{\circ}{u} + A_e \int_0^t u(s)ds, \quad t > 0. \tag{3.7}$$

It is clear then in such a case $U(t) = \int_0^t u(s)ds$ is a classical solution of the nonhomogeneous problem

$$\begin{aligned} \partial_t U &= A_e U + \overset{\circ}{u}, \quad t > 0, \\ \lim_{t \to 0+} U(t) &= 0. \end{aligned} \tag{3.8}$$

In particular, if u is a mild nul solution to (3.4) of normal type ω, then U is a nul solution to (3.4) of the same type. We can prove the following minor modification of Theorem 3.1.

Corollary 3.2 *Let A_e be a closed operator. If (3.4) has a mild nul solution of type $\leq \omega$, then the characteristic equation*

$$A_e x_\lambda = \lambda x_\lambda \tag{3.9}$$

has a solution $x_\lambda \neq 0$, which is a bounded and holomorphic function of λ in each half-plane $Re\lambda \geq \omega + \epsilon$, $\epsilon > 0$. As before, x_λ in (3.9) can be taken as

$$x_\lambda = \int_0^\infty e^{-\lambda t} u(t)dt. \tag{3.10}$$

Proof. If u is a mild nul solution of the type ω, then $U(t) = \int_0^t u(s)ds$ is a nul solution of the same type, thus by Theorem 3.1 the first part of the proposition is proved with x_λ of Eq. (3.9) given by

$$x_\lambda = \int_0^\infty e^{-\lambda t} U(t)dt.$$

Easy calculation shows that $\|x_\lambda\| = O(\lambda^{-1})$. Moreover,

$$X_\lambda = \int_0^\infty e^{-\lambda} u(t)dt = \lambda x_\lambda$$

hence X_λ is a bounded holomorphic function for $Re\lambda \geq \omega + \epsilon$, $\epsilon > 0$. Since multiplication by λ does not influence (3.9), Eq. (3.2) is proved. ■

4 Fragmentation model revisited

We shall apply the considerations of the previous section to the operator $T_{\alpha,max}$ where in the definition of $T_{\alpha,max}$ we put

$$F(x) = \int_x^\infty y^{\alpha-1} f(y)dy = \lim_{r\to\infty} \int_x^r y^{\alpha-1} f(y)dy,$$

and the integral over $[x, r]$ is understood as the Lebesgue integral. This definition is correct as $f \in L_1(\mathbb{R}, xdx)$, thus $y \to y^{\alpha-1}f$ has finite Lebesgue integral over any subinterval $[x, r]$, $0 < x < r < +\infty$, for any $\alpha > 0$. It is clear that then for any function f for which the integral exists, F is a continuous function on $]0, \infty[$ with $\lim\limits_{x\to\infty} F(x) = 0$, possibly infinite at $x = 0$ (only if $\alpha < 2$).

Lemma 4.1 *The operator $T_{\alpha,max}$ is closed.*

Proof. Let us suppose that $u \in X_1$ is a solution to

$$2\int_x^\infty y^{\alpha-1} u(y)dy = (x^\alpha + \lambda)u(x). \tag{4.1}$$

By the discussion preceding the lemma, the function F is continuous for any $x > 0$ whenever u is such that the integral exists. This shows that u is a continuous function on $]0, \infty[$ and, repeating the argument, differentiable there. Hence, any solution of (4.1) is also a solution to the differential equation

$$(\lambda + x^\alpha)u' = -(\alpha + 2)x^{\alpha-1}u,$$

which has the general solution

$$u_{\lambda,\alpha}(x) = \frac{C(\lambda)}{(\lambda + x^\alpha)^{1+2/\alpha}}. \tag{4.2}$$

Hence, for any fixed λ with $Re\lambda > 0$, the null-space $N(\lambda I - T_{\alpha,max})$ is one dimensional and thus closed in X_1. By Proposition 3.1 this yields the closedness of $T_{\alpha,max}$. ■

Proposition 4.1 *If $\alpha < 0$, then $T_{\alpha,max} = T_\alpha$*

Proof. The argument leading to (4.2) is valid also for $\alpha < 0$, thus any element of $N(\lambda I - T_{\alpha,max})$ must be of the form (4.2). However, for $\alpha < 0$ we have

$$u_{\lambda,\alpha}(x) = O(1), \quad \text{as } x \to \infty$$

thus $u_{\lambda,\alpha} \notin X_1$. ■

As an example of application we shall present another proof of the fact that $T_\alpha \neq \overline{K_\alpha}$ for $\alpha < 0$ (see [4], Corollary 5.1).

Proposition 4.2 *Let $\alpha < 0$. Then*

$$T_\alpha \neq \overline{K_\alpha}.$$

Proof. Let us consider the function

$$f(x) = \begin{cases} x^{-2-\alpha} & \text{for} \quad 0 < x < 1, \\ 0 & \text{for} \quad x > 1. \end{cases}$$

Since $\alpha < 0$, we have clearly $\int\limits_0^\infty xf(x)dx < +\infty$, thus $f \in X_1$. Next we have

$$\int\limits_x^\infty y^{\alpha-1}f(y)dy = \begin{cases} \frac{1}{2}(x^{-2}-1) & \text{for} \quad 0 < x < 1, \\ 0 & \text{for} \quad x > 1, \end{cases}$$

therefore the function

$$-x^\alpha f(x) + 2\int\limits_x^\infty y^{\alpha-1}f(y)dy = \begin{cases} -1 & \text{for} \quad 0 < x < 1, \\ 0 & \text{for} \quad x > 1, \end{cases}$$

is in X_1, thus $f \in D(T_{\alpha,max}) = D(T_\alpha)$ by Proposition 4.1. For this function we have

$$\begin{aligned} &< T_\alpha f, 1 > \\ &= \int\limits_0^\infty (-x^\alpha f(x) + 2\int\limits_x^\infty y^{\alpha-1}f(y)dy)xdx = \lim_{\epsilon\to 0}\int\limits_\epsilon^\infty (-x^\alpha f(x) + 2\int\limits_x^\infty y^{\alpha-1}f(y)dy)xdx \\ &= \lim_{\epsilon\to 0}\left(\int\limits_\epsilon^\infty (-x^\alpha f(x)xdx + 2\int\limits_\epsilon^\infty \left(\int\limits_\epsilon^y xdx\right) y^{\alpha-1}f(y)dy\right) = -\lim_{\epsilon\to 0}\epsilon^2\int\limits_\epsilon^1 y^{-3}dy \\ &= -\frac{1}{2} < 0. \end{aligned}$$

On the other hand, if $T_\alpha = \overline{K_\alpha}$, then the semigroup would be conservative on non-negative data, which would yield

$$< T_\alpha g, 1 >=< \overline{K_\alpha} g, 1 >= 0$$

for all $g \in D(\overline{K_\alpha}) = D(T_\alpha)$. This contradiction proves the proposition. ■

Before we formulate and prove the main result of this paper, we have to specify what we mean by the mild solution to (1.2). Firstly, let us note that since X_1 is the space of type L (see [8], pp. 69–70), for any $u \in C([0,\infty[, X_1)$ there exists a measurable on $[0,\infty[\times[0,\infty[$ function ψ such that $u(t) = \psi(t,\cdot)$ for any $t \geq 0$, and such that

$$\left[\int\limits_0^t u(s)ds\right](x) = \int\limits_0^t \psi(s,x)ds$$

for almost all $x \in [0,\infty[$, where the first integral is the Riemann integral in X_1 and the second integral is the Lebesgue integral of scalar functions. In what follows we shall use the same symbol for both functions u and ψ.

According to the definition of mild solution adopted in Section 3, u is a mild solution to

$$\begin{aligned} u_t &= T_{\alpha,max}u, \quad t > 0, \\ u(0) &= \overset{\circ}{u} \end{aligned} \tag{4.3}$$

if $u \in C([0,\infty[, X_1)$, $\int\limits_0^t u(s)ds \in D(T_{\alpha,max})$ and

$$u(t) = \overset{\circ}{u} + T_{\alpha,max}\left(\int\limits_0^t u(s)ds\right), \qquad t > 0 \tag{4.4}$$

which translates, by the definition of $T_{\alpha,max}$, into

$$u(t,x) = \overset{\circ}{u}(x) - x^\alpha \int\limits_0^t u(s,x)ds + 2\int\limits_x^\infty y^{\alpha-1}\int\limits_0^t u(s,y)dsdy \tag{4.5}$$

for almost all $x \geq 0$ and all $t \geq 0$. For non-negative solutions the above formula may by rewritten by Tonelli's theorem as

$$u(t,x) = \overset{\circ}{u}(x) - x^\alpha \int\limits_0^t u(s,x)ds + 2\int\limits_0^t\int\limits_x^\infty y^{\alpha-1}u(s,y)dyds \tag{4.6}$$

Conversely, if $u \in C([0,\infty[, X)$ (resp. $u \in C([0,\infty[, X)$ and is non-negative) and satisfies (4.5) (resp. (4.6)) for almost all $(t,x) \in [0,\infty[\times[0,\infty[$, then u is a mild solution to (4.3). Indeed, we can convert then all the integrals with respect to s into Riemann integrals in X_1 and by the definition of $T_{\alpha,max}$ we obtain from (4.5) that $\int\limits_0^t u(s)ds \in D(T_{\alpha,max})$ and Eq. (4.4) holds for almost any t. By the closedness of $T_{\alpha,max}$ and continuity of the left-hand side we obtain that the equation holds for all $t \geq 0$, thus u is a mild solution of (4.3).

Definition 4.1 *We say that u is a mild solution to (1.2) if $u \in C([0,\infty[, X_1)$ and satisfies (4.5) almost everywhere in $[0,\infty[\times[0,\infty[$.*

Now we can formulate the main result of the paper.

Theorem 4.1 *All physically acceptable (positive for positive initial values and with non-increasing mass) mild solutions to (1.2) are given by $u(t) = G_\alpha(t)\overset{\circ}{u}$, $\overset{\circ}{u}\in X_1$.*

Proof. Let $u(t,\overset{\circ}{u})$ be a positive mild solution to (1.2) with non-increasing mass, that is, for all $t \geq 0$

$$\|u(t,\overset{\circ}{u})\| = \int\limits_0^\infty x[u(t,\overset{\circ}{u})](x)dx \leq \int\limits_0^\infty x\overset{\circ}{u}(x)dx = \|\overset{\circ}{u}\|. \tag{4.7}$$

Since $G_\alpha(t)\overset{\circ}{u}$ is also a solution to the same initial value problem, we must have

$$u(t,\overset{\circ}{u}) = G_\alpha(t)\overset{\circ}{u} + u_{nul}(t), \tag{4.8}$$

where $u_{nul}(t)$ is a mild nul solution. u_{nul} must be bounded, thus of normal type 0. Otherwise, there would be a sequence $(t_n)_{n\in\mathbb{N}}$ such that $\|u_{nul}(t_n)\| \to \infty$ as $n \to \infty$, which by

$$\|u(t_n,\overset{\circ}{u})\| \geq |\|u_{nul}(t_n)\| - \|G_\alpha(t_n)\overset{\circ}{u}\|| = |\|u_{nul}(t_n)\| - \|\overset{\circ}{u}\|| \to \infty,$$

would contradict (4.7).

Applying the Laplace transform to (4.8), and using Corollary 3.2 and Lemma 4.1 we obtain that for $\lambda > 0$

$$\int\limits_0^\infty e^{-\lambda t}u(t,\overset{\circ}{u})dt = \int\limits_0^\infty e^{-\lambda t}G_\alpha(t)\overset{\circ}{u}\,dt + \frac{C(\lambda)}{(\lambda + x^\alpha)^{1+2/\alpha}}. \tag{4.9}$$

By (4.7)

$$\int\limits_0^\infty \left(\int\limits_0^\infty e^{-\lambda t}u(t,\overset{\circ}{u})dt\right)xdx = \int\limits_0^\infty e^{-\lambda t}\left(\int\limits_0^\infty u(t,\overset{\circ}{u})xdx\right)dt \leq \lambda^{-1}\|\overset{\circ}{u}\|,$$

and similarly, since the semigroup is conservative on non-negative data, we obtain

$$\int\limits_0^\infty \left(\int\limits_0^\infty e^{-\lambda t}G_\alpha(t)\overset{\circ}{u}\,dt\right)xdx = \lambda^{-1}\|\overset{\circ}{u}\|.$$

Thus the integration of (4.9) with respect to the measure xdx yields

$$C(\lambda) \int_0^\infty \frac{xdx}{(\lambda + x^\alpha)^{1+2/\alpha}} \le 0.$$

Since the integral doesn't vanish for any $\lambda > 0$, we get $C(\lambda) \le 0$ for $\lambda > 0$.

Let us fix λ for which $C(\lambda) = -a < 0$ and denote

$$\mathcal{L}(\lambda) = \int_0^\infty e^{-\lambda t} u(t, \mathring{u}) dt.$$

For any $\lambda > 0$ we have $\mathcal{L}(\lambda) \ge 0$ almost everywhere. Using (1.5) we can write (4.9) as

$$[\mathcal{L}(\lambda)](x) = \frac{\mathring{u}(x)}{\lambda + x^\alpha} + \frac{2}{(\lambda + x^\alpha)^{1+2/\alpha}} \int_x^\infty \mathring{u}(s)(\lambda + s^\alpha)^{-1+2/\alpha} s^{\alpha-1} ds + \frac{C(\lambda)}{(\lambda + x^\alpha)^{1+2/\alpha}}.$$

Multiplying both sides by $(\lambda + x^\alpha)^{1+2/\alpha}$ we obtain

$$(\lambda + x^\alpha)^{1+2/\alpha} \mathcal{L}(\lambda) = \frac{\mathring{u}(x)(\lambda + x^\alpha)^{1+2/\alpha}}{\lambda + x^\alpha} + 2 \int_x^\infty \mathring{u}(s)(\lambda + s^\alpha)^{-1+2/\alpha} s^{\alpha-1} ds + C(\lambda).$$

Since $\mathring{u} \in X_1$, we know that the integral tends to zero as $x \to \infty$. Let us fix x_0 for which

$$2 \int_{x_0}^\infty \mathring{u}(s)(\lambda + s^\alpha)^{-1+2/\alpha} s^{\alpha-1} ds < a/2.$$

This, together with $\mathcal{L}u \ge 0$, yields for almost all $x > x_0$

$$\frac{\mathring{u}(x)(\lambda + x^\alpha)^{1+2/\alpha}}{\lambda + x^\alpha} \ge a/2.$$

However, as in the proof of Proposition 2.1, we have $(\lambda + x^\alpha)^{1+2/\alpha}/(\lambda + x^\alpha) = O(x^2)$ for large x which would yield $\lim\limits_{x\to\infty} x^2 \mathring{u}(x) \ge a/2$, contradicting the assumption $\mathring{u} \in X_1$.

Hence, $C(\lambda) = 0$ for $\lambda > 0$ and therefore $C(\lambda)(\lambda + x^\alpha)^{-1-2/\alpha} = 0$ there. Since the Laplace transform of the nul solution is a holomorphic function for $Re\,\lambda > 0$ by Corollary 3.2, $C(\lambda)(\lambda + x^\alpha)^{-1-2/\alpha} = 0$ in the right half-plane. This yields $u_{nul}(t) = 0$. ■

A Appendix: Relation to Arlotti extensions

As we shall see below, the maximal operators discussed in the paper are closely related to the extensions introduced in [2] and further developed in [3, 4]. This allows to give e.g. an alternative proof of the result of [11] that the generator in the fragmentation equation is a restriction of the maximal operator. This result has been already used in Section 2.

Let (Ω, μ) be a σ-finite measure space. By X we denote the Banach space $L_1(\Omega, \mu)$ endowed with the standard norm $\|\cdot\|$. For any subspace $Z \subset X$ by Z_+ we denote the cone of nonnegative (a.e.) elements of Z. Let $(G(t))_{t\geq 0}$ be a strongly continuous semigroup on X. We say that $(G(t))_{t\geq 0}$ is a *substochastic semigroup* if for each $t \geq 0$, $G(t) \geq 0$ and $\|G(t)\| \leq 1$. It is called a *stochastic* semigroup if additionally $\|G(t)f\| = \|f\|$ for $f \in X_+$.

We shall consider two linear operators in X: A and B that have the following properties:

1. $(A, D(A))$ generates a substochastic semigroup denoted by $(G_A(t))_{t\geq 0}$,
2. $D(B) \supseteqq D(A)$ and $Bf \geq 0$ for any $f \in D(B)_+$,
3. for any $f \in D(A)_+$

$$\int_\Omega (Af + Bf)d\mu = 0. \tag{A.1}$$

It can be checked [4] that the following two properties hold

(i) for any $\lambda > 0$ the operator $B(\lambda I - A)^{-1}$ is a bounded positive operator in X,

(ii) if $Af = -mf$ for some a.e. positive measurable function m, then for any $f \in D(A)$

$$\|Bf\| \leq \|mf\| \tag{A.2}$$

Let E denote the set of all the extended real-valued measurable functions defined on Ω; clearly $X \subset \mathsf{E}$. We define the subset $\mathsf{F} \subset \mathsf{E}$ by the following condition: $f \in \mathsf{F}$ if and only if for every nonnegative and non-decreasing sequence of functions $(f_n)_{n\in\mathbb{N}}$ satisfying $\sup_n f_n = |f|$ we have $\sup_n (I - A)^{-1} f_n \in X$.

Before proceeding any further we adopt the assumption that $f \in D(A)$ if and only if $f_+ = \max\{f, 0\}$, $f_- = \max\{-f, 0\}$ both belong to $D(A)$.

Through B we construct another subset of E, say G, defined as the set of all functions $f \in X$ such that for any nonnegative, non-decreasing sequence $(f_n)_{n\in\mathbb{N}}$ of elements of $D(B)$ such that $\sup_n f_n = |f|$, we have $\sup_n Bf_n < +\infty$ almost everywhere. It is easy to check that $D(A) \subseteq \mathsf{G} \subseteq X \subseteq \mathsf{F} \subseteq \mathsf{E}$. To define the extension of B to G in general case, the following assumption is needed

(B) If $f \in \mathsf{G}_+$ and $(f'_n)_{n\in\mathbb{N}}$, $(f''_n)_{n\in\mathbb{N}}$ are two non-decreasing sequences of elements of $D(B)_+$ satisfying $f = \sup_n f'_n = \sup_n f''_n$, then $\sup_n Bf'_n = \sup_n Bf''_n$.

Then for $f \in \mathsf{G}_+$ we can define

$$\mathsf{B}f = \sup_n Bf_n$$

and for arbitrary $f \in \mathsf{G}$ with $f = f_+ - f_-$ we put

$$\mathsf{B}f = \mathsf{B}f_+ - \mathsf{B}f_-.$$

Though the range of applications of the above construction is much wider (see [2, 3]) we shall focus here on the case of immediate interest where Ω is a subset of $\mathbb{R}^N$, $Au = -mu$ where m is a measurable non-negative function which is finite everywhere except possibly countably many isolated points of Ω, and B is an integral operator, that is

$$[Bf](x) = \int_\Omega k(x,y)f(y)d\mu_y, \tag{A.3}$$

for some measurable $0 \leq k(x,y) : \Omega \times \Omega \to \mathsf{E}$ with the domain

$$D(B) := \{f \in X;\ Bf \in X\}.$$

By Eq. (A.1) we obtain immediately that

$$m(y) = \int_\Omega k(x,y)d\mu_x.$$

Using the monotone convergence theorem we see that the assumption (B) is satisfied, thus the extension B is well-defined.

We also consider another extension of B, denoted by B_{max}, defined by the formula (A.3) on the domain

$$D(B_{max}) = \{f \in X;\ [Bf](x) \text{ is finite almost everywhere}\}$$

We have the following result:

Lemma A.1

$$\mathsf{B} = B_{max} \tag{A.4}$$

Proof. Let first $f \in \mathsf{G}$, then $f = f_+ - f_-$ and

$$\mathsf{B}f = \mathsf{B}f_+ - \mathsf{B}f_- = \sup_n Bf_{n,+} - \sup_n Bf_{n,-},$$

where $\{f_{n,+}\}_{n\in\mathbb{N}}$ and $\{f_{n,-}\}_{n\in\mathbb{N}}$ are non-decreasing sequences in $D(B)_+$ with $\sup_n f_{n,+} = f_+$ and $\sup_n f_{n,-} = f_-$. Let us concentrate on one of these sequences. For (almost) any $x \in \Omega$ we have $\sup_n k(x,y)f_{n,+} = k(x,y)f_+$ and by the monotone convergence theorem the function $y \to k(x,y)f_+(y)$ we have

$$\sup_n [Bf_{n,+}(x)] = \sup_n \int_\Omega k(x,y)f_{n,+}(y)d\mu_y = \int_\Omega k(x,y)f_+(y)d\mu_y$$

By the definition of G the last integral is finite for almost every x, thus $y \to k(x,y)f_+$ is Lebesgue integrable for almost every x. Since the same holds for $\{f_{n,-}\}_{n\in\mathbb{N}}$, we obtain from the definition of the Lebesgue integral that $y \to k(x,y)f$ is Lebesgue integrable with $\int_\Omega k(x,y)f(y)dy$ finite almost everywhere, hence $f \in D(B_{max})$.

Conversely, if $f \in D(B_{max})$, then again from the definition of the Lebesgue integral both integrals

$$\int_\Omega [k(x,y)f(y)]_+ dy = \int_\Omega k(x,y)f_+(y)d\mu_y$$

and

$$\int_\Omega [k(x,y)f(y)]_- dy = \int_\Omega k(x,y)f_-(y)d\mu_y$$

are finite for almost every $x \in \Omega$.

Let $S = \{x_1, x_2, \ldots, x_n, \ldots\}$ be a set of singular points of m. Because of the assumption, if S is infinite, then $|x_n| \to \infty$, where $|\cdot|$ is the norm in $\mathbb{R}^N$. Let $V_{n,l}$ be the ball centered at $x_n \in S$ with radius $1/l$ and $V_\infty = \{x \in \mathbb{R}^N, |x| > l\}$. For a given f_+ we construct a sequence $(f_n)_{n\in\mathbb{N}}$ by

$$f_l(x) = \begin{cases} f_+(x) \quad \text{for} \quad x \notin \left(\bigcup_{n=1}^{\infty} V_{n,l} \cup V_\infty\right), \\ 0 \qquad\qquad\qquad \text{otherwise.} \end{cases}$$

Since $f_+ \in X$ we have $\int_\Omega f_l(x)m(x)d\mu_x < \infty$, thus by Eq. (A.2), $f_l \in D(B)$. Moreover, as x_n's are isolated, $f_l \nearrow f_+$ almost everywhere, thus by the dominated convergence theorem

$$\sup_{l\in\mathbb{N}} \int_\Omega k(x,y)f_l(y)d\mu_y = \int_\Omega k(x,y)f_+(y)d\mu_y.$$

As the same argument holds for f_-, we see that $f \in \mathsf{G} = D(\mathsf{B})$ and the lemma is proved. ■

It can be proved that we can correctly define the mapping $\mathsf{L} : \mathsf{F}_+ \to X_+$ by

$$\mathsf{L}f = \sup_n (I - A)^{-1} f_n,$$

where $0 \le f_n \le f_{n+1}$ for any n, and $\sup_n f_n = f$ and extend it by linearity onto the whole of F. In our case it is clear that

$$\mathsf{F} = \{f \in \mathsf{E};\ (1+m)^{-1} f \in X\}, \tag{A.5}$$

and $\mathsf{L}f = (1+m)^{-1}f$. It follows [2, 4] that there exists an extension T of $A+B$ which generates a positive semigroup in X. Let us also consider the "maximal" operator

$$[T_{max}u](x) = -m(x)u(x) + [B_{max}u](x) = -m(x)u(x) + [\mathsf{B}u](x)$$

defined on the domain

$$D(T_{max}) = \{u \in X;\ x \to [T_{max}u](x) \in X\}.$$

In [11] it was proved that under certain additional assumptions $T \subset T_{max}$. Here we shall show, following a part of the proof of Theorem 1 in [2], that this holds in general situation discussed in this Appendix.

Proposition A.1

$$T \subset T_{max}.$$

Proof. By [2, 4] we have that for any $g \in X$

$$(I-T)^{-1}g = \sum_{k=0}^{\infty}(I-A)^{-1}(B(I-A)^{-1})^k g.$$

For $h \in D(T)$ denote $g = (I-T)h \in X$. Using the above equation and replacing the operators involved there by their extensions we see that h can be expressed by

$$h = \sum_{k=0}^{\infty}\mathsf{L}(\mathsf{BL})^k g \tag{A.6}$$

where the limit is taken in X. Let us denote, for any $g \in X$

$$f_n = \sum_{k=0}^{n}(\mathsf{BL})^k g$$

and

$$h_n = \mathsf{L}f_n = (I-A)^{-1}\sum_{k=0}^{n}(B(I-A)^{-1})^k g$$

Let $g \geq 0$. Then all the sequences above are monotone and we can define $f = \sup_n f_n$; then $h = \sup_n (I-A)^{-1}f_n \in X$, thus $h = Lf$ and $f = (1+m)h$. Since $h \in X$ and m is finite almost everywhere, f is finite almost everywhere. Thus

$$h(x) = (1+m(x))^{-1}f(x)$$

for a.a. x and $m(x)h(x) = -h(x) + f(x)$.

On the other hand, $h_n \in D(A) \subset D(B)$, thus

$$Bh_n = B(I-A)^{-1}\sum_{k=0}^{n}(B(I-A)^{-1})^k g = f_{n+1} - g$$

and therefore $\sup_n Bh_n = f - g$ is finite almost everywhere, thus $h \in \mathsf{G}$ and $\mathsf{B}h = B_{max}h = f - g$. Now,

$$Th = h - g$$

and at the same time, for almost all x,

$$-m(x)h(x) + [B_{max}h](x) = h(x) - g(x)$$

which proves the proposition for $g \geq 0$. The extension for arbitrary g is done by linearity. ■

References

[1] M. Aizenman, T. A. Bak, Convergence to Equilibrium in a System of Reacting Polymers, *Comm. Math. Phys.*, **65**, (1979), 203–230.

[2] L. Arlotti, A Perturbation Theorem for Positive Contraction Semigroups on L^1-Spaces with Applications to Transport Equations and Kolmogorov's Differential Equations, *Acta Applicandae Mathematicae*, **23** (1991), 129–144.

[3] J. Banasiak, On a diffusion-kinetic equation arising in extended kinetic theory, *Math. Meth. in the Appl. Sci.*, **23**, (2000), 1237–1256.

[4] J. Banasiak, On an extension of Kato–Voigt perturbation theorem for substochastic semigroups and its applications, *Taiwanese Journal of Mathematics*, **5** (1), 2001, in print.

[5] P.B. Dubovskii, I. W. Stewart, Existence, Uniqueness and Mass Conservation for the Coagulation–Fragmentation Equation, *Math. Meth. in the Appl. Sci.*, **19**, (1996), 571–591.

[6] K.-J. Engel, R. Nagel, *One-parameter Semigroups for Linear Evolution Equations*, Graduate Texts in Mathematics, Springer Verlag, 1999.

[7] H. O. Fattorini, *The Cauchy Problem*, Encyclopedia of Mathematics and its Applications, vol. 18, Addison–Wesley Publ. Comp. Reading, Massachusetts, 1983.

[8] E. Hille, R. S. Phillips, *Functional Analysis and Semi-groups*, Colloquium Publications, v. 31, American Mathematical Society, Providence, 1957.

[9] T. Kato *Perturbation theory for linear operators*, Springer Verlag, Berlin, 1966.

[10] E. D. McGrady, R. M. Ziff, "Shattering" Transition in Fragmentation, *Phys. Rev. Lett.*, **58**(9), (1987), 892–895.

[11] D.J. McLaughlin, W. Lamb, A. C. McBride, A semigroup approach to fragmentation models, *SIAM J. Math. Anal.*, **28** (5), (1997), 1158–1172.

[12] R. M. Ziff, E. D. McGrady, The kinetics of cluster fragmentation equation and polymerization, *J. Phys. A: Math. Gen.* **18**, (1985), 3027–3037.

[13] R. M. Ziff, E. D. McGrady, Kinetics of Polymer Degradation, *Macromolecules*, **19** (1986), 2513–2519.