BOUNDED LAPLACE TRANSFORMS, PRIMITIVES AND SEMIGROUP ORBITS

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ABSTRACT. Let $f : \mathbb{R}_+ \to \mathbb{C}$ be an exponentially bounded, measurable function whose Laplace transform has a bounded holomorphic extension to the open right half-plane. It is known that there is a constant C such that $\left|\int_0^t f(s)\,ds\right| \le C(1+t)$ for all $t \ge 0$. We show that this estimate is sharp. Furthermore, the corresponding estimates for orbits of C_0-semigroups are also sharp.

1. Introduction

Let $f : \mathbb{R}_+ \to \mathbb{C}$ be an exponentially bounded, measurable function whose Laplace transform $\hat{f}$ has a bounded holomorphic extension to the open right half-plane $\mathbb{C}_+$. It is known [2], [1, Theorem 4.4.12] that the primitive of f grows at most linearly, i.e., there is a constant C such that

$$\left|\int_0^t f(s)\,ds\right| \le C(1+t) \tag{1.1}$$

for all $t \ge 0$. A consequence of (1.1) is that the abscissa of convergence, $\mathrm{abs}(f)$, of $\hat{f}$ does not exceed the abscissa of boundedness, $\mathrm{hol}_0(\hat{f})$, whenever $f : \mathbb{R}_+ \to \mathbb{C}$ is exponentially bounded and measurable (see [2] or [1, Sections 1.4, 1.5] for the definitions).

The question has been raised [6], [2] whether (1.1) can be improved. If f is polynomially bounded and $\hat{f}$ is bounded on $\mathbb{C}_+$, then (1.1) can be improved to a logarithmic estimate. For positive real-valued f, a simple consequence of the Pringsheim-Landau Theorem is that the primitive of f is bounded if $\hat{f}$ is bounded on $\mathbb{C}_+$ [7]. On the other hand, Bloch [4] gave an example of a Laplace transformable function f such that $\mathrm{abs}(f) > \mathrm{hol}_0(\hat{f})$.

The corresponding result for sequences $(a_r)_{r\ge 0}$ states that if $h(z) = \sum_{r=0}^{\infty} a_r z^r$ is a bounded holomorphic function on the unit disc, then

$$\sum_{r=0}^{n} a_r = O(\log n) \quad \text{as } n \to \infty. \tag{1.2}$$

This is easily deduced from the fact that the L^1-norm of the Dirichlet kernel $D_n(t) = \sin((n+\frac{1}{2})t)/\sin(t/2))$ grows like $\log n$ [9, Theorem II.11.9]. Moreover, this estimate is

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sharp; this is shown in [9, Theorem VIII.1.2] for two-sided sequences $(a_r)_{r\in\mathbb{Z}}$ and it is not difficult to deduce the sharpness for one-sided sequences (the author is very grateful to Tom Ransford for informing him about this and pointing out its significance). By putting $f(t) = a_r$ $(r \le t < r+1; r = 0,1,2,\dots)$, one obtains examples of bounded functions f such that $\hat{f}$ is bounded on $\mathbb{C}_+$ and the primitive of f is unbounded but with growth which is at most logarithmic. In Section 2, we give a construction which shows that the estimate (1.1) is sharp for exponentially bounded functions, in the sense that $1+t$ cannot be replaced by any function which grows more slowly.

The estimates above extend to the case of functions with values in a complex Banach space X (to see this, one may apply linear functionals and use the Uniform Boundedness Theorem). A special case of particular interest occurs when $f(t) = T(t)x$ for some C_0-semigroup $\{T(t) : t \ge 0\}$ on X, with generator A, and some $x \in X$. Then $\hat{f}(\lambda) = R(\lambda, A)x := (\lambda I - A)^{-1}x$ whenever $\operatorname{Re}\lambda > \omega(T)$ (the growth bound of T). For $\mu > \omega(T)$, one has

$$T(t)R(\mu,A)x = R(\mu,A)x + (\mu R(\mu,A) - I)\int_0^t T(s)x\,ds. \tag{1.3}$$

It follows that if x has a bounded local resolvent in the sense that $R(\cdot,A)x$ has a bounded holomorphic extension to $\mathbb{C}_+$, then there exists C such that

$$\|T(t)R(\mu,A)x\| \le C(1+t) \tag{1.4}$$

for all $t \ge 0$. This was first proved by van Neerven [6]; a corollary is the result, originally due to Weis and Wrobel [8], that the growth bound of classical solutions, $\omega_1(T) := \omega(T(\cdot)R(\mu,A))$, does not exceed the pseudospectral bound $s_0(A)$ (see [1, Section 5.1] and [3]). Stronger results than (1.4) have been obtained when X has non-trivial Fourier type [5] and when X is an ordered space, T is a positive semigroup and $x \in X_+$ [7]. Examples of functions showing that (1.1) is sharp can easily be converted into semigroup examples showing that the estimate $\left\|\int_0^t T(s)x\,ds\right\| \le C(1+t)$ is also sharp when x has a bounded local resolvent. However, there are many examples where $T(t)R(\mu,A)x$ grows more slowly than $\int_0^t T(s)x\,ds$; moreover, the discrete analogue of (1.4), for powers of a single operator, is not sharp. Nevertheless, we shall show in Section 3 that (1.4) is sharp, by constructing functions with a further more delicate property.

Our construction in Section 2 uses basically the same building blocks as Bloch used in [4]. In order to obtain an exponentially bounded function f, we spread the blocks out by inserting long intervals on which f is zero. It turns out to be possible to choose the lengths of these intervals and the values of other parameters in such a way that f has all the properties which we seek.

2. The inequality for functions

In this section, we shall show that (1.1) is sharp by showing that $C(1+t)$ cannot be replaced by $C(1+th(t))$ if $\lim_{t\to\infty} h(t) = 0$. We first show the following result, in which vi) is not needed for this purpose but it will be required in Section 3 to show that (1.4) is sharp.

Proposition 2.1. *Let $(k_m)_{m\geq 1}$ and $(q_m)_{m\geq 1}$ be sequences of real numbers such that*

a) $k_m > 1$ *for all* $m \geq 1$ *and* $\lim_{m\to\infty} k_m = \infty$,
b) $q_1 \geq 1$ *and* $q_{m+1} \geq q_m + 1$ *for all* $m \geq 1$, *and*
c) for each $a > 0$, $\lim_{m\to\infty} e^{-m} k_m e^{-aq_m} = 0$.

Then there is a function $f : \mathbb{R}_+ \to \mathbb{R}$ *such that*

i) $|f(t)| \leq k_m e^{-m}$ *if* $q_m \leq t < q_m + 1; m = 1, 2, \ldots,$
ii) $f(t) = 0$ *for all other values of* t,
iii) f *is continuous except for a discrete set of jump discontinuities,*
iv) $\hat{f}(\lambda) := \lim_{t\to\infty} \int_0^t f(s)e^{-\lambda s}\, ds$ *exists whenever* $\operatorname{Re}\lambda > 0$ *and* $\sup_{\operatorname{Re}\lambda>0} \left|\hat{f}(\lambda)\right| < \infty$,
v) for all $m \geq 1$,

$$\left|\int_{q_m+\frac{1}{2}}^{q_m+1} f(t)\, dt\right| \geq e^{-m} \log k_m,$$

vi) for each $\mu > 0$,

$$\left|\int_0^\infty f(t + q_m + \tfrac{1}{2})e^{-\mu t}\, dt\right| \geq e^{-\mu} e^{-m} \log k_m$$

for all sufficiently large m.

Remark 2.2. The integral in vi) exists in the improper sense and equals

$$e^{\mu(q_m+\frac{1}{2})} \left(\hat{f}(\mu) - \int_0^{q_m+\frac{1}{2}} f(s)e^{-\mu s}\, ds \right).$$

Proof of Proposition 2.1. For $k > 2$ and $0 \leq t < 1$, let

$$\Psi_k(t) = \begin{cases} \dfrac{2}{1-2t} & \left(0 \leq t \leq \dfrac{k-2}{2k-2} \text{ or } \dfrac{k}{2k-2} \leq t < 1\right), \\ 0 & \text{otherwise.} \end{cases} \tag{2.1}$$

Then

$$|\Psi_k(t)| \leq k - 1, \tag{2.2}$$

$$\begin{aligned} \left|\int_0^1 \Psi_k(t)e^{-\lambda t}\, dt\right| &= \left|\int_1^{k-1} \frac{\exp\left(-\frac{\lambda(u-1)}{2k-2}\right)}{k-u}\, du + \int_{k+1}^{2k-1} \frac{\exp\left(-\frac{\lambda(u-1)}{2k-2}\right)}{k-u}\, du\right| \\ &\leq \Gamma e^{\operatorname{Re}\lambda/(2k-2)} \qquad (\operatorname{Re}\lambda > 0), \end{aligned} \tag{2.3}$$

where $\Gamma = 2\int_0^\pi \frac{\sin u}{u}\, du$ [4, Hilfssatz C]. We now define

$$f(t) = \begin{cases} e^{-m}\Psi_{k_m+1}(t - q_m) & (q_m \leq t < q_m + 1; m = 1, 2, \ldots), \\ 0 & \text{otherwise.} \end{cases}$$

It is immediate from (2.2) and (2.1) that i), ii) and iii) hold. Moreover, for $\operatorname{Re}\lambda > 0$ and $m \geq 1$, it follows from (2.3) that

$$\begin{aligned} \left| \int_{q_m}^{q_m+1} f(t)e^{-\lambda t}\, dt \right| &= e^{-m} \left| \int_0^1 \Psi_{k_m+1}(t) e^{-\lambda(t+q_m)}\, dt \right| \\ &\leq \Gamma \exp\left((2k_m)^{-1} \operatorname{Re}\lambda - q_m \operatorname{Re}\lambda - m \right) \qquad (2.4) \\ &\leq \Gamma e^{-m}, \qquad (2.5) \end{aligned}$$

and from (2.2) that

$$\int_{q_m}^{q_m+1} \left| f(t)e^{-\lambda t} \right| dt \leq e^{-m} k_m e^{-q_m \operatorname{Re}\lambda} \to 0 \quad \text{as } m \to \infty.$$

It follows that $\hat{f}(\lambda)$ exists whenever $\operatorname{Re}\lambda > 0$, and

$$\left| \hat{f}(\lambda) \right| \leq \sum_{m=1}^{\infty} \left| \int_{q_m}^{q_m+1} f(t)e^{-\lambda t}\, dt \right| \leq \sum_{m=1}^{\infty} \Gamma e^{-m} \leq \Gamma.$$

Now, let $\mu \geq 0$ and $m \geq 1$. Then

$$\begin{aligned} \left| \int_0^{\frac{1}{2}} f(t + q_m + \tfrac{1}{2}) e^{-\mu t}\, dt \right| &= \left| e^{-m} \int_0^{\frac{1}{2}} \Psi_{k_m+1}(t + \tfrac{1}{2}) e^{-\mu t}\, dt \right| \\ &= e^{-m} \int_{\frac{1}{2k_m}}^{\frac{1}{2}} \frac{e^{-\mu t}}{t}\, dt \\ &\geq e^{-m} e^{-\mu/2} \log k_m. \end{aligned}$$

Taking $\mu = 0$ gives v). Taking $\mu > 0$ and using (2.4) gives:

$$\begin{aligned} &\left| \int_0^{\infty} f(t + q_m + \tfrac{1}{2}) e^{-\mu t}\, dt \right| \\ &\geq \left| \int_0^{\frac{1}{2}} f(t + q_m + \tfrac{1}{2}) e^{-\mu t}\, dt \right| - \sum_{j=m+1}^{\infty} \left| \int_{q_j}^{q_j+1} f(t) e^{-\mu(t - q_m - \frac{1}{2})}\, dt \right| \\ &\geq e^{-m} e^{-\mu/2} \log k_m - \sum_{j=m+1}^{\infty} \Gamma e^{\mu/2} e^{-\mu q_j} e^{-j} e^{\mu(q_m + \frac{1}{2})} \\ &\geq e^{-m} e^{-\mu/2} \log k_m - \frac{\Gamma e^{\mu}}{e-1} e^{-m} \\ &\geq e^{-\mu} e^{-m} \log k_m \end{aligned}$$

for all sufficiently large m, since $\lim_{m\to\infty} k_m = \infty$. □

Now we give the main result for functions. In Theorem 2.3, it is the properties ii) and iii) which show that (1.1) is sharp. Property iv) (and the continuity of f) will be needed in the next section. Recall that the growth bound $\omega(f)$ of a function $f : \mathbb{R}_+ \to \mathbb{C}$ is defined by

$$\omega(f) = \limsup_{t\to\infty} \frac{\log |f(t)|}{t}.$$

Theorem 2.3. *Let $h : (0,\infty) \to (0,\infty)$ satisfy $\lim_{t\to\infty} h(t) = 0$. There is a continuous, exponentially bounded function $f : \mathbb{R}_+ \to \mathbb{R}$ such that*

i) $\omega(f) = 0$,
ii) $\hat{f}$ *is bounded on* $\mathbb{C}_+$,
iii)

$$\limsup_{t\to\infty} \frac{\left|\int_0^t f(s)\,ds\right|}{th(t)}\,dt = \infty,$$

iv) for $\mu > 0$,

$$\limsup_{t\to\infty} \frac{\left|\int_0^\infty f(t+s)e^{-\mu s}\,ds\right|}{th(t)} = \infty.$$

Proof. First, we construct a function f with all these properties, except that f has a discrete set of jump discontinuities. For this, we take f as in Proposition 2.1, where k_m and q_m are chosen so that the following hold:

$$\begin{aligned} q_1 &\geq 1, \\ q_{m+1} &\geq 2q_m, \\ h(t) &\leq m^{-2}e^{-m} \qquad (t > q_m), \qquad (2.6)\\ k_m &= e^{q_m/m}. \qquad (2.7) \end{aligned}$$

Then $q_m \geq 2^{m-1}$, so

$$k_m \geq e^{2^{m-1}/m^2} \to \infty,$$
$$e^{-m}k_m e^{-aq_m} \leq e^{-q_m(a-1/m)} \to 0$$

as $m \to \infty$, for all $a > 0$.

By Proposition 2.1 i) and (2.7),

$$|f(t)| \leq e^{-m}k_m \leq e^{t/m} \quad (q_m \leq t < q_{m+1}).$$

It follows that $\omega(f) = 0$. Property ii) is immediate from Proposition 2.1 iv).

Proposition 2.1 v), (2.6) and (2.7) give:

$$\begin{aligned} &\frac{\left|\int_0^{q_m+1} f(s)\,ds\right|}{(q_m+1)h(q_m+1)} + \frac{\left|\int_0^{q_m+\frac{1}{2}} f(s)\,ds\right|}{(q_m+\frac{1}{2})h(q_m+\frac{1}{2})} \\ &\qquad \geq \frac{m^2e^m}{q_m+1}\left|\int_0^{q_m+1} f(s)\,ds - \int_0^{q_m+\frac{1}{2}} f(s)\,ds\right| \\ &\qquad \geq \frac{m^2e^m}{q_m+1}e^{-m}\frac{q_m}{m} \\ &\qquad = m\frac{q_m}{q_m+1}, \end{aligned}$$

and iii) follows.

For $\mu > 0$, Proposition 2.1 vi), (2.7) and (2.6) give:

$$\left|\int_0^\infty f(s + q_m + \tfrac{1}{2})e^{-\mu s}\, ds\right| \geq e^{-\mu}e^{-m}\frac{q_m}{m} \geq e^{-\mu}mh(q_m + \tfrac{1}{2})q_m,$$

for all sufficiently large m. This establishes iv).

So far, our function f has jump discontinuities at a discrete set of points. Then we can replace f by a continuous function by replacing the jumps by arbitrarily steep line-segments. Thus, there exists $g \in L^1(\mathbb{R}_+)$ such that $f+g$ is continuous and $\omega(g) \leq \omega(f) = 0$. Then replacing f by $f+g$ gives all the required properties, if $\lim_{t\to\infty} th(t) = \infty$ (which we may assume without loss of generality). □

For polynomially bounded functions, one can obtain sharper estimates than (1.1). If f is measurable, $|f(t)| \leq c(1+t^n)$ for all $t \geq 0$ and $\hat{f}$ is bounded on $\mathbb{C}_+$, then there is a constant C such that

$$\left|\int_0^t f(s)\, ds\right| \leq C(1 + \log t) \tag{2.8}$$

for all $t \geq 2$. This can be seen by modifying the existing proofs of (1.1) by choosing some parameters differently; for example, by taking $\alpha = t^{-n}$ in the proof given in [1, Theorem 4.4.12].

The estimate (2.8) is sharp. One way to see this is to proceed as in Theorem 2.3, but replace (2.7) by

$$k_m = q_m.$$

Then

$$|f(t)| \leq q_m e^{-m} \leq t \quad (q_m \leq t < q_{m+1}),$$

so $|f(t)| \leq t$ for all $t \geq 0$. Moreover,

$$\left|\int_{q_m+\frac{1}{2}}^{q_m+1} f(s)\, ds\right| \geq e^{-m}\log q_m \geq m^2 h(t)\log q_m$$

whenever $t > q_m$, and it follows that

$$\limsup_{t\to\infty} \frac{\left|\int_0^t f(s)\, ds\right|}{h(t)\log t} = \infty.$$

A more striking example is obtained by taking a function $\sum_{r=0}^\infty a_r z^r$ in the disc algebra with $\limsup_{n\to\infty} \sum_{r=0}^n a_r/(h(n)\log n) = \infty$ (see [9, Theorem VIII.1.2]), and then putting $f(t) = a_r$ $(r \leq t < r+1; r = 0,1,2,\dots)$. This shows that (2.8) is sharp even for *bounded* functions f with $\hat{f}$ bounded in $\mathbb{C}_+$. Note, however, that properties such as Theorem 2.3 iv) are incompatible with f being bounded.

3. The inequality for semigroups

We will now show that the inequality (1.4) is sharp, by converting the example in Theorem 2.3 into a semigroup example.

Theorem 3.1. *There is a separable Banach space X and a C_0-semigroup T on X, with generator A, such that for each function $h : (0,\infty) \to (0,\infty)$ with $\lim_{t\to\infty} h(t) = 0$, there exists $x \in X$ such that $R(\cdot, A)x$ has a bounded holomorphic extension to $\mathbb{C}_+$ and*

$$\limsup_{t\to\infty} \frac{\|T(t)R(\mu, A)x\|}{th(t)} = \infty. \tag{3.1}$$

Proof. Fix $\omega > 0$. Let

$$X = \left\{ g : \mathbb{R}_+ \to \mathbb{C} : g \text{ continuous}, \lim_{t\to\infty} g(t)e^{-\omega t} = 0 \right\}.$$

Then X is a Banach space with the norm

$$\|g\|_X = \sup_{t\geq 0} |g(t)|e^{-\omega t}.$$

Define a C_0-semigroup T on X by

$$(T(t)g)(s) = g(s+t).$$

Then $\|T(t)\| = e^{\omega t}$. Let A be the generator of T.

For a given h, take f as in Theorem 2.3. Then $f \in X$ and for $\operatorname{Re}\lambda > \omega$,

$$(R(\lambda, A)f)(t) = \int_0^\infty e^{-\lambda s} f(s+t)\, ds \tag{3.2}$$

$$= e^{\lambda t}\left(\hat{f}(\lambda) - \int_0^t e^{-\lambda s} f(s)\, ds\right). \tag{3.3}$$

For $0 < \operatorname{Re}\lambda \leq \omega$, we define $(R(\lambda, A)f)(t)$ by (3.2) or (3.3). Then $R(\cdot, A)f$ is a holomorphic function from $\mathbb{C}_+$ to X.

Since $\omega(f) = 0$, there exists M such that $|f(s)| \leq Me^{\omega s/4}$ for all $s \geq 0$. Also, there exists K such that $|\hat{f}(\lambda)| \leq K$ whenever $\operatorname{Re}\lambda > 0$. By (3.2),

$$|(R(\lambda, A)f)(t)| \leq \frac{M}{\operatorname{Re}\lambda - \omega/4} \qquad (\operatorname{Re}\lambda > \omega/4).$$

For $0 < \operatorname{Re}\lambda < \omega/2$, (3.3) gives:

$$\begin{aligned} |(R(\lambda, A)f)(t)| &\leq e^{\omega t/2}\left(K + \int_0^t e^{-s\operatorname{Re}\lambda} Me^{\omega s/4}\, ds\right) \\ &\leq Ke^{\omega t/2} + \frac{4M}{\omega} e^{3\omega t/4}. \end{aligned}$$

These estimates show that $R(\cdot, A)f : \mathbb{C}_+ \to X$ is bounded. However, for $\mu > 0$,

$$\begin{aligned} \|T(t)R(\mu, A)f\|_X &\geq |(T(t)R(\mu, A)f)(0)| \\ &= |(R(\mu, A)f)(t)| = \left|\int_0^\infty e^{-\mu s} f(t+s)\, ds\right|. \end{aligned}$$

Now the claim follows from Theorem 2.3 iv). $\square$

The semigroup constructed in the proof of Theorem 3.1 has growth bound $\omega(T) = \omega$, where ω is strictly positive (but arbitrarily small). For a given function h, it is possible to construct a semigroup T on a space X (depending on h) with $\omega(T) = 0$ and a vector x such that $R(\cdot,A)x$ is bounded on $\mathbb{C}_+$ and (3.1) holds. We describe such an example here, without giving details.

Given h, choose k_m, q_m and f as in the proof of Theorem 2.3. Then define $w : \mathbb{R}_+ \to [1,\infty)$ by

$$w(t) = \begin{cases} e^{q_m/m} & ((m-1)q_m/m \le t \le q_m;\ m = 1.2,\dots), \\ e^{t/m} & (q_m \le t \le mq_{m+1}/(m+1);\ m = 1.2,\dots). \end{cases}$$

Then w is a continuous increasing function with $\omega(w) = 0$, satisfying

$$\begin{aligned} w(s+t) &\le w(s)w(t) \quad (s,t \ge 0), \\ \lim_{t\to\infty} \frac{f(t)}{w(t)} &= 0, \\ \sup_{\substack{\operatorname{Re}\lambda>0 \\ t\ge 0}} \frac{\left|\int_0^\infty e^{-\lambda s} f(s+t)\,ds\right|}{w(t)} &< \infty. \end{aligned}$$

Then take

$$\begin{aligned} X &= \left\{ g : \mathbb{R}_+ \to \mathbb{C} : g \text{ continuous}, \lim_{t\to\infty} g(t)/w(t) = 0 \right\}, \\ \|g\|_X &= \sup_{t\ge 0} \frac{|g(t)|}{w(t)}, \\ (T(t)g)(s) &= g(s+t). \end{aligned}$$

Finally, we contrast Theorem 3.1 with the situation for discrete semigroups, i.e., powers of a single bounded linear operator T on X. For $x \in X$, the local resolvent is given by

$$(\lambda I - T)^{-1}x = \sum_{r=0}^\infty \lambda^{-r-1} T^r x$$

for $|\lambda| > \|T\|$. Suppose that this has a bounded holomorphic extension for $|\lambda| > 1$. Then the power series $\sum_{r=0}^\infty z^r T^r x$ defines a bounded holomorphic function on the unit disc, and (1.2) shows that

$$\left\| \sum_{r=0}^n T^r x \right\| \le C \log n$$

for all $n \ge 2$. This inequality is sharp: one may convert the scalar-valued examples into operator examples by a method analogous to Theorem 3.1. However, the inequality $\left\|T^n(\mu I - T)^{-1}x\right\| \le C\log n$, corresponding to (1.4), is not sharp. Indeed, Cauchy's Theorem implies that $\sup_n \|T^n x\| < \infty$ and hence $\sup_n \left\|T^n(\mu I - T)^{-1}x\right\| < \infty$ whenever $|\mu| > \|T\|$.

References

[1] W. Arendt, C.J.K. Batty, M. Hieber and F. Neubrander, *Vector-valued Laplace transforms and Cauchy problems*, Birkhäuser, Basel, 2001.

[2] C.J.K. Batty and M.D. Blake, Convergence of Laplace integrals, *C.R. Acad. Sci. Paris Ser. I* 330 (2000), 71–75.

[3] C.J.K.Batty, R.Chill and J.M.A.M.van Neerven, Asymptotic behaviour of C_0-semigroups with bounded local resolvents, *Math. Nachr.* 219 (2000), 65–83.

[4] P.H.Bloch, Über den Zusammenhang zwischen den Konvergenzabszissen, der Holomorphie- und der Beschränktheitsabszisse bei der Laplace-Transformation, *Comment. Math. Helv.* 22 (1949), 34–47.
[5] S.Z. Huang and J.M.A.M. van Neerven, B-convexity, the analytic Radon-Nikodym property, and individual stability of C_0-semigroups, *J. Math. Anal. Appl.* 231 (1999), 1–20.
[6] J.M.A.M. van Neerven, Individual stability of C_0-semigroups with uniformly bounded local resolvent, *Semigroup Forum* 53 (1996), 155–161.
[7] J.M.A.M. van Neerven, Individual stability of positive C_0-semigroups, preprint, 2000.
[8] L. Weis and V. Wrobel, Asymptotic behavior of C_0-semigroups in Banach spaces, *Proc. Amer. Math. Soc.* 124 (1996), 3663–3671.
[9] A. Zygmund, *Trigonometric Series.* Vol. I, 2nd ed., Cambridge Univ. Press, Cambridge, 1959.

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