Chaos for a class of linear kinetic models

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Abstract. In the recent years it was observed that the chaotic behaviour can occur in some infinite–dimensional linear systems. One of the first examples of this type, related to death processes, was discussed in [PA]. In this paper we generalize the results of [PA] to variable coefficients case showing that the property of being chaotic can be in certain sense stable. On the other hand we show that the "opposite" birth process cannot be chaotic.

Key words: chaos, birth and death process, hypercyclic operators, kinetic model

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1. Introduction

In the recent decade it has been observed that chaos, typically associated with nonlinear phenomena, can occur also in linear infinite-dimensional systems. To make this statement more precise, let us recall that the (semi-)dynamical system $\{T(t)\}_{t\geq 0}$ on a metric space X is said to be topologically chaotic (in the sense of Devaney [D]) if it is *topologically transitive* (meaning that for any two non-empty open sets $U, V \subset X$ there is $t_0 \geq 0$ such that $T(t_0)U \cap V \neq \emptyset$) and its set of periodic points is dense in X.

Historically speaking, the original Devaney's definition contained also the intuitive notion of sensitive dependence on initial conditions (sdic) but it was proved in [Bal] that in fact the two conditions given above yield sdic (the original proof for a discrete in time systems can be repeated with slight modifications for continuous in time systems — see [B]).

A link between chaos theory and linear operator theory has been established when it was realized that the topological transitivity was really equivalent to the notion of

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hypercyclicity: a strongly continuous semigroup of bounded linear operators $\{T(t)\}_{t\geq 0}$ on a Banach space X is called *hypercyclic* if for some $x \in X$ the orbit $\{T(t)x\}_{t\geq 0}$ is dense in X (see e.g. [DSW], [GS]). This path was followed later in [H] (for discrete in time systems) and several other papers.

The significance of these results for the applied mathematics (and continuous in time systems) was probably first noted in [PA] where the authors showed that a class of linear systems of kinetic type (described in more detail in Section 2) that can model e.g. the death part of the birth-and-death process, scattering with energy loss, and many other phenomena, is chaotic. However, it should be mentioned that in order to be chaotic, the system must contain a "creation" part ($\beta > \alpha$ in the model), as no conservative system (such as e.g. the standard birth-and-death process) can be chaotic in the whole Banach space.

This paper was followed by [W] where it was shown that a linear transport equation modeling certain cell population dynamics is chaotic.

The recent paper [DSW] summarizes and considerably extends many of the existing results. The authors show, among others, that the convection-diffusion on the half-line is chaotic, rendering thus another important example of linear chaotic system. They also provide two important theorems allowing the identification of certain classes of chaotic and non-chaotic systems. As these two theorems will be the main tool in our investigations here we shall quote them for readers' convenience.

Theorem 1 ([DSW] — Theorem 3.1). *Let X be a separable Banach space and let A be the infinitesimal generator of a strongly continuous semigroup $\{T(t)\}_{t\geq 0}$ on X. Suppose that the point spectrum of A, $\sigma_p(A)$, has a nonempty interior U such that $U \cap i\mathbf{R} \neq \emptyset$. For each $\lambda \in U$ we denote by x_λ a corresponding nonzero eigenvector: $Ax_\lambda = \lambda x_\lambda$ and define the function $F_\Phi(\lambda) =< \Phi, x_\lambda >$ where $\Phi \in X^*$. Assume that for any $\Phi \in X^*$ the function F_Φ is analytic in U and doesn't vanish identically on U unless $\Phi = 0$. Then $\{T(t)\}_{t\geq 0}$ is chaotic.*

Theorem 2 ([DSW] — Theorem 3.3). *Let $\{T(t)\}_{t\geq 0}$ be a hypercyclic semigroup generated by A in a Banach space X. Then the adjoint A^* of A and the dual semigroup $\{T^*(t)\}_{t\geq 0}$ have the following properties:*
a. *for any $0 \neq \phi \in X^*$ the orbit $\{T^*(t)\phi\}_{t\geq 0}$ is unbounded;*
b. *the point spectrum $\sigma_p(A^*)$ is empty.*

A survey of chaotic models coming from the applied sciences together with some discussion of links between discrete and continuous time theory can be found also in [B].

2. Death model with constant coefficients

Following [PA] we consider an immobile medium that hosts particles characterized by an internal degree of freedom, indexed by a positive integers $n \in \mathbb{N}_0 = \{0, 1, 2 \ldots\}$, and related to internal levels of energetic excitation. These particles collide and interact with the host medium as follows. Upon collision, particles of internal energy level $n \geq 1$ are absorbed by the medium at a rate $\alpha > 0$ and re-emitted as particles of internal energy level $n-1$ at a rate $\beta > \alpha$. Particles with internal energy $n = 0$ are absorbed and cease to exist. The process underlying this interaction can be viewed as a chemical reaction, a biological process, or as a generalized automaton rule.

Particles of internal energy n are then described by a one-particle distribution function, $f_n(t)$, that satisfies the linear rate equation

$$\frac{\mathrm{d}f_n}{\mathrm{d}t} = (Lf)_n = -\alpha f_n + \beta f_{n+1}, \qquad n \in \mathbb{N}. \tag{1}$$

The proper space for this process is l^1 so that the norm of any elements in the positive cone: $f = (f_0, f_1 \ldots) \geq 0$, given by

$$\|f\| = \sum_{n=0}^{\infty} f_n ,$$

represents the total number of particles.

The operator L on the right-hand side is bounded so that there exists a semigroup $\{T(t)\}_{t\geq 0}$ solving the above equation. It follows that the point spectrum of L is given by

$$\sigma_p(L) = \{-\alpha + \beta\mu\ ; \quad |\mu| < 1\} ,$$

with eigenvectors given by $h_\mu = (\mu, \mu^2, \mu^3, \ldots)$, which satisfies the assumptions of Theorem 1.

3. Death model with variable coefficients

In this section we consider a generalization of the model discussed earlier. The generalization consists in allowing variable coefficients $\alpha_n, \beta_n, n \in \mathbb{N}_0$.

We consider in l^1 the infinite system of equations

$$\frac{\mathrm{d}f_n}{\mathrm{d}t} = (Lf)_n = -\alpha_n f_n + \beta_n f_{n+1} , \qquad n \in \mathbb{N}_0 . \tag{2}$$

Note that some motivation for studing Eq. (2) comes also from the theory of kinetic integral equations (generalized Boltzmann models or nonlocal models) — see [BL] and [LW]. In fact consider the followig linear equation

$$\partial_t f = \gamma * f - \eta f , \tag{3}$$

in $L^2(\mathbb{T})$, where $\mathbb{T}$ is the 1–dimensional torus, γ, η are given function on $\mathbb{T}$ and $*$ is the convolution. The bilinear version of this type were studied in [LW]. Eq. (3) in Fourier transform reads

$$\partial_t f_n = \gamma_n f_n - \sum_{m=-\infty}^{+\infty} \eta_{n-m} f_m , \qquad n = 0, \pm 1, \pm 2, \ldots , \tag{4}$$

where f_n is the n-th Fourier coefficient of function f, Eq. (4), under the assumption $\eta_n = 0$, $\forall\, n \neq -1, 0$ leads to Eq. (2) with $\alpha_n = \eta_0 - \gamma_n$ and $\beta_n = -\eta_{-1}$ for $n \in \mathbb{N}_0$.

In this paper we assume that $0 < \alpha_n < \beta_n$ for any $n \in \mathbb{N}_0$ and

(A1) $\alpha_n = \alpha + a'_n$, $(n \in \mathbb{N}_0)$, for some $\alpha \geq 0$ and with $\lim\limits_{n\to\infty} a'_n = 0$;

(A2) $\beta_n = \beta b_n$, $(n \in \mathbb{N}_0)$, for some $\beta > \alpha$ and with $\lim\limits_{n\to\infty} b_n = 1$.

The assumption $\lim\limits_{n\to\infty} b_n = 1$ is largely for technical convenience.

Solving the eigenvalue problem

$$\lambda f_n = -\alpha_n f_n + \beta_n f_{n+1}\,, \qquad n \in \mathbb{N}_0\,, \tag{5}$$

gives formally

$$h(\lambda) = \{h_n(\lambda)\}_{n\in\mathbb{N}_0}\,, \tag{6}$$

where $h_0(\lambda) = 1$, and for $n \geq 1$

$$h_n(\lambda) = \prod_{i=0}^{n-1} \frac{\lambda + \alpha_i}{\beta_i}\,. \tag{7}$$

Lemma 1. *Under the assumptions* (A1) *and* (A2) *the circle* $\{\lambda \in \mathbb{C}\,;\ |\lambda+\alpha| < \beta\}$ *belongs to the point spectrum* $\sigma_p(L)$ *of the operator* L *and* $\lambda = 0$ *belongs to the interior of* $\sigma_p(L)$.

Proof. Assume that $|\lambda + \alpha| < \beta$; then for some $\varepsilon > 0$ we have $|\lambda + \alpha| = \beta - \varepsilon$. There is n_0 such that for all $n > n_0$ we have $|\beta_n - \beta| < \varepsilon/2$ so that

$$\left|\frac{\lambda + \alpha_n}{\beta_n}\right| \leq \left|\frac{\lambda + \alpha}{\beta_n}\right| + \left|\frac{a'_n}{\beta_n}\right| \leq \frac{\beta - \varepsilon}{\beta - \varepsilon/2} + \left|\frac{a'_n}{\beta_n}\right|\,.$$

Since the first term is strictly smaller than 1 and the second tends to 0, by picking possibly larger n_0 we obtain that for some $q < 1$ and $n > n_0$ we have

$$|\frac{\lambda + \alpha_n}{\beta_n}| \leq q\,,$$

thus $h(\lambda) \in l^1$ and $\lambda \in \sigma_p(L)$. Assumption (A2) gives that $\lambda = 0$ is an interior point of this circle. ■

Using the ideas of Theorem 1 we take an arbitrary sequence $\{\tilde{\Phi}_k\}_{k\in\mathbb{N}_0} \in l^\infty$ and consider the function

$$F_{\tilde{\Phi}}(\lambda) = \sum_{k=0}^{\infty} \tilde{\Phi}_k h_k(\lambda)\,,$$

where the series is uniformly convergent at least in the circle $|\lambda+\alpha| < \beta$. We shall rearrange the above expression as

$$F_\Phi(\mu) = F_{\tilde{\Phi}}(\lambda) = \sum_{k=0}^{\infty} \frac{\tilde{\Phi}_k}{\prod\limits_{i=0}^{k-1} b_i} \prod_{i=0}^{k-1} \left(\frac{\lambda+\alpha}{\beta} + \frac{a_i'}{\beta} \right) = \sum_{k=0}^{\infty} \Phi_k \prod_{i=0}^{k-1} (\mu + a_i) \, ,$$

where we introduced the notation

$$\Phi_k = \frac{\tilde{\Phi}_k}{\prod\limits_{i=0}^{k-1} b_i} \, , \qquad \mu = \frac{\lambda+\alpha}{\beta} \, , \qquad a_n = \frac{a_n'}{\beta} \, ,$$

and the convention that $\prod\limits_{i=0}^{-1} = 1$. The function $F_\Phi(\mu)$ is analytic for $|\mu| < 1$. By Assumption (A2), if $\{\tilde{\Phi}_k\}_{k\in\mathbb{N}_0} \in l^\infty$, then also $\{\Phi_k\}_{k\in\mathbb{N}_0} \in l^\infty$. Thus, showing that $F_\Phi \equiv 0$ yields $\Phi_k = 0$ for $k \in \mathbb{N}_0$ for any bounded $\{\Phi_k\}_{k\in\mathbb{N}_0}$ proves the same statement for $F_{\tilde{\Phi}}$. Consequently we shall concentrate on F_Φ.

To be able to use the properties of analytic functions we shall write $F_\Phi(\mu)$ as a power series in μ by multiplying out the brackets and rearranging the terms - this is possible as the series is absolutely convergent (see e.g. [K], p. 142). By induction one can prove that

$$\prod_{j=0}^{n} (\mu + \alpha_j) = \sum_{j=0}^{n} \mu^j A_{jn} \, ,$$

where

$$A_{jn} = \begin{cases} 0 & \text{for} \quad j > n \, , \\ 1 & \text{for} \quad n = j \, , \\ \sum\limits_{0 \le i_1 < \ldots < i_{n-j} \le n-1} a_{i_1} \cdot \ldots \cdot a_{i_{n-j}} & \text{otherwise} \quad . \end{cases}$$

Thus we have

$$F_\Phi(\mu) = \sum_{j=0}^{\infty} \mu^j \sum_{n=j}^{\infty} \Phi_n A_{jn} \, ,$$

and to be able to use Theorem 1 we have to show that the only bounded solution of the infinite upper-triangular linear system

$$\begin{array}{ccccccccc} \Phi_0 & + & A_{01}\Phi_1 & + & A_{02}\Phi_2 & + & \ldots & = 0, \\ & & \Phi_1 & + & A_{12}\Phi_2 & + & \ldots & = 0, \\ & & & & \Phi_2 & + & \ldots & = 0, \\ & & & & & & \ldots & = 0, \end{array}$$

is the zero solution.

Let us denote by $\mathcal{A}$ the operator generated by the matrix $\{A_{ij}\}_{0\le i,j\le\infty}$ in l^∞. We have the following result.

Lemma 2. *Let us assume that there exist: $q < 1$, and k_0 such that $|a_k| \leq q^{k+1}$ for $k \geq k_0$. Then $\mathcal{A}$ is a bounded operator in l^∞.*

Proof. Let $f = (f_1, f_2, \ldots) \in l^\infty$; then $\mathcal{A}f = (g_1, g_2, \ldots)$ where $g_n = \sum_{j=n}^{\infty} A_{nj} f_j$ and

$$\|\mathcal{A}f\|_\infty = \sup_{n\in\mathbb{N}_0} \Big|\sum_{j=n}^{\infty} A_{nj} f_j\Big| \leq \|f\|_\infty \sup_{n\in\mathbf{N}_0} \sum_{j=n}^{\infty} |A_{nj}| = \|f\|_\infty \sup_{n\in\mathbb{N}_0} S_n \,,$$

where

$$S_n = 1 + \Big|\sum_{0\leq i\leq n} a_i\Big| + \Big|\sum_{0\leq i_1<i_2\leq n+1} a_{i_1} a_{i_2}\Big| + \ldots + \Big|\sum_{0\leq i_1<\ldots<i_{j-n}\leq j-1} a_{i_1} \cdot \ldots \cdot a_{i_{j-n}}\Big| + \ldots .$$

It is clear that we can assume that for $k < k_0$ we have $|a_k| \leq Mq^{k+1}$ for some constant M. With this assumption we get the estimate

$$S_n \leq M^{k_0}\Big(1 + \sum_{1\leq i\leq n+1} q^i + \sum_{1\leq i_1<i_2\leq n+2} q^{i_1+i_2} + \ldots \sum_{1\leq i_1<\ldots<i_{j-n}\leq j} q^{i_1+\ldots+i_{j-n}} + \ldots\Big) .$$

Each power q^k with $k > 2$ can appear several times in the above sum. However, the number of occurences cannot exceed the number $P(k)$ of all possible integer solutions to $i_1 + \ldots + i_r = k$ (it is clear that any particular sequence $(i_1, \ldots, i_r)$ may appear at most once). Thus we can write

$$S_n \leq M^{k_0} \sum_{k=0}^{\infty} P(k) q^k$$

and using the result from combinatorics (see e.g. [VW], p. 140) that

$$P(k) < \frac{\pi}{\sqrt{6(k-1)}} \exp(\pi\sqrt{2k/3}) \,, \qquad \text{for } k > 2$$

we obtain that

$$\overline{\lim_{k\to\infty}} \sqrt[k]{q^k P(k)} < 1 \,.$$

Thus the Cauchy criterion yields the convergence of each S_n with the common bound independent of n. This gives the boundedness of $\mathcal{A}$. ∎

Lemma 3. *There is $q < 1$ such that if $|a_k| \leq q^{k+1}$ for all $k \in \mathbb{N}_0$, then $\mathcal{A}$ is an isomorphism in l^∞.*

Proof. We split $\mathcal{A}$ as

$$\mathcal{A} = \mathcal{I} + \begin{pmatrix} 0 & A_{01} & A_{02} & A_{03} & \ldots \\ 0 & 0 & A_{12} & A_{13} & \ldots \\ 0 & 0 & 0 & A_{23} & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ldots \end{pmatrix} = \mathcal{I} + \mathcal{A}' \,,$$

where $\mathcal{I}$ is the infinite identity matrix. As in the proof of Lemma B we obtain for any $f = (f_1, f_2, \ldots) \in l^\infty$

$$\|\mathcal{A}' f\|_\infty = \sup_{n\in\mathbb{N}_0} \Big| \sum_{j=n+1}^{\infty} A_{nj} f_j \Big| \leq \|f\|_\infty \sup_{n\in\mathbb{N}_0} \sum_{j=n+1}^{\infty} |A_{nj}| = \|f\|_\infty \sup_{n\in\mathbb{N}_0} S'_n \,.$$

where this time we have the estimate

$$S'_n(q) \leq q + q^2 + \ldots + q^{n+1} + \sum_{1\leq i_1 < i_2 \leq n+2} q^{i_1+i_2} + \ldots \sum_{1\leq i_1 < \ldots < i_{j-n} \leq j} q^{i_1+\ldots+i_{j-n}} + \ldots \,.$$

Using the combinatorial argument of Lemma 2 we obtain that there exists a continuous function of $q \in [0,1)$, say $\mathcal{S}(q)$ such that $S'_n(q) \leq \mathcal{S}(q)$ for any n. Thus

$$\|\mathcal{A}' f\|_\infty \leq \mathcal{S}(q) \|f\|_\infty \,.$$

Since $\mathcal{S}$ is continuous with $\mathcal{S}(0) = 0$ we obtain the existence of q_0 such that if $q < q_0$, then $\mathcal{S}(q) < 1$. Using the Neumann series theorem we obtain that $\mathcal{A}$ is an isomorphism. ∎

Combining the above results we obtain the main result of this paper.

Theorem 3. *Suppose that the sequences $\{\alpha_n\}_{n\in\mathbb{N}_0}$ and $\{\beta_n\}_{n\in\mathbb{N}_0}$ satisfy the assumptions (A1) and (A2), and the assumption of* Lemma 3. *Then the semigroup generated by Eq.* (2) *is chaotic in any l_p, $1 \leq p < \infty$, and in c_0.*

Proof. From Lemma 1 we see that $\sigma_p(L)$ contains the open circle $|\lambda + \alpha| < \beta$ with $\lambda = 0$ being an interior point of $\sigma_p(L)$, that is, $\sigma_p(L)$ contains an open subsets $V_+ \subset \mathbb{C}_+ = \{\lambda;\ \Re\lambda > 0\}$, $V_- \subset \mathbb{C}_- = \{\lambda;\ \Re\lambda < 0\}$ and an open interval V_0 of the imaginary axis. Using the representation of $F_\Phi(\mu)$ in the form of the power series we obtain that if $F_\Phi(\mu)$ vanishes on either set: $\beta^{-1}(V_\pm + \alpha)$ or $\beta^{-1}(V_0 + \alpha)$, then the coefficients of the power series must vanish, which is equivalent to $\mathcal{A}\Phi = 0$. Since we proved in Lemma 3 that $\mathcal{A}$ is an isomorphism of l^∞, the only bounded solution is $\Phi = 0$ which, by Theorem 1, yields that the semigroup $\{T(t)\}_{t\geq 0}$ is chaotic in l^1.

Let us consider the same problem in c_0 and l^p with $0 < p < +\infty$. The constructed eigenvectors belong to each of the above spaces. We have $c_0^* = l^1$ (see e.g. [KF], p. 185), and $(l^p)^* = l^r$ with $1/p + 1/r = 1$. Hence, if the only solution $\Phi \in l^r$, $1 \leq r < +\infty$ of the equation $\mathcal{A}\Phi = 0$ is the zero solution, then the semigroup $\{T(t)\}_{t\geq 0}$ is chaotic in each of these spaces. But any element of l^r is necessarily bounded, thus the statement follows from the unique solvability of $\mathcal{A}\Phi = 0$ in l^∞. ∎

3. The birth process

We can consider the "opposite" process to the one considered above, that is related to birth (if the former is related to death), or to heating with creation [PA]. The rate equations are

$$
\begin{aligned}
\frac{\mathrm{d}f_0}{\mathrm{d}t} &= -\alpha_0 f_0\,, \\
\frac{\mathrm{d}f_n}{\mathrm{d}t} &= -\alpha_n f_n + \beta_{n-1} f_{n-1}\,, \qquad n \in \mathbb{N}\,.
\end{aligned} \tag{8}
$$

We adopt the same assumptions on the sequences $\{\alpha_n\}_{n\in\mathbb{N}_0}$ and $\{\beta_n\}_{n\in\mathbb{N}_0}$ as in the previous section. Then it is clear that the matrix

$$
K^T = \begin{pmatrix} -\alpha_0 & 0 & 0 & 0 & \dots \\ \beta_0 & -\alpha_1 & 0 & 0 & \dots \\ 0 & \beta_1 & -\alpha_2 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}
$$

defines a bounded operator $\mathcal{K}_0^T$ in c_0, and $\mathcal{K}_p^T$ in l^p, $1 \le p \le \infty$, respectively. Moreover, the transpose matrix K defines a bounded operator $\mathcal{K}_1$ in l^1 and, for $p \ne \infty$, $\mathcal{K}_r$ in l^r, where $1/p + 1/r = 1$. Define the semigroup generated by $\mathcal{K}_p^T$, $p \in \{0\} \cup [1, \infty]$, in the respective space, by $\{S_p(t)\}_{t\ge0}$. Let us also recall that a semigroup is chaotic if it is hypercyclic (topologically transitive) and has a dense set of periodic points. Then we have

Theorem 4. *The semigroup $\{S_p(t)\}_{t\ge0}$ is not hypercyclic for any $p \in 0 \cup [1, \infty)$.*

Proof. First we note that since the generators are bounded, the semigroups are uniformly continuous and therefore the adjoint of the generator generates the adjoint semigroup. Thus $\{S_p^*(t)\}_{t\ge0} = \{T_r(t)\}_{t\ge0}$ for $p \in [1, \infty), 1/p + 1/r = 1$ and $\{S_0^*(t)\}_{t\ge0} = \{T_1(t)\}_{t\ge0}$. Using Theorem 2 we know that if $\{S_p(t)\}_{t\ge0}$ was hypercyclic, then all the orbits if the adjoint semigroup would be unbounded. Let first $p \ne 1$. Then we know from Theorem 3 that $\{T_r(t)\}_{t\ge0}$ is chaotic so, in particular, it has periodic orbits which cannot be unbounded. Thus in this case $\{S_p(t)\}_{t\ge0}$ is not hypercyclic.

If $p = 1$, then the adjoint semigroup acts in l^∞ and we do not have any result about its chaoticity. However, since clearly $\mathcal{K}_0$ is the restriction of $\mathcal{K}_\infty$ to the closed invariant subspace $c_0 \subset l^\infty$, we obtain that $\{T_0(t)\}_{t\ge0}$ is a restriction of $\{T_\infty(t)\}_{t\ge0}$, ([EN], p. 43). From the proof of Theorem 3 we know that there exists a nonempty subspace $X_0 \subset c_0$ (as $\overline{X}_0 = l^1 \subset c_0$) such that for $x \in X_0$ we have $\lim\limits_{t\to\infty} \|T_0(t)x\|_\infty = 0$. Since $X_0 \subset c_0 \subset l^\infty$, for $x \in X_0$ we get $\|T_\infty(t)x\|_\infty = \|T_0(t)x\|_\infty$ so that the trajectories of $\{T_\infty(t)\}_{t\ge0}$ starting from X_0 tend to 0 as $t \to \infty$. This shows that $\{S_1(t)\}_{t\ge0}$ cannot be hypercyclic. ∎

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