CLASSICAL SOLUTION FOR PARTIAL FUNCTIONAL DIFFERENTIAL EQUATIONS WITH NONAUTONOMOUS PAST

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Abstract. We study abstract Cauchy problems associated to partial differential equations with infinite delay where the history function is modified by a backward evolution family. Under appropriate assumptions and using semigroup techniques we prove the existence of a unique classical solution.

Mathematics Subject Classification (1991): 47D06, 34K06

1. Introduction

This is a contribution to the recent paper of S. Brendle and R. Nagel [1]. They study the combination of the following two equations

$$\frac{\partial}{\partial t}u(t,s) = \frac{\partial}{\partial s}u(t,s) + A(s)u(t,s), \quad s \le 0,\ t \ge 0, \tag{1.1}$$

$$\frac{\partial}{\partial t}u(t,0) = Bu(t,0) + \Phi u(t,\cdot), \qquad t \ge 0, \tag{1.2}$$

where $A(s)$ are (unbounded) operators on a Banach space X, B is the generator of a strongly continuous semigroup $(S(t))_{t\geq 0}$ on X and Φ, the delay operator, is a linear operator from a space of $X-$valued functions on $\mathbb{R}_-$ into X. In [1], they find mild solutions of the above equations by constructing an appropriate semigroup.
In this paper we will prove that, under appropriate assumptions, the same semigroup yields the classical solution of (1.1) and (1.2).

2. Backward Evolution Family and Backward Evolution Semigroups

In this section we give the basic definitions and notations in order to discuss backward nonautonomous Cauchy problems in terms of backward evolution families and evolution semigroups. We refer to [2], [3], Chapter IV.9, [4], [6], [7], [8], [9] and [10], where the "forward" Cauchy problem has been treated.

Definition 2.1. A family $\mathcal{U} := (U(r,s))_{r\leq s\leq 0}$ of bounded linear operators on a Banach space X is called an **exponentially bounded backward evolution family** on $\mathbb{R}_-$ if

(i) $U(r,t)U(t,s) = U(r,s),\ U(t,t) = Id$ for all $r \le t \le s \le 0$,
(ii) the mapping $(t,s) \mapsto U(t,s)$ is strongly continuous,
(iii) $\|U(t,s)\| \le Me^{\omega(s-t)}$ for some $M \ge 1, \omega \in \mathbb{R}$ and all $t \le s \le 0$.

[0]The author is grateful for the support of the Italian "Istituto Nazionale di Alta Matematica" (INdAM). She also wants to thank Roland Schaubelt, Rainer Nagel and Gregor Nickel for many helpful discussions and Livianna Fragnelli and Dimitri Mugnai for their moral support.

Remark 2.2. We observe that if $(U(r,s))_{r\le s\le 0}$ is a backward evolution family, then the family $(V(\tau,\sigma))_{\tau\ge\sigma\ge 0}$, defined as

$$V(\tau,\sigma) := U(-\tau,-\sigma) \quad \text{for} \quad \tau \ge \sigma \ge 0,$$

is a forward evolution family. So all results for the forward evolution family can be transfered to the backward evolution family as well.

In this section we reformulate the most important results.

The exponential bound in *(iii)* is needed if we want to associate to every backward evolution family a C_0-semigroup on $X-$valued function spaces, see [2] or the contribution by R. Schnaubelt in [3], Section IV.9, for a systematic treatment. These semigroups characterize many features of the backward evolution family and, consequently, will be called *backward evolution semigroups*. To that purpose we choose the Banach space

$$\mathcal{C}_0 := C_0(\mathbb{R}_-, X) = \{f : \mathbb{R}_- \to X : f \text{ is continuous and } \lim_{|t|\to\infty} f(t) = 0\}$$

normed by

$$\|f\| := \sup_{t\in\mathbb{R}_-} \|f(t)\|, \quad f \in \mathcal{C}_0.$$

Definition 2.3. For every backward evolution family $(U(r,s))_{r\le s\le 0}$ we define the corresponding **backward evolution semigroup** $(T(t))_{t\ge 0}$ on the space $\mathcal{C}_0$ by

$$(T(t)f)(s) := \begin{cases} U(s,s+t)f(s+t) & \text{for } s+t \le 0 \\ 0 & \text{for } s+t > 0 \end{cases}$$

for $f \in \mathcal{C}_0$, $s \in \mathbb{R}_-$, and $t \ge 0$.

As in [3], Section IV.9, one proves that the semigroup $(T(t))_{t\ge 0}$ is strongly continuous on $\mathcal{C}_0$. We denote its generator by $(G, D(G))$.

In order to proceed, we denote by

$$\mathcal{C}^1 := C_0^1(\mathbb{R}_-, X) = \{f \in C^1(\mathbb{R}_-, X) : f, f' \in \mathcal{C}_0\}$$

the domain of the generator $\left(\frac{d}{dx}, D\left(\frac{d}{dx}\right)\right)$ of the left translation semigroup $(T_l(t))_{t\ge 0}$ in $\mathcal{C}_0$ defined as

$$(T_l(t)f)(s) := \begin{cases} f(s+t) & \text{for } s+t \le 0 \\ 0 & \text{for } s+t > 0 \end{cases}.$$

So, the above evolution semigroup can be written as

$$T(t)f = U(\cdot, \cdot + t)T_l(t)f \quad \text{for } f \in \mathcal{C}_0.$$

In general, no differentiability holds for the backward evolution family, while the backward evolution semigroup is always differentiable on the domain of its generator. The wellposedness of the backward nonautonomous Cauchy problem ($bACP$) (see Def. 3.2) will now be characterized by some properties of the generator $(G, D(G))$ of $(T(t))_{t\ge 0}$.

Let $(U(t,s))_{t\le s\le 0}$ be a backward evolution family and $(T(t))_{t\ge 0}$ the corresponding evolution semigroup with generator $(G, D(G))$ on the space $\mathcal{C}_0$. Then, we define the operator $(\mathcal{A}, D(\mathcal{A}))$ as

$$\mathcal{A}f := \lim_{t\searrow 0} \frac{[U(\cdot,\cdot+t)-I]}{t} f(\cdot)$$

for $f \in D(\mathcal{A}) := \left\{ f \in \mathcal{C}_0 : \lim_{t\searrow 0} \frac{[U(\cdot,\cdot+t)-Id]}{t} f(\cdot) \text{ exists in } \mathcal{C}_0 \right\}$. In [5] it is proved that $(\mathcal{A}, D(\mathcal{A}))$ is contained in a multiplication operator $A(\cdot)$ on $\mathcal{C}_0$.
Moreover the following relation between the spaces $\mathcal{C}^1$, $D(G)$ and $D(\mathcal{A})$ holds. This shows in which sense the family $(U(t,s))_{t\le s\le 0}$ is differentiable with respect to s and t (see Lemma 3.5).

Proposition 2.4. *With the above notation and*

$$\Delta := \mathcal{C}^1 \cap D(\mathcal{A}) = \mathcal{C}^1 \cap D(G)$$

we have

$$Gf = \mathcal{A}f + f' \quad \textit{for} \quad f \in \Delta. \tag{2.1}$$

Moreover, if $f \in \Delta$, then the differential equation

$$\frac{\partial}{\partial s} U(t,s)f(s) = U(t,s)[A(s)f(s) + f'(s)] \tag{2.2}$$

holds for all $t \le s \le 0$, and we have

$$\lim_{|s|\to\infty} \|A(s)f(s) + f'(s)\| = 0. \tag{2.3}$$

If we now assume that the orbit of $f \in \Delta$ under $(T(t))_{t\ge 0}$ remains in Δ, we arrive at the desired differential equation.

Proposition 2.5. *With the above notations the following assertions for a function $f \in \mathcal{C}_0$ are equivalent.*

(i) We have $T(t)f \in \Delta = \mathcal{C}^1 \cap D(G)$ for all $t \ge 0$.
(ii) For the function $f \in \mathcal{C}^1$ we have $U(t,s)f(s) \in D(A(t))$ for all $t \le s \le 0$, and the differential equation

$$-\frac{\partial}{\partial t} U(t,s)f(s) = A(t)U(t,s)f(s) \tag{2.4}$$

holds for all $t \le s \le 0$. Moreover, we have

$$A(\cdot)U(\cdot,\cdot+t)f(\cdot+t) \in \mathcal{C}_0 \tag{2.5}$$

for all $t \ge 0$.

The proofs of both propositions are analogous to the proofs of Proposition 2.7 and 2.8 in [5].

3. Wellposedness of Backward Nonautonomous Cauchy Problems

In this section we discuss the wellposedness of the following backward nonautonomous Cauchy problem

$$(bACP) \quad \begin{cases} \dot{u}(t) = -A(t)u(t), & t \le s \le 0, \\ u(s) = x, \end{cases}$$

where $(A(t), D(A(t)))_{t\in\mathbb{R}_-}$ is a family of linear operator on X, and characterize it by the corresponding backward evolution semigroup.

Definition 3.1. A continuous function $u : (-\infty, s] \to X$ is called a **classical solution** of $(bACP)$ if $u_s(\cdot, x) = u(\cdot) \in C^1((-\infty, s], X), u(t) \in D(A(t))$ for all $t \le s \le 0$, $u(s) = x$, and $\dot{u}(t) = -A(t)u(t)$ for $t \le s \le 0$.

Definition 3.2. For a family $(A(t), D(A(t)))_{t\in\mathbb{R}_-}$ of linear operators on the Banach space X the backward nonautonomous Cauchy problem $(bACP)$ is called **wellposed** with *regularity subspaces* $(Y_s)_{s\in\mathbb{R}_-}$ if

(i) **(Existence)** For all $s \in \mathbb{R}_-$ the subspace

$$Y_s := \{x \in X : \text{ there exists a classical solution for } (bACP)\} \subseteq D(A(s))$$

is dense in X.

(ii) **(Uniqueness)** For every $x \in Y_s$ the solution $u_s(\cdot, x)$ is unique.

(iii) **(Continuous dependence)** The solution depends continuously on s and x, i.e. if $s_n \to s$ and $Y_{s_n} \ni x_n \to x \in Y_s$, we have $\widetilde{u}_{s_n}(t, x_n) \to \widetilde{u}_s(t, x)$ uniformly for t in compact intervals in $\mathbb{R}_-$, where

$$\widetilde{u}_s(t, x) := \begin{cases} u_s(t, x) & \text{if } t \le s \le 0 \\ x & \text{if } s \le t \le 0. \end{cases}$$

Moreover, if there exist constants $M \ge 1$ and $\omega \in \mathbb{R}$ such that

$$\|u_s(t, x)\| \le Me^{\omega(s-t)}\|x\|$$

for all $x \in Y_s$ and $t \le s \le 0$, the backward nonautonomous Cauchy problem $(bACP)$ is called **wellposed with exponentially bounded solutions**.

Definition 3.3. A backward evolution family $(U(t, s))_{t\le s\le 0}$ is called a **backward evolution family solving** $(bACP)$ if there are dense subspaces Y_s of X such that the function $t \mapsto u(t) = U(t, s)x$ is a classical solution of $(bACP)$ for $s \in \mathbb{R}_-$ and $x \in Y_s$.

Remark 3.4. We can observe that the wellposedness of $(bACP)$ on *regularity subspaces* $(Y_s)_{s\in\mathbb{R}_-}$, see [3], Section IV.9, is equivalent to the wellposedness of the following forward Cauchy problem

$$(fACP) \quad \begin{cases} \dot{v}(\tau) = B(\tau)v(\tau), & \tau \ge \sigma \ge 0, \\ v(\sigma) = x \end{cases}$$

on *regularity subspaces* $Z_\tau := Y_{-\tau}$, with solutions $v(\tau) := V(\tau, \sigma)x$, where the family $(V(\tau, \sigma))_{\tau\ge\sigma\ge0}$ is the corresponding forward evolution family of $(U(t, s))_{t\le s\le 0}$ (see Remark 2.2) and the operators $B(\tau)$ are defined as $B(\tau) := A(-\tau)$, $\tau \in \mathbb{R}_+$.

We want to connect the Definition 3.2 of wellposedness to the existence of a backward evolution family solving the backward nonautonomous Cauchy problem.

In Definition 3.3, we supposed that the backward evolution family $(U(t,s))_{t\le s\le 0}$ maps the subspaces Y_s of initial values into the domain $D(A(t))$. However, as in [8], Proposition 1.2, we can prove that these spaces are mapped into each other.

Lemma 3.5. *If $(bACP)$ is wellposed and the backward evolution family $(U(r,s))_{r\le s\le 0}$ solves $(bACP)$, then we have*

$$U(r,s)Y_s \subseteq Y_r \subseteq D(A(r))$$

for all $r \le s \le 0$. Moreover, we have

$$\frac{\partial^+}{\partial s}U(r,s)y = U(r,s)A(s)y \qquad \text{for all } y \in Y_s.$$

We can now characterize wellposedness of a backward nonautonomous Cauchy problem by the existence of a unique backward evolution family solving $(bACP)$.

Proposition 3.6 (see [5], Proposition 2.5)**.** *Let $(A(t),D(A(t)))_{t\in\mathbb{R}_-}$ be a family of linear operators on X and consider the corresponding backward nonautonomous Cauchy problem $(bACP)$. The following assertions are equivalent.*

(i) The backward nonautonomous Cauchy problem $(bACP)$ is wellposed with exponentially bounded solutions.
(ii) There exists a unique exponentially bounded backward evolution family $(U(t,s))_{t\le s\le 0}$ solving $(bACP)$.

As the main result of this section we now relate properties of the domain $D(G)$ to the wellposedness of the backward nonautonomous Cauchy problem $(bACP)$. Roughly speaking, $(bACP)$ is wellposed if there are sufficiently many orbits of $(T(t))_{t\ge 0}$ staying in the space of differentiable functions $\mathcal{C}^1$.

Theorem 3.7 (see [5], Theorem 2.9)**.** *Let X be a Banach space and $(A(t),D(A(t)))_{t\in\mathbb{R}_-}$ a family of linear operators on X. The following assertions are equivalent.*

(i) The backward nonautonomous Cauchy problem $(bACP)$ for the family $(A(t),D(A(t)))_{t\in\mathbb{R}_-}$ is wellposed with exponentially bounded solutions.
(ii) There exists a unique evolution semigroup $(T(t))_{t\ge 0}$ with generator $(G,D(G))$ and an invariant core $\mathcal{D} \subseteq \Delta := \mathcal{C}^1 \cap D(G)$ such that

$$Gf - f' = A(\cdot)f$$

for $f \in \mathcal{D}$.

In particular, we have the following result (see [9], Proposition 1.14).

Theorem 3.8. *Let $(bACP)$ be well-posed on Y_s with an exponentially bounded backward evolution family $\mathcal{U}$ and regularity subspaces Y_s. Then*

$$\mathcal{D} := \{f \in C^1(\mathbb{R}_-,X) : f(s) \in Y_s \text{ for } s \in \mathbb{R}_-,\ f, f', A(\cdot)f \in \mathcal{C}_0\}$$

is a core of the generator G of the corresponding evolution semigroup $(T(t))_{t\ge 0}$ on $\mathcal{C}_0$, and

$$Gf = f' + A(\cdot)f(\cdot)$$

for $f \in \mathcal{D}$.

4. Classical Solution

We now return to equations (1.1) and (1.2) and try to find a unique classical solution $u(t,s)$, i.e. u is in the domain of the operator, is continuously differentiable and satisfies the two equations.
In [1], S. Brendle and R. Nagel suppose that there is a family of linear operator $(A(t), D(A(t)))_{t\in\mathbb{R}_-}$ such that the Cauchy problem $(bACP)$ is well-posed, so there exists a unique backward evolution family $(U(t,s))_{t\le s\le 0}$ solving $(bACP)$. Using $(U(t,s))_{t\le s\le 0}$, they consider an evolution semigroup $(\widetilde{T}(t))_{t\ge 0}$ on $C_0(\mathbb{R}, X)$ with generator $(\widetilde{G}, D(\widetilde{G}))$. Finally, they define a "backward evolution semigroup" on $\mathcal{C}_0 = C_0(\mathbb{R}_-, X)$, denoted by $(T_{B,\Phi}(t))_{t\ge 0}$, with generator $G_{B,\Phi}$, so defined

$$G_{B,\Phi} f := Gf$$

with domain

$$D(G_{B,\Phi}) := \{f \in D(G) : f(0) \in D(B) \text{ and } (Gf)(0) = Bf(0) + \Phi f\},$$

where $(G, D(G))$ is the restriction of $(\widetilde{G}, D(\widetilde{G}))$ to $\mathcal{C}_0$. They prove the following proposition (see [1], Proposition 2.14).

Proposition 4.1. *The semigroup $(T_{B,\Phi}(t))_{t\ge 0}$ satisfies*

$$(T_{B,\Phi}(t)f)(s) = \begin{cases} U(s, s+t)f(s+t) & for\ s+t \le 0, \\ U(s,0)e^{(s+t)B}f(0) + \int_0^{s+t} U(s,0)e^{(s+t-\tau)B}\Phi T_{B,\Phi}(\tau)f\, d\tau & for\ s+t \ge 0. \end{cases}$$

Remark 4.2. This identity leads the authors of [1] to interpret, for each $f \in \mathcal{C}_0$, the function $u(t,s) := (T_{B,\Phi}(t)f)(s)$ as a mild solution of (1.1) and (1.2).

We will now give assumptions in order to have solutions of (1.1) and (1.2) in a classical sense. To that purpose, we have to show that $u(t,s) := (T_{B,\Phi}(t)f)(s)$, for appropriate f, belongs to the the domain $D(G_{B,\Phi})$ of the generator $G_{B,\Phi}$, is continuously differentiable, and satisfies the equations (1.1) and (1.2).

As appropriate subspace of $D(G_{B,\Phi})$ we consider

$$\begin{aligned} \widetilde{D} := \{f \in C^1(\mathbb{R}_-, X) \quad : \quad & f(s) \in Y_s \text{ for } s \in \mathbb{R}_-, f, f', A(\cdot)f \in \mathcal{C}_0, \\ & f(0) \in D(B) \text{ and } (f' + A(\cdot)f)(0) = Bf(0) + \Phi f\}. \end{aligned}$$

Lemma 4.3. *Let $(A(t), D(A(t)))_{t\in\mathbb{R}_-}$ be a family of linear operators such that the associated backward Cauchy problem $(bACP)$ is well-posed with regularity subspaces $(Y_s)_{s\in\mathbb{R}_-}$ and such that condition (2.5) holds. Let $(T_{B,\Phi}(t))_{t\ge 0}$ be the evolution semigroup corresponding to the exponentially bounded backward evolution family $(U(r,s))_{r\le s\le 0}$, which solves $(bACP)$ and assume that the function $s \mapsto A(s)U(s,0)g(s)$ is continuous for all $g(\cdot) \in C([-t,0], D(B))$, where B is the generator of a strongly continuous semigroup on X with $D(B) \hookrightarrow Y_0$. Finally, take $\Phi \in \mathcal{L}(\mathcal{C}_0, X)$. Then the space $\widetilde{D}$ defined above is a $T_{B,\Phi}(t)-$invariant subspace of $D(G_{B,\Phi})$.*

Proof. Let $\mathcal{D} := \{f \in C^1(\mathbb{R}_-, X) : f(s) \in Y_s \text{ for } s \in \mathbb{R}_-, f, f', A(\cdot)f \in \mathcal{C}_0\}$ be defined as in Theorem 3.9.
We then obtain

$$\widetilde{D} = D(G_{B,\Phi}) \cap \mathcal{D} \quad \text{for all } t \ge 0.$$

It always holds that

$$T_{B,\Phi}(t)D(G_{B,\Phi}) \subseteq D(G_{B,\Phi}).$$

In order to prove that $T_{B,\Phi}(t)f \in \mathcal{D}$ for $f \in \mathcal{D}$, we consider two cases.
First case: For $s \geq -t$, by Proposition 4.1, we can write $(T_{B,\Phi}(t)f)(s)$ as

$$(T_{B,\Phi}(t)f)(s) = U(s,0)g_t(s),$$

where $g_t(s) := e^{(s+t)B}f(0) + \int_0^{s+t} e^{(s+t-\tau)B}\Phi T_{B,\Phi}(\tau)f\, d\tau$. Since $f \in D(G_{B,\Phi})$, $f(0) \in D(B)$ and the function

$$\mathbb{R}_+ \ni \tau \mapsto \Phi T_{B,\Phi}(\tau)f \in X \tag{4.1}$$

is continuously differentiable, it follows that $g_t(\cdot) \in C^1([-t,0],X) \cap C([-t,0],D(B))$. Hence $g_t(s) \in Y_0$ for $s \in [-t,0]$ because $D(B) \subseteq Y_0$. By assumption, we have that $U(s,0)Y_0 \subseteq Y_s$, so $(T_{B,\Phi}(t)f)(s) \in Y_s$. Moreover, we observe that

$$\begin{aligned} \partial_s(T_{B,\Phi}(t)f)(s) = &- A(s)U(s,0)g_t(s) + U(s,0)Bg_t(s) \\ &+ U(s,0)\Phi T_{B,\Phi}(s+t)f, \end{aligned} \tag{4.2}$$

hence $T_{B,\Phi}(t)f$ is differentiable. In order to prove the assertion, it remains to show that the functions

(i) $[-t,0] \ni s \mapsto (T_{B,\Phi}(t)f)(s)$,
(ii) $[-t,0] \ni s \mapsto (T_{B,\Phi}(t)f)'(s)$,
(iii) $[-t,0] \ni s \mapsto (A(\cdot)T_{B,\Phi}(t)f)(s)$

are continuous. Since (i) is a consequence of the definition and since (ii) follows by (4.2) from (iii) and from the fact that $f(0) \in D(B)$, it suffices to prove (iii).
It is obvious that $(A(\cdot)T_{B,\Phi}(t)f)(s) \in X$, because $(A(\cdot)T_{B,\Phi}(t)f)(s) = A(s)(T_{B,\Phi}(t)f)(s)$, and $(T_{B,\Phi}(t)f)(s) \in Y_s \subseteq D(A(s)) \in X$. Since $(T_{B,\Phi}(t)f)(s) = U(s,0)g_t(s)$, it follows by the assumption in the theorem that $s \mapsto A(s)(T_{B,\Phi}(t)f)(s)$ is continuous.
Second case: For $s < -t$, by Proposition 4.1, we can write $(T_{B,\Phi}(t)f)(s)$ as

$$(T_{B,\Phi}(t)f)(s) = U(s,s+t)f(s+t)$$

and obtain

$$\begin{aligned} \partial_s(T_{B,\Phi}(t)f)(s) = &- A(s)U(s,s+t)f(t+s) + U(s,s+t)A(s+t)f(s+t) \\ &+ U(s,s+t)f'(s+t). \end{aligned} \tag{4.3}$$

As before and using condition (2.5), we can show that the functions in (i), (ii) and (iii) are continuous for $s < -t$.
Combining the two cases we conclude that $\widetilde{D}$ is a $T_{B,\Phi}(t)-$invariant.

□

This leads to our main result.

Theorem 4.4. *Let $(A(t),D(A(t)))_{t\in\mathbb{R}_-}$ be a family of linear operators such that the associated backward Cauchy problem ($bACP$) is well-posed with regularity subspaces $(Y_s)_{s\in\mathbb{R}_-}$ and such that contidion (2.5) holds. Let $(T_{B,\Phi}(t))_{t\geq 0}$ be the semigroup corresponding to the exponentially bounded backward evolution family $(U(r,s))_{r\leq s\leq 0}$, which solves ($bACP$) and assume that the function $s \mapsto A(s)U(s,0)g(s)$ is continuous for all $g(\cdot) \in C([-t,0],D(B))$, where B is the generator of a strongly continuous semigroup on X*

with $D(B) \hookrightarrow Y_0$*. Finally, take* $\Phi \in \mathcal{L}(\mathcal{C}_0, X)$*. Then the function* $u(t,s) := (T_{B,\Phi}(t)f)(s)$ *is the unique classical solution of* (1.1) *and* (1.2) *whenever* $f \in \widetilde{D}$*.*

Proof. By the previous lemma we have that if $f \in \widetilde{D}$, then $T_{B,\Phi}(t)f \in \widetilde{D}$ and $G_{B,\Phi}T_{B,\Phi}(t)f = (T_{B,\Phi}(t)f)' + A(\cdot)T_{B,\Phi}(t)f$. Moreover the function $u(t,s) := (T_{B,\Phi}(t)f)(s)$ is continuously differentiable respect to t and to s such that

$$G_{B,\Phi}u(t,\cdot) = \frac{d}{dt}u(t,\cdot),$$
$$G_{B,\Phi}u(t,0) = Bu(t,0) + \Phi u(t,\cdot)$$

and

$$G_{B,\Phi}u(t,\cdot) = \frac{\partial}{\partial \cdot}u(t,\cdot) + A(\cdot)u(t,\cdot).$$

So, $u(t,s)$ satisfies the two equations (1.1) and (1.2).
For the uniqueness, it suffices observe that $f \equiv 0$ implies $u(t,s) = (T_{B,\Phi}(t)f)(s) \equiv 0$. □

As particular cases, we can use the so called hyperbolic and parabolic assumptions on the operator family $A(\cdot)$ (see the contribution by R. Schnaubelt in [3], Section IV.9 or [6], Chapter 5).

Corollary 4.5 (Hyperbolic case)**.** *If the operators* $(A(t), D(A(t)))_{t\in\mathbb{R}_-}$ *generate contraction semigroups and satisfy* $D(A(t)) = D(A(0))$ *and* $A(\cdot)x \in C^1(\mathbb{R}_-, X)$ *for* $x \in D(A(0))$*, the backward Cauchy problem is well-posed. Let* $(U(r,s))_{r\le s\le 0}$ *be the exponentially bounded backward evolution family solving* $(bACP)$*,* B *the generator of a strongly continuous semigroup on* X *with* $D(B) \hookrightarrow Y_0$ *and* $\Phi \in \mathcal{L}(\mathcal{C}_0, X)$*. Under these assumptions the assertion of Theorem 4.4 holds.*

Proof. In order to apply Theorem 4.4 it suffices to prove that the function $s \mapsto A(s)U(s,0)g_t(s)$ is continuous, because the other assumptions are obviously satisfied. To this end we can observe that $A(s)U(s,0)$ is strongly continuous on $Y := Y_0$ (see [6], Section 5.4) and $g_t(\cdot) : \mathbb{R}_- \to Y$ is continuous since $D(B) \hookrightarrow Y$. So the assertion follows. □

Corollary 4.6 (Parabolic case)**.** *If the operators* $(A(t), D(A(t)))_{t\in\mathbb{R}_-}$ *generate bounded analytic semigroups of the same type* (M,δ) *such that* $A(t)$ *is invertible and*

$$\|A(t)R(\lambda, A(t))(A(t)^{-1} - A(s)^{-1})\| \le L|t-s|^{\mu}|\lambda|^{-\sigma}$$

for $t,s \in \mathbb{R}$*,* $|arg\lambda| \le \phi$*, and constants* $\phi \in (\frac{\pi}{2}, \pi)$*,* $L \ge 0$*, and* $\mu, \sigma \in (0,1]$ *with* $\mu + \sigma > 1$*, then the backward Cauchy problem is well-posed. Let* $(U(r,s))_{r\le s\le 0}$ *be the exponentially bounded backward evolution family solving* $(bACP)$*,* B *the generator of a strongly continuous semigroup on* X *with* $D(B) \hookrightarrow Y_0$ *and* $\Phi \in \mathcal{L}(\mathcal{C}_0, X)$*. Under these assumptions the assertion of Theorem 4.4 holds.*

Proof. As in the previous case we have only to prove that the function $s \mapsto A(s)U(s,0)g_t(s)$ is continuous. To this purpose we write $A(s)U(s,0)g_t(s)$ as $A(s)U(s,0)A(0)^{-1}A(0)g_t(s)$. Now, $A(s)U(s,0)A(0)^{-1}$ is strongly continuous in s and $A(0)g_t(\cdot) : \mathbb{R}_- \to Y_0$ is continuous since $D(B) \hookrightarrow Y_0 \subseteq D(A(0))$. Again, the assertion follows. □

Example 4.7. Let X be the Hilbert space $L^2[-1,0]$ and let B be the operator defined as $Bf := f''$ with domain $D(B) := \{f \in H^2([-1,0]) : f(0) = f(-1) = 0\}$. As in [1], Example 5, we define the operators $A(s)$ as

$$A(s) := a(s)B, \quad s \leq 0,$$

where $0 < a(\cdot) \in C(\mathbb{R}_-)$. These operators generate a backward evolution family $(U(t,s))_{t \leq s \leq 0}$ given by

$$U(t,s) = e^{(\int_t^s a(\sigma)d\sigma)B}.$$

Since B generates an analytic semigroup $(T(t))_{t\geq 0}$ on X, we obtain

$$e^{(\int_t^s a(\sigma)d\sigma)B}x \in \begin{cases} D(B), & t < s \\ X, & t = s, \end{cases}$$

for every $x \in X$. It follows that $Y_t = D(B)$ for all $t \leq 0$.
Let $\eta : \mathbb{R}_- \to \mathcal{L}(X)$ be of bounded variation and define $\Phi : C_0(\mathbb{R}_-, X) \to X$ as

$$\Phi f := \int_{-\infty}^0 d\eta f$$

for all $f \in C_0(\mathbb{R}_-, X)$. Then the semigroup $(T_{B,\Phi}(t))_{t\geq 0}$ is well defined and, applying Theorem 4.4, the functions $u(t,s) := (T_{B,\Phi}f)(s)$ are classical solutions of (1.1) and (1.2) for $f \in \widetilde{D}$.

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