Domination of semigroups associated with sectorial forms

Amir Manavi*, Hendrik Vogt, and Jürgen Voigt†

Fachrichtung Mathematik, Technische Universität Dresden, D-01062 Dresden, Germany

Abstract

Let τ be a closed sectorial form in a Hilbert space H, and let $T = (T(t); t \geqslant 0)$ be the C_0-semigroup associated with τ. We generalize a criterion of Ouhabaz on T-invariance of a closed convex set $K \subseteq H$ in terms of τ and the Hilbert space projection onto K. Using this criterion, we generalize known criteria for domination properties between two C_0-semigroups associated with closed sectorial forms in an L_2-space.

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1 Introduction

Let $\mathfrak{a}$, $\mathfrak{b}$ be closed sectorial forms in a Hilbert space $H = L_2(\Omega, \mu)$, and denote by S, T the C_0-semigroups associated with $\mathfrak{a}$, $\mathfrak{b}$, respectively. Our aim is characterizing when S is dominated by T, i.e.,

$$|S(t)u| \leqslant T(t)|u|$$

for all $u \in \overline{D(\mathfrak{a})}$, $t \geqslant 0$. The results we are going to present generalize previous results in two ways: On the one hand we will study the case where the forms are not densely defined, and on the other hand we will present core results.

Our work is motivated by the paper of E. M. Ouhabaz [8]—to which we refer for the history and earlier literature—as well as by the desire to find manageable criteria for applications. The application we have in mind is the theory of Schrödinger operators with magnetic fields.

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†partly supported by the DFG

manavi@math.tu-dresden.de, vogt@math.tu-dresden.de, voigt@math.tu-dresden.de

In this case, Ω is an open subset of $\mathbb{R}^d$, $\mu = \sigma\,dx$ with a (singular) weight $\sigma\colon \Omega \to [0,\infty)$, and $\mathfrak{a}$, $\mathfrak{b}$ are associated with the formal differential operators

$$\sigma^{-1}(\nabla - ib)^* a(\nabla - ib) \ \Big(= -\sigma^{-1}\sum_{j,k=1}^{d}(\partial_j - ib_j)a_{jk}(\partial_k - ib_k)\Big),$$

$\sigma^{-1}\nabla^* a\nabla$, respectively. Here, $b\colon \Omega \to \mathbb{R}^d$ is a (singular) vector potential and $a(x) = (a_{jk}(x))_{j,k=1,\ldots,d}$ is a nonnegative real symmetric matrix for all $x \in \Omega$. It was this situation in which the first named author was faced with the two problems mentioned above: In general, the forms $\mathfrak{a}$ and $\mathfrak{b}$ are not densely defined, and they are explicitly given only on cores (cf. [3], [4]).

In order to illustrate the type of characterization we obtain, we mention here the result that S is dominated by T if and only if $D(\mathfrak{a})$ is a generalized ideal of $D(\mathfrak{b})$, and

$$\operatorname{Re}\mathfrak{a}(u,v) \geqslant \mathfrak{b}(|u|,|v|)$$

for all $u, v \in D(\mathfrak{a})$ satisfying $u\overline{v} \geqslant 0$; cf. Theorem 4.1. Our results on domination are derived from a result characterizing semigroups leaving invariant a closed convex set in a Hilbert space (Theorem 2.1). This is parallel to the treatment in [8], the difference being that we push this analysis further. As a consequence, our proof of the characterization of domination is in all parts based on the result of invariance. It is by this method that we are able to treat the case where $D(\mathfrak{a})$ and $D(\mathfrak{b})$ are not densely defined and where inequalities are assumed only on suitable cores of $D(\mathfrak{a})$ and $D(\mathfrak{b})$.

As far as possible, we treat the real case and the complex case simultaneously. Throughout the paper, we assume (Ω,μ) to be a measure space, and by $M(\mu;\mathbb{K})$ we denote the set (in fact a lattice) of measurable $\mathbb{K}$-valued functions, where $\mathbb{K} \in \{\mathbb{R},\mathbb{C}\}$.

The main result of Section 2 is our result presenting various characterizations for invariance of a closed convex set in a Hilbert space under a semigroup which is associated with a sectorial form. This result is applied to characterizing when a semigroup is real, positive, or sub-Markovian, respectively.

Section 3 contains preparations for the application of the main result of Section 2 to domination. We point out the importance of the convex cone

$$K := \big\{(u,v) \in M(\mu;\mathbb{K})^2;\ v \text{ real-valued},\ |u| \leqslant v\big\}$$

for domination of operators, we compute a distinguished projection P from $M(\mu;\mathbb{K})^2$ onto this cone, and we characterize subspaces $U, V \subseteq M(\mu;\mathbb{K})$ for which $U \times V$ is invariant under P. It is in this context that generalized ideals appear as a natural concept.

In Section 4, the main result of Section 2 is applied to characterizing domination of a semigroup S by a positive semigroup T. As special situations we study the cases (a) that both S and T are symmetric, and (b) that S is a positive semigroup, too.

In Section 5 we supply some additional information concerning generalized ideals.

2 The invariance criterion

The aim of this section is to generalize Ouhabaz' criterion on invariance of closed convex sets ([8; Thm. 2.1]) in two directions. On the one hand, we want to state the criterion in the setting of not necessarily densely defined forms; this is in fact an easy task. On the other hand we want to present a core version which is useful in applications.

Throughout this section let τ be a non-negative closed sectorial form in a (real or complex) Hilbert space H. For the theory of sectorial forms we refer to [1; Ch. VI], [2; Ch. I]. Recall that τ is said to be non-negative if $\operatorname{Re}\tau(u) \geqslant 0$ for all $u \in D(\tau)$. We do not assume τ to be densely defined. Let $H_\tau := \overline{D(\tau)}^H$, A the m-sectorial operator in H_τ associated with τ and $T = (T(t); t \geqslant 0)$ the (contractive) C_0-semigroup on H_τ generated by $-A$.

Before stating the main theorem of this section we introduce the following notation. If $\varnothing \neq K$ is a closed convex subset of H, we denote by $P_K^H \colon H \to K$ the projection (= Hilbert space projection) from H onto K, i.e., for $u \in H$ the element $P_K^H u$ is the unique element of K satisfying

$$\|u - P_K^H u\| = \min_{v \in K} \|u - v\|.$$

It is known that $P_K^H u$ is characterized by

$$\operatorname{Re}(u - P_K^H u | v - P_K^H u) \leqslant 0 \quad \text{for all } v \in K. \tag{1}$$

We say that a set $M \subseteq H$ is invariant under a semigroup S on H if $S(t)M \subseteq M$ for all $t \geqslant 0$.

In the case that the form τ is densely defined, equivalence '(i)$\Leftrightarrow$(iv)' of the following theorem is due to E. M. Ouhabaz ([8; Thm. 2.1]).

Theorem 2.1. *Let $\varnothing \neq K \subseteq H$ be a closed convex set, $P := P_K^H$. Then the following are equivalent:*

(i) *H_τ is invariant under P, and $K \cap H_\tau$ is invariant under T.*

(ii) *$D(\tau)$ is invariant under P, and*

$$\operatorname{Re}\tau(Pu, u - Pu) \geqslant 0 \tag{2}$$

for all $u \in D(\tau)$.

(iii) *There exists a dense subset D of $D(\tau)$ (i.e., D is a core for τ) such that $P(D) \subseteq D(\tau)$ and (2) holds for all $u \in D$.*

(iv) $D(\tau)$ *is invariant under* P*, and*

$$\operatorname{Re}\tau(u, u - Pu) \geqslant 0 \tag{3}$$

for all $u \in D(\tau)$*.*

(v) *There exists* $D \subseteq D(\tau)$ *dense, convex and invariant under* P *such that* (3) *holds for all* $u \in D$*.*

Proof. (i) $\Leftrightarrow$ (iv). If (iv) holds then $H_\tau = \overline{D(\tau)}$ is invariant under P since P is continuous. Therefore, either of the conditions implies that $P\restriction_{H_\tau} = P^{H_\tau}_{K\cap H_\tau}$. Since τ is densely defined in H_τ, the asserted equivalence reduces to [8; Thm. 2.1].

(iv) $\Rightarrow$ (v) is trivial.

(v) $\Rightarrow$ (iii). (cf. [8; proof of Cor. 2.5]) Let $u \in D$. Then $u_\varepsilon := Pu + \varepsilon(u - Pu) \in D$ for all $\varepsilon \in (0,1)$, and $u_\varepsilon \to Pu$ in $D(\tau)$ as $\varepsilon \to 0$. Moreover, $Pu_\varepsilon = Pu$ by characterization (1). Thus

$$0 \leqslant \frac{1}{\varepsilon}\operatorname{Re}\tau(u_\varepsilon, u_\varepsilon - Pu_\varepsilon) = \operatorname{Re}\tau(u_\varepsilon, u - Pu) \to \operatorname{Re}\tau(Pu, u - Pu) \quad (\varepsilon \to 0).$$

(iii) $\Rightarrow$ (ii). Let $u \in D(\tau)$. Let $(u_n) \subseteq D$ such that $u_n \to u$ in $D(\tau)$ as $n \to \infty$. By (iii) we have

$$\operatorname{Re}\tau(Pu_n) \leqslant \operatorname{Re}\tau(Pu_n, u_n) \quad (n \in \mathbb{N}).$$

As in [8; proof of Thm. 2.1] one deduces that there exists $c \geqslant 1$ such that

$$\operatorname{Re}\tau(Pu_n) \leqslant c\big(\operatorname{Re}\tau(u_n) + \|u_n\|^2\big) + \|Pu_n\|^2 \quad (n \in \mathbb{N}).$$

Therefore, (Pu_n) is a bounded sequence in $D(\tau)$. Since $Pu_n \to Pu$ in H we obtain that $Pu_n \to Pu$ weakly in $D(\tau)$ (see, e.g., [2; Lemma I.2.12]). Thus, the lower semicontinuity of $\operatorname{Re}\tau$ implies

$$\operatorname{Re}\tau(Pu) \leqslant \liminf_{n\to\infty}\operatorname{Re}\tau(Pu_n) \leqslant \liminf_{n\to\infty}\operatorname{Re}\tau(Pu_n, u_n) = \operatorname{Re}\tau(Pu, u),$$

so (ii) holds.

(ii) $\Rightarrow$ (iv) follows from the equality

$$\tau(u, u - Pu) = \tau(Pu, u - Pu) + \tau(u - Pu)$$

since τ is non-negative. □

The next result states an equivalent reformulation of property (i) of the above theorem, which holds in a slightly more general context.

Proposition 2.2. *Let* S *be a* C_0*-semigroup on a closed subspace* H_S *of* H*. Let* S_0 *be the semigroup on* H *defined by* $S_0(t) := S(t)P^H_{H_S}$ *for all* $t \geqslant 0$*. Let* $\varnothing \neq K \subseteq H$ *be a closed convex set,* $P := P^H_K$*. Then the following are equivalent:*

(i) K *is invariant under* S_0*.*

(ii) H_S *is invariant under* P*, and* $K \cap H_S$ *is invariant under* S*.*

Proof. It is immediate that one only has to show that K is invariant under $S_0(0) = P^H_{H_S}$ if and only if H_S is invariant under $P = P^H_K$. This equivalence is a special case of the following lemma. $\square$

Lemma 2.3. *Let $K_1, K_2 \subseteq H$ be closed convex sets, $P_j := P^H_{K_j}$ $(j = 1, 2)$. Then K_1 is invariant under P_2 if and only if K_2 is invariant under P_1.*

Proof. We only need to show necessity. Let K_1 be invariant under P_2, $x \in K_2$. We have to show that $y := P_1 x \in K_2$. Since $x \in K_2$, i.e. $P_2 x = x$, we have

$$\|x - P_2 y\| = \|P_2 x - P_2 y\| \leqslant \|x - y\| = \|x - P_1 x\|.$$

Note that $P_2 y \in K_1$ since $y \in K_1$. Since $P_1 x$ is the best approximation of x in K_1, we conclude that $y = P_1 x = P_2 y \in K_2$. $\square$

In the remainder of the section we apply Theorem 2.1 and Proposition 2.2 to real, positive and sub-Markovian semigroups. Let τ, H_τ, T be as above and define T_0 like S_0 in Proposition 2.2.

For the first result we assume that H is the complexification of a real Hilbert space H_r. Let X be a subspace of H, and $X_r := H_r \cap X$. We say that X is *real* if $X = \operatorname{lin}(X_r)$. If X is real then a semigroup on X is called *real* if it leaves X_r invariant.

Corollary 2.4. *(cf. [7; Prop. 2.2]) The following are equivalent:*

(i) *T_0 is real.*

(ii) *H_τ is real and T is real.*

(iii) *$D(\tau)$ is real, and $\tau(u, v) \in \mathbb{R}$ for all $u, v \in D(\tau)_r$.*

(iv) *There exists a dense subset D of $D(\tau)$ such that $u \in D$ implies $\operatorname{Re} u \in D(\tau)$ and $\tau(\operatorname{Re} u, \operatorname{Im} u) \in \mathbb{R}$.*

Proof. Note that $P\colon x \mapsto \operatorname{Re} x$ is the projection from H onto $K := H_r$, and that a subspace of H is real if and only if it is invariant under P. Thus, (i) $\Leftrightarrow$ (ii) is immediate from Proposition 2.2. Moreover, $\operatorname{Re} \tau(Pu, u - Pu) = \operatorname{Im} \tau(\operatorname{Re} u, \operatorname{Im} u)$ for all $u \in D(\tau)$, so (iv) $\Rightarrow$ (ii) follows from Theorem 2.1, (iii)$\Rightarrow$(i).

(iii) $\Rightarrow$ (iv) is trivial, so it remains to show (ii) $\Rightarrow$ (iii). Let $u, v \in D(\tau)_r$, $w := u + iv$. By Theorem 2.1, (i)$\Rightarrow$(ii), $\operatorname{Im} \tau(u, v) = \operatorname{Re} \tau(Pw, w - Pw) \geqslant 0$ as well as $-\operatorname{Im} \tau(u, v) = \operatorname{Re} \tau(P\overline{w}, \overline{w} - P\overline{w}) \geqslant 0$, so (iii) holds. $\square$

For the next result let H be a real Hilbert lattice. It is easy to show that the projection from H onto $K := H_+$ is given by $x \mapsto x^+$. Applying Proposition 2.2 and Theorem 2.1, (i)$\Leftrightarrow$(ii)$\Leftrightarrow$(iii), we obtain

Corollary 2.5. *(cf. [7; Thm. 2.4], [8; Cor. 2.5]) The following are equivalent:*

(i) *T_0 is positive.*

(ii) *H_τ is a sublattice of H, and T is positive.*

(iii) *$D(\tau)$ is a sublattice of H, and $\tau(u^+, u^-) \leqslant 0$ for all $u \in D(\tau)$.*

(iv) *There exists a dense subset D of $D(\tau)$ such that $u \in D$ implies $u^+ \in D(\tau)$ and $\tau(u^+, u^-) \leqslant 0$.*

Finally, let H be a *Stonian* Hilbert sublattice of $M(\mu;\mathbb{R})$ (i.e., $f \wedge 1 \in H$ for all $f \in H$). A semigroup S on H is called *sub-Markovian* if S is positive and $|S(t)f| \leqslant 1$ for all $f \in H$ such that $|f| \leqslant 1$. Note that S is sub-Markovian if and only if $S(t)f \leqslant 1$ for all $f \in H$ such that $f \leqslant 1$. Moreover, the projection from H onto $K := \{f \in H;\, f \leqslant 1\}$ is given by $f \mapsto f \wedge 1$. Applying Proposition 2.2, Theorem 2.1, and Corollary 2.5, we obtain

Corollary 2.6. *(cf. [8; Cor. 2.7]) The following are equivalent:*

(i) T_0 *is sub-Markovian.*

(ii) H_τ *is a Stonian sublattice of* H*, and* T *is sub-Markovian.*

(iii) $D(\tau)$ *is a Stonian sublattice of* H*, and* $\tau(u\wedge 1, (u-1)^+) \geqslant 0$ *for all* $u \in D(\tau)$.

(iv) *There exists a dense subset* D *of* $D(\tau)$ *such that* $u \in D$ *implies* $u \wedge 1 \in D(\tau)$ *and* $\tau(u \wedge 1, (u-1)^+) \geqslant 0$.

3 Generalized ideals

In this section we provide the tools needed to formulate and prove the domination criterion in Section 4.

We define the convex cone

$$K := \{(u,v) \in M(\mu;\mathbb{K})^2;\, v \text{ real-valued, } |u| \leqslant v\} \tag{4}$$

(cf. [8; formula (3.2)]). The following elementary observations establish the relationship between K and positivity of operators as well as domination properties of operators.

Remarks 3.1. Let V be a sublattice of $M(\mu;\mathbb{K})$, $T\colon V \to V$ a linear operator.

(a) Then T is positive if and only if $T\times T$, defined on $V\times V$, leaves $K\cap(V\times V)$ invariant.

(b) Let U be a subspace of $M(\mu;\mathbb{K})$ with $|u| \in V$ for all $u \in U$, and $S\colon U \to U$ a linear operator. Then T is positive and S is dominated by T (i.e., $|Su| \leqslant T|u|$ for all $u \in U$) if and only if $K \cap (U \times V)$ is invariant under $S \times T$.

In view of Remark 3.1(b) we can expect that Theorem 2.1 is applicable to domination of semigroups. In order to apply Theorem 2.1, we have to study the projection from $L_2(\mu;\mathbb{K})^2$ onto $K \cap L_2(\mu;\mathbb{K})^2$ and the invariant subspaces for this projection.

Let $P\colon M(\mu;\mathbb{K})^2 \to K$ be defined by

$$P(u,v) := \frac{1}{2}\big((|u| + |u| \wedge \operatorname{Re} v)^+ \operatorname{sgn} u, (|u| \vee \operatorname{Re} v + \operatorname{Re} v)^+\big). \tag{5}$$

Lemma 3.2. (a) *Let* $\tilde{K} := \{(x,y) \in \mathbb{K} \times [0,\infty);\, |x| \leqslant y\}$*. Then* $\tilde{K}$ *is a closed convex subset of* $\mathbb{K}^2$*, and for* $(x,y) \in \mathbb{K}^2$ *one has*

$$P_{\tilde{K}}^{\mathbb{K}^2}(x,y) = \frac{1}{2}\big((|x| + |x| \wedge \operatorname{Re} y)^+ \operatorname{sgn} x, (|x| \vee \operatorname{Re} y + \operatorname{Re} y)^+\big)$$

($= P(x,y)$ if $\mathbb{K}$ is interpreted as $M(\mu;\mathbb{K})$ with a one-point measure space).

(b) *The restriction of P to $L_2(\mu;\mathbb{K})^2$ is the projection from $L_2(\mu;\mathbb{K})^2$ onto $K \cap L_2(\mu;\mathbb{K})^2$.*

Proof. (a) A simple geometric consideration shows that on the half-plane $[0,\infty)\times\mathbb{R}$ the projection $P_{\tilde K}^{\mathbb{R}^2}$ is given by

$$\begin{aligned} P_{\tilde K}^{\mathbb{R}^2}(x,y) &= \begin{cases} (x,y) & \text{for } y \geqslant x, \\ \frac{1}{2}(x+y, x+y) & \text{for } |y| \leqslant x, \\ (0,0) & \text{for } y \leqslant -x \end{cases} \\ &= \frac{1}{2}\big((x + x\wedge y)^+, (x \vee y + y)^+\big). \end{aligned}$$

Moreover, in the complex case, $\tilde K$ is a subset of the real three-dimensional subspace $\mathbb{C}\times\mathbb{R}$, and therefore

$$P_{\tilde K}^{\mathbb{C}^2}(x,y) = P_{\tilde K}^{\mathbb{C}^2}(x, \operatorname{Re} y) \quad \big((x,y)\in\mathbb{C}^2\big).$$

Finally, the fact that $\tilde K$ is invariant under multiplication of the first coordinate by $\gamma\in\mathbb{K}$, $|\gamma| = 1$, implies that $P_{\tilde K}^{\mathbb{K}^2}$ commutes with this operation. Putting these observations together one obtains the desired formula.

(b) is a direct consequence of (a). □

In order to describe P-invariant subspaces of $M(\mu;\mathbb{K})^2$, we need the following notion.

Definition 3.3. Let U, V be subspaces of $M(\mu;\mathbb{K})$. We call U a *generalized ideal* of V if

(i) $u\in U$ implies $|u|\in V$,

(ii) $u\in U$, $v\in V$, $|v|\leqslant|u|$ implies $v \operatorname{sgn} u \in U$.

Note that, if $V_r := V\cap M(\mu;\mathbb{R})$ is a sublattice of $M(\mu;\mathbb{R})$ and $V = \operatorname{lin}(V_r)$ (or equivalently, $(\operatorname{Re} v)^+\in V$ for all $v\in V$) then it is sufficient to require (ii) for $v\in V_+$.

Remarks 3.4. (a) Ouhabaz [8] introduced the object defined in Definition 3.3 under the name of ideal. We reserve the name of ideal for lattice ideals.

(b) Note that we do not assume the inclusion $U\subseteq V$ in the above definition. The prominent example where this inclusion need not be valid is in the theory of Schrödinger operators with magnetic field where $V = W_2^1(\mathbb{R}^n)$, and U is the form domain of the magnetic operator (cf. [4]). See also Remark 4.2(d).

(c) In the above definition, the inclusion $U\subseteq V$ is valid if $(\operatorname{Re} u)^+\in U$ for all $u\in U$ since then $(\operatorname{Re} u)^+ = |(\operatorname{Re} u)^+|\in V$ for all $u\in U$.

The following proposition establishes the relationship between the projection P and the notion of generalized ideal.

Proposition 3.5. *Let U, V be subspaces of $M(\mu;\mathbb{K})$. Then the following are equivalent:*

(i) *$U \times V$ is invariant under P.*

(ii) *V_r is a sublattice of $M(\mu;\mathbb{R})$, $V = \operatorname{lin}(V_r)$, and U is a generalized ideal of V.*

Proof. (i) $\Rightarrow$ (ii). Let $v \in V$. Then $\big(0, (\operatorname{Re} v)^+\big) = P(0,v) \in U \times V$, which proves the first two properties of (ii).

Let $u \in U$. Then $\frac{1}{2}\big(u, |u|\big) = P(u,0) \in U \times V$, so $|u| \in V$. Further, let $v \in V$, $|v| \leqslant |u|$. Then $\frac{1}{2}\big(u + (\operatorname{Re} v)\operatorname{sgn} u, |u| + \operatorname{Re} v\big) = P(u,v) \in U \times V$, and therefore $(\operatorname{Re} v)\operatorname{sgn} u = \big(u + (\operatorname{Re} v)\operatorname{sgn} u\big) - u \in U$. Applying the last statement to $-iv$ in place of v (if $\mathbb{K} = \mathbb{C}$) one also obtains $(\operatorname{Im} v)\operatorname{sgn} u \in U$. This shows the last property of (ii).

(ii) $\Rightarrow$ (i). Let $(u,v) \in U \times V$. Then $|u| \in V$ since U is a generalized ideal of V. From the hypothesis on V we conclude

$$\frac{1}{2}(|u| + |u| \wedge \operatorname{Re} v)^+, \ \frac{1}{2}(|u| \vee \operatorname{Re} v + \operatorname{Re} v)^+ \in V.$$

The inequality $0 \leqslant \frac{1}{2}(|u| + |u| \wedge \operatorname{Re} v)^+ \leqslant |u|$ and U being a generalized ideal of V then imply $\frac{1}{2}(|u| + |u| \wedge \operatorname{Re} v)^+ \operatorname{sgn} u \in U$. Hence $P(u,v) \in U \times V$. $\square$

Proposition 3.6. *Let $V \subseteq M(\mu;\mathbb{K})$ be a generalized ideal of itself. Then V is a sublattice of $M(\mu;\mathbb{K})$.*

Proof. For $\mathbb{K} = \mathbb{R}$ there is nothing to show. In the case $\mathbb{K} = \mathbb{C}$ we only have to show that $v \in V$ implies $(\operatorname{Re} v)^+ \in V$. So, let $v \in V$. Applying alternatingly conditions (i) and (ii) of Definition 3.3, we obtain $\overline{v}\operatorname{sgn} v = |v| \in V$, $v \operatorname{sgn} v \in V$, thus $(\operatorname{Re} v)\operatorname{sgn} v \in V$, $|\operatorname{Re} v| = |(\operatorname{Re} v)\operatorname{sgn} v| \in V$, $|\operatorname{Re} v|\operatorname{sgn} v \in V$, and finally $(\operatorname{Re} v)^+ \operatorname{sgn} v = \frac{1}{2}\big((\operatorname{Re} v)\operatorname{sgn} v + |\operatorname{Re} v|\operatorname{sgn} v\big) \in V$, $(\operatorname{Re} v)^+ = |(\operatorname{Re} v)^+ \operatorname{sgn} v| \in V$. $\square$

Remarks 3.7. (a) For $\mathbb{K} = \mathbb{R}$, the converse implication in the above proposition is always true, whereas for $\mathbb{K} = \mathbb{C}$ it is true under an additional hypothesis. This will be shown in Proposition 5.1.

(b) It is immediate from Propositions 3.5 and 3.6 that a subspace V of $M(\mu;\mathbb{K})$ is a generalized ideal of itself if and only if $V \times V$ is invariant under P.

Proposition 3.8. *Let V be a closed subspace of $L_2(\mu;\mathbb{K})$. Then the following are equivalent:*

(i) *$P_V^{L_2(\mu;\mathbb{K})}$ is positive.*

(ii) *V is a generalized ideal of itself.*

(iii) *$(\operatorname{Re} v)^+ \in V$ for all $v \in V$.*

Proof. (i) $\Leftrightarrow$ (ii): By Remark 3.1(a), $P_V^{L_2(\mu;\mathbb{K})}$ is positive if and only if $K \cap L_2(\mu;\mathbb{K})^2$ is invariant under $P_V^{L_2(\mu;\mathbb{K})} \times P_V^{L_2(\mu;\mathbb{K})}$ $(= P_{V\times V}^{L_2(\mu;\mathbb{K})^2})$, whereas V is a generalized ideal of itself if and only if $V \times V$ is invariant under $P\!\restriction_{L_2(\mu;\mathbb{K})^2}$ $(= P_{K\cap L_2(\mu;\mathbb{K})^2}^{L_2(\mu;\mathbb{K})^2})$, by Remark 3.7(b). Thus, the equivalence follows from Lemma 2.3.

(i) $\Leftrightarrow$ (iii) is a special case of Lemma 2.3, with $H = L_2(\mu;\mathbb{K})$, $K_1 = V$, $K_2 = H_+$, and $P_2 v = (\operatorname{Re} v)^+$ $(v \in V)$. □

Proposition 3.9. *(cf. [8; Prop. 3.6]) Let $U \subseteq V$ be subspaces of $M(\mu;\mathbb{K})$.*

(a) *Assume that V is a sublattice of $M(\mu;\mathbb{K})$, and that U is a generalized ideal of V. Then U is an ideal of V.*

(b) *Assume that V is a generalized ideal of itself. Then U is an ideal of V if and only if U is a generalized ideal of V.*

Proof. (a) If $u \in U$, $v \in V$, $|v| \leqslant |u|$ then $\overline{u} \in V$, $|\overline{u}| \leqslant |u|$, therefore $|u| = \overline{u} \operatorname{sgn} u \in U$, $v = v \operatorname{sgn} |u| \in U$.

(b) By (a) and Proposition 3.6 we only have to show necessity. For $u \in U \subseteq V$ we have $|u| \in V$, by Definition 3.3(i). Let moreover $v \in V$, $|v| \leqslant |u|$. Then $v \operatorname{sgn} u \in V$ by Definition 3.3(ii). But also $|v \operatorname{sgn} u| \leqslant |u|$, so U being an ideal of V implies $v \operatorname{sgn} u \in U$. □

Obviously, $V = \operatorname{lin}(V_r)$ is the only property of a lattice used in the proof of part (a) of the above proposition. The assumption of V being a lattice is needed for expressing that U is an ideal of V.

4 The domination criterion

In this section we assume $\mathfrak{a}$, $\mathfrak{b}$ to be two closed sectorial forms in $L_2(\mu;\mathbb{K})$. Denote by A the m-sectorial operator associated with $\mathfrak{a}$ in the Hilbert space $H_\mathfrak{a} := \overline{D(\mathfrak{a})}^{L_2(\mu;\mathbb{K})}$, by S the C_0-semigroup on $H_\mathfrak{a}$ generated by $-A$, and define the semigroup S_0 on $L_2(\mu;\mathbb{K})$ as in Proposition 2.2. Similarly for $\mathfrak{b}$, with the operator B, the C_0-semigroup T on $H_\mathfrak{b}$, and the semigroup T_0 on $L_2(\mu;\mathbb{K})$. Let K and P be as in Section 3, (4) and (5).

Theorem 4.1. *Assume that $H_\mathfrak{b}$ is a sublattice of $L_2(\mu;\mathbb{K})$ and that T is positive. Then the following statements* (i) – (iv) *are equivalent:*

(i) *S_0 is dominated by T_0, i.e., $|S_0(t)u| \leqslant T_0(t)|u|$ for all $u \in L_2(\mu;\mathbb{K})$, $t \geqslant 0$.*

(ii) *$H_\mathfrak{a}$ is a generalized ideal of $H_\mathfrak{b}$, and S is dominated by T.*

(iii) *$D(\mathfrak{a})$ is a generalized ideal of $D(\mathfrak{b})$, and*

$$\operatorname{Re} \mathfrak{a}(u, v) \geqslant \mathfrak{b}(|u|, |v|) \quad \big(u, v \in D(\mathfrak{a}),\ u\overline{v} \geqslant 0\big). \tag{6}$$

(iv) *There exist dense subspaces $D_\mathfrak{a} \subseteq D(\mathfrak{a})$, $D_\mathfrak{b} \subseteq D(\mathfrak{b})$ such that $D_\mathfrak{a} \times D_\mathfrak{b}$ is invariant under P (i.e., by Proposition 3.5, $D_\mathfrak{a}$ is a generalized ideal of $D_\mathfrak{b}$ and*

$v \in D_{\mathfrak{b}}$ implies $(\operatorname{Re} v)^+ \in D_{\mathfrak{b}}$*) and*

$$\operatorname{Re} \mathfrak{a}(u, v \operatorname{sgn} u) \geqslant \mathfrak{b}(|u|, v) \quad \big(u \in D_{\mathfrak{a}},\ v \in D_{\mathfrak{b}},\ 0 \leqslant v \leqslant |u|\big).$$

Moreover, if (ii) *holds then*

(v) *$D(\mathfrak{a})$ is a generalized ideal of $D(\mathfrak{b})$, and*

$$\mathfrak{b}(|u|) - \mathfrak{b}(v) \leqslant \operatorname{Re}\big(\mathfrak{a}(u) - \mathfrak{a}(v \operatorname{sgn} u)\big) \quad \big(u \in D(\mathfrak{a}),\ v \in D(\mathfrak{b})_r,\ |v| \leqslant |u|\big).$$

Conversely, if $\mathfrak{a}$ and $\mathfrak{b}$ are symmetric and

(vi) *there exist dense subspaces $D_{\mathfrak{a}} \subseteq D(\mathfrak{a})$, $D_{\mathfrak{b}} \subseteq D(\mathfrak{b})$ such that $D_{\mathfrak{a}} \times D_{\mathfrak{b}}$ is invariant under P, and*

$$\mathfrak{b}(|u|) - \mathfrak{b}(v) \leqslant \mathfrak{a}(u) - \mathfrak{a}(v \operatorname{sgn} u) \quad \big(u \in D_{\mathfrak{a}},\ v \in D_{\mathfrak{b}},\ 0 \leqslant v \leqslant |u|\big)$$

then (iv) *holds.*

Remark 4.2. (a) Observe that $H_{\mathfrak{b}}$ is a sublattice of $L_2(\mu; \mathbb{K})$ and T is positive if and only if T_0 is positive. This is a direct consequence of Proposition 3.8.

(b) Assume that $H_{\mathfrak{b}}$ is a sublattice of $L_2(\mu; \mathbb{K})$ and that T is positive. Then T_0 is dominated by itself, by part (a). By Theorem 4.1, (i)⇒(ii),(iii), we obtain that $H_{\mathfrak{b}}$ is a generalized ideal of itself (already known from Proposition 3.8), $D(\mathfrak{b})$ is a generalized ideal of itself, and

$$\operatorname{Re} \mathfrak{b}(u) \geqslant \mathfrak{b}(|u|) \quad \big(u \in D(\mathfrak{b})\big).$$

If $\mathfrak{b}$ is symmetric and densely defined, then this inequality is just one of the equivalences of the first Beurling-Deny criterion.

(c) Observe that in case $H_{\mathfrak{a}} \subseteq H_{\mathfrak{b}}$, the word ‘generalized’ can be dropped in statement (ii) of Theorem 4.1. This is immediate from Propositions 3.8, (iii)⇒(ii), and 3.9(b).

(d) The following elementary example illustrates that, in the situation of Theorem 4.1, the domination property is possible without $H_{\mathfrak{a}}$ being a subset of $H_{\mathfrak{b}}$. Let Ω consist of two points, $L_2(\mu; \mathbb{K}) = \mathbb{K}^2$, $\mathfrak{b} = 0$ on $D(\mathfrak{b}) := \{(x, x);\ x \in \mathbb{K}\}$. Let $\gamma \in \mathbb{K}$, $|\gamma| = 1$, $\gamma \neq 1$, $\mathfrak{a} = 0$ on $D(\mathfrak{a}) := \{(x, \gamma x);\ x \in \mathbb{K}\}$. Then $H_{\mathfrak{a}} = D(\mathfrak{a})$ is a generalized ideal of $H_{\mathfrak{b}} = D(\mathfrak{b})$, and $S = (\mathrm{id}_{H_{\mathfrak{a}}}; t \geqslant 0)$ is dominated by $T = (\mathrm{id}_{H_{\mathfrak{b}}}; t \geqslant 0)$.

Proof of Theorem 4.1. Without restriction $\mathfrak{a}, \mathfrak{b}$ are non-negative. Following the proof of [8; Thm. 3.3, 1⇒2], we define the sesquilinear form $\tau =: \mathfrak{a} \times \mathfrak{b}$ by

$$D(\tau) := D(\mathfrak{a}) \times D(\mathfrak{b}), \quad \tau\big((u_1, v_1), (u_2, v_2)\big) := \mathfrak{a}(u_1, u_2) + \mathfrak{b}(v_1, v_2).$$

Then τ is a non-negative closed sectorial form in $H := L_2(\mu; \mathbb{K})^2$. Moreover, the C_0-semigroup $W = S \times T$ on $H_\tau = H_{\mathfrak{a}} \times H_{\mathfrak{b}}$ associated with τ is given by

$$W(t)(f, g) = \big(S(t)f, T(t)g\big) \quad \big((f, g) \in H_\tau,\ t \geqslant 0\big),$$

and we have $W_0 = S_0 \times T_0$. By Remark 3.1(b), statement (i) of the theorem holds if and only if $K \cap H$ is invariant under W_0.

Since $H_{\mathfrak{b}}$ is a lattice, Proposition 3.5 implies that $H_{\mathfrak{a}}$ is a generalized ideal of $H_{\mathfrak{b}}$ if and only if $H_\tau = H_{\mathfrak{a}} \times H_{\mathfrak{b}}$ is invariant under P. We can thus reformulate statement (ii) of the theorem as follows.

(ii') H_τ is invariant under P, and $K \cap H_\tau$ is invariant under W.

(For the following recall from Lemma 3.2(b) that $P\restriction_H = P^H_{K\cap H}$.)

(i) $\Leftrightarrow$ (ii') is a direct consequence of Proposition 2.2.

(ii') $\Rightarrow$ (iii), (v). Theorem 2.1, (i)$\Rightarrow$(ii), implies that $D(\tau)$ is invariant under P, and hence, by Proposition 3.5, that $D(\mathfrak{a})$ is a generalized ideal of $D(\mathfrak{b})$. Moreover,

$$\operatorname{Re} \tau\big(P(u,v), (1-P)(u,v)\big) \geqslant 0 \quad \big((u,v) \in D(\tau)\big).$$

Let $u \in D(\mathfrak{a})$, $v \in D(\mathfrak{b})_r$, $|v| \leqslant |u|$. Then $P(u,v) = \frac{1}{2}(u + v \operatorname{sgn} u, |u| + v)$, $(1-P)(u,v) = \frac{1}{2}(u - v \operatorname{sgn} u, v - |u|)$, and we obtain

$$\operatorname{Re} \mathfrak{a}(u + v \operatorname{sgn} u, u - v \operatorname{sgn} u) \geqslant \mathfrak{b}(|u| + v, |u| - v). \tag{7}$$

Replacing v by $-v$ in (7) and adding the resulting inequality to (7) yields (v).

In order to show (iii), let $u, v \in D(\mathfrak{a})$ with $u\overline{v} \geqslant 0$. Let $\tilde{u} := \frac{1}{2}(u+v)$, $\tilde{v} := \frac{1}{2}(|u| - |v|)$. Then $\tilde{u} \in D(\mathfrak{a})$, $\tilde{v} \in D(\mathfrak{b})_r$, $|\tilde{v}| \leqslant |\tilde{u}|$. Moreover,

$$u = \tilde{u} + \tilde{v} \operatorname{sgn} \tilde{u}, \quad v = \tilde{u} - \tilde{v} \operatorname{sgn} \tilde{u}, \quad |u| = |\tilde{u}| + \tilde{v}, \quad |v| = |\tilde{u}| - \tilde{v}.$$

Hence (7) applied to $\tilde{u}, \tilde{v}$ yields (iii).

(iii) $\Rightarrow$ (iv). We only have to show that $v \in D(\mathfrak{b})$ implies $(\operatorname{Re} v)^+ \in D(\mathfrak{b})$. Since T is positive, this is a consequence of Corollaries 2.4 and 2.5.

(iv) $\Rightarrow$ (ii'). Let $u \in D_{\mathfrak{a}}$, $v \in D_{\mathfrak{b}}$. By Theorem 2.1, (v)$\Rightarrow$(i), we only have to show that

$$\alpha := \operatorname{Re} \tau\big((u,v), (1-P)(u,v)\big) \geqslant 0$$

Let $P =: (P_1, P_2)$. Then $P_1(u,v) \in D_{\mathfrak{a}}$. Further, note that $|P_1(u,v)| = P_1(|u|, v) \leqslant |u|$ and $P_1(u,v) = P_1(|u|,v) \operatorname{sgn} u$. Hence $\tilde{v} := |u| - P_1(|u|,v) \in D_{\mathfrak{b}}$, $0 \leqslant \tilde{v} \leqslant |u|$, and $\tilde{v} \operatorname{sgn} u = u - P_1(u,v)$. Therefore,

$$\begin{aligned}
\alpha &= \operatorname{Re} \mathfrak{a}(u, u - P_1(u,v)) + \operatorname{Re} \mathfrak{b}(v, v - P_2(u,v)) \\
&= \operatorname{Re} \mathfrak{a}(u, \tilde{v} \operatorname{sgn} u) + \operatorname{Re} \mathfrak{b}(v, v - P_2(|u|, v)) \\
&\geqslant \mathfrak{b}(|u|, \tilde{v}) + \operatorname{Re} \mathfrak{b}(v, v - P_2(|u|,v)) = \operatorname{Re}(\mathfrak{b} \times \mathfrak{b})\big((|u|, v), (1-P)(|u|, v)\big).
\end{aligned}$$

The latter quantity is non-negative by Theorem 2.1, (i)$\Rightarrow$(iv), since $H_{\mathfrak{b}} \times H_{\mathfrak{b}}$ is invariant under P, by Remark 3.7(b), and $K \cap (H_{\mathfrak{b}} \times H_{\mathfrak{b}})$ is invariant under $T \times T$.

(vi) $\Rightarrow$ (iv). If $\mathfrak{a}$ and $\mathfrak{b}$ are symmetric then (vi) is equivalent to the validity of (7) for all $u \in D_{\mathfrak{a}}$, $v \in D_{\mathfrak{b}}$, $0 \leqslant v \leqslant |u|$. Defining $\tilde{u} := \frac{1}{2}(u + v \operatorname{sgn} u)$, $\tilde{v} := \frac{1}{2}(|u| - v)$, we deduce (iv) as in the proof of (ii') $\Rightarrow$ (iii). $\square$

Corollary 4.3. *Assume that $H_{\mathfrak{a}}$, $H_{\mathfrak{b}}$ are sublattices of $L_2(\mu;\mathbb{K})$ and that S, T are positive. Then the following are equivalent:*

(i) *$H_{\mathfrak{a}}$ is an ideal of $H_{\mathfrak{b}}$, and S is dominated by T.*

(ii) *$D(\mathfrak{a})$ is an ideal of $D(\mathfrak{b})$, and*

$$\mathfrak{a}(u,v) \geqslant \mathfrak{b}(u,v) \quad \big(0 \leqslant u, v \in D(\mathfrak{a})\big). \tag{8}$$

Proof. By Remark 4.2(b), $H_{\mathfrak{b}}$ is a generalized ideal of itself, and so is $D(\mathfrak{b})$. Moreover, $H_{\mathfrak{a}}$ as well as $D(\mathfrak{a})$ satisfy the assumption on U in Remark 3.4(c), by Corollaries 2.4 and 2.5.

In order to use Theorem 4.1, (ii)$\Leftrightarrow$(iii), it remains to apply Proposition 3.9(b) and to show that (8) implies (6). Since S_0 is dominated by S_0 (see Remark 4.2(a)), we obtain by Theorem 4.1, (i)$\Rightarrow$(iii), and (8) that

$$\operatorname{Re}\mathfrak{a}(u,v) \geqslant \mathfrak{a}(|u|,|v|) \geqslant \mathfrak{b}(|u|,|v|) \quad \big(u,v \in D(\mathfrak{a}),\ u\overline{v} \geqslant 0\big). \qquad \square$$

In the following theorem we show the connection between the domination results of Theorem 4.1 and the well-established 'generalized Kato's inequality for operators'.

Theorem 4.4. *Assume that $H_{\mathfrak{b}}$ is a sublattice of $L_2(\mu;\mathbb{K})$, that T is positive, and that $H_{\mathfrak{a}}$ is a generalized ideal of $H_{\mathfrak{b}}$. Then the following are equivalent:*

(i) *S is dominated by T.*

(ii) *$u \in D(A)$ implies $|u| \in D(\mathfrak{b})$, and*

$$\operatorname{Re}(Au|v\operatorname{sgn}u) \geqslant \mathfrak{b}(|u|,v) \quad \big(u \in D(A),\ 0 \leqslant v \in D(\mathfrak{b})\big).$$

(iii) $\operatorname{Re}(Au, v\operatorname{sgn}u) \geqslant (|u|, B^*v)$ *for all $u \in D(A)$, $0 \leqslant v \in D(B^*)$, where B^* is the adjoint of B in the Hilbert space $H_{\mathfrak{b}}$.*

Proof. (i) $\Rightarrow$ (ii). The first assertion of (ii) follows from Theorem 4.1, (ii)$\Rightarrow$(iii). Let $u \in D(A)$, $v \in D(\mathfrak{b})_+$. The domination hypothesis implies

$$\begin{aligned}\operatorname{Re}\frac{1}{t}\big(u - S(t)u\big|v\operatorname{sgn}u\big) &= \frac{1}{t}\big(|u|\big|v\big) - \frac{1}{t}\big(\operatorname{Re}(\overline{\operatorname{sgn}u}\cdot S(t)u)\big|v\big) \\ &\geqslant \frac{1}{t}\big(|u| - T(t)|u|\big|v\big) \quad (t>0).\end{aligned}$$

By [8; Lemma 1.1] we obtain

$$\frac{1}{t}\big(|u| - T(t)|u|\big|v\big) \longrightarrow \mathfrak{b}(|u|,v) \quad (t \to 0),$$

and thus

$$\operatorname{Re}\big(Au\big|v\operatorname{sgn}u\big) = \lim_{t\to 0}\operatorname{Re}\frac{1}{t}\big(u - S(t)u\big|v\operatorname{sgn}u\big) \geqslant \mathfrak{b}(|u|,v).$$

(ii) $\Rightarrow$ (iii). Recall that $D(B^*) \subseteq D(\mathfrak{b})$, and $\mathfrak{b}(w,v) = (w|B^*v)$ for all $w \in D(\mathfrak{b})$, $v \in D(B^*)$; cf. [1; Thm. VI.2.5]. This shows that (iii) is a special case of (ii).

(iii) $\Rightarrow$ (i) is shown as in [6; C-II, proof of Thm. 4.2, (ii)$\Rightarrow$(i)]. $\square$

Remarks 4.5. (a) Theorem 4.1 is the generalization of Ouhabaz' criterion announced in the introduction. The equivalence '(i)$\Leftrightarrow$(ii)' in Theorem 4.4 is due to Simon [9; Thm. 1] for the special case of densely defined symmetric forms. In the case that both semigroups act on the same space, equivalence '(i)$\Leftrightarrow$(iii)' is true in the more general framework of countably order complete Banach lattices (see [6; C-II, Thm. 4.2]).

(b) Corollary 4.3, (ii)$\Rightarrow$(i), was proved in [10; Cor. B.3] for symmetric forms. In [8; Thm. 3.7], Corollary 4.3 was proved for the case $H_{\mathfrak{a}} = H_{\mathfrak{b}}$.

5 Supplement on generalized ideals

In the first result of this section we present sufficient conditions for V to be a generalized ideal of itself.

Proposition 5.1. *Let V be a subspace of $M(\mu;\mathbb{K})$.*

(a) *If $\mathbb{K} = \mathbb{R}$ then V is a generalized ideal of itself if and only if V is a sublattice of $M(\mu;\mathbb{R})$.*

(b) *Assume that $\mathbb{K} = \mathbb{C}$, $V = \operatorname{lin}(V_r)$, and V_r is a uniformly complete sublattice of $M(\mu;\mathbb{R})$. Then V is a generalized ideal of itself. (We recall that for an Archimedean vector lattice E,* uniform completeness *means: For all $x \in E_+$, the principal ideal E_x, endowed with the norm*

$$\|y\|_x := \inf\{c \geqslant 0;\ |y| \leqslant cx\},$$

is complete; cf. [5; Def. 1.1.7 and Prop. 1.2.13].)

Proof. (a) Necessity is clear; sufficiency holds since $u, v \in V$, $|v| \leqslant |u|$ one has $v \operatorname{sgn} u = |u+v| - |u| \in V$.

(b) The uniform completeness of V_r implies that V is a (uniformly complete) complex sublattice of $M(\mu;\mathbb{C})$. Observe that, for all $w, z \in \mathbb{C}$, $|w| \leqslant |z|$, $0 < \lambda \leqslant 1$, one has

$$\left|\frac{1}{\lambda}\big(|\overline{z} + \lambda w| - |\overline{z}|\big) - \operatorname{Re}(w \operatorname{sgn} z)\right| \leqslant \lambda |w|.$$

(It is easy to see that it is sufficient to prove the inequality for the special case $w = 1$. For the proof of this case one estimates $\frac{1}{\lambda}\int_0^\lambda \left|\frac{d}{dt}|\overline{z} + t| - \operatorname{Re}(\operatorname{sgn} z)\right| dt$ suitably.)

Let $u, v \in V$ with $|v| \leqslant |u|$. Then $\big||\overline{u} + \lambda v| - |\overline{u}|\big| \leqslant \lambda|v|$ and hence $\frac{1}{\lambda}\big(|\overline{u} + \lambda v| - |\overline{u}|\big) \in V_{|v|}$ for all $\lambda > 0$, where $V_{|v|}$ is the principal ideal in V generated by $|v|$. The above inequality shows that $\frac{1}{\lambda}\big(|\overline{u} + \lambda v| - |\overline{u}|\big) \to \operatorname{Re}(v \operatorname{sgn} u)$ as $\lambda \to 0$, in the $|v|$-norm on $M(\mu;\mathbb{C})_{|v|}$. The uniform completeness of V implies that $V_{|v|}$ is closed in $M(\mu;\mathbb{C})_{|v|}$ and therefore $\operatorname{Re}(v \operatorname{sgn} u) \in V$. Applying this result to $-iv$ in place of v, we obtain $\operatorname{Im}(v \operatorname{sgn} u) = \operatorname{Re}(-iv \operatorname{sgn} u) \in V$, $v \operatorname{sgn} u \in V$. This shows property (ii) of Definition 3.3. $\square$

If $V \subseteq M(\mu;\mathbb{K})$ is a generalized ideal of itself one may ask whether $v \operatorname{sgn} u \in V$ for all $u, v \in V$. This is evidently true if V is an ideal of $M(\mu;\mathbb{K})$. With $(\Omega, \mathcal{A}, \mu) = (\mathbb{N}, 2^{\mathbb{N}}, \text{counting measure})$ and V the set of all convergent sequences one obtains a generalized ideal of itself, but there exist $u, v \in V$ such that $v \operatorname{sgn} u \notin V$.

Proposition 5.2. *Let V be a sublattice of $M(\mu;\mathbb{K})$. Assume that, for each countable order bounded set $A \subseteq V_r$, the supremum of A taken in $M(\mu;\mathbb{R})$ belongs to V. Then $v \operatorname{sgn} u \in V$ for all $u, v \in V$.*

Proof. The assumption implies that V is countably order complete and hence uniformly complete (cf. [5; Prop. 1.1.8]). Thus we obtain from Proposition 5.1 that V is a generalized ideal of itself.

Since V is a lattice it is sufficient to show the assertion for $v \geqslant 0$. Note that $|ku| \wedge v \in V$ for all $k \in \mathbb{N}$. Thus property (ii) of Definition 3.3 implies $(|ku| \wedge v) \operatorname{sgn} u \in V$ $(k \in \mathbb{N})$. Since $(|ku| \wedge v) \operatorname{sgn} u \to v \operatorname{sgn} u$ $(k \to \infty)$ pointwise and boundedly (by v), the assumption on V implies $v \operatorname{sgn} u \in V$. ☐

Remarks 5.3. (a) The assumptions of Proposition 5.2 are fulfilled, in particular, if V is a closed sublattice of $L_p(\mu;\mathbb{K})$, with $1 \leqslant p < \infty$ (or more generally, of a Banach function space with order continuous norm).

(b) Proposition 5.2 can be interpreted as the statement that in a sublattice V of $M(\mu;\mathbb{K})$ satisfying the assumption of Proposition 5.2, the signum operator S_u with respect to an element $u \in V$ is explicitly given by

$$S_u v = v \operatorname{sgn} u \quad (v \in V)$$

(cf. [6; C-I, 8.] for the signum operator).

(c) Incidentally, Proposition 5.2 also supplies a detail which seems to have been neglected in [8; proof of Prop. 3.2]. Indeed, if H is a closed sublattice of $L_2(\Omega, \mathcal{A}, \mu)$, then $H \cong L_2(\tilde{\Omega}, \tilde{\mathcal{A}}, \tilde{\mu})$ (as a Hilbert space and a lattice) for a suitable measure space. Now, if $u \in L_2(\mu)$ belongs to H and $\tilde{u}$ is the corresponding element of $L_2(\tilde{\mu})$, then part (a) above shows that the signum operator S_u is given by $S_u v = v \operatorname{sgn} u$, but evidently in the representation $L_2(\tilde{\mu})$ one has $S_{\tilde{u}} \tilde{v} = \tilde{v} \operatorname{sgn} \tilde{u}$. Since the signum operator is unique this implies that $v \operatorname{sgn} u$ is represented by the function $\tilde{v} \operatorname{sgn} \tilde{u}$ in $L_2(\tilde{\mu})$, for arbitrary $u, v \in H$.

(d) In Proposition 5.2, the hypothesis on V may not simply be replaced by assuming V_r countably order complete. Indeed, let $\mathcal{F}$ be a free ultrafilter on $\mathbb{N}$, and define $V \subseteq M(\mathbb{N};\mathbb{R})$ ($\mathbb{N}$ with counting measure) by

$$V := \big\{ u \in \ell_\infty(\mathbb{N});\ u(1) = \lim_{\mathcal{F}} u \big\}.$$

Then V is an order complete sublattice of $M(\mathbb{N};\mathbb{R})$, but there exist $u, v \in V$ such that $v \operatorname{sgn} u \notin V$. (Note that, for $u \in V$, the signum operator in V exists and is given by multiplication with

$$\tilde{u}(n) := \begin{cases} \lim\limits_{\mathcal{F}} \operatorname{sgn} u & \text{for } n = 1, \\ \operatorname{sgn} u(n) & \text{for } n \geqslant 2. \end{cases})$$

References

[1] T. Kato, *Perturbation theory for linear operators*, 2nd edition, Springer-Verlag, Berlin, 1980.

[2] Z.-M. Ma and M. Röckner, *Introduction to the theory of (non-symmetric) Dirichlet forms*, Springer-Verlag, Berlin, 1992.

[3] A. Manavi, *Zur Störung von C_0-Halbgruppen auf Banachfunktionenräumen mit ordnungsstetiger Norm und sektoriellen Formen mit singulären komplexen Potentialen*, doctoral dissertation, 2001.

[4] A. Manavi, *Dominated semigroups with singular complex potentials and applications to Schrödinger operators with magnetic fields*, in preparation.

[5] P. Meyer-Nieberg, *Banach lattices*, Springer-Verlag, Berlin, 1991.

[6] R. Nagel (ed.), *One-parameter semigroups of positive operators*, Lecture Notes in Math. **1184**, Springer-Verlag, Berlin, 1986.

[7] E.-M. Ouhabaz, L^∞-contractivity of semigroups generated by sectorial forms, *J. London Math. Soc. (2)* **46** (1992), no. 3, 529–542.

[8] E. M. Ouhabaz, Invariance of closed convex sets and domination criteria for semigroups, *Potential Anal.* **5** (1996), 611–625.

[9] B. Simon, Kato's inequality and the comparison of semigroups, *J. Func. Anal.* **32** (1977), 97–101.

[10] P. Stollmann and J. Voigt, Perturbation of Dirichlet forms by measures, *Potential Anal.* **5** (1996), 109–138.