A Hille – Yosida theorem for bi–continuous semigroups

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Abstract

In order to treat one–parameter semigroups of linear operators on Banach spaces which are not strongly continuous, we introduce the concept of bi-continuous semigroups defined on Banach spaces with an additional locally convex topology τ. On such spaces we define bi-continuous semigroups as semigroups consisting of bounded linear operators which are locally bi-equicontinuous for τ and such that the orbit maps are τ-continuous. We give a characterization of the generators of these semigroups in form of a Hille-Yosida theorem and apply the result to semigroups induced by flows on a metric space as studied by J. R. Dorroh and J. W. Neuberger [Neu72], [Neu73], [DN93], [DN96], [DN00], [Neu00].

1 Introduction

For many applications of operator semigroups, strong continuity with respect to the norm of a Banach space is a too strong requirement. Instead, a "weaker" strong continuity with respect to some locally convex topology holds in many interesting cases as, e.g., semigroups induced by flows (e.g., [DN96]), the Ornstein–Uhlenbeck semigroup on $C_b(H)$ (e.g., [DPZ92], [Cer94], [DPL95]), adjoint semigroups (e.g., [Nee92]), and implemented semigroups (e.g., [BR79, Section 3.2], [ARS94]). We take this observation as motivation to introduce *bi–continuous semigroups* on a Banach space X endowed with an additional locally convex Hausdorff topology τ which is coarser than the norm topology and such that the topological dual $(X, \tau)'$ is *norming* for $(X, \|\cdot\|)$. On such spaces we define *bi–continuous semigroups* $(T(t))_{t\geq 0}$ as semigroups consisting of bounded linear operators which are *locally bi–equicontinuous for* τ (see Definition 2) and such that the orbit maps

$$\mathbb{R}_+ \ni t \mapsto T(t)x \in X$$

are τ–continuous. For such a bi–continuous semigroup $(T(t))_{t\geq 0}$ we show the existence of its τ–Laplace transform

$$R(\lambda)x := \int_0^\infty e^{-\lambda t} T(t)x dt = \|\cdot\| - \lim_{a\to\infty} \left(\tau\text{-} \int_0^a e^{-\lambda t} T(t)x dt \right), \quad x \in X. \tag{1}$$

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From $R(\lambda)$ we obtain the *generator* of $(T(t))_{t\geq 0}$ as a Hille – Yosida operator defined on a τ–dense subspace of the Banach space X. Moreover, corresponding to this generator there exists a τ–differentiable integrated semigroup F whose generator coincides with the generator of $(T(t))_{t\geq 0}$. The relation between bi–continuous semigroups, integrated semigroups and Hille–Yosida operators yields a characterization of the generator of a bi–continuous semigroup in form of a generalized Hille–Yosida theorem.
Finally, we reproduce results by J. R. Dorroh and J. W. Neuberger [DN96], [DN93] by verifying that semigroups on $C_b(\Omega)$ which are induced by flows are bi–continuous for the topology of compact convergence. In particular, we give a simplified proof for their generation theorem for such semigroups on $C_b(\Omega)$ applying our Hille-Yosida theorem.

2 Bi–continuous semigroups

In order to define our new concept of semigroups, we assume that our underlying space X satisfies the following conditions.

Assumptions 1. *Let $(X, \|\cdot\|)$ be a Banach space with topological dual X', and let τ be a locally convex topology on X with the following properties.*

1. *The space (X, τ) is sequentially complete on $\|\cdot\|$–bounded sets, i.e., every $\|\cdot\|$–bounded τ–Cauchy sequence converges in (X, τ).*
2. *The topology τ is Hausdorff and coarser than the $\|\cdot\|$–topology.*
3. *The space $(X, \tau)'$ is norming for $(X, \|\cdot\|)$, i.e.,*

$$\|x\| = \sup\{| < x, \phi > | \ : \phi \in (X, \tau)' \text{ and } \|\phi\|_{(X,\|\cdot\|)'} \leq 1\} \quad \text{for all } x \in X.$$

We remark that the notion of a *norming space* was introduced by J. Lindenstrauss and L. Tzafriri in [LT70, p.29].
In the following we denote by $\mathcal{L}(X)$ the space of bounded linear operators on $(X, \|\cdot\|)$, and P_τ denotes a family of seminorms inducing the locally convex topology τ on X. Since τ is coarser than the $\|\cdot\|$–topology, we assume without loss of generality that $p(x) \leq \|x\|$ for all $x \in X$ and $p \in P_\tau$. For the definition of *bi–continuous semigroups*, we require a specific relation between the semigroup operators and the τ–topology.

Definition 2. *An operator family $\{T(t) : \ t \geq 0\} \subseteq \mathcal{L}(X)$ is called* **(globally) bi–equicontinuous** *if for every $\|\cdot\|$–bounded sequence $(x_n)_{n\in\mathbb{N}} \subseteq X$ which is τ–convergent to $x \in X$ we have*

$$\tau\text{–}\lim_{n\to\infty} T(t)(x_n - x) = 0$$

uniformly for all $t \geq 0$.
It is called **locally bi–equicontinuous** *if for every $t_0 \geq 0$ the subset $\{T(t) : \ 0 \leq t \leq t_0\}$ is bi–equicontinuous.*

With this terminology we now introduce a new class of semigroups.

Definition 3. *An operator family $\{T(t) : \ t \geq 0\} \subseteq \mathcal{L}(X)$ is called a* **bi–continuous semigroup** *(with respect to τ and of type ω) if the following conditions hold.*

(i) $T(0) = Id$ *and* $T(t+s) = T(t)T(s)$ *for all* $s, t \geq 0$.

(ii) The operators $T(t)$ are exponentially bounded, i.e., $\|T(t)\|_{\mathcal{L}(X)} \leq Me^{\omega t}$ for all $t \geq 0$ and some constants $M \geq 1$ and $\omega \in \mathbb{R}$.

(iii) $(T(t))_{t\geq 0}$ is strongly τ–continuous, i.e., the map

$$\mathbb{R}_+ \ni t \longmapsto T(t)x \in X$$

is τ–continuous for each $x \in X$.

(iv) $(T(t))_{t\geq 0}$ is locally bi–equicontinuous.

Clearly, every strongly continuous semigroup on a Banach space is a bi–continuous semigroup with respect to $\tau = \|\cdot\|$ (see [EN00, Ch. I, Def. 5.1]). A variety of examples of bi–continuous semigroups as mentioned in the Introduction are discussed in detail in [Küh01a], [Küh01b] and in Section 5 below.

We now state some important consequences of the above definition.

Proposition 4. *Let $(T(t))_{t\geq 0}$ be a bi-continuous semigroup of type ω on X. Then the following properties hold.*

(a) For every $a \geq 0$ and $\lambda \in \mathbb{C}$ there exists the operator $R_a(\lambda) : X \longrightarrow X$ defined as

$$R_a(\lambda)x := \int_0^a e^{-\lambda t}T(t)x dt \tag{2}$$

for all $x \in X$. The integral has to be understood as a τ–Riemann integral.

(b) The rescaled semigroup $(e^{-\alpha t}T(t))_{t\geq 0}$ is globally bi–equicontinuous for every $\alpha > \omega$.

Proof. Assertion (a) is an immediate consequence of Assumptions 1 and Definition 3 (cf. [Kom64, Prop. 1.1]).
To prove property (b), let $\alpha > \omega, \epsilon > 0, p \in P_\tau$, and $(x_n)_{n\in\mathbb{N}} \subseteq X$ be a $\|\cdot\|$–bounded sequence which is τ–convergent to $x \in X$. Then there exists $t_0 \geq 0$ such that

$$\begin{aligned}\sup_{t>t_0} p(e^{-\alpha t}T(t)(x_n - x)) &\leq \sup_{t>t_0} e^{-\alpha t}\|T(t)(x_n - x)\| \\ &\leq \sup_{t>t_0} e^{(\omega-\alpha)t}M(\|x_n\| + \|x\|) \\ &\leq \frac{\epsilon}{2}\end{aligned}$$

for all $n \in \mathbb{N}$.

Further, by Definition 3(iv), there exists $n_0 \in \mathbb{N}$ such that

$$\sup_{0 \le t \le t_0} p(e^{-\alpha t} T(t)(x_n - x)) \le \frac{\epsilon}{2}$$

for all $n \ge n_0$. Therefore,

$$\begin{aligned}\sup_{t \ge 0} p(e^{-\alpha t} T(t)(x_n - x)) &\le \sup_{0 \le t \le t_0} p(e^{-\alpha t} T(t)(x_n - x)) + \sup_{t > t_0} p(e^{-\alpha t} T(t)(x_n - x)) \\ &\le \frac{\epsilon}{2} + \sup_{t > t_0} e^{-\alpha t} \|T(t)(x_n - x)\| \\ &\le \epsilon\end{aligned}$$

for all $n \ge n_0$. □

Remark 5. (a) The semigroup law, the τ–continuity of the map $t \mapsto T(t)x$ at 0 and local bi–equicontinuity imply the τ–continuity at every point in $\mathbb{R}_+$. To see this, let $t_0 > 0$, $x \in X$ and $p \in P_\tau$. Then $(T(1/n)x)_{n \in \mathbb{N}} \subseteq X$ is a $\|\cdot\|$–bounded sequence which is τ–convergent to x by the τ–continuity at 0. By Definition 3(ii),(iv), we obtain that $p(T(t)(T(1/n)x - x)) \to 0$ uniformly for $0 \le t \le t_0$ as $n \to \infty$, and therefore $p(T(t_0 + h)x - T(t_0)x)$ converges to 0 as $h \searrow 0$, and by the same argument as $h \nearrow 0$ which implies the continuity at t_0.
(b) In [Kōm68, Prop. 1.1] (cf. [Sch80, Ch. III, Thm. 4.2]) it is shown that on a barreled * locally convex vector space (X, τ) conditions (i)–(iii) in Definition 3 automatically imply that $(T(t))_{t \ge 0}$ is **locally equicontinuous**, i.e., for any fixed $t_0 > 0$ and for any continuous seminorm $p \in P_\tau$ there exists a continuous seminorm $q \in P_\tau$ such that

$$p(T(t)x) \le q(x)$$

for all $x \in X$ and uniformly for $0 \le t \le t_0$. Therefore, condition (iv) in Definition 3 is satisfied automatically.

In the following we give first an example of a bi–continuous semigroup which is not locally equicontinuous in the sense of the definition given in Remark 5(b). Further, we show that the translation semigroup on $C_b(\mathbb{R})$ is not bi–continuous with respect to the topology of pointwise convergence. However, bi–continuity holds if we use the topology of uniform convergence on compact intervals.

Examples 6. (a) Let X be the space $C_b(\mathbb{R})$ endowed with the supremum norm $\|\cdot\|_\infty$ and the topology τ_c of uniform convergence on compact subsets of $\mathbb{R}$. Clearly, $(C_b(\mathbb{R}), \tau_c)$ is sequentially complete on $\|\cdot\|_\infty$–bounded sets, τ_c is coarser than the $\|\cdot\|_\infty$–topology, Hausdorff, and, since the topological dual $(C_b(\mathbb{R}), \tau_c)'$ contains the point measures, it is norming for $(C_b(\mathbb{R}), \|\cdot\|_\infty)$. On this space we consider the diffusion semigroup $(T(t))_{t \ge 0}$ defined as

$$T(t)f(x) = \int_{\mathbb{R}} f(y) \mathcal{N}(x, t) dy, \quad x \in \mathbb{R}, f \in C_b(\mathbb{R}), t > 0,$$

*A locally convex vector space is **barreled** if each absorbing, absolutely convex and closed subset is a neighborhood of zero.

where $\mathcal{N}(x,t)$ denotes the Gauss measure with mean x and variance t defined via the probability density $g_{x,t}$ on $\mathbb{R}$ defined as

$$g_{x,t}(y) := \frac{1}{\sqrt{4\pi t}} e^{\frac{-(y-x)^2}{4t}}$$

for all $y \in \mathbb{R}$. This semigroup is a bi–continuous semigroup with respect to τ_c. In fact, $(T(t))_{t\geq 0}$ is a contraction semigroup on $C_b(\mathbb{R})$ and for $f \in C_b(\mathbb{R}), \epsilon > 0$, and a compact subset $K \subseteq \mathbb{R}$ there exists $\delta_{\epsilon,K} > 0$ such that $|y| < \delta_{\epsilon,K}$ implies $|f(x+y) - f(x)| \leq \epsilon$ for all $x \in K$. Therefore, by the Chebyshev inequality, we have

$$\begin{aligned}
&\sup_{x\in K}|T(t)f(x) - f(x)| \\
&\quad \leq \sup_{x\in K} \int_{\{|y|<\delta_{\epsilon,K}\}} |f(x+y) - f(x)|\mathcal{N}(0,t)dy + 2\|f\|_\infty \frac{1}{\sqrt{4\pi t}} \int_{\{|y|\geq\delta_{\epsilon,K}\}} e^{\frac{-|y|^2}{4t}} dy \\
&\quad \leq \epsilon + t\frac{2\|f\|_\infty}{\delta^2_{\epsilon,K}},
\end{aligned}$$

which yields the strong τ–continuity of $(T(t))_{t\geq 0}$ at 0. Next, we show that $(T(t))_{t\geq 0}$ is locally bi–equicontinuous. To that purpose, let $K \subset \mathbb{R}$ be compact, $t_0 \geq 0$, and $\epsilon > 0$. Then there exists a compact subset $K_\epsilon \subseteq \mathbb{R}$ such that

$$\mathcal{N}(x,t)(K_\epsilon) \geq 1 - \epsilon$$

uniformly for $x \in K$ and $0 \leq t \leq t_0$.

Fix a function $f \in C_b(\mathbb{R})$ and a $\|\cdot\|_\infty$–bounded sequence $(f_n)_{n\in\mathbb{N}} \subseteq C_b(\mathbb{R})$ which is τ_c–convergent to f. Therefore, there exists $n_0 \in \mathbb{N}$ such that

$$\sup_{x\in K_\epsilon} |f_n(x) - f(x)| \leq \epsilon$$

for all $n \geq n_0$. Thus

$$\begin{aligned}
&\sup_{x\in K}|T(t)(f_n(x) - f(x))| \\
&\quad \leq \sup_{x\in K} \int_{K_\epsilon} |f_n(y) - f(y)|\mathcal{N}(x,t)dy + \sup_{x\in K} \int_{\complement K_\epsilon} |f_n(y) - f(y)|\mathcal{N}(x,t)dy \\
&\quad \leq \epsilon + (\|f_n\|_\infty + \|f\|_\infty)\epsilon
\end{aligned}$$

uniformly for $0 \leq t \leq t_0$, and hence $(T(t))_{t\geq 0}$ is locally bi–equicontinuous.
By Definition 3 it follows that $(T(t))_{t\geq 0}$ is bi–continuous.
However, $(T(t))_{t\geq 0}$ is not locally equicontinuous in the sense of the definition given in Remark 5. Suppose the contrary, then for every $t_0 \geq 0$ and $K \subset \mathbb{R}$ compact there would exist a compact subset $K_0 \subset \mathbb{R}$ such that

$$\sup_{x\in K} |T(t)f(x)| \leq \sup_{x\in K_0} |f(x)|$$

for all $f \in C_b(\mathbb{R})$ and uniformly for $0 \le t \le t_0$. This must also be true for any function $g \in C_b(\mathbb{R})$ such that $g(x) > 0$ for all $x \in \mathbb{R}$ and $g(x) = 0$ for all x in the interval $[a, b]$ containing K_0. Therefore $\sup_{x \in K_0} |g(x)| = 0$, but

$$\sup_{x \in K} |T(t)g(x)| = \sup_{x \in K} |\int_{\mathbb{R}\setminus[a,b]} g(y)\mathcal{N}(x,t)dy| > 0$$

because of the strict positivity of the integrand. This is a contradiction to our assumption.
(b) Let X be the space $C_b(\mathbb{R})$ endowed with the supremum norm $\|\cdot\|_\infty$ and the topology τ_p of pointwise convergence. By the same argument as in (a) $(C_b(\mathbb{R}), \tau_p)$ satisfies Assumptions 1. We consider the (left)translation semigroup $(T(t))_{t\ge0}$ defined as

$$T(t)f(x) := f(x+t), \quad x \in \mathbb{R}, f \in C_b(\mathbb{R}), t \ge 0.$$

It does not satisfy the property of local bi–equicontinuity. This can be easily seen by taking a sequence $(f_n)_{n\in\mathbb{N}} \subseteq C_b(\mathbb{R})$ defined for $n \ge 2$ as

$$f_n(x) := \begin{cases} \max\left(1 - n^2|x - \frac{1}{n}|, 0\right) & \text{if } x \in [0,1], \\ 0 & \text{else.} \end{cases}$$

Then $(f_n)_{n\in\mathbb{N}}$ is a $\|\cdot\|_\infty$–bounded sequence, and $\lim_{n\to\infty} f_n(x) = 0$ for all $x \in \mathbb{R}$. However, since $T(\frac{1}{n})f_n(0) = 1$ for all $n \ge 2$, the semigroup $(T(t))_{t\ge0}$ is not locally bi–equicontinuous. Hence $(T(t))_{t\ge0}$ is not bi–continuous with respect to τ_p. However, if we take the finer topology τ_c as in (a), then it is easy to see that conditions (i)–(iv) in Definition 3 hold, hence $(T(t))_{t\ge0}$ is bi–continuous with respect to τ_c.

3 Generators and resolvents

In this section we assume that $(T(t))_{t\ge0}$ is a bi–continuous semigroup on a Banach space X satisfying Assumptions 1. Since the space $(X,\tau)'$ is norming for $(X, \|\cdot\|)$, we obtain that the τ–Laplace transform $R(\lambda)$ defined as in (1) becomes a $\|\cdot\|$–bounded operator satisfying the Hille–Yosida estimates. This observation will lead us to the generator of a bi–continuous semigroup whose resolvent coincides with $R(\lambda)$.
First, we collect some elementary properties of these resolvents.
For $\omega \in \mathbb{R}$ we set $\Lambda_\omega := \{\lambda \in \mathbb{C} : Re\lambda > \omega\}$ and let ω_0 denote the growth bound of $(T(t))_{t\ge0}$.

Lemma 7. *Let $(T(t))_{t\ge0}$ be a bi–continuous semigroup on X. Then the following properties hold.*

(a) Let $\lambda \in \mathbb{C}$ and $a \ge 0$. Then $R_a(\lambda) \in \mathcal{L}(X)$ and

$$R(\lambda) := \lim_{a\to\infty} R_a(\lambda) \tag{3}$$

for $\lambda \in \Lambda_{\omega_0}$ exists with respect to the operator norm and satisfies the estimate

$$\|R(\lambda)\|_{\mathcal{L}(X)} \leq \frac{M}{Re\lambda - \omega}$$

for all $\lambda \in \Lambda_\omega$, $\omega > \omega_0$, and some constant $M \geq 1$.

(b) For every $x \in X$ we have

$$\tau\text{-}\lim_{\lambda \to \infty} \lambda R(\lambda)x = x.$$

Proof. (a) Let $a \geq 0$, $\lambda \in \mathbb{C}$, and $x \in X$. Since $(X, \tau)'$ is norming for $(X, \|\cdot\|)$, we have with $\Phi := \{\phi \in (X,\tau)' : \|\phi\|_{(X,\|\cdot\|)'} \leq 1\}$ that

$$\|R_a(\lambda)x\| = \sup_{\phi \in \Phi} |\int_0^a e^{-\lambda t} < T(t)x, \phi > dt| \leq \frac{M}{Re\lambda - \omega}\|x\|.$$

Therefore assertion (a) holds.
(b) Let $x \in X, p \in P_\tau$, and $\epsilon > 0$. There exists $\delta_\epsilon > 0$ such that $0 \leq t < \delta_\epsilon$ implies $p(T(t)x - x) < \epsilon$. Thus, we have

$$\begin{aligned}
p(\lambda R(\lambda)x - x) &= p\left(\int_0^\infty \lambda e^{-\lambda t} T(t)x dt - \int_0^\infty \lambda e^{-\lambda t} x dt\right) \\
&\leq p\left(\int_0^{\delta_\epsilon} \lambda e^{-\lambda t}(T(t)x - x)dt\right) + p\left(\int_{\delta_\epsilon}^\infty \lambda e^{-\lambda t}(T(t)x - x)dt\right) \\
&=: T_1 + T_2 \ .
\end{aligned}$$

For the term T_2 we obtain with Φ as above that

$$T_2 \leq \sup_{\phi \in \Phi} \int_{\delta_\epsilon}^\infty \lambda e^{-\lambda t} | < T(t)x - x, \phi > |dt \leq \|x\| \left[M \frac{\lambda}{\lambda - \omega} e^{(\omega - \lambda)\delta_\epsilon} + e^{-\lambda \delta_\epsilon}\right],$$

which converges to zero as λ tends to infinity. For the term T_1 we obtain

$$T_1 \leq \int_0^{\delta_\epsilon} \lambda e^{-\lambda t} p(T(t)x - x)dt \leq \epsilon \int_0^\infty \lambda e^{-\lambda t} dt = \epsilon,$$

which concludes the proof. □

Let $(T(t))_{t \geq 0}$ be a bi–continuous semigroup on X and ω_0 its growth bound. To the operators $(R(\lambda))_{\lambda \in \Lambda_{\omega_0}}$ we can now associate an operator whose resolvent coincides with $(R(\lambda))_{\lambda \in \Lambda_{\omega_0}}$.

Proposition 8. *The family of operators $(R(\lambda))_{\lambda \in \Lambda_{\omega_0}}$ is the resolvent of a closed linear operator.*

Proof. Let $\lambda \neq \mu \in \Lambda_{\omega_0}$. We assume without loss of generality that $\lambda > \mu$ and obtain

$$
\begin{aligned}
&\frac{R(\mu)x - R(\lambda)x}{\lambda - \mu} \\
&\quad = \int_0^\infty e^{(\mu-\lambda)t} dt\ R(\mu)x - \int_0^\infty \frac{e^{(\mu-\lambda)t}}{(\lambda-\mu)} e^{-\mu t} T(t)x\ dt \\
&\quad = \int_0^\infty e^{(\mu-\lambda)t} \int_0^\infty e^{-\mu s} T(s)x\ ds\ dt - \left[\frac{e^{(\mu-\lambda)t}}{\lambda-\mu} \int_0^t e^{-\mu s} T(s)x\ ds \right]_{t=0}^{t=\infty} \\
&\qquad - \int_0^\infty e^{(\mu-\lambda)t} \int_0^t e^{-\mu s} T(s)x\ ds\ dt
\end{aligned}
$$

$$
\begin{aligned}
&= \int_0^\infty e^{(\mu-\lambda)t} \int_t^\infty e^{-\mu s} T(s)x\ ds\ dt \\
&= \int_0^\infty e^{-\lambda t} \int_0^\infty e^{-\mu s} T(s)T(t)x\ ds\ dt \\
&= R(\lambda)R(\mu)x
\end{aligned}
$$

for all $x \in X$. Therefore, $(R(\lambda))_{\lambda \in \Lambda_{\omega_0}}$ is a pseudoresolvent by [EN00, Ch. III, Def. 4.3]. By Lemma 7(b) we obtain the injectivity of the operators $R(\lambda)$. Applying [EN00, Ch. III, Prop. 4.6] we conclude the proof. □

We observe that, by Proposition 8, the map $\Lambda_{\omega_0} \ni \lambda \mapsto R(\lambda) \in \mathcal{L}(X)$ is holomorphic and

$$\frac{d^k}{d\lambda^k} R(\lambda)x = (-1)^k k! R(\lambda)^{k+1} x \tag{4}$$

for all $x \in X, k \in \mathbb{N}$, and $\lambda \in \Lambda_{\omega_0}$ (see [EN00, Ch. IV, Prop. 1.3]).

The above observations allow the definition of the generator of a bi–continuous semigroup.

Definition 9. *The* **generator** $A : D(A) \subseteq X \longrightarrow X$ *of a bi–continuous semigroup* $(T(t))_{t\geq 0}$ *on* X *is the unique operator on* X *such that its resolvent* $R(\lambda, A)$ *is*

$$R(\lambda, A)x = \int_0^\infty e^{-\lambda t} T(t)x dt \tag{5}$$

for all $\lambda \in \Lambda_{\omega_0}$ *and* $x \in X$.

As a first consequence, we obtain that these generators satisfy the Hille–Yosida estimates.

Proposition 10. *Let* $(A, D(A))$ *be the generator of a bi–continuous semigroup* $(T(t))_{t\geq 0}$ *of type* ω *on* X. *Then we have*

$$\frac{d^k}{d\lambda^k} R(\lambda, A)x = (-1)^k \int_0^\infty t^k e^{-\lambda t} T(t)x dt \tag{6}$$

for all $x \in X, k \in \mathbb{N}$ and $\lambda \in \Lambda_{\omega_0}$. In particular, there exists for each $\omega > \omega_0$ a constant $M \geq 1$ such that

$$\|R(\lambda, A)^k\| \leq \frac{M}{(Re\lambda - \omega)^k}$$

for all $k \in \mathbb{N}$ and $\lambda \in \Lambda_\omega$.

Proof. Let $x \in X$, $\lambda \in \Lambda_\omega$, $\omega > \omega_0$, and $\Phi := \{\phi \in (X,\tau)' : \|\phi\|_{X'} \leq 1\}$. Since the space $(X,\tau)'$ is norming for $(X, \|\cdot\|)$, for every $\mu \in \Lambda_\omega$ we have

$$\begin{aligned} &\left\| \frac{R(\mu)x - R(\lambda)x}{\mu - \lambda} + \int_0^\infty te^{-\lambda t} T(t)x dt \right\| \\ &\qquad \leq \sup_{\phi\in\Phi} \int_0^\infty \left| \frac{e^{-\mu t} - e^{-\lambda t}}{\mu - \lambda} + te^{-\lambda t} \right| | < T(t)x, \phi > | dt \qquad (7) \\ &\qquad \leq M\|x\| \int_0^\infty \left| \frac{e^{-\mu t} - e^{-\lambda t}}{\mu - \lambda} + te^{-\lambda t} \right| e^{\omega t} dt, \end{aligned}$$

which converges to zero as μ tends to λ as a consequence of Lebesgue's dominated convergence theorem. Via induction we obtain the desired equality. Further, (4) and (6) imply

$$\|R(\lambda)^k x\| \leq \sup_{\phi\in\Phi} \frac{1}{(k-1)!} \int_0^\infty t^{k-1} |e^{-\lambda t} < T(t)x, \phi > | dt \leq \frac{M}{(Re\lambda - \omega)^k} \|x\| \qquad (8)$$

for all $x \in X$ and $\lambda \in \Lambda_\omega$. □

Proposition 10 says that generators of bi–continuous semigroups are *Hille–Yosida* operators (see [EN00, Ch. II, Def. 3.22]).
Following, e.g., [Yos74, Ch. IX, Sec. 3], [Kōm68, p. 260], [DS57, Ch. VIII, Def. 1.6], [EN00, Ch. II, Def. 1.2], another way to introduce the generator $(A_\tau, D(A_\tau))$ of a bi–continuous semigroup $(T(t))_{t\geq 0}$ would be to define

$$\begin{aligned} A_\tau x &:= \tau\text{–}\lim_{t\searrow 0} \frac{T(t)x - x}{t} \\ \text{for all } x \in D(A_\tau) &:= \{x \in X : \tau\text{–}\lim_{t\searrow 0} \frac{T(t)x - x}{t} \text{ exists in } X\}. \end{aligned} \qquad (9)$$

In the setting of bi–continuous semigroups, the operators $(A_\tau, D(A_\tau))$ and $(A, D(A))$ coincide. To see this, we first look at the following fundamental properties of the operator $(A_\tau, D(A_\tau))$ stated in [Kōm68, Prop. 1.2, 1.4].

Proposition 11. *Let $(T(t))_{t\geq 0}$ be a bi–continuous semigroup on X and $(A_\tau, D(A_\tau))$ as above. Then the following properties hold.*

(a) If $x \in D(A_\tau)$, then $T(t)x \in D(A_\tau)$ for all $t \geq 0$, $T(t)x$ is continuously differentiable in t with respect to the topology τ, and

$$\frac{d}{dt} T(t)x = A_\tau T(t)x = T(t)A_\tau x$$

for all $t \geq 0$.

(b) An element $x \in X$ belongs to $D(A_\tau)$ and satisfies $A_\tau x = y$ if and only if

$$T(t)x - x = \int_0^t T(s)yds \tag{10}$$

for all $t \geq 0$.

(c) The operator $(A_\tau, D(A_\tau))$ is **bi–closed**, *i.e., for all sequences $(x_n)_{n\in\mathbb{N}} \subseteq D(A_\tau)$ with $(x_n)_{n\in\mathbb{N}}$ and $(A_\tau x_n)_{n\in\mathbb{N}}$ $\|\cdot\|$–bounded, $x_n \xrightarrow{\tau} x \in X$ and $A_\tau x_n \xrightarrow{\tau} y \in X$ we have $x \in D(A_\tau)$ and $A_\tau x = y$.*

Theorem 12. *Let $(T(t))_{t\geq 0}$ be a bi–continuous semigroup on X with generator $(A, D(A))$ and define $(A_\tau, D(A_\tau))$ as in (9). Then $A = A_\tau$.*

Proof. We show first that $A \subset A_\tau$. For $x \in X$ and $\lambda \in \Lambda_{\omega_0}$ we have

$$\frac{T(h) - Id}{h} R(\lambda, A)x = \frac{(e^{\lambda h} - 1)}{h} \int_0^\infty e^{-\lambda t} T(t)xdt - \frac{e^{\lambda h}}{h} \int_0^h e^{-\lambda t} T(t)xdt,$$

which converges to $\lambda R(\lambda, A)x - x = AR(\lambda, A)x$ as $h \searrow 0$. Thus $A \subset A_\tau$.
On the other hand, for $x \in D(A_\tau)$ we define $y := (\lambda - A_\tau)x$. By Proposition 11 we have

$$A_\tau \int_0^\infty e^{-\lambda t} T(t)xdt = \int_0^\infty e^{-\lambda t} T(t)A_\tau xdt.$$

Therefore, we obtain

$$R(\lambda, A)y = (\lambda - A_\tau) \int_0^\infty e^{-\lambda t} T(t)xdt = (\lambda - A_\tau)R(\lambda, A)x = x,$$

and hence $A_\tau \subset A$. □

Proposition 13. *Let $(A, D(A))$ be the generator of a bi–continuous semigroup $(T(t))_{t\geq 0}$ of type ω on X. Then the following properties hold.*

(a) The generator $(A, D(A))$ is bi–closed.

(b) The domain of A is bi–dense in X, i.e., for every $x \in X$ there exists a $\|\cdot\|$–bounded sequence $(x_n)_{n\in\mathbb{N}} \subseteq D(A)$ which is τ–convergent to x.

(c) The subspace $X_0 := \overline{D(A)}^{\|\cdot\|} \subseteq X$ is $(T(t))_{t\geq 0}$–invariant and $(T(t)_{|X_0})_{t\geq 0}$ is the strongly continuous semigroup on X_0 generated by the part of A in X_0.

Proof. Assertion (a) follows directly from Proposition 11(c) and Theorem 12.
(b) Let $x \in X$. By Lemma 7 the sequence $(x_n)_{n\in\mathbb{N}} \subseteq D(A)$ defined as

$$x_n := \begin{cases} nR(n, A)x & \text{if } n > \omega\ , \\ 0 & else, \end{cases}$$

is $\|\cdot\|$–bounded and τ–convergent to x.
Assertion (c) is a consequence of Proposition 10 and [EN00, Ch. II, Cor. 3.21]. $\square$

Combining Formulas (4) and (6) we obtain the following additional properties of the powers of the resolvent operators $R(\lambda, A)$, $\lambda \in \Lambda_{\omega_0}$ (cf. [Cer94, Prop. 3.5] in the context of her "weakly continuous semigroups on $C_{ub}(H)$").

Proposition 14. *Let $(T(t))_{t\geq 0}$ be a bi–continuous semigroup of type ω on X with generator $(A, D(A))$. Then the operators $\lambda^k R(\lambda, A)^k$, $\lambda > \omega_0$, $k \in \mathbb{N}$, have the following properties.*

(a) τ–$\lim_{\lambda\to\infty} \lambda^k R(\lambda, A)^k x = x$ for all $x \in X$ and $k \in \mathbb{N}$.

(b) Let $(x_n)_{n\in\mathbb{N}} \subseteq X$ be a $\|\cdot\|$–bounded sequence which is τ–convergent to $x \in X$. Let $\alpha > \omega$. Then
$$\tau\text{–}\lim_{n\to\infty} (\lambda - \alpha)^k R(\lambda, A)^k (x_n - x) = 0 \tag{11}$$
uniformly for $k \in \mathbb{N}$ and $\lambda > \alpha$.

Proof. (a) Without loss of generality we suppose that $(T(t))_{t\geq 0}$ is $\|\cdot\|$–bounded. Let $x \in X, k \in \mathbb{N}, \epsilon > 0$, and $p \in P_\tau$. There exists $\delta_\epsilon > 0$ such that $0 \leq t < \delta_\epsilon$ implies $p(T(t)x - x) < \frac{\epsilon}{2}$. Formula (4) and (6) imply that there exists $\lambda_0 > \omega_0$ such that
$$\begin{aligned}
&p(\lambda^k R(\lambda, A)^k x - x)\\
&\qquad \leq \frac{\epsilon}{2}\frac{\lambda^k}{(k-1)!}\int_0^{\delta_\epsilon} t^{k-1}e^{-\lambda t}dt + (1+M)\|x\|\frac{\lambda^k}{(k-1)!}\int_{\delta_\epsilon}^{\infty} t^{k-1}e^{-\lambda t}dt\\
&\qquad \leq \frac{\epsilon}{2} + (1+M)\|x\|\left(\frac{\lambda^k}{(k-1)!}\delta_\epsilon^{k-1}e^{-\lambda\delta_\epsilon} + \ldots + \lambda\delta_\epsilon e^{-\lambda\delta_\epsilon} + e^{-\lambda\delta_\epsilon}\right)\\
&\qquad \leq \epsilon
\end{aligned}$$
for all $\lambda \geq \lambda_0$ and some constant $M \geq 1$.
(b) Let $\epsilon > 0$, $p \in P_\tau$, and $\alpha > \omega$. By Proposition 4(b) we obtain that the rescaled semigroup $(e^{-\alpha t}T(t))_{t\geq 0}$ is globally bi–equicontinuous. Hence, by Formula (4) and (6), there exists $n_0 \in \mathbb{N}$ such that
$$\begin{aligned}
p((\lambda-\alpha)^k R(\lambda, A)^k (x_n - x)) &\leq \frac{(\lambda-\alpha)^k}{(k-1)!}\int_0^\infty t^{k-1}e^{-(\lambda-\alpha)t}p(e^{-\alpha t}T(t)(x_n - x))dt\\
&\leq \epsilon
\end{aligned}$$
for all $n \geq n_0$ and uniformly for $k \in \mathbb{N}$ and $\lambda > \alpha$. $\square$

To complete our basic considerations we recall that a Hille–Yosida operator is always the generator of an integrated semigroup with the following additional properties (see [ABHN01, Section 3.3], [HN93]).

Proposition 15. *Let A be a Hille–Yosida operator of type ω on X and denote $X_0 = \overline{D(A)}^{\|\cdot\|}$. Then there exists an integrated semigroup F on X possessing the following properties.*

(a) The map $\mathbb{R}_+ \ni t \mapsto F(t)x \in X$ is continuously differentiable with respect to the norm of X for all $x \in X_0$. The operator family $(F'(t)_{|X_0})_{t\geq 0}$ is a strongly continuous semigroup on X_0.

(b) The integrated semigroup F is given by

$$F(t) = \lim_{k\to\infty} (-1)^k(k+1) \int_{\frac{k}{t}}^{\infty} s^k R(s,A)^{k+2} ds \tag{12}$$

for all $t > 0$, $F(0) = 0$, and $\|F(t+h) - F(t)\| \leq M \int_t^{t+h} e^{\omega r} dr$ for all $t, h \geq 0$.

4 A generation theorem

We are now able to state the desired relation between bi–continuous semigroups, integrated semigroups and Hille–Yosida operators in form of a generalized Hille–Yosida theorem for bi–continuous semigroups. This theorem puts in a general framework the Hille–Yosida type theorems due to S. Cerrai for *weakly continuous semigroups* ([Cer94]), due to J. R. Dorroh and J. W. Neuberger for *semigroups induced by flows* ([DN96], see Section 5), and due to O. Bratelli and D. W. Robinson for *adjoint semigroups* ([BR79, Thm. 3.1.10]).
Let X satisfy Assumptions 1.

Theorem 16. *Let $A : D(A) \subseteq X \longrightarrow X$ be a linear operator and denote $X_0 = \overline{D(A)}^{\|\cdot\|}$. Then the following assertions are equivalent.*

(a) $(A, D(A))$ generates a bi–continuous semigroup $(T(t))_{t\geq 0}$ (of type ω) on X.

(b) $(A, D(A))$ is a bi–densely defined Hille–Yosida operator of type ω, and the family $\{(s-\alpha)^k R(s,A)^k : k \in \mathbb{N}, s > \alpha\}$ is bi–equicontinuous for every $\alpha > \omega$.

(c) There exists an integrated semigroup F satisfying the following conditions.

(i) $F(\cdot)x \in C^1(\mathbb{R}_+, (X, \|\cdot\|))$ for all $x \in D(A)$, $(F'(t)_{|X_0})_{t\geq 0}$ exists and is a strongly continuous semigroup on X_0.

(ii) $F(\cdot)x \in C^1(\mathbb{R}_+, (X, \tau))$ for all $x \in X$.

(iii) The operator family $(F'(t))_{t\geq 0}$ is locally bi–equicontinuous.

(iv) $F'(t)$, $t \geq 0$, is exponentially bounded on X.

(v) For all $\lambda > \omega$ and $x \in X$ we have

$$R(\lambda, A)x = \int_0^\infty e^{-\lambda t} F'(t) x dt.$$

Proof. *(b)* $\Rightarrow$ *(c)* First, we assume $\omega < 0$.
As a consequence of Proposition 15 there exists an integrated semigroup F satisfying assertion (i).
To prove (ii) we consider, for $x \in X$ and $t \geq 0$, the sequence $(D_m(x,t))_{m \in \mathbb{N}} \subseteq X$ defined as

$$D_m(x,t) := \frac{F(t+1/m)x - F(t)x}{1/m}$$

for all $m \in \mathbb{N}$. As a consequence of Proposition 15(b) this sequence is $\|\cdot\|$–bounded. It remains to prove that it is a τ–Cauchy sequence. Let $x \in X, \epsilon > 0, p \in P_\tau$. By the assumptions there exists a $\|\cdot\|$–bounded sequence $(x_n)_{n \in \mathbb{N}} \subseteq D(A)$ which is τ–convergent to x and there exists $n_0 \in \mathbb{N}$ such that

$$p(s^k R(s,A)^k (x - x_n)) \leq \frac{\epsilon}{3} \tag{13}$$

for all $n \geq n_0$ and uniformly for $s \geq 0$ and $k \in \mathbb{N}$. Applying estimate (13) and Proposition 15(b) we obtain

$$\begin{aligned} p\left(D_m(x - x_n, t)\right) &= p\left(\lim_{k \to \infty} m[(k+1) \int_{\frac{k}{t}}^{\infty} s^k R(s,A)^{k+2}(x - x_n) ds \right. \\ &\qquad \left. -(k+1) \int_{\frac{k}{t+1/m}}^{\infty} s^k R(s,A)^{k+2}(x - x_n) ds]\right) \\ &= p\left(m \lim_{k \to \infty} (k+1) \left[\int_{\frac{k}{t+1/m}}^{\frac{k}{t}} s^k R(s,A)^{k+2}(x - x_n) ds\right]\right) \\ &\leq \frac{\epsilon}{3} m \lim_{k \to \infty} (k+1) \left[-\frac{1}{s}\right]_{s=\frac{k}{t+1/m}}^{\frac{k}{t}} \\ &= \frac{\epsilon}{3} \lim_{k \to \infty} (1 + \frac{1}{k}) \\ &= \frac{\epsilon}{3} \end{aligned} \tag{14}$$

for all $n \geq n_0$ and uniformly for $m \in \mathbb{N}$ and $t \geq 0$.
Since (i) is valid, $(D_m(x_n,t))_{m \in \mathbb{N}}$ is a τ–Cauchy sequence for all $n \in \mathbb{N}$. Therefore, there exists $m_0 \in \mathbb{N}$ such that

$$p(D_m(x,t) - D_l(x,t)) \leq p(D_m(x - x_{n_0}, t)) + p(D_m(x_{n_0}, t) - D_l(x_{n_0}, t)) + p(D_l(x - x_{n_0}, t)) \leq \epsilon$$

for all $m, l \geq m_0$. Since (X, τ) is sequentially complete on $\|\cdot\|$–bounded sets, the map $(t \mapsto F(t)x)$ is differentiable with respect to τ for all $x \in X$.

Before proving that its derivative is continuous, we show assertion (iii).
Clearly, by estimate (14), the operator family $\{F'(t) : t \geq 0\}$ is globally bi–equicontinuous. Next, we show the τ–continuity of $F'(\cdot)x$ for all $x \in X$. To that purpose, let $x \in X$, $t_0 \geq 0$, $\epsilon > 0$, and $p \in P_\tau$. Further, let $(x_n)_{n\in\mathbb{N}} \subseteq D(A)$ be a $\|\cdot\|$–bounded sequence which is τ–convergent to $x \in X$. Since $F'(\cdot)x$ is τ–continuous for all $x \in D(A)$, we obtain that there exists $\delta_\epsilon > 0$ depending on n such that $|t - t_0| \leq \delta_\epsilon$ implies

$$p(F'(t)x_n - F'(t_0)x_n) \leq \frac{\epsilon}{3}.$$

Again with estimate (14) there exists $n_0 \in \mathbb{N}$ such that $|t - t_0| \leq \delta_\epsilon$ implies

$$\begin{aligned} p(F'(t)x - F'(t_0)x) &\leq p(F'(t)(x - x_{n_0})) + p(F'(t)x_{n_0} - F'(t_0)x_{n_0}) \\ &\quad + p(F'(t_0)(x_{n_0} - x)) \\ &\leq \epsilon. \end{aligned}$$

Therefore, assertion (ii) is shown.
Property (iv) holds by Proposition 15(b) and the fact that $(X,\tau)'$ is norming for $(X, \|\cdot\|)$. In fact, for $\Phi := \{\phi \in (X,\tau)' : \ \|\phi\| \leq 1\}$ we have

$$\begin{aligned} \|F'(t)x\| &= \sup_{\phi\in\Phi} | < \tau\text{–} \lim_{m\to\infty} D_m(x,t), \phi > | \\ &= \sup_{\phi\in\Phi} | \lim_{m\to\infty} < D_m(x,t)x, \phi > | \\ &\leq M\|x\| \lim_{m\to\infty} m \int_t^{t+1/m} e^{\omega s} ds \\ &= Me^{\omega t}\|x\| \end{aligned}$$

for all $x \in X$.
To prove (v), we note first that the sequential completeness of (X,τ) on $\|\cdot\|$–bounded sets implies that

$$\tau - \int_0^a e^{-\lambda t} F'(t)x dt$$

exists for all $a \geq 0, \lambda > \omega$, and $x \in X$.
Since $(X,\tau)'$ is norming for $(X, \|\cdot\|)$, we obtain as in the proof of Lemma 7(a) that

$$\tau - \int_0^\infty e^{-\lambda t} F'(t)x dt$$

exists.

It remains to prove that

$$R(\lambda, A)x = \int_0^\infty e^{-\lambda t} F'(t)x dt \tag{15}$$

for all $x \in X$ and $\lambda > \omega$. To show this, let $x \in X$, $\epsilon > 0$. By assumption there exists a $\|\cdot\|$–bounded sequence $(x_n)_{n\in\mathbb{N}} \subseteq D(A)$ and $n_0 \in \mathbb{N}$ such that $p(x - x_n) \leq \frac{\epsilon}{2}$ and hence,

$$p(R(\lambda, A)(x - x_n)) \leq \frac{\epsilon}{2}$$

for all $n \geq n_0$. Since F is an integrated semigroup, and by (iii), we obtain

$$\begin{aligned}
& p\left(R(\lambda, A)x - \int_0^\infty e^{-\lambda t} F'(t)x dt\right) \\
& \leq p(R(\lambda, A)(x - x_{n_0})) + p\left(\int_0^\infty e^{-\lambda t} F'(t)(x_{n_0} - x) dt\right) \\
& \leq \epsilon.
\end{aligned}$$

Therefore, assertion (v) holds.
By a rescaling argument we obtain the desired properties (i)–(v) for arbitrary $\omega \in \mathbb{R}$.
$(c) \Rightarrow (a)$ We define first

$$T(t)x := F'(t)x = \tau\text{--}\lim_{h \searrow 0} \frac{F(t+h)x - F(t)x}{h}\ , \quad x \in X, t \geq 0.$$

By assumption (c) it remains to show that the semigroup law on X holds. By (i) the semigroup law already holds on X_0. Further, let $x \in X$ and $t, s \geq 0$. By the same arguments as in the proof of Lemma 7(b) and equality (iv) one can see that $\tau - \lim_{\lambda \to \infty} \lambda R(\lambda, A)x = x$. Hence, there exists a $\|\cdot\|$–bounded sequence $(x_n)_{n \in \mathbb{N}} \subseteq D(A)$ which is τ–convergent to x. Since $(x_n)_{n \in \mathbb{N}} \subseteq D(A)$, we have

$$T(t+s)x_n = T(t)T(s)x_n$$

for all $t, s \geq 0$ and $n \in \mathbb{N}$, and

$$\tau\text{--}\lim_{n \to \infty} T(t+s)x_n = T(t+s)x.$$

Since $(T(s)x_n)_{n \in \mathbb{N}}$ is $\|\cdot\|$–bounded and τ–convergent, we also obtain

$$\tau\text{--}\lim_{n \to \infty} T(t)T(s)x_n = T(t)T(s)x.$$

The topology τ is Hausdorff, therefore the limit is unique and the semigroup law holds for $(T(t))_{t \geq 0}$. By the same way we obtain $T(0) = Id$.
$(a) \Rightarrow (b)$ If we have a bi–continuous semigroup $(T(t))_{t \geq 0}$, assertion (b) follows directly from Proposition 10, Proposition 13 and Proposition 14. □

In the next section we show how this theorem can be applied. Other applications of Theorem 16 can be found in [Küh01a].

5 An application to semigroups induced by flows

Around 1970, J. W. Neuberger [Neu72], [Neu73] and D. L. Lovelady [Lov75] started a theory of nonlinear, jointly continuous semigroups (flows) on a metric space Ω in terms of their *Lie generator*. In a certain analog to ideas of S. Lie, they define an associated linear semigroup, the *semigroup induced by the flow*, on the Banach space X of bounded, continuous functions on Ω. This semigroup, in general, is not strongly continuous for the supremum norm. However, there exists a locally convex topology on X such that strong continuity holds. To treat this class of semigroups, J. W. Neuberger proved a generation theorem and a representation formula.

20 years later, in joint work by J. R. Dorroh and J. W. Neuberger [DN93], [DN96] a complete characterization of the Lie generator of a jointly continuous flow on a complete, separable, metric space Ω was established. The main step towards these results was, using a result by F. D. Sentilles [Sen72], to introduce the "strict" locally convex topology β on the space $C_b(\Omega)$ of bounded, continuous functions on Ω, for which the associated linear semigroup becomes strongly continuous. We remark that by a result of F. D. Sentilles [Sen72] a sequence in $C_b(\Omega)$ converges with respect to β if and only if it is $\|\cdot\|$–bounded and convergent with respect to the compact–open topology τ_c.

In the following we show, using the norm and the compact–open topology τ_c on $C_b(\Omega)$, that these Dorroh–Neuberger semigroups fit into the theory of bi–continuous semigroups. Further, we give a short proof of the generation result stated in [DN96] using our Theorem 16.

Clearly, the space $C_b(\Omega)$ endowed with $\|\cdot\|_\infty$ and τ_c satisfies Assumptions 1.

We now give some basic definitions taken from [DN96] and [Nag86, Ch. B.II, Sec. 3].

Definition 17. *a) A* **jointly continuous flow** *(or semigroup of jointly continuous transformations) on a topological space Ω is a mapping $\phi : (t, x) \mapsto \phi_t(x)$ from $\mathbb{R}_+ \times \Omega \longrightarrow \Omega$ such that*

(i) $\phi_0 = id$, $\phi_t\phi_s = \phi_{t+s}$ for all $t, s \geq 0$.

(ii) The mapping $(t, x) \mapsto \phi_t(x)$ is jointly continuous from $\mathbb{R}_+ \times \Omega$ into Ω.

b) The **Lie generator** *of a jointly continuous flow ϕ is the linear operator A in $C_b(\Omega)$ consisting of all ordered pairs (f, g) such that $f, g \in C_b(\Omega)$ and*

$$g(x) = \lim_{t \searrow 0} \frac{f(\phi_t(x)) - f(x)}{t}$$

for all $x \in \Omega$.

c) A linear operator $(A, D(A))$ on $C_b(\Omega)$ is called a derivation if $f, g \in D(A)$ implies $fg \in D(A)$ and $A(fg) = fAg + gAf$.

d) The (linear) semigroup $(T(t))_{t\geq 0}$ on $C_b(\Omega)$ defined as

$$T(t)f := f \circ \phi_t \tag{16}$$

for all $f \in C_b(\Omega)$ and $t \geq 0$ is called the **semigroup induced by the (jointly continuous) flow ϕ**.

It is clear that $\{T(t): t \geq 0\} \subseteq \mathcal{L}(C_b(\Omega))$ and $\|T(t)\| \leq 1$ for all $t \geq 0$.
Moreover, $(T(t))_{t\geq 0}$ is a semigroup which, if Ω is compact, is strongly continuous on $(C_b(\Omega), \|\cdot\|_\infty)$ (see [Nag86, Ch. B.II, Lemma 3.2]). In the general case, using the joint continuity of the flow ϕ, we easily obtain the following (cf.[DN93, Thm. 2.1, 2.2]).

Proposition 18. *Let ϕ be a jointly continuous flow on Ω and $(T(t))_{t\geq 0}$ be the semigroup induced by ϕ on $C_b(\Omega)$. Then $(T(t))_{t\geq 0}$ is a bi–continuous contraction semigroup with respect to the topology τ_c.*

We note that, by Theorem 12, the generator $(A, D(A))$ of $(T(t))_{t\geq 0}$ is given by

$$\begin{aligned} Af &= \tau_c\text{–}\lim_{t\searrow 0} \frac{T(t)f - f}{t} \\ \text{for all } f \in D(A) &= \{f \in C_b(\Omega):\ \tau_c\text{–}\lim_{t\searrow 0} \frac{T(t)f - f}{t} \text{ exists in } C_b(\Omega)\}. \end{aligned} \tag{17}$$

It follows from the next proposition that this generator $(A, D(A))$ coincides with the Lie generator of the flow ϕ (see [DN96, Prop. 2.4]).

Proposition 19. *Let ϕ be a jointly continuous flow and let $f, g \in C_b(\Omega)$. Then the following assertions are equivalent.*

(i) $g(x) = \lim_{t\searrow 0} \frac{f(\phi_t(x)) - f(x)}{t}$ *for all $x \in \Omega$.*

(ii) $g = \tau_c\text{–}\lim_{t\searrow 0} \frac{f\circ\phi_t - f}{t}$.

We can now apply Theorem 16 to obtain a generation theorem for bi–continuous contraction semigroups induced by jointly continuous flows. Moreover, by Proposition 19, this yields a characterization of the Lie generator of a jointly continuous flow thereby reproving the results in [DN96, Thm. 3.1, 3.2].

Theorem 20. *Let $A : D(A) \subseteq C_b(\Omega) \longrightarrow C_b(\Omega)$ be a linear operator. Then the following assertions are equivalent.*

(a) The operator A is the Lie generator of a jointly continuous flow ϕ on Ω.

(b) $(A, D(A))$ is a derivation and generates a bi–continuous contraction semigroup $(T(t))_{t\geq 0}$ on $C_b(\Omega)$ with respect to τ_c which is induced by a jointly continuous flow ϕ.

(c) $(A, D(A))$ is a bi–densely defined Hille–Yosida operator of type 0 and a derivation, and the family $\{(s-\alpha)^k R^k(s, A):\ k \in \mathbb{N}, s \geq \alpha\}$ is bi–equicontinuous for every $\alpha > 0$.

Proof. $(a) \Rightarrow (b)$ It is easy to see that the Lie generator of a jointly continuous flow ϕ on Ω is a derivation. Defining $T(t)f := f \circ \phi_t$ for all $f \in C_b(\Omega)$ and $t \geq 0$, we obtain, by Proposition 18, that $(T(t))_{t\geq 0}$ is a bi–continuous contraction semigroup on $C_b(\Omega)$ with

respect to τ_c. Let $(B, D(B))$ denote its generator. Combining (17) and Proposition 19, we have $A = B$. Therefore, (b) holds.
$(b) \Rightarrow (c)$ If (b) holds, Theorem 16 leads to assertion (c).
$(c) \Rightarrow (a)$ By Theorem 16 we obtain that there exists a bi–continuous semigroup $(T(t))_{t\geq 0}$ on $C_b(\Omega)$ with respect to τ_c. We show that each operator $T(t)$ is an algebra homomorphism, i.e., we have

$$T(t)(fg) = T(t)f \cdot T(t)g \tag{18}$$

for all $t > 0$ and $f, g \in C_b(\Omega)$. Indeed, for $f, g \in D(A)$ we define a function $\eta : [0, t] \longrightarrow C_b(\Omega)$ by

$$\eta(s) := T(t - s)[T(s)f \cdot T(s)g]$$

which is differentiable with respect to τ_c by Proposition 11. Then $\eta(0) = T(t)(f \cdot g)$ and $\eta(t) = T(t)f \cdot T(t)g$. Since A is a derivation, we have $\eta'(s) = 0$ for $s \in [0, t]$. Hence, $\eta(0) = \eta(t)$ and $T(t)(fg) = T(t)f \cdot T(t)g$. The bi–density of $D(A)$ yields equation (18) for all $f, g \in C_b(\Omega)$.
Since $(T(t))_{t\geq 0}$ consists of τ_c–continuous algebra homomorphisms, there exists, by [DN96, Prop. 2.1], for each $x \in \Omega$ a unique $y \in \Omega$ such that $T(t)f(x) = f(y)$ for all $f \in C_b(\Omega)$. Thus, $(T(t))_{t\geq 0}$ is a contraction semigroup, and, for every $t \geq 0$, we can define a flow ϕ from Ω into Ω as

$$f(\phi_t(x)) := T(t)f(x)$$

for all $f \in C_b(\Omega), x \in \Omega$. This flow is jointly continuous from $\mathbb{R}_+ \times \Omega \to \Omega$ by Theorem 3.3 stated in [DN93]. Further, by Theorem 12, the generator of $(T(t))_{t\geq 0}$ is given by

$$Af = \tau_c\text{–}\lim_{t\searrow 0} \frac{f \circ \phi_t - f}{t},$$

and, by Proposition 19, this is equivalent to the fact that A is the Lie generator of the flow ϕ. This concludes the proof of the equivalent assertions. □

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