FEEDBACKS FOR NON–AUTONOMOUS REGULAR LINEAR SYSTEMS

ROLAND SCHNAUBELT

Abstract. We introduce non–autonomous well–posed and (absolutely) regular linear systems as quadrupels consisting of an evolution family and output, input and input–output maps subject to natural hypotheses. In the spirit of G. Weiss' work these maps are represented in terms of admissible observation and control operators (the latter in an approximative sense) in the time domain. In this setting the closed–loop system exists for a canonical class of 'admissible' feedbacks, and it inherits the absolute regularity and other properties of the given system. In particular, one can iterate feedbacks. We study a second order parabolic partial differential equation in non–divergence form with point control and observation in space dimension $n \geq 3$.

1. Introduction

As a motivation, we first look at the finite dimensional non–autonomous linear system

$$\begin{aligned} x'(t) &= A(t)x(t) + B(t)u(t), \qquad t \geq s \geq 0, \\ y(t) &= C(t)x(t), \qquad t \geq s \geq 0, \\ x(s) &= x_0, \end{aligned} \tag{1.1}$$

on the state space X with control operators $B(t) : U \to X$, observation operators $C(t) : X \to Y$, the control space U, and the observation space Y. Let $T(t,s)$, $t \geq s \geq 0$, be the evolution family (propagator) on X generated by $A(\cdot)$. Then the output of (1.1) with $u = 0$, the state of (1.1) with $x_0 = 0$, and the input–output operator of (1.1) are given by

$$\begin{aligned} (\Psi_s x_0)(t) &= C(t)T(t,s)x_0, \qquad \Phi_{t,s}u = \int_s^t T(t,\tau)B(\tau)u(\tau)d\tau, \\ (\mathbb{F}_s u)(t) &= C(t)\int_s^t T(t,\tau)B(\tau)u(\tau)d\tau, \qquad t \geq s. \end{aligned} \tag{1.2}$$

If one feeds back the output via $u(t) = \Delta(t)y(t)$, the resulting closed–loop system is described by the perturbed evolution equation

$$x'(t) = [A(t) + B(t)\Delta(t)C(t)]\,x(t), \quad t \geq s \geq 0, \qquad x(s) = x_0. \tag{1.3}$$

Of course, the function $x(t) = T_\Delta(t,s)x_0$ solves (1.3), where $A(t)+B(t)\Delta(t)C(t)$ generates $T_\Delta(t,s)$. This evolution family also satisfies the 'variation of constants formulas'

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I started this project together with Yuri Latushkin and Timothy Randolph from University of Missouri, and I profited greatly from many joint discussions.

$$T_\Delta(t,s)x = T(t,s)x + \int_s^t T(t,\tau)B(\tau)\Delta(\tau)C(\tau)T_\Delta(\tau,s)x\,d\tau, \tag{1.4}$$

$$T_\Delta(t,s)x = T(t,s)x + \int_s^t T_\Delta(t,\tau)B(\tau)\Delta(\tau)C(\tau)T(\tau,s)x\,d\tau \tag{1.5}$$

for $t \geq s$ and $x \in X$. The first equation is just the integrated version of (1.3). The second one occurs if we perturb T_Δ by $-B(t)\Delta(t)C(t)$. The formula (1.5) and related identities are in particular needed if one wants to repeat feedbacks or to deduce properties of the unperturbed system from the perturbed one as we do in Section 5.

If we pass to an infinite dimensional state space X, it is not clear anymore that (1.3) possesses differentiable solutions for 'many' initial values even if the Cauchy problem for $A(\cdot)$ is well–posed, cf. [7], [9, §VI.9], [11]. Nevertheless, as long as $B(t)$, $C(t)$, and $\Delta(t)$ are bounded operators, the formulas (1.2) still work and there is an evolution family T_Δ fulfilling (1.4) and (1.5). Thus $x(t) = T_\Delta(t,s)x_0$ is the 'mild' solution of (1.3), [7]. However, point or boundary control and observation lead to input and output operators $B(t) : U \to \overline{X}_t$ and $C(t) : \underline{X}_t \to Y$ for spaces $\underline{X}_t \subsetneqq X \subsetneqq \overline{X}_t$, where $C(t)$ usually is not closable, see e.g. [3], [18]. In order to solve (1.4) in this more general setting, one may restrict oneself to a class of 'admissible' observation and control operators – roughly speaking those for which the expressions (1.2) make sense. Then one is also faced with the question whether the operators $B(t)$ and $C(t)$ are again admissible for the perturbed evolution family T_Δ, which is necessary to verify (1.5) or to iterate feedbacks.

The resulting perturbation problem (1.3) generalizes the settings of both the Desch–Schappacher theorem (where $\Delta(t) = C(t) = I$) and the Miyadera theorem (where $\Delta(t) = B(t) = I$) from semigroup theory, [9, §III.3], [20]. In the control literature there is a rich perturbation theory for the autonomous case (i.e., $A(t) = A$, $B(t) = B$, $C(t) = C$, $\Delta(t) = \Delta$). Linear systems belonging to the *Pritchard–Salamon class*, [19], were exhaustively treated in [6]. G. Weiss introduced the larger class of *well–posed linear systems* in [29]–[32]. Here the semigroup T is given and the operators Φ, Ψ, and $\mathbb{F}$ are defined in an abstract way by certain algebraic relations. One can then construct admissible control and observation operators B and C and obtain formulas such as (1.2) if the system satisfies the rather weak hypothesis of *regularity*. Weiss established a powerful feedback theory for regular systems in the Hilbert space situation, [33]. We refer to Section 4, [3, §3.3], [22], and in particular to O. Staffan's monograph [26] for further information and literature.

For non–autonomous systems in variational form, there is the well–known approach due to J.L. Lions [18], see also [3, Chap.2]. In a general setting, D. Hinrichsen, B. Jacob, and A.J. Pritchard, [11], [13], [15], constructed an evolution family solving (1.4) for initial values x contained in a dense subspace $\underline{X}$ of X under rather weak assumptions covering autonomous regular systems. But (1.5) and the admissibility of the perturbed problem was only studied in [13] requiring stronger hypotheses of Pritchard–Salamon type.

In the present work we combine the direct approach of Hinrichsen, Jacob, and Pritchard with some of Weiss' ideas: In 2.6 we define 'Lebesgue extensions' of given observation operators $C(t)$, cf. [29], which allow to study (1.4) and (1.5) for all $x \in X$ and simplify several technical details of the proofs considerably. For similar reasons, we mostly work with *non–autonomous (absolutely) regular systems*, which are defined in the spirit of Weiss' work (see Definitions 3.6 and 3.10), as opposed to *admissible systems*, which have been used in [11], [13], [15] and are given directly by operators $B(t)$ and $C(t)$ (see Definition 3.8). In Theorem 2.7, Proposition 3.5, and Theorem 3.11 we represent a given regular system similar as in (1.2). It is known, see e.g. [28, Exa.6], that (1.3) can only be solved if the feedback is not 'too large'. We thus introduce *admissible* feedbacks in Definition 4.1, cf. [26, §7.1], [33, §3]. This property is quite flexible and also necessary for some of our results. Theorem 4.4 then establishes the existence of an absolutely regular closed–loop system for a given absolutely regular non–autonomous system with admissible time varying feedback.

However, the extension of Weiss' theory to the non–autonomous case is limited by two serious obstacles: One cannot apply transform methods and, in contrast to semigroups (see e.g. [2, Chap.V], [9, §II.5]), we do not have a general extrapolation theory for evolution families. The first point excludes the use of transfer functions (being crucial in [33]), but leads us to arguments which work in a Banach space setting (as in [26, Chap.7]). The second point forces us to employ approximation formulas for the representation of control systems in Proposition 3.5. A similar problem occurs in the computation of the feedback system and in the context of (1.5). These matters are discussed after Theorem 4.4.

In Section 5 the closed–loop system is studied in greater detail. We derive analogues of (1.4) and (1.5) for the operators given in (1.2). It is also seen that the feedback system is controllable (or observable) if and only if the given system is controllable (or observable). Moreover, iterated feedbacks behave as one would expect. We further prove that the feedback system inherits the exponential dichotomy (or stability) of T. Results of this type are important tools to investigate the long term behaviour of evolution equations, but have not yet been obtained for perturbations mapping from a subspace of X to a larger space. Finally, the equivalence of internal stability with input–output stability, detectability, and stabilizability is established, extending results from [5], [6], [21] to the present setting. As a sample of possible applications we treat in Section 6 a parabolic problem with point observation and control in space dimension $n \geq 3$.

2. Non–autonomous observation systems

We denote the space of bounded linear operators from X to Y by $\mathcal{L}(X,Y)$, and put $\mathcal{L}(X) := \mathcal{L}(X,X)$, where X,Y,U,Z always designate Banach spaces. $C_b(\mathbb{R}_+,\mathcal{L}_s(X,Y))$ and $L^\infty(\mathbb{R}_+,\mathcal{L}_s(X,Y))$ are the spaces of (essentially) bounded strongly continuous and strongly measurable operator–valued functions, respectively. We set $a \vee b = \max\{a,b\}$ and $a \wedge b = \min\{a,b\}$ for $a,b \in \mathbb{R}$ and write $\mathbf{1}_N$ for the characteristic function of $N \subset M$. Unless otherwise stated, p is a number contained in $[1,\infty)$. The spaces $L^p_{loc}([s,\infty),Z)$ and $C([s,\infty),Z)$ are endowed with their standard Fréchet topologies. We usually do not distinguish in notation between a function on $J \subset \mathbb{R}$ and its restrictions to subintervals.

Definition 2.1. *We call a set* $T = \{T(t,s), t \geq s \geq 0\} \subseteq \mathcal{L}(X)$ *an* evolution family *if*

(E1) $T(t,s) = T(t,r)T(r,s),\ \ T(s,s) = I,$
(E2) $(t,s) \mapsto T(t,s)$ *is strongly continuous, and*
(E3) $\|T(t,s)\| \leq Me^{w(t-s)}$

for $t \geq r \geq s \geq 0$ *and constants* $M \geq 1$ *and* $w \in \mathbb{R}$. *We also define* $(\mathbb{K}_s f)(t) = \int_s^t T(t,\tau)f(\tau)\,d\tau$ *for* $t \geq s \geq 0$ *and* $f \in L^1_{loc}([s,\infty),X)$, *and put* $\mathbb{K} = \mathbb{K}_0$.

Evolution families arise as solution operators of non–autonomous evolution equations, although not every evolution family solves such a problem. We refer to [4], [9, §VI.9], and the references therein for further information. We have included the condition (E3) of exponential boundedness in view of the study of asymptotic properties in Section 5. All other results can easily be refomulated and shown without assuming (E3). Some of our proofs remain valid if $T(t,s)$ is only required to be strongly continuous in t and s separately (as in [11] or [13]), see Remark 4.5.

The following definition (essentially contained in [14]) is parallel to [29, Def.2.1], but we prefer to work on $[s,\infty)$ without using bounded time intervals initially.

Definition 2.2. *Let T be an evolution family on X and $\Psi_s : X \to L^p_{loc}([s,\infty),Y)$, $s \geq 0$, be linear operators satisfying*

$$\Psi_s x = \Psi_t T(t,s)x \quad on \quad [t,\infty) \quad and \tag{2.1}$$

$$\int_s^{s+t_0} \|(\Psi_s x)(t)\|^p \, dt \leq \gamma^p \, \|x\|^p \tag{2.2}$$

for $t \geq s \geq 0$, $x \in X$, and constants $t_0, \gamma = \gamma(t_0) > 0$. Then $(T,\Psi) = (T, \{\Psi_s : s \geq 0\})$ is called a non–autonomous observation system. *We extend the function $\Psi_s x$ by 0 to $\mathbb{R}$ using the same symbol.*

Lemma 2.3. *Let (T,Ψ) be a non–autonomous observation system. Then one can replace in (2.2) the constant t_0 by every $t_1 > 0$ and $\gamma = \gamma(t_0)$ by $\gamma(t_1) = M\gamma(t_0)c(t_1,w,p)$, where*

$$c(t,w,p) = \begin{cases} (wt_0p)^{-\frac{1}{p}} \, e^{wt_0} \, e^{wt}, & w > 0, \\ (1+\frac{t}{t_0})^{\frac{1}{p}}, & w = 0, \\ (1-e^{wt_0p})^{-\frac{1}{p}}, & w < 0. \end{cases}$$

Proof. The case $t_1 \leq t_0$ is obvious. So let $t_1 = nt_0 + \tau$ for some $n \in \mathbb{N}$ and $\tau \in [0,t_0)$. Setting $I_k = [s+kt_0, s+(k+1)t_0]$, we deduce from Definition 2.2

$$\|\Psi_s x\|^p_{L^p([s,s+t_1],Y)} \leq \sum_{k=0}^{n} \|\Psi_{s+kt_0} T(s+kt_0,s)x\|^p_{L^p(I_k,Y)} \leq M^p \, \gamma(t_0)^p \sum_{k=0}^{n} e^{wpt_0k}$$

for $x \in X$ and $s \geq 0$. The assertion then follows easily. □

Non–autonomous observation systems are typically given by admissible observation operators which we define in slightly greater generality than in [11], [13, §1.3], [14], [15].

Definition 2.4. *Let T be an evolution family on X and $C(s) : D(C(s)) \subseteq X \to Y$, $s \geq 0$, be densely defined linear operators such that $T(\cdot,s)x \in D(C(\cdot),s)$ and*

$$\int_s^{s+t_0} \|C(t)T(t,s)x\|^p \, dt \leq \gamma^p \, \|x\|^p \tag{2.3}$$

for $s \geq 0$, $x \in D(C(s))$, and some constants $\gamma, t_0 > 0$. Then we say that $C(s)$, $s \geq 0$, are (T–)admissible observation operators. *Here, $D(C(\cdot),s) = \{f \in L^p_{loc}([s,\infty),X) : f(t) \in D(C(t))$ for a.e. $t \geq s$, $C(\cdot)f(\cdot) \in L^p_{loc}([s,\infty),Y)\}$ for $s \geq 0$.*

Lemma 2.5. *Let $C(s)$, $s \ge 0$, be T–admissible observation operators. Then* (2.3) *holds for all $t_0 > 0$ with a possibly different γ. Let $\Psi_s : X \to L^p_{loc}([s,\infty), Y)$, $s \ge 0$, be the continuous extension of the map $D(C(s)) \ni x \mapsto C(\cdot)T(\cdot, s)x$. Then (T, Ψ) is a non–autonomous observation system.*

Proof. The first claim can be established as Lemma 2.3; one only has to replace $s + kt_0$ by points $s_k \approx s + kt_0$ such that $T(s_k, s)x \in D(C(s_k))$, see [24, Lem.4.13]. So we can define Ψ_s as in the assertion. Given $t \ge s \ge 0$ and $x \in D(C(s))$, we take $z_n \in D(C(t))$ converging in X to $T(t,s)x$ and $t_n \searrow t$ such that $T(t_n, s)x, T(t_n, t)z_n \in D(C(t_n))$. Using

$$\Psi_t T(t,s)x = \lim_{n\to\infty} \mathbf{1}_{[t_n, t+t_0]}\, C(\cdot)T(\cdot, t)z_n$$

in $L^p([t, t+t_0], Y)$, we obtain

$$\begin{aligned}
&\|\Psi_t T(t,s)x - C(\cdot)T(\cdot, s)x\|^p_{L^p([t,t+t_0],Y)} \\
&= \lim_{n\to\infty} \Big(\int_{t_n}^{t+t_0} \|C(\tau)T(\tau, t_n)\,[T(t_n,t)z_n - T(t_n,s)x]\|^p \, d\tau + \int_t^{t_n} \|C(\tau)T(\tau,s)x\|^p \, d\tau \Big) \\
&\le \gamma^p \lim_{n\to\infty} \|T(t_n,t)z_n - T(t_n,s)x\|^p = 0.
\end{aligned}$$

Therefore (2.1) holds for $x \in D(C(s))$ and thus for $x \in X$ by approximation. $\square$

We note that different admissible observation operators $C_1(s)$ and $C_2(s)$ may yield the same observation system as shown in [29, Ex.1.2]. However, if the observation operators $C(s)$ are closable, then one easily verifies that $\Psi_s x = \overline{C(\cdot)}T(\cdot, s)x$ for the induced observation system. We now proceed in the converse direction and represent a given observation system by admissible observation operators, cf. [29, Def.4.1].

Definition 2.6. *For a non–autonomous observation system (T, Ψ) and $s \ge 0$, we define*

$$\underline{X}_s = \Big\{ x \in X : \exists\, C(s)x := Y - \lim_{\tau \searrow 0} \frac{1}{\tau} \int_s^{s+\tau} (\Psi_s x)(\sigma)\, d\sigma \Big\} \quad \textit{and}$$

$$\|x\|_{\underline{X}_s} = \|x\| + \sup_{0<\tau\le 1} \Big\| \frac{1}{\tau} \int_s^{s+\tau} (\Psi_s x)(\sigma)\, d\sigma \Big\|_Y \quad \textit{for } \ x \in \underline{X}_s.$$

The space $D(C(\cdot), s)$ is defined as in 2.4 by replacing $D(C(t))$ with $\underline{X}_t$.

It is clear that $\|\cdot\|_{\underline{X}_s}$ is a norm on the subspace $\underline{X}_s$ and that $C(s) : \underline{X}_s \to Y$ is linear and continuous. The space $(\underline{X}_s, \|\cdot\|_{\underline{X}_s})$ is complete. This is a consequence of the closedness of the operator $M_s : \underline{X}_s \subseteq X \to C([0,1], Y)$ given by

$$(M_s x)(\tau) = \begin{cases} \frac{1}{\tau}\int_s^{s+\tau} (\Psi_s x)(\sigma)\, d\sigma, & \tau \in (0,1], \\ C(s)x, & \tau = 0, \end{cases}$$

which can easily be checked as in [29, Prop.4.3].

We say that $t \in \mathbb{R}$ is a *Lebesgue point* of the function $f \in L^p_{loc}(\mathbb{R}, Z)$ if

$$\lim_{|J|\to 0} \frac{1}{|J|} \int_J \|f(s) - f(t)\|^p \, ds = 0,$$

where the limit is taken over compact intervals J containing t (of length $|J|$). Recall that almost every t is a Lebesgue point of f, see e.g. [32, Lem.6.1] or [27, §I.1.8]. The next representation theorem extends [29, Thm.4.5] to non–autonomous observation systems.

A different representation of output functions was given in [12] applying Weiss' theory to the 'evolution semigroup' on $L^p([0,t_0],X)$ associated with T, cf. [4].

Theorem 2.7. *Let (T,Ψ) be a non–autonomous observation system and $C(s)\in\mathcal{L}(\underline{X}_s,Y)$ be defined as in 2.6. Let $x\in X$ and $t\geq s\geq 0$. Then $T(t,s)x\in\underline{X}_t$ if and only if $1/\tau\int_0^\tau(\Psi_s x)(t+\sigma)\,d\sigma$ converges as $\tau\searrow 0$. If this is the case, then the limit is equal to $C(t)T(t,s)x$. In particular, $(\Psi_s x)(t)=C(t)T(t,s)x$ for all Lebesgue points t of $\Psi_s x$.*

Proof. The theorem follows from the identity

$$\frac{1}{\tau}\int_t^{t+\tau}[\Psi_s x](\sigma)\,d\sigma=\frac{1}{\tau}\int_t^{t+\tau}[\Psi_t T(t,s)x](\sigma)\,d\sigma. \qquad \square$$

Therefore we have defined in 2.6 admissible observation operators $C(t)$. These operators then yield an observation system $(\tilde{\Psi},T)$ by Lemma 2.5, but of course $(\Psi_s x)(t)=(\tilde{\Psi}_s x)(t)$ for a.e. $t\geq s$. Hence, $C(t)$ are maximal in this sense. We say that the operators $C(t)$ from Definition 2.6 *represent the observation system* (T,Ψ).

In the remainder of this section we establish several properties of Ψ_s which will be important for our main perturbation result.

Lemma 2.8. *Let (T,Ψ) be a non–autonomous observation system, $t_0>0$, and $f\in L^p_{loc}(\mathbb{R}_+,X)$. Then the mapping $[0,t_0]\ni s\mapsto\Psi_s f(s)\in L^p([0,t_0],Y)$ is measurable and*

$$\int_0^{t_0}\|\Psi_s f(s)\|^p_{L^p([0,t_0],Y)}\,ds\leq\gamma(t_0)^p\,\|f\|^p_{L^p([0,t_0],X)}. \tag{2.4}$$

Proof. First, let $f\in C(\mathbb{R}_+,X)$. Then

$$\|\Psi_s f(s)-\Psi_r f(r)\|^p_{L^p([0,t_0],Y)}=\|\Psi_s(f(s)-T(s,r)f(r))\|^p_{L^p([s,t_0],Y)}+\|\Psi_r f(r)\|^p_{L^p([r,s],Y)}$$

for $0\leq r\leq s\leq t_0$. Hence, $s\mapsto\Psi_s f(s)$ is continuous from the right. Functions $f\in L^p_{loc}(\mathbb{R}_+,X)$ can be treated by approximation. The estimate (2.4) follows from (2.2). $\square$

The non–closedness of $C(t)$ is a major obstacle for the analysis of observation systems and input–output operators; for instance, it is a priori not clear whether $C(t)$ can be taken out of an integral. As in the autonomous case, see e.g. [32, §4], such problems can be overcome employing the operators

$$C_\tau(s)x=\frac{1}{\tau}\int_s^{s+\tau}(\Psi_s x)(\sigma)\,d\sigma, \tag{2.5}$$

$x\in X$, $s\geq 0$, and $\tau\in(0,1]$. Due to this definition, $C_\tau(s)$ belongs to $\mathcal{L}(X,Y)$ with norm less or equal than $\gamma(1)\tau^{-\frac{1}{p}}$, $C_\tau(s)x$ converges as $\tau\to 0$ if and only if $x\in\underline{X}_s$, and then the limit is equal to $C(s)x$. We also define

$$\mathcal{D}_s=\text{lin}\,\{f=\varphi(\cdot)T(\cdot,r)x : x\in X,\ r\geq s,\ \varphi\in C_c(\mathbb{R}_+),\ \varphi(t)=0\text{ for }s\leq t<r\} \tag{2.6}$$

for $s\geq 0$ (setting $T(t,s):=0$ for $t<s$), and put $\mathcal{D}=\mathcal{D}_0$. This space is dense in $L^p([s,\infty),X)$ and in $C_0([s,\infty),X)$ because of (E2), cf. [4, p.64].

Lemma 2.9. *Let (T,Ψ) be a non–autonomous observation system represented by $C(s)$ and let $C_\tau(s)$ be given by (2.5). Then $(s,\tau)\mapsto C_\tau(s)x$ is continuous,*

$$\|C_\tau(\cdot)T(\cdot,s)x\|_{L^p([s,s+t_0],Y)} \le \gamma(t_0+1)\,\|x\|, \quad \textit{and} \tag{2.7}$$
$$\Psi_s x = \lim_{\tau\to 0} C_\tau(\cdot)T(\cdot,s)x \quad \textit{in} \quad L^p_{loc}([s,\infty),Y) \tag{2.8}$$

for $x\in X$, $s\ge 0$, $\tau\in(0,1]$, and $t_0>0$.

Proof. If $f\in\mathcal{D}$, then $(t,\tau)\mapsto C_\tau(t)f(t)$ is continuous since

$$C_\tau(t)f(t)=\sum_{k=1}^n \varphi_k(t)\,\frac{1}{\tau}\int_t^{t+\tau}(\Psi_{r_k}x_k)(\sigma)\,d\sigma$$

for $\tau>0$, $t\ge 0$, and suitable $n\in\mathbb{N}$, $r_k\ge 0$, $x_k\in X$, $\varphi_k\in C_c(\mathbb{R}_+)$. The first assertion follows by approximation. We further estimate

$$\begin{aligned}\|C_\tau(\cdot)T(\cdot,s)x\|^p_{L^p([s,s+t_0],Y)} &\le \int_s^{s+t_0}\frac{1}{\tau}\int_t^{t+\tau}\|(\Psi_s x)(\sigma)\|^p\,d\sigma\,dt\\ &\le \int_s^{s+t_0+\tau}\frac{1}{\tau}\int_{\sigma-\tau}^{\sigma}\|(\Psi_s x)(\sigma)\|^p\,dt\,d\sigma\\ &\le \gamma(t_0+1)^p\,\|x\|^p\end{aligned}$$

using Hölder's inequality and Fubini's theorem. In a similar way, we deduce (2.8) from

$$\|C_\tau(\cdot)T(\cdot,s)x-\Psi_s x\|^p_{L^p([s,s+t_0],Y)} \le \frac{1}{\tau}\int_0^\tau\int_s^{s+t_0}\|(\Psi_s x)(t+\sigma)-(\Psi_s x)(t)\|^p\,dt\,d\sigma. \qquad \square$$

We want to show that $C(\cdot)\mathbb{K}_s : L^p([s,s+t_0],X)\to L^p([s,s+t_0],Y)$ is well–defined and bounded. To that purpose, we introduce the functions

$$\varphi(t;\,\tau,\sigma,f)=(C_\tau(t)-C_\sigma(t))\int_0^t T(t,s)f(s)\,ds,$$
$$o_f(t)=\overline{\lim_{\tau,\sigma\to 0}}\,\|\varphi(t;\,\tau,\sigma,f)\| = \overline{\lim_{n\to\infty}}\,\sup_{m\ge n}\,\sup_{\tau,\sigma\in[1/m,1/n]}\|\varphi(t;\,\tau,\sigma,f)\|$$

for $f\in L^1_{loc}(\mathbb{R}_+,X)$, $t\ge 0$, $\tau,\sigma\in(0,1]$. Observe that o_f is measurable. We further need the *maximal operator* given by

$$M\psi(t)=\sup_{\tau>0}\frac{1}{\tau}\int_t^{t+\tau}|\psi(s)|\,ds$$

for all $t\in\mathbb{R}$ and $\psi\in L^1_{loc}(\mathbb{R})$. Recall that

$$\|M\psi\|_{L^p(\mathbb{R})}\le c_p\,\|\psi\|_{L^p(\mathbb{R})} \tag{2.9}$$

for $\psi\in L^p(\mathbb{R})$, $1<p\le\infty$, and a constant c_p, see [27, Thm.I.1].

Lemma 2.10. *Let (T,Ψ) be a non–autonomous observation system, $p\in(1,\infty)$, and $f\in L^p_{loc}(\mathbb{R}_+,X)$. Then $C_\tau(t)(\mathbb{K}f)(t)$ converges to $C(t)(\mathbb{K}f)(t)$ as $\tau\searrow 0$ for a.e. $t\ge 0$.*

Proof. Take $g\in\mathcal{D}$ and $f\in L^p_{loc}(\mathbb{R}_+,X)$. Observe that $o_g=0$ a.e. since

$$C_\tau(t)(\mathbb{K}g)(t)=\sum_{k=1}^n\int_0^t\varphi_k(s)\,ds\;\frac{1}{\tau}\int_t^{t+\tau}(\Psi_{r_k}x_k)(\sigma)\,d\sigma$$

for $\tau > 0$, $t \geq 0$, and suitable $n \in \mathbb{N}$, $r_k \geq 0$, $x_k \in X$, $\varphi_k \in C_c(\mathbb{R}_+)$. Due to Lemma 2.8, there is a measurable function $\psi_{f-g} : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\text{for a.e. } \; s \geq 0 \qquad \|[\Psi_s(f(s) - g(s))](t)\|_Y = \psi_{f-g}(t,s) \quad \text{for a.e. } \; t \geq s.$$

(Here we set $[\Psi_s(f(s) - g(s))](t) = \psi_{f-g}(t,s) = 0$ for $t < s$, $t > t_0$, or $s > t_0$, where $t_0 > 0$ is fixed but arbitrary.) Employing these facts, we estimate

$$\begin{aligned} o_f(t) \leq o_{f-g}(t) &\leq \sup_{\tau,\sigma \in (0,1]} \int_0^t \|(C_\tau(t) - C_\sigma(t))\, T(t,s)(f(s) - g(s))\|\, ds \\ &\leq 2 \sup_{\tau \in (0,1]} \int_0^t \frac{1}{\tau} \int_t^{t+\tau} \|[\Psi_s(f(s) - g(s))](\rho)\|\, d\rho\, ds \\ &\leq 2 \int_0^t [M\psi_{f-g}(\cdot, s)](t)\, ds. \end{aligned} \tag{2.10}$$

for t not contained in a set of measure 0 depending on g. Let us check that the function $(t,s) \mapsto [M\phi(\cdot,s)](t)$ is measurable if $0 \leq \phi \in L^1_{loc}(\mathbb{R}^2)$. Notice that

$$[M\phi(\cdot,s)](t) = \sup_{n\in\mathbb{N}} \sup_{\tau \geq 1/n} \frac{1}{\tau} \int_t^{t+\tau} \phi(\rho, s)\, d\rho =: \sup_{n\in\mathbb{N}} \phi_n(t,s)$$

for a.e. $s \in \mathbb{R}$ and all $t \in \mathbb{R}$, so that it suffices to verify the measurability of ϕ_n. If ϕ is continuous, then ϕ_n is continuous. The general case thus follows by approximation. We can now use (2.10), Fubini's theorem, and the maximal inequality (2.9) to derive

$$\begin{aligned} |\{t \in [0,t_0] : o_f(t) > \varepsilon\}| &\leq \frac{2}{\varepsilon} \int_0^{t_0} \int_0^t [M\psi_{f-g}(\cdot,s)]\,(t)\, ds\, dt \\ &= \frac{2}{\varepsilon} \int_0^{t_0} \int_s^{t_0} [M\psi_{f-g}(\cdot,s)]\,(t)\, dt\, ds \\ &\leq \frac{c}{\varepsilon} \int_0^{t_0} \|M\psi_{f-g}(\cdot,s)\|_{L^p(\mathbb{R})}\, ds \\ &\leq \frac{c'}{\varepsilon} \int_0^{t_0} \|\psi_{f-g}(\cdot,s)\|_{L^p([s,t_0])}\, ds \\ &= \frac{c'}{\varepsilon} \int_0^{t_0} \|\Psi_s(f(s) - g(s))\|_{L^p([s,t_0],Y)}\, ds \\ &\leq \frac{\gamma c'}{\varepsilon} \|f - g\|_{L^1([0,t_0],X)} \end{aligned}$$

for each $\varepsilon > 0$ and constants c, c' not depending on f, g, ε. Since g is arbitrary, the set $\{o_f > \varepsilon\}$ has Lebesgue measure 0. This fact implies the assertion. $\square$

Proposition 2.11. *Let (T, Ψ) be a non–autonomous observation system represented by $C(t)$, $p \in (1, \infty)$, and $C_\tau(t)$ be given by (2.5). Then $\mathbb{K}_s f \in D(C(\cdot), s)$,*

$$\|C(\cdot)\mathbb{K}_s f\|_{L^p([s,s+t_0],Y)} \leq c(t_0)\, \|f\|_{L^p([s,s+t_0],X)}\,,$$

and $C_\tau(\cdot)\mathbb{K}_s f \to C(\cdot)\mathbb{K}_s f$ in $L^p_{loc}([s,\infty), Y)$ as $n \to \infty$ for $s \geq 0$, $t_0 > 0$, $f \in L^p_{loc}(\mathbb{R}_+, X)$, and a constant $c(t_0)$ independent of f and s.

Proof. By the previous lemma $C(\cdot)\mathbb{K}_s f$ is a well-defined measurable function. Further,

$$\begin{aligned}\int_s^{s+t_0} \|C_\tau(t)(\mathbb{K}_s f)(t)\|^p \, dt &\le c \int_s^{s+t_0} \int_s^t \|C_\tau(t)T(t,r)f(r)\|^p \, dr \, dt \\ &= c \int_s^{s+t_0} \int_r^{s+t_0} \|C_\tau(t)T(t,r)f(r)\|^p \, dt \, dr \\ &\le c\tilde{\gamma}^p \, \|f\|^p_{L^p([s,s+t_0],X)} \end{aligned} \tag{2.11}$$

for a constant c by Hölder's inequality, Fubini's theorem, and Lemma 2.9. Similarly,

$$\|(C_\tau(\cdot) - C_\sigma(\cdot))\mathbb{K}_s f\|_p^p \le c \int_s^{s+t_0} \int_r^{s+t_0} \|(C_\tau(t) - C_\sigma(t))T(t,r)f(r)\|^p \, dt \, dr,$$

and the right side tends to 0 as $\tau, \sigma \to 0$ by Lemma 2.9 and the dominated convergence theorem. Hence, $C_\tau(\cdot)\mathbb{K}_s f$ also converges in $L^p([s, s+t_0], Y)$ to $C(\cdot)\mathbb{K}_s f$. The asserted estimate then follows from (2.11). □

3. Well–posed and regular non–autonomous systems

Again we follow [14] and generalize a definition due to Weiss to the non–autonomous case, cf. [30, Def.1.1].

Definition 3.1. *Let T be an evolution family on X and $\Phi_{t,s} = \Phi(t,s) : L^p_{loc}([s,\infty), U) \to X$, $t \ge s \ge 0$, be linear operators satisfying*

$$\Phi_{t,s}u = \Phi_{t,r}(u|[r,\infty)) + T(t,r)\Phi_{r,s}u, \quad t \ge r \ge s \ge 0, \quad \text{and} \tag{3.1}$$

$$\|\Phi_{t,s}u\|_X \le \beta \, \|u\|_{L^p([s,t],U)}, \quad 0 \le t-s \le t_0, \tag{3.2}$$

for $u \in L^p(\mathbb{R}_+, U)$ and constants $t_0, \beta = \beta(t_0) > 0$. Then $(T, \Phi) = (T, \{\Phi_{t,s} : t \ge s \ge 0\})$ is called a non–autonomous control system.

Observe that the above definition implies that $\Phi_{t,t} = 0$ and $\Phi_{t,s}u = \Phi_{t,r}u$ if $u = 0$ on $[s,r] \subseteq [s,t]$. Thus the control system is *causal*.

Lemma 3.2. *Let (T, Φ) be a non–autonomous control system. Then*

$$\|\Phi_{t,s}u\|_X \le M \, \beta(t_0) \, e^{w^- t_0} \, c(t-s, w, q) \, \|u\|_{L^p([s,t],U)}, \tag{3.3}$$

$$\|\Phi(\cdot, s)u\|_{L^p([s,t],X)} \le M \, \beta(t_0) \, t_0^{\frac{1}{p}} \, e^{w^- t_0} \, c(t-s, w, 1) \, \|u\|_{L^p([s,t],U)} \tag{3.4}$$

for $t \ge s \ge 0$ and $u \in L^p([s,t], U)$, where $w^- = 0 \vee (-w)$, $\frac{1}{p} + \frac{1}{q} = 1$, and $c(t, w, r)$ is defined in Lemma 2.3.

Proof. In Lemma 3.4 we show the measurability of $\Phi(\cdot, s)u$ (of course without refering to (3.4)). The assertion is clear for $s \le t \le s + t_0$. Let $s_k = s + kt_0$ for $k \in \mathbb{N}_0$, $t \in [s_n, s_{n+1}]$ for some $n \in \mathbb{N}$, and u_k be the restriction of u to $[s_k, s_{k+1}] \cap [s,t]$ for $k = 0, \cdots, n$. Then

$$\Phi(t,s)u = \Phi(t, s_n)u_n + \sum_{k=1}^n T(t, s_k)\Phi(s_k, s_{k-1})u_{k-1}\,, \tag{3.5}$$

$$\|\Phi(t,s)u\| \le \beta \, \|u_n\|_p + M\beta \sum_{k=1}^n e^{w(t-s_k)} \, \|u_{k-1}\|_p \le M\beta e^{w^- t_0} \, (a * b)_n\,,$$

where $a_k = e^{wt_0 k}$ if $k = 0, \cdots, n$ and $a_k = 0$ otherwise, $b_k = \|u_k\|_p$, $a = (a_k)_k$, and $b = (b_k)_k$. Young's inequality now implies the lemma. □

The next definition is a slight extension of those given in [11], [13, §1.3], and [15].

Definition 3.3. *Let T be an evolution family on X and let $\overline{X}_t$, $t \geq 0$, be Banach spaces in which X is densely and continuously embedded. Assume that $T(t,s)$ has a locally uniformly bounded extension $\overline{T}(t,s) : \overline{X}_s \to \overline{X}_t$ (which then satisfies (E1) and is strongly continuous w.r.t. s). We call $B(t) \in \mathcal{L}(U, \overline{X}_t)$, $t \geq 0$, (T–)*admissible control operators *if the function $\overline{T}(t,\cdot)B(\cdot)u(\cdot)$ is integrable in $\overline{X}_t$,*

$$(\overline{\mathbb{K}}_s B(\cdot)u)(t) := \int_s^t \overline{T}(t,\tau)B(\tau)u(\tau)\,d\tau \in X,$$

and there are constants $t_0, \beta > 0$ such that

$$\|(\overline{\mathbb{K}}_s B(\cdot)u)(t)\|_X \leq \beta\,\|u\|_{L^p([s,t],U)} \tag{3.6}$$

for all $0 \leq s \leq t \leq s + t_0$ and $u \in L^p([s,t],U)$. (We omit the subscript s if $s = 0$.)

Setting $\Phi_{t,s}u := (\overline{\mathbb{K}}_s B(\cdot)u)(t)$, we obtain of course a non–autonomous control system (T,Φ) if $B(t)$, $t \geq 0$, are admissible control operators. Every autonomous control system is given by a T–admissible control operator due to [30, Thm3.9], where $\overline{X}_t$, $t \geq 0$, coincide with the extrapolation space X_{-1} of X w.r.t. the semigroup T (see e.g. [2, Chap.V], [9, §II.5]). We can extend this result to the time dependent setting only in an approximate sense because of the lack of an extrapolation theory for evolution families. The next lemma serves as a preliminary step and will be improved in Proposition 3.5.

Lemma 3.4. *Let (T,Φ) be a non–autonomous control system and $u \in L^p_{loc}(\mathbb{R}_+,U)$. Then $t \mapsto \Phi_{t,s}u \in X$ is continuous from the right for $t \geq s$, $s \mapsto \Phi_{t,s}u \in X$ is continuous for $s \in [0,t]$ (locally uniformly in t), and $(t,s) \mapsto \Phi_{t,s}u \in X$ is measurable.*

Proof. Definition 3.1 implies the estimates

$$\begin{aligned}
\|\Phi(t',s)u - \Phi(t,s)u\| &\leq \|\Phi(t',t)u\| + \|(T(t',t) - I)\Phi(t,s)u\| \\
&\leq \beta\,\|u\|_{L^p([t,t'],U)} + \|(T(t',t) - I)\Phi(t,s)u\|, \\
\|\Phi(t,s)u - \Phi(t,s')u\| &= \|T(t,s')\Phi(s',s)u\| \leq M\beta e^{|w|(t-s)}\,\|u\|_{L^p([s,s'],U)}\,,
\end{aligned}$$

where $t' \geq t \geq s' \geq s$. Thus the lemma is established. ☐

Let $u \in L^p_{loc}(\mathbb{R},U)$, $t \geq 0 \geq s$, and $n \in \mathbb{N}$. We define $(B_n u)(t) = n\,\Phi(t, t - \frac{1}{n})u$, where $\Phi(t,s)u := \Phi(t,0)u$. Note that $B_n u \in L^\infty(\mathbb{R}_+,X)$ because of the above lemma. To approximate Φ, we introduce

$$\Phi^n(t,s)u = \Phi^n_{t,s}u := \int_s^t T(t,\tau)(B_n u)(\tau)\,d\tau = (\mathbb{K}_s B_n u)(t) \tag{3.7}$$

for $t \geq s \geq 0$, $n \in \mathbb{N}$, and $u \in L^p_{loc}(\mathbb{R},U)$. These operators can be expressed by

$$\begin{aligned}
\Phi^n(t,s)u &= n\int_s^t (\Phi(t,\tau - \tfrac{1}{n})u - \Phi(t,\tau)u)\,d\tau \\
&= n\int_{s-\frac{1}{n}}^s \Phi(t,\tau)u\,ds - n\int_{t-\frac{1}{n}}^t \Phi(t,\tau)u\,d\tau \\
&= \Phi(t,s)u + nT(t,s)\int_{s-\frac{1}{n}}^s \Phi(s,\tau)u\,d\tau - n\int_{t-\frac{1}{n}}^t \Phi(t,\tau)u\,d\tau
\end{aligned} \tag{3.8}$$

due to (3.1). If we take a function $u \in L^p_{loc}([s,\infty),U)$ and extend it by 0 to $\mathbb{R}$, then

$$\Phi(t,s)u - \Phi^n(t,s)u = n\int_{t-\frac{1}{n}}^{t} \Phi(t,\tau)u\,d\tau =: r_n(t;\,u). \tag{3.9}$$

Inserting the constant function $u_z \equiv z$, we obtain operators $B_n(t) \in \mathcal{L}(U,X)$ given by

$$B_n(t)z := (B_n u_z)(t) = n\Phi(t,t-\tfrac{1}{n})u_z.$$

The next result shows that we can approximate every observation system by observation systems represented by bounded input operators.

Proposition 3.5. *Let (T,Φ) be a non–autonomous control system, $n \in \mathbb{N}$, $0 \le s \le t \le s+t_0$, $t_0 > 0$, $z \in U$, and $u \in L^p_{loc}(\mathbb{R},U)$. Then the following assertion hold.*

(a) *$\Phi^n(t,s)u \to \Phi(t,s)u$ in X as $n \to \infty$ locally uniformly in (t,s), and $\|\Phi^n(t,s)u\|_X \le 2\beta(t_0)\|u\|_{L^p([s,t],U)}$.*
(b) *$(t,s) \mapsto \Phi(t,s)u$ and $t \mapsto B_n(t)z$ are continuous in X.*
(c) *$[\mathbb{K}_s B_n(\cdot)u](t) \to \Phi(t,s)u$ in X locally uniformly in (t,s) and $\|[\mathbb{K}_s B_n(\cdot)u](t)\| \le \beta(t_0+1)\,\|u\|_{L^p([s,t],U)}$.*

Proof. For $u \in L^p_{loc}(\mathbb{R},U)$ we estimate

$$\|n\int_{t-\frac{1}{n}}^{t} \Phi(t,\tau)u\,d\tau\|_X \le \sup_{t-\frac{1}{n}\le\tau\le t} \|\Phi(t,\tau)u\|_X \le \beta\,\|u\|_{L^p([t-\frac{1}{n},t],U)}\,,$$

$$\|nT(t,s)\int_{s-\frac{1}{n}}^{s} \Phi(s,\tau)u\,d\tau\|_X \le M\beta e^{w(t-s)}\,\|u\|_{L^p([s-\frac{1}{n},s],U)}\,,$$

which yields the first part of (a) because of (3.8). This fact implies (b). The second part of (a) follows from (3.9) if we extend $u \in L^p([s,\infty),X)$ by 0 to $\mathbb{R}$. To show (c), we set $\tilde{u}(\tau,\sigma) = u(\tau)$ for $\sigma \ge \tau \ge 0$ and $u^{(n)}(\sigma) = n\int_\sigma^{\sigma+\frac{1}{n}} u(\tau)\,d\tau$ for $\sigma \ge 0$. Taking first $u \in W^{1,p}_{loc}(\mathbb{R},U)$, using Hölder's inequlity, and interchanging integrals, we estimate

$$\begin{aligned}
\|B_k u - B_k(\cdot)u\|_{L^1([s,s+t_0],X)} &\le k\int_s^{s+t_0} \|\Phi(\tau,\tau-\tfrac{1}{k})[u-\tilde{u}(\tau,\cdot)]\|_X\,d\tau \\
&\le \beta k \int_s^{s+t_0} \Big(\int_{\tau-\frac{1}{k}}^{\tau} \Big\| \int_\sigma^\tau u'(\rho)\,d\rho\Big\|_U^p d\sigma\Big)^{\frac{1}{p}} d\tau \\
&\le \beta k^{1-\frac{1}{p}} \int_s^{s+t_0}\int_0^{\frac{1}{k}} \|u'(\tau-\rho)\|_U\,d\rho\,d\tau \\
&\le c\Big(\int_0^{\frac{1}{k}}\int_s^{s+t_0} \|u'(\tau-\rho)\|_U^p\,d\tau\,d\rho\Big)^{\frac{1}{p}}
\end{aligned}$$

so that $[\mathbb{K}_s B_k(\cdot)u](t) \to \Phi(t,s)u$ as $k \to \infty$ locally uniformly in this case. We now fix $t > s \ge 0$, extend $u \in L^p([s,t],X)$ by 0 to $\mathbb{R}$. Then (3.1) and the previous results imply

$$\begin{aligned}
[\mathbb{K}_s B_n(\cdot)u](t) &= n\int_{s-\frac{1}{n}}^{t} \Phi(t,\tau)[\tilde{u}(\tau+\tfrac{1}{n},\cdot) - \tilde{u}(\tau,\cdot)]\,d\tau \\
&= \lim_{k\to\infty} n\int_{s-\frac{1}{n}}^{t}\int_\tau^t T(t,\sigma)B_k(\sigma)[u(\tau+\tfrac{1}{n}) - u(\tau)]\,d\sigma\,d\tau
\end{aligned}$$

$$= \lim_{k\to\infty} n \int_{s-\frac{1}{n}}^{t} T(t,\sigma)B_k(\sigma) \int_{s-\frac{1}{n}}^{\sigma} [u(\tau+\tfrac{1}{n}) - u(\tau)]\, d\tau\, d\sigma$$
$$= \Phi(t, s-\tfrac{1}{n})u^{(n)},$$

Observing that $\|u^{(n)}\|_{L^p([s,t],U)} \le \|u\|_{L^p([s,t],U)}$, we deduce (c). □

We give as in [14] the non–autonomous analogue of G. Weiss' definition of *well–posed system*, see [31, Def.1.1].

Definition 3.6. *Let (T,Φ) and (T,Ψ) be non–autonomous control and observation systems. If there are linear operators $\mathbb{F}_s : L^p_{loc}([s,\infty),U) \to L^p_{loc}([s,\infty),Y)$, $s \ge 0$, satisfying*

$$\mathbb{F}_s u = \Psi_t \Phi_{t,s} u + \mathbb{F}_t(u|[t,\infty)) \quad on\ [t,\infty) \quad and \tag{3.10}$$
$$\|\mathbb{F}_s u\|_{L^p([s,s+t_0],Y)} \le \kappa \, \|u\|_{L^p([s,s+t_0],U)} \tag{3.11}$$

for $u \in L^p_{loc}([s,\infty),U)$, $t \ge s \ge 0$, and constants $t_0, \kappa = \kappa(t_0) > 0$, then $\Sigma = (T,\Phi,\Psi,\mathbb{F}) = (T,\Phi_{t,s},\Psi_s,\mathbb{F}_s)_{t\ge s\ge 0}$ is called a well–posed non–autonomous system*, and $\mathbb{F}_s$, $s \ge 0$, are called* input-output operators*. We put $\mathbb{F} = \mathbb{F}_0$.*

Observe that the above definition implies that $\mathbb{F}_s u = 0$ on $[s,t]$ and $\mathbb{F}_s u = \mathbb{F}_t(u|[t,\infty))$ on $[t,\infty)$ if u vanishes on $[s,t]$. Hence, $\mathbb{F}_s$ is *causal* and we can define its restriction

$$\mathbb{F}_{t,s} = \mathbb{F}(t,s) : L^p([s,t],U) \to L^p([s,t],Y), \qquad t \ge s \ge 0.$$

Lemma 3.7. *A well–posed non–autonomous linear system Σ satisfies (3.11) with t_0 replaced by each $t_1 > 0$ and $\kappa = \kappa(t_0)$ by $\kappa(t_1) = (\kappa(t_0) \vee M\beta(t_0)\gamma(t_0))\, e^{w^- t_0}\, c(t_1,w,1)$, where $c(t,w,1)$ was defined in Lemma 2.3.*

Proof. The assertion is clear for $t_1 \le t_0$. So let $t_1 \in [s_n, s_{n+1})$ for some $n \in \mathbb{N}$, $s_k = s + kt_0$, $I_k = [s_k, s_{k+1}]$, and $u_k = u|I_k$ for $k \in \mathbb{N}_0$, $s \ge 0$, and $u \in L^p_{loc}([s,\infty),U)$. From Definition 3.6 and (3.5) we deduce

$$\mathbb{F}_s u = \mathbb{F}_{s_k} u_k + \sum_{j=1}^{k} \Psi_{s_k} T(s_k,s_j)\Phi(s_j,s_{j-1})u_{j-1} \quad \text{on} \quad I_k,$$

$$\begin{aligned}
\|\mathbb{F}_s u\|_{L^p(I_k,Y)} &\le \kappa(t_0)\, \|u_k\|_p + M\beta(t_0)\gamma(t_0) \sum_{j=1}^{k} e^{wt_0(k-j)}\, \|u_{j-1}\|_p \\
&\le (\kappa(t_0) \vee M\beta(t_0)\gamma(t_0))\, e^{w^- t_0}\, (a*b)_k\,,
\end{aligned}$$
$$\|\mathbb{F}_s u\|_{L^p([s,s+t_1],Y)} \le \Big(\sum_{k=0}^{n} \|\mathbb{F}_s u\|^p_{L^p(I_k,Y)} \Big)^{\frac{1}{p}} \le (\kappa(t_0) \vee M\beta(t_0)\gamma(t_0))\, e^{w^- t_0}\, \|a*b\|_{\ell^p}\,,$$

where the sequences a and b were defined in the proof of Lemma 3.2. The asserted estimate then follows from Young's inequality. □

Also Definition 3.6 is complemented by a concept involving admissible input and output operators, cf. [11], [13, §1.3], [15].

Definition 3.8. *Let $B(s)$ and $C(s)$, $s \ge 0$, be T–admissible control and observation operators. We call the triple $(T, B(\cdot), C(\cdot))$ an* admissible non–autonomous system *if $\overline{\mathbb{K}}_s B(\cdot)u \in D(C(\cdot),s)$ and $\|C(\cdot)\overline{\mathbb{K}}_s B(\cdot)u\|_{L^p([s,s+t_0],Y)} \le \kappa\, \|u\|_{L^p([s,s+t_0],U)}$ for $s \ge 0$, $u \in L^p_{loc}([s,\infty),U)$, and constants $\kappa, t_0 > 0$.*

Lemma 3.9. *Let $(T, B(\cdot), C(\cdot))$ be an admissible non–autonomous system. Define Ψ_s as in Lemma 2.5, $\Phi_{t,s}u := (\overline{\mathbb{K}}_s B(\cdot)u)(t)$, and $\mathbb{F}_s := C(\cdot)\overline{\mathbb{K}}_s B(\cdot)u$. Then $(T, \Phi, \Psi, \mathbb{F})$ is a well–posed non–autonomous system.*

Proof. Taking into account Lemma 2.5, we only have to verify (3.10) for $u \in L^p_{loc}([s,\infty), U)$ and $t \geq s \geq 0$. There are $t_n \searrow t$ such that $\Phi_{t_n,t}u, \Phi_{t_n,s}u \in D(C(t_n))$, and hence

$$\mathbb{F}_s u = \mathbb{F}_t u - \Psi_{t_n}\Phi_{t_n,t}u + \Psi_{t_n}\Phi_{t_n,s}u$$

a.e. on $[t_n, \infty)$. The third term on the right hand side converges in L^p to $\Psi_t\Phi_{t,s}u$ due to Proposition 3.5 and the proof of Lemma 2.8. The assertion then follows from

$$\|\Psi_{t_n}\Phi_{t_n,t}u\|_{L^p([t_n,s+t_0],Y)} \leq \beta\gamma\,\|u\|_{L^p([t,t_n],U)}\,. \qquad \square$$

In order to prove a converse to the above lemma, we need the first one of the following notions which extends the corresponding concept due to Weiss, [31, Def.4.1]. The second condition should be satisfied in many applications, cf. [31, p.410] or Section 6. It is used below at two instances, but see Remark 4.6.

Definition 3.10. *A well–posed non–automonous system $\Sigma = (T, \Phi, \Psi, \mathbb{F})$ is called* regular (with feedthrough $D = 0$) *if*

$$Y - \lim_{\tau\searrow 0}\frac{1}{\tau}\int_t^{t+\tau}(\mathbb{F}_t u_z)(\sigma)\,d\sigma = 0 \tag{3.12}$$

and absolutely regular *if*

$$\lim_{\tau\searrow 0}\frac{1}{\tau}\int_t^{t+\tau}\|(\mathbb{F}_t u_z)(\sigma)\|_Y^p\,d\sigma = 0 \tag{3.13}$$

for all $t \geq 0$ and $z \in U$, where $u_z(s) := z$ for $s \geq 0$.

We derive several useful properties of a well–posed system Σ. First, (3.11) yields

$$\left\|\frac{1}{\tau}\int_t^{t+\tau}(\mathbb{F}_t u_z)(\sigma)\,d\sigma\right\|_Y^p \leq \frac{1}{\tau}\int_t^{t+\tau}\|(\mathbb{F}_t u_z)(\sigma)\|_Y^p\,d\sigma \leq \kappa^p\,\|z\|^p \tag{3.14}$$

for $0 < \tau \leq t_0$, $t \geq 0$, and $z \in U$. Take $u \in L^p_{loc}(\mathbb{R}_+, U)$ and set $\tilde{u}(t,\sigma) = u(t)$ for $\sigma \geq t$ and $t \geq 0$. Then the functions

$$t \mapsto F_\tau(t) = \frac{1}{\tau}\int_t^{t+\tau}(\mathbb{F}_t u)(\sigma)\,d\sigma \quad \text{and} \quad t \mapsto \tilde{F}_\tau(t) = \frac{1}{\tau}\int_t^{t+\tau}(\mathbb{F}_t\tilde{u}(t,\cdot))(\sigma)\,d\sigma$$

are measurable for a fixed $\tau > 0$. Indeed, using (3.10) we can write

$$\begin{aligned}\tau\tilde{F}_\tau(t) - \tau\tilde{F}_\tau(r) = &\int_{r+\tau}^{t+\tau}[\mathbb{F}_t\tilde{u}(t,\cdot)](\sigma)\,d\sigma + \int_t^{r+\tau}[\mathbb{F}_t(\tilde{u}(t,\cdot) - \tilde{u}(r,\cdot))](\sigma)\,d\sigma \\ &- \int_t^{r+\tau}[\Psi_t\Phi_{t,r}\tilde{u}(r,\cdot)](\sigma)\,d\sigma - \int_r^t[\mathbb{F}_r\tilde{u}(r,\cdot)](\sigma)\,d\sigma\end{aligned}$$

for $t \geq r \geq t - \tau$. This identity and straightforward estimates imply the left continuity of $\tilde{F}_\tau$ if u is continuous. Thus $\tilde{F}_\tau$ is measurable by approximation and (3.14). The function F_τ can be handled in the same way. If Σ is regular, we deduce from Lebesgue's theorem and (3.14) that

$$\lim_{\tau\searrow 0}\int_0^{t_0}\|\tilde{F}_\tau(t)\|_Y^p\,dt = 0. \tag{3.15}$$

Similarly, $\varphi_u(\cdot,\tau)^{\frac{1}{p}}$ is measurable and absolute regularity yields

$$\lim_{\tau\searrow 0}\int_0^{t_0} \tfrac{1}{\tau}\,\varphi_u(t,\tau)\,dt := \lim_{\tau\searrow 0}\int_0^{t_0} \frac{1}{\tau}\int_t^{t+\tau} \|[\mathbb{F}_t\,\tilde{u}(t,\cdot)](\sigma)\|_Y^p\,d\sigma\,dt = 0. \tag{3.16}$$

We now show a non–autonomous version of Weiss' representation theorem 4.5 in [31].

Theorem 3.11. *Let $\Sigma = (T,\Phi,\Psi,\mathbb{F})$ be a regular non–autonomous system and let $C(s)$ and $C_\tau(s)$ be given by Definition 2.6 and (2.5). Then $\Phi(\cdot,s)u \in D(C(\cdot),s)$ and $\mathbb{F}_s u = C(\cdot)\Phi(\cdot,s)u$ for $s\ge 0$ and $u \in L^p_{loc}([s,\infty),U)$. Moreover, $C_\tau(\cdot)\Phi(\cdot,s)u \to \mathbb{F}_s u$ in $L^p_{loc}([s,\infty),Y)$ as $\tau\searrow 0$ and $\|C_\tau(\cdot)\Phi(\cdot,s)u\|_{L^p([s,s+t_0],Y)} \le c\,\|u\|_{L^p([s,s+t_0],U)}$ for $\tau\in(0,1]$ and a constant $c = c(t_0)$ independent of u and s.*

Proof. Let $t\in[s,\infty)$ be a Lebesgue point of u and $\mathbb{F}_s u$ such that the regularity condition (3.12) holds at this point t. Setting $o_t(\sigma) = u(\sigma)-u(t)$ for $\sigma\ge t$, we have

$$\mathbb{F}_s u = \mathbb{F}_t\,\tilde{u}(t,\cdot) + \mathbb{F}_t\,o_t + \Psi_t\Phi_{t,s}u \quad \text{on} \quad [t,\infty) \quad \text{and} \tag{3.17}$$

$$\Big\|\frac{1}{\tau}\int_t^{t+\tau} (\mathbb{F}_t\,o_t)(\sigma)\,d\sigma\Big\|^p \le \kappa^p\,\frac{1}{\tau}\int_t^{t+\tau} \|u(\sigma)-u(t)\|^p d\sigma. \tag{3.18}$$

Consequently, $C_\tau(t)\Phi_{t,s}u$ converges in Y to $(\mathbb{F}_s u)(t)$ as $\tau\to 0$, so that the first assertion holds. The estimate (3.18) and Fubini's theorem further yield

$$\int_s^{s+t_0} \Big\|\frac{1}{\tau}\int_t^{t+\tau}(\mathbb{F}_t\,o_t)(\sigma)\,d\sigma\Big\|^p dt \le \frac{\kappa^p}{\tau}\int_0^{\tau}\int_s^{s+t_0} \|u(t+\sigma)-u(t)\|^p dt\,d\sigma, \tag{3.19}$$

$$\int_s^{s+t_0} \Big\|\frac{1}{\tau}\int_t^{t+\tau}[(\mathbb{F}_s u)(\sigma)-(\mathbb{F}_s u)(t)]\,d\sigma\Big\|^p dt \le \frac{\kappa^p}{\tau}\int_0^{\tau}\int_s^{s+t_0} \|(\mathbb{F}_s u)(t+\sigma)-(\mathbb{F}_s u)(t)\|^p dt\,d\sigma,$$

where both terms on the right hand side converge to 0 as $\tau\to 0$. Combining these facts with (3.15) and (3.17), we establish that $\mathbb{F}_s u = L^p - \lim_\tau C_\tau(\cdot)\Phi(\cdot,s)u$. The asserted estimate follows in a similar way. □

The next approximation result complements Proposition 3.5 for absolutely regular systems. Observe that only regularity is used in the proof of estimate (3.20).

Proposition 3.12. *Let Σ be an absolutely regular non–autonomous system, $p\in(1,\infty)$, and $C(s)$ and $\Phi^n_{t,s}$ be given as in Definition 2.6 and (3.7). Then $\Phi^n(\cdot,s)u \in D(C(\cdot),s)$, $C(\cdot)\Phi^n(\cdot,s)u \to \mathbb{F}_s u$ in $L^p_{loc}([s,\infty),Y)$ as $n\to\infty$, and*

$$\|C(\cdot)\Phi^n(\cdot,s)u\|_{L^p([s,s+t_0],Y)} \le 2\kappa(t_0)\,\|u\|_{L^p([s,s+t_0],U)} \tag{3.20}$$

for $u\in L^p_{loc}([s,\infty),U)$, $s\ge 0$, $n\in\mathbb{N}$, and $t_0>0$.

Proof. Due to Proposition 2.11 we have $\Phi^n(\cdot,s)u = \mathbb{K}_s B_n u \in D(C(\cdot),s)$. Formula (3.9), Proposition 2.11, and Theorem 3.11 further yield

$$\mathbb{F}_s u - C(\cdot)\Phi^n(\cdot,s)u = C(\cdot)r_n(\cdot;\,u) = L^p - \lim_{\tau\to 0} C_\tau(\cdot)r_n(\cdot;\,u).$$

Using Hölder's inequality, Fubini's theorem, and Theorem 3.11, we now derive

$$\begin{aligned} \|C(\cdot)r_n(\cdot;\,u)\|^p_{L^p([s,s+t_0],Y)} &\le \lim_{\tau\to 0} n\int_s^{s+t_0}\int_{t-\frac{1}{n}}^{t}\|C_\tau(t)\Phi_{t,\sigma}u\|_Y^p\,d\sigma\,dt \\ &\le n\int_{s-\frac{1}{n}}^{s+t_0}\|\mathbb{F}_\sigma u\|^p_{L^p([\sigma,\sigma+1/n],Y)}\,d\sigma \\ &\le n\kappa(t_0)\int_0^{\frac{1}{n}}\int_{s-\frac{1}{n}}^{s+t_0}\|u(t+\sigma)\|^p\,d\sigma\,dt \le \kappa(t_0)\,\|u\|^p_{L^p([s,s+t_0],U)}. \end{aligned} \tag{3.21}$$

(Here we have considered a function $u\in L^p([s,s+t_0],U)$ and extended it by 0 to $\mathbb{R}$.) Thus (3.20) holds. The estimate (3.21) also gives

$$\begin{aligned} &\|C(\cdot)r_n(\cdot;\,u)\|_{L^p([s,s+t_0],Y)} \\ &\quad\le\Big(\int_{s-\frac{1}{n}}^{s+t_0} n\,\|\mathbb{F}_\sigma\, o_\sigma\|^p_{L^p([\sigma,\sigma+\frac{1}{n}],Y)}\,d\sigma\Big)^{\frac{1}{p}}+\Big(\int_{s-\frac{1}{n}}^{s+t_0} n\,\|\mathbb{F}_\sigma\,\tilde u(\sigma,\cdot)\|^p_{L^p([\sigma,\sigma+\frac{1}{n}],Y)}\,d\sigma\Big)^{\frac{1}{p}}. \end{aligned}$$

The right hand side tends to 0 as in (3.19) and (3.16). □

4. The main result and discussion

Let Σ be a regular non–autonomous system, $\Delta(\cdot)\in L^\infty(\mathbb{R}_+,\mathcal{L}_s(Y,U))$, and $C(s)$ and $\Phi^n_{t,s}$ be defined by 2.6 and (3.7). Given $x\in X$ and $s\ge 0$, we are looking for functions $x(\cdot)\in C([s,\infty),X)\cap D(C(\cdot),s)$ satisfying

$$x(t)=T(t,s)x+\Phi_{t,s}\Delta(\cdot)C(\cdot)x(\cdot) \tag{4.1}$$

or, if $\Phi(\cdot,s)u=\overline{\mathbb{K}}_sB(\cdot)u(\cdot)$ for admissible control operators $B(s)$,

$$x(t)=T(t,s)x+\int_s^t\overline{T}(t,\tau)B(\tau)\Delta(\tau)C(\tau)x(\tau)\,d\tau \tag{4.2}$$

for $t\ge s$. As shown by [28, Exa.6] one cannot allow for every bounded feedback, in general. (We note that this example gives rise to an absolutely regular autonomous system with $p=1$ and $\Delta=B=I$.) This fact motivates the next concept.

Definition 4.1. *Let $\Sigma=(T,\Phi,\Psi,\mathbb{F})$ be a well-posed non–autonomous system. We call $\Delta(\cdot)\in L^\infty(\mathbb{R}_+,\mathcal{L}_s(Y,U))$ an* admissible feedback *for Σ if there is $t_0>0$ such that the operators $I-\mathbb{F}(s+t_0,s)\Delta(\cdot)$, $s\ge 0$, on $L^p([s,s+t_0],Y)$ have uniformly bounded inverses.*

Of course, $\Delta(\cdot)$ is admissible if

$$\|\Delta(\cdot)\|_\infty<\Big[\inf_{t_0>0}\sup_{s\ge 0}\|\mathbb{F}(s+t_0,s)\|\Big]^{-1}=:q. \tag{4.3}$$

The right hand side in this estimate equals ∞ if $B(t)$ and $C(t)$ are of 'lower order', see e.g. [6] or [24, Exa.4.11]. We point out that the invertibility of $I-\mathbb{F}(s+t_0,s)\Delta(\cdot)$ is in fact necessary for some properties of the feedback system as shown by Lemma 4.3 and Proposition 5.1. Moreover, the following lemmas indicate that our notion of admissibility is quite flexible, see [22, Lem.4.1], [26, §7.1], [33, §3] for autonomous analogues.

Lemma 4.2. *Let $\Delta(\cdot) \in L^\infty(\mathbb{R}_+, \mathcal{L}_s(Y,U))$ and Σ be a well–posed non–autonomous system. If $I - \mathbb{F}(s+t_0, s)\Delta(\cdot)$ is invertible on $L^p([s, s+t_0], Y)$ for all $s \geq 0$ and some $t_0 > 0$, then $I - \mathbb{F}(s+t_1, s)\Delta(\cdot)$ is invertible on $L^p([s, s+t_1], U)$ for all $s \geq 0$ and $t_1 > 0$. Moreover, the definition of an admissible feedback is independent of $t_0 > 0$.*

Proof. We assume that $U = Y$ and $\Delta(s) = I$ for simplicity. First let $t_1 \leq t_0$. Extend a given $v \in L^p([s, s+t_1], Y)$ by 0 to $\tilde{v} \in L^p([s, s+t_0], Y)$ and set $\tilde{u} = (I - \mathbb{F}(s+t_0, s))^{-1}\tilde{v}$. Then $(I - \mathbb{F}(s+t_1, s))u = v$ for the restriction u of $\tilde{u}$. If $u = \mathbb{F}(s+t_1, s)u$, then there is a function $u_1 \in L^p([s+t_1, s+t_0], U)$ such that $(I - \mathbb{F}(s+t_0, s+t_1))u_1 = \Psi_{s+t_1}\Phi_{s+t_1,s}u$. Set $\tilde{u} = u$ on $[s, s+t_1]$ and $\tilde{u} = u_1$ on $[s+t_1, s+t_0]$. Hence, $\tilde{u} = \mathbb{F}(s+t_0, s)\tilde{u}$, so that $u = 0$. This shows the assertion in this case.

It remains to consider $t_1 = nt_0$. Proceeding by induction, we assume that the assertion is true for $t_1 = nt_0$. It is then clear that $I - \mathbb{F}(s+(n+1)t_0, s)$ is injective. For $v \in L^p([s, s+(n+1)t_0], Y)$, we set $u_1 = (I - \mathbb{F}(s+nt_0, s))^{-1}(v|[s, s+nt_0])$ and

$$u_2 = (I - \mathbb{F}(s+(n+1)t_0, s+nt_0))^{-1}\left\{v|[s+nt_0, s+(n+1)t_0] + \Psi_{s+nt_0}\Phi_{s+nt_0,s}u_1\right\}.$$

Putting u_1 and u_2 together, one sees that $I - \mathbb{F}(s+(n+1)t_0, s)$ is also surjective. $\square$

Lemma 4.3. *For maps $T : E \to F$ and $V : F \to E$ the following are equivalent.*

(a) *$I - VT$ is bijective on E.*
(b) *$I - TV$ is bijective on F.*
(c) *There is a map $S : E \to F$ such that $S - T = TVS = SVT$.*

Then we have $(I - TV)^{-1} = I + T(I - VT)^{-1}V = I + SV$, $(I - VT)^{-1} = I + VS$, and S in (c) is uniquely given by $S = (I - TV)^{-1}T = T(I - VT)^{-1}$.

In particular, in Definition 4.1 the feedback $\Delta(\cdot)$ is admissible if and only if $I - \Delta(\cdot)\mathbb{F}(s+t_0, s)$, $s \geq 0$, have a uniformly bounded inverse for some/all $t_0 > 0$ if and only if for some/all $t_0 > 0$ there are uniformly bounded operators $\mathbb{F}^\Delta(s+t_0, s)$, $s \geq 0$, such that

$$\mathbb{F}^\Delta(s+t_0, s) - \mathbb{F}(s+t_0, s) = \mathbb{F}^\Delta(s+t_0, s)\Delta(\cdot)\mathbb{F}(s+t_0, s) = \mathbb{F}(s+t_0, s)\Delta(\cdot)\mathbb{F}^\Delta(s+t_0, s).$$

We now solve (4.1) by constructing an evolution family T_Δ on X and show that the feedback system Σ^Δ is again absolutely regular if the unperturbed system is absolutely regular. Proposition 5.1 describes the relations between Σ and Σ^Δ in greater detail.

Theorem 4.4. *Assume that $\Sigma = (T, \Phi, \Psi, \mathbb{F})$ is a regular non–autonomous system and $\Delta(\cdot) \in L^\infty(\mathbb{R}_+, \mathcal{L}_s(Y,U))$ is an admissible feedback. Then the following assertions hold.*

(a) There is an evolution family $(T_\Delta(t,s))_{t \geq s \geq 0}$ on X such that $T_\Delta(\cdot, s)x \in D(C(\cdot), s)$,

$$\|C(\cdot)T_\Delta(\cdot, s)x\|_{L^p([s,s+t_0],Y)} \leq \gamma' \|x\|, \tag{4.4}$$

$x(\cdot) = T_\Delta(\cdot, s)x$ is the unique solution of (4.1), and

$$T_\Delta(t,s)x = T(t,s)x + \Phi_{t,s}\Delta(\cdot)C(\cdot)T_\Delta(\cdot, s)x \tag{4.5}$$

for $t \geq s \geq 0$, $x \in X$, and a constant γ'. If, in addition, $\Phi(\cdot, s)u = \overline{\mathbb{K}}_s B(\cdot)u(\cdot)$ for T–admissible control operators $B(t)$, then

$$T_\Delta(t,s)x = T(t,s)x + \int_s^t \overline{T}(t,\tau)B(\tau)\Delta(\tau)C(\tau)T_\Delta(\tau, s)x\, d\tau. \tag{4.6}$$

(b) If the system is absolutely regular and $p \in (1, \infty)$, then

$$T_\Delta(t,s)x = T(t,s)x + \lim_{n\to\infty} \int_s^t T_\Delta(t,\tau)[B_n(\Delta(\cdot)\Psi_s x)](\tau)\,d\tau \tag{4.7}$$

for $t \geq s \geq 0$ and $x \in X$, where the limit is taken in X and is locally uniform in t. Moreover, $\Sigma^\Delta = (T_\Delta, \Phi^\Delta, \Psi^\Delta, \mathbb{F}^\Delta)$ is an absolutely regular system, where we set

$$\Psi_s^\Delta x = C(\cdot)T_\Delta(\cdot,s)x, \quad \Phi_{t,s}^\Delta u = X - \lim_{n\to\infty} [\mathbb{K}_s^\Delta B_n u](t), \quad \mathbb{F}_s^\Delta u = L^p - \lim_{n\to\infty} C(\cdot)\mathbb{K}_s^\Delta B_n u,$$

$$\mathbb{K}_s^\Delta f(t) = \int_s^t T_\Delta(t,\tau)f(\tau)\,d\tau$$

for $t \geq s \geq 0$, $x \in X$, $u \in L^p_{loc}([s,\infty),U)$, and $f \in L^p_{loc}([s,\infty),X)$.

Proof. (a) We first prove the uniqueness of solutions to (4.1). If v solves (4.1) with $x = 0$, then $C(\cdot)v = \mathbb{F}_s\Delta(\cdot)C(\cdot)v$ by Theorem 3.11. Since $I - \mathbb{F}(s+t_1,s)\Delta(\cdot)$ is injective, $C(\cdot)v$ has to vanish a.e. on $[s, s+t_1]$, where $t_1 > 0$ can be chosen arbitrary large by Lemma 4.2. Hence, (4.1) implies that $v = 0$. To solve (4.1), we define

$$T_\Delta(t,s)x = T(t,s)x + \Phi_{t,s}\Delta(\cdot)(I - \mathbb{F}(s+t_1,s)\Delta(\cdot))^{-1}\Psi_s x \tag{4.8}$$

for $0 \leq t-s \leq t_1$ and $x \in X$. Clearly, $T_\Delta(t,s)$ is an exponentially bounded linear operator on X and $T_\Delta(\cdot,s)x$ is continuous in X by Proposition 3.5. Theorems 2.7 and 3.11 further show that $T_\Delta(\cdot,s)x \in D(C(\cdot),s)$ and

$$\begin{aligned} C(\cdot)T_\Delta(\cdot,s)x &= \Psi_s x + \mathbb{F}_s\Delta(\cdot)(I - \mathbb{F}(s+t_1,s)\Delta(\cdot))^{-1}\Psi_s x \\ &= (I - \mathbb{F}(s+t_1,s)\Delta(\cdot))^{-1}\Psi_s x, \end{aligned} \tag{4.9}$$

which yields the estimate (4.4). Inserting (4.9) into (4.8), we obtain (4.5) and thus have solved (4.1). (We note that the uniqueness of solutions implies that T_Δ does not depend on t_1.) One verifies (E1) for T_Δ using the uniqueness of (4.1), formula (3.1), and a standard argument. It remains to establish the strong continuity of T_Δ. We first take $(t_n, s_n) \to (s_0, s_0)$ with $t_n \geq s_n \geq 0$. For $\varepsilon > 0$, $x \in X$, and large n, there exists $r \in [0, s_0] \cap [0, s_n]$ such that $\|T(s_0,r)x - x\| \leq \varepsilon$. Employing (4.8) and (2.1), we estimate

$$\begin{aligned} \|T_\Delta(t_n,s_n)x - x\| &\leq \|T_\Delta(t_n,s_n)(x - T(s_n,r)x)\| + \|T_\Delta(t_n,s_n)T(s_n,r)x - x\| \\ &\leq c\,\|x - T(s_n,r)x\| + \|T(t_n,r)x - x\| + c\Big(\int_{s_n}^{t_n} \|[\Psi_r x](\sigma)\|^p\,d\sigma\Big)^{\frac{1}{p}}, \end{aligned}$$

$$\overline{\lim_{n\to\infty}}\, \|T_\Delta(t_n,s_n)x - x\| \leq (c+1)\varepsilon$$

for a constant c. Therefore T_Δ is strongly continuous at (s_0,s_0). If $(t_n,s_n) \to (t_0,s_0)$ for some $t_0 > s_0$, we may assume that $t_n > s_n, s_0$. We take $t_n \geq r_n \geq s_n, s_0$ with $r_n \to s_0$ and derive (E2) from the above results and the expression

$$T_\Delta(t_n,s_n)x - T_\Delta(t_0,s_0)x = T_\Delta(t_n,r_n)(T_\Delta(r_n,s_n)x - T_\Delta(r_n,s_0)x) + T_\Delta(t_n,s_0)x - T_\Delta(t_0,s_0)x.$$

It is obvious that (4.6) holds under the extra assumption.

(b) For $f \in \mathcal{D}_{\Delta,s}$ and $s \geq 0$, formula (4.5) and Proposition 3.5 imply that

$$\mathbb{K}_s^\Delta f(t) = \mathbb{K}_s f(t) + \lim_{n\to\infty} \int_s^t \int_\tau^t T(t,\sigma)B_n(\sigma)\Delta(\sigma)C(\sigma)T_\Delta(\sigma,\tau)f(\tau)\,d\sigma\,d\tau$$

since the integrand is the sum of functions of the form

$$(\tau,\sigma)\mapsto \varphi(\tau)\,T(t,\sigma)B_n(\sigma)\Delta(\sigma)C(\sigma)T_\Delta(\sigma,r)x.$$

For the same reason we can apply Fubini's theorem and take $T(t,\sigma)B_n(\sigma)\Delta(\sigma)C(\sigma)\in\mathcal{L}(\underline{X}_\sigma,X)$ out of the inner integral. So we arrive at

$$\mathbb{K}_s^\Delta f=\mathbb{K}_s f+\lim_{n\to\infty}\mathbb{K}_s B_n(\cdot)\Delta(\cdot)C(\cdot)\mathbb{K}_s^\Delta f=\mathbb{K}_s f+\Phi(\cdot,s)\Delta(\cdot)C(\cdot)\mathbb{K}_s^\Delta f \tag{4.10}$$

for $f\in\mathcal{D}_{\Delta,s}$ due to Proposition 3.5. Proposition 2.11 and Theorem 3.11 now show that $\mathbb{K}_s^\Delta f\in D(C(\cdot),s)$ and $C(\cdot)\mathbb{K}_s^\Delta f=C(\cdot)\mathbb{K}_s f+\mathbb{F}_s\Delta(\cdot)C(\cdot)\mathbb{K}_s^\Delta f$. Hence,

$$C(\cdot)\mathbb{K}_s^\Delta f=(I-\mathbb{F}(s+t_1,s)\Delta(\cdot))^{-1}C(\cdot)\mathbb{K}_s f \tag{4.11}$$

on $[s,s+t_1]$ for each $t_1>0$. Inserting (4.11) into (4.10), we conclude that

$$\mathbb{K}_s^\Delta f=\mathbb{K}_s f+\Phi(\cdot,s)\Delta(\cdot)(I-\mathbb{F}(s+t_1,s)\Delta(\cdot))^{-1}C(\cdot)\mathbb{K}_s f \tag{4.12}$$

on $[s,s+t_1]$ for $f\in\mathcal{D}_{\Delta,s}$. This identity holds for all $f\in L^p_{loc}([s,\infty),X)$ by Proposition 2.11. So we may take $f=B_n u$ for $u\in L^p_{loc}([s,\infty),U)$ and $n\in\mathbb{N}$, and thus obtain

$$\mathbb{K}_s^\Delta B_n u=\mathbb{K}_s B_n u+\Phi(\cdot,s)\Delta(\cdot)(I-\mathbb{F}(s+t_1,s)\Delta(\cdot))^{-1}C(\cdot)\mathbb{K}_s B_n u. \tag{4.13}$$

As a consequence, $\mathbb{K}_s^\Delta B_n u\in D(C(\cdot),s)$ and

$$C(\cdot)\mathbb{K}_s^\Delta B_n u=(I-\mathbb{F}(s+t_1,s)\Delta(\cdot))^{-1}C(\cdot)\mathbb{K}_s B_n u \tag{4.14}$$

by Proposition 2.11 and Theorem 3.11. In view of Propositions 3.5 and 3.12, we can take the limit as $n\to\infty$ in the formulas (4.13) and (4.14) (in $C([s,s+t_1],X)$ and $L^p([s,s+t_1],Y)$, respectively). It is then easy to see that the quadrupel $(T_\Delta,\Phi^\Delta,\Psi^\Delta,\mathbb{F}^\Delta)$ defined in the assertion is a well–posed non–autonomous system. Equation (4.14) and Proposition 3.12 further yield

$$\int_t^{t+\tau}\|(\mathbb{F}_t^\Delta u_z)(\sigma)\|^p\,d\sigma\le c\int_t^{t+\tau}\|(\mathbb{F}_t u_z)(\sigma)\|^p\,d\sigma$$

for $t\ge 0$, $\tau>0$, $u_z\equiv z$, $z\in U$, so that the perturbed system is also absolutely regular.

We now choose $u=\Delta(\cdot)\Psi_s x$ for $x\in X$ and deduce from (4.13), Propositions 3.5 and 3.12, and (4.8) that

$$\begin{aligned}\lim_{n\to\infty}\mathbb{K}_s^\Delta B_n\Delta(\cdot)\Psi_s x&=\Phi(\cdot,s)\Delta(\cdot)\Psi_s x+\Phi(\cdot,s)\Delta(\cdot)(I-\mathbb{F}(s+t_1,s)\Delta(\cdot))^{-1}\mathbb{F}_s\Delta(\cdot)\Psi_s x\\&=\Phi(\cdot,s)\Delta(\cdot)(I-\mathbb{F}(s+t_1,s)\Delta(\cdot))^{-1}\Psi_s x\\&=T_\Delta(\cdot,s)x-T(\cdot,s)x,\end{aligned}$$

where the limit is taken in X and is locally uniform in t. Thus (4.7) holds. □

We state several variants of Theorem 4.4 and compare it to related results.

Remark 4.5. The above theorem remains valid if we do not assume (E3), require β,γ,κ and $\|(I-\mathbb{F}(s+t_0,s)\Delta(\cdot))^{-1}\|$ only to be uniform w.r.t. $s\in[0,a]$ for every $a>0$, and assert for the perturbed problem only the analogous properties. The proof of part (a) also works in the case that T (and then T_Δ) is only strongly continuous in t and s separately. Part (a) can be verified for admissible systems, too, if one only considers $x\in D(C(s))$ in (4.2), (4.4), and (4.6), see [24, Thm.4.18] or (c) below.

Remark 4.6. Let Σ be a non–autonomous regular system with $p \in (1,\infty)$. It can be shown that $\langle C(\cdot)\mathbb{K}_s B_n u, v\rangle \to \langle \mathbb{F}_s u, v\rangle$ as $n \to \infty$ for all $v \in L^q([s,s+t_0],Y^*)$, $\frac{1}{p}+\frac{1}{q}=1$. Thus, if Y is reflexive, the conclusions of Theorem 4.4(b) hold for merely regular systems except that the limits exist only weakly and that it is not clear whether Σ^Δ is regular again. In the autonomous case the regularity of Σ^Δ for regular Σ was established in [33, Thm.4.5, 4.7], but the proof given there relies on Laplace transforms and a Tauberian theorem, [32, Thm.5.2], not available here; see also [26, §7.5].

(a) Perturbation theory of evolution equations. First, let $B(t)$ be T–admissible control operators and define $Y = X$, $\Psi_s = T(\cdot,s)$, and $\mathbb{F}_s = \overline{\mathbb{K}}_s B(\cdot)$, i.e., $C(t) \equiv I$. This clearly gives an absolutely regular non–autonomous system.

Second, let (T,Ψ) be a non–autonomous observation system for $p \in (1,\infty)$ represented by $C(t)$. Setting $U = X$, $\Phi_{t,s}u = (\mathbb{K}_s u)(t)$, and $\mathbb{F}_s = C(\cdot)\mathbb{K}_s$, i.e., $B(t) \equiv I$, we obtain a well–posed non–autonomous system thanks to Proposition 2.11. Moreover,

$$\begin{aligned}\Big(\frac{1}{\tau}\int_t^{t+\tau} \|[\mathbb{F}_t u_z](s)\|^p\,ds\Big)^{\frac{1}{p}}&\\ &\le \Big(\frac{1}{\tau}\int_t^{t+\tau} \|[\mathbb{F}_t (u_z - T(\cdot,t)z)](s)\|^p\,ds\Big)^{\frac{1}{p}} + \Big(\frac{1}{\tau}\int_t^{t+\tau} \|[\mathbb{F}_t T(\cdot,t)z](s)\|^p\,ds\Big)^{\frac{1}{p}}\\ &\le \kappa \sup_{t\le s\le t+\tau} \|z - T(s,t)z\| + \Big(\int_t^{t+\tau} \tau^{p-1}\,\|[\Psi_t z](s)\|^p\,ds\Big)^{\frac{1}{p}}\end{aligned}$$

for $z \in X$ so that the system is absolutely regular.

As a result, Theorem 4.4 is a joint non–autonomous extension of the Desch–Schappacher and Miyadera perturbation theorem from semigroup theory, see e.g. [9, §III.3]. For suitable $\underline{X}$ and $\overline{X}$, perturbations $B : \underline{X} \to \overline{X}$ of a generator A of an analytic semigroup can be treated via resolvent estimates, cf. [16, Thm.8]. A non–autonomous Miyadera theorem for closable perturbations $C(t)$ and $p \ge 1$ was proved in [20] by other methods.

(b) Autonomous controlled systems. Let $T(t-s) = T(t,s)$ be a C_0–semigroup generated by A and $\Delta(t) \equiv \Delta$. We say that (T,B,C) belongs to the *Pritchard–Salamon class*, [19], if (2.3) holds with $\|x\|_X$ replaced by $\|x\|_{\overline{X}}$ and (3.6) holds with $\|\cdot\|_X$ replaced by $\|\cdot\|_{\underline{X}}$. Thus one is working in a two space setting (without X). The perturbation theory for this class was developed in detail in [6]. In this case one can extend $T_\Delta(t)$ to $\overline{X}$ and the number q in (4.3) is equal to ∞.

G. Weiss introduced autonomous regular systems in [29]–[32] similarly as in the above definitions by only considering the initial time $s = 0$. He solved the feedback problem in [33, Thm.6.1] for a well–posed system with $p = 2$ on Hilbert spaces X, Y, U allowing for non-trivial feedthrough D and assuming that (roughly speaking) $I - CR(\lambda, A_{-1})B\Delta$ is invertible on a right halfspace, see also [22]. If the system is regular, the feedback system is again regular and can be represented almost in the natural way, see [33, §7]. The feedback theory for several classes of (non–)regular systems is exhaustively studied in Chapter 7 of Staffan's monograph [26] in a Banach space setting and also for $p = 1, \infty$.

The remaining difficulties come from the fact that in general $T_\Delta(t)$ cannot be continuously extended to the extrapolation space X_{-1}^A corresponding to T, see [24, Exa.4.20]; in particular, the extrapolation space X_{-1}^Δ of T_Δ may differ from X_{-1}^A. Weiss constructed subspaces W and W_Δ of X_{-1}^A and X_{-1}^Δ, respectively, such that $Jx := X_{-1}^\Delta -$

$\lim_{\lambda\to\infty} \lambda R(\lambda, A_{-1})x$ defines an isomorphism $J : W \to W_\Delta$, see [33, Thm.7.7]. Note that $Jx = x$ for $x \in X$. Then

$$T_\Delta(t)x = T(t)x + \int_0^t T_{\Delta,-1}(t-\tau)JB\Delta CT(\tau)x\,d\tau$$

by (6.11), (6.1), and p.55 of [33]. In other words, Weiss managed to put the limes in (4.7) inside the integral using a different regularization. Identifying B and JB, he represented the feedback system in terms of B and C and computed the generator of T_Δ, [33, §7]; see [26, §7.4] for a somewhat different approach.

(c) Non–autonomous controlled systems. Part (a) of Theorem 4.4 was proved by D. Hinrichsen, B. Jacob, and A.J. Pritchard for non–autonomous admissible systems in a slightly differing setting, see [11, Thm.3.2] and [13], [15] also for non–linear feedback. They work with separately strongly continuous evolution families and have some additional technical assumptions (see e.g. Hypotheses 4 and 7 of [11]). Moreover, they obtain (4.6) with a pointwise representation of $C(\cdot)T(\cdot,s)x$ only for $x \in D(C(s))$. The issues investigated in Theorem 4.4(b) were not considered in [11], and in [13, Thm.3.4.7] only for systems of Pritchard–Salamon type.

Remark 4.7. In the situation of Theorem 4.4(b), we additionally assume that $\Phi_{t,s}$ is given by admissible observation operators $B(\cdot) \in L^\infty(\mathbb{R}_+, \mathcal{L}_s(U, \overline{X}))$ for $\overline{X}_t \equiv \overline{X}$ and that $T_\Delta(t,s)$ has a locally uniformly bounded extension $\overline{T}_\Delta(t,s) : \overline{X} \to \overline{X}$. Thus, $(t,s) \mapsto \overline{T}_\Delta(t,s)$ is strongly continuous, and we can define

$$(\overline{\mathbb{K}}_s^\Delta f)(t) = \int_s^t \overline{T}_\Delta(t,\tau)f(\tau)\,d\tau, \qquad t \ge s \ge 0,$$

for $f \in L^1_{loc}([s,\infty), \overline{X})$. Then we obtain $\Phi^\Delta(\cdot,s)u = \overline{\mathbb{K}}_s^\Delta B(\cdot)u$, $\mathbb{F}_s^\Delta u = C(\cdot)\overline{\mathbb{K}}_s^\Delta B(\cdot)u$, and

$$T_\Delta(t,s)x = T(t,s)x + \int_s^t \overline{T}_\Delta(t,\tau)B(\tau)\Delta(\tau)C(\tau)T(\tau,s)x\,d\tau. \tag{4.15}$$

for $u \in L^p_{loc}([s,\infty), U)$, $x \in X$, and $t \ge s \ge 0$.

Proof. Observe that $B_n u \to B(\cdot)u$ as $n \to \infty$ in $L^p_{loc}(\mathbb{R}_+, \overline{X})$ for $u \in L^p_{loc}(\mathbb{R}_+, \overline{X})$ because of the inequality

$$\|B_n u - B(\cdot)u\|^p_{L^p([0,t_0],\overline{X})} \le n \int_0^{\frac{1}{n}} \int_0^{t_0} \|\overline{T}(\tau+\sigma,\tau)B(\tau)u(\tau) - B(\tau+\sigma)u(\tau+\sigma)\|^p_{\overline{X}}\,d\tau\,d\sigma$$

which is a consequence of Hölder's inequality and Fubini's theorem. Thus $\Phi^\Delta(\cdot,s)u = \overline{\mathbb{K}}_s^\Delta B(\cdot)u$ and (4.15) holds. The identities (4.13) and (4.14) then imply that

$$C(\cdot)\overline{\mathbb{K}}_s^\Delta B(\cdot)u = (I - \mathbb{F}(s+t_1,s)\Delta(\cdot))^{-1}\mathbb{F}_s u = \mathbb{F}_s^\Delta u. \qquad \square$$

The above remark and paragraph (b) indicate that an (absolutely) regular non–autonomous system and the corresponding feedback system can be represented by operators $B(t)$ (and not just approximately by $B_n(t)$) whenever we have a decent extrapolation theory for the given problem. It seems to be reasonable to study first the case that T is generated by operators $A(t)$ and consider spaces $\overline{X}_t$ related to $A(t)$. For various results on parabolic evolution equations and extrapolation spaces we refer to [1], [2, Chap.V], [24, Prop.2.1].

5. Further properties of the feedback system

In the setting of Theorem 4.4 we study the relationship between the open and the closed–loop system in more detail, see [33, §6] for similar results in the autonomous case. To put the formulas in a concise form, we define $\Psi(t,s)x = \mathbf{1}_{[s,t]}\Psi_s x$ and

$$\Sigma(t,s) = \begin{pmatrix} T(t,s) & \Phi(t,s) \\ \Psi(t,s) & \mathbb{F}(t,s) \end{pmatrix} : X \times L^p([s,t],U) \to X \times L^p([s,t],Y), \quad t \geq s \geq 0.$$

Proposition 5.1. *Let Σ be an absolutely regular non–autonomous system, $p \in (1,\infty)$, $\Delta(\cdot)$ be an admissible feedback, and Σ^Δ be the feedback system from Theorem 4.4. Then*

$$\mathbb{F}_s^\Delta = (I - \mathbb{F}_s\Delta(\cdot))^{-1}\mathbb{F}_s = \mathbb{F}_s(I - \Delta(\cdot)\mathbb{F}_s)^{-1} = C(\cdot)\Phi^\Delta(\cdot,s), \tag{5.1}$$

$$\Sigma^\Delta(t,s) - \Sigma(t,s) = \Sigma(t,s)\begin{pmatrix} 0 & 0 \\ 0 & \Delta(\cdot) \end{pmatrix}\Sigma^\Delta(t,s) = \Sigma^\Delta(t,s)\begin{pmatrix} 0 & 0 \\ 0 & \Delta(\cdot) \end{pmatrix}\Sigma(t,s). \tag{5.2}$$

Proof. The first equality in (5.1) is an immediate consequence of (4.14) and Proposition 3.12. Lemma 4.3 then yields the second equality in (5.1) and the expressions for $\mathbb{F}^\Delta - \mathbb{F}$ in the lower right corner of (5.2). Taking the limit in (4.13) and using the formulas for $\mathbb{F}^\Delta$, we deduce the last equality in (5.1). The identities for $T^\Delta - T$ in the upper left corner of (5.2) were established in Theorem 4.4, and they imply the formulas for $\Psi^\Delta - \Psi$ in the lower left corner in (5.2). The first equality in the upper right corner follows from (4.13). Employing the previous results, we finally obtain

$$\begin{aligned} \Phi^\Delta(\cdot,s)\Delta(\cdot)\mathbb{F}_s &= \Phi(\cdot,s)\Delta(\cdot)\mathbb{F}_s + \Phi(\cdot,s)\Delta(\cdot)\mathbb{F}_s^\Delta\Delta(\cdot)\mathbb{F}_s \\ &= \Phi(\cdot,s)\Delta(\cdot)\mathbb{F}_s^\Delta = \Phi^\Delta(\cdot,s) - \Phi(\cdot,s). \end{aligned}$$ □

The above result allows to prove that the following control theoretic properties remain unchanged under feedback, see e.g. [7] for the non–autonomous case.

Definition 5.2. *(a) A non–autonomous control system (T,Φ) is called* exactly (approximately) controllable on $[s,t]$ *if $\Phi(t,s)$ is surjective (has dense range) and it is called* exactly (approximately) null controllable on $[s,t]$ *if $T(t,s)X$ is contained in the (closure of) $\Phi(t,s)L^p([s,t],U)$.*
(b) A non–autonomous observation system (T,Ψ) is called (continuously) initially observable on $[s,t]$ *if $\Psi(t,s)$ is injective (bounded from below) and* (continuously) finally observable on $[s,t]$ *if $\ker\Psi(t,s) \subset \ker T(t,s)$ (if $\|T(t,s)x\| \leq c\,\|\Psi(t,s)x\|_p$ for a constant $c > 0$ and $x \in X$).*

Proposition 5.3. *Let Σ be an absolutely regular non–autonomous system, $p \in (1,\infty)$, $\Delta(\cdot)$ be an admissible feedback, and Σ^Δ be the corresponding feedback system. Then Σ possesses one of the properties in Definition 5.2 if and only if Σ^Δ has the same property.*

Proof. (1) The assertions concerning exact (approximate) controllability and (continuous) initial observability follow from the formulas

$$\begin{aligned} \Phi^\Delta(t,s) &= \Phi(t,s)(I + \Delta(\cdot)\mathbb{F}_s^\Delta), & \Phi(t,s) &= \Phi^\Delta(t,s)(I - \Delta(\cdot)\mathbb{F}_s), \\ \Psi^\Delta(t,s) &= (I + \mathbb{F}_s^\Delta\Delta(\cdot))\Psi(t,s), & \Psi(t,s) &= (I - \mathbb{F}_s\Delta(\cdot))\Psi^\Delta(t,s), \end{aligned}$$

which are immediate conseqences of (5.2).

(2) Assume that T is null controllable. For $x \in X$ there is $u \in L^p([s,t],U)$ such that $T(t,s)x = \Phi_{t,s}u$. Thus (5.2) yields $T_\Delta(t,s)x = \Phi^\Delta(t,s)[u - \Delta(\cdot)\mathbb{F}_s u + \Delta(\cdot)\Psi_s x]$ and T_Δ is null controllable. The converse implication and the equivalence for approximate null controllability are shown in the same way.

(3) Assume that T is continuously finally observable. Using (5.2), we estimate

$$\begin{aligned} \|T^\Delta(t,s)x\| &\le \|T(t,s)x\| + \|\Phi_{t,s}\Delta(\cdot)\Psi^\Delta(t,s)x\| \\ &\le c\,\|\Psi(t,s)x\|_p + c_1\,\|\Psi^\Delta(t,s)x\|_p \\ &\le (c+c_1)\,\|\Psi^\Delta(t,s)x\|_p + \|\mathbb{F}_s\Delta(\cdot)\Psi^\Delta(t,s)x\|_p \le c_2\,\|\Psi^\Delta(t,s)x\|_p \end{aligned} \tag{5.3}$$

so that T^Δ is continuously finally observable. If T is finally observable and $\Psi^\Delta(t,s)x = 0$, then $\Psi(t,s)x = -\mathbb{F}_s\Delta(\cdot)\Psi^\Delta(t,s)x = 0$. Hence, $T(t,s)x = 0$, and (5.3) yields $T^\Delta(t,s)x = 0$. The converse implications are proved similarly. ◻

Theorem 4.4 also guarantees that repeated feedbacks behave nicely.

Proposition 5.4. *Let Σ be an absolutely regular non–autonomous system with $p \in (1,\infty)$, $\Delta(\cdot)$ be an admissible feedback, Σ^Δ be the corresponding feedback system, and $\tilde\Delta(\cdot) \in L^\infty(\mathbb{R}_+,\mathcal{L}_s(Y,U))$. Then $\tilde\Delta(\cdot)$ is admissible for Σ^Δ if and only if $\Delta(\cdot)+\tilde\Delta(\cdot)$ is admissible for Σ. If this is the case, then $\Sigma^{\Delta+\tilde\Delta} = (\Sigma^\Delta)^{\tilde\Delta}$.*

Proof. Proposition 5.1 implies that

$$\begin{aligned} \mathbb{F}_s\,[I - \tilde\Delta(\cdot)\mathbb{F}_s^\Delta] &= [I - \mathbb{F}_s(\Delta(\cdot)+\tilde\Delta(\cdot))]\,\mathbb{F}_s^\Delta, \\ [I - \mathbb{F}_s^\Delta\,\tilde\Delta(\cdot)]\,\mathbb{F}_s &= \mathbb{F}_s^\Delta\,[I - (\Delta(\cdot)+\tilde\Delta(\cdot))\mathbb{F}_s]. \end{aligned} \tag{5.4}$$

Assume that $\Delta(\cdot)+\tilde\Delta(\cdot)$ is admissible for Σ. We then deduce from (5.4) and (5.1) that

$$\mathbb{F}_s^{\Delta+\tilde\Delta} - \mathbb{F}_s^\Delta = \mathbb{F}_s^{\Delta+\tilde\Delta}\,\tilde\Delta(\cdot)\mathbb{F}_s^\Delta = \mathbb{F}_s^\Delta\,\tilde\Delta(\cdot)\mathbb{F}_s^{\Delta+\tilde\Delta},$$

so that $\tilde\Delta(\cdot)$ is admissible for Σ^Δ by Lemma 4.3. The converse implication is proved in the same way. The second assertion follows similarly from (5.2) and Lemma 4.3. ◻

We introduce a basic asymptotic property of evolution equations, see e.g. [4], [9].

Definition 5.5. *An evolution family T has an* exponential dichotomy *(or is called* hyperbolic*) if there are projections $P(t)$, $t \ge 0$, and constants $N, \delta > 0$ such that $P(\cdot) \in C_b(\mathbb{R}_+,\mathcal{L}_s(X))$ and, for $t \ge s \ge 0$,*

(a) *$T(t,s)P(s) = P(t)T(t,s)$,*
(b) *the restriction $T_Q(t,s) : Q(s)X \to Q(t)X$ of $T(t,s)$ has the inverse $T_Q(s,t)$ (where $Q(t) = I - P(t)$),*
(c) *$\|T(t,s)P(s)\| \le Ne^{-\delta(t-s)}$ and $\|T_Q(s,t)Q(t)\| \le Ne^{-\delta(t-s)}$.*

If $P(t) \equiv I$, then T is called exponentially stable.

Persistence of dichotomy under perturbations mapping from spaces $\underline{X}_t$ into X has been studied intensively, see [4, §5.2], [23, §5], and the references given there. The next result also holds for admissible systems, see [24, Thm.4.23].

Theorem 5.6. *Assume that $(T,\Phi,\Psi,\mathbb{F})$ is a regular non–autonomous system and that $\Delta(\cdot) \in L^\infty(\mathbb{R}_+,\mathcal{L}_s(Y,U))$ is admissible with $k(t_0) := \sup_s \|(I - \mathbb{F}(s+t_0,s)\Delta(\cdot))^{-1}\|$.*

Suppose that T has an exponential dichotomy with constants $N, \delta > 0$ and projections $P(t)$. Then there is a number $\varepsilon_0 = \varepsilon_0(N, \delta, t_0) > 0$ such that

$$k(t_0)\beta(t_0)\gamma(t_0)\,\|\Delta(\cdot)\|_\infty \le \varepsilon_0$$

implies that T_Δ is hyperbolic with projections having the same rank as $P(t)$ and $Q(t)$.

Proof. We extend the evolution families T and T_Δ to the time interval $\mathbb{R}$ by setting $T(t,s) = T_\Delta(t,s) = \exp[(t-s)\delta(Q(0) - P(0))]$ for $0 \ge t \ge s$ and $T_{(\Delta)}(t,s) = T_{(\Delta)}(t,0)\exp[-s\delta(Q(0) - P(0))]$. Observe that we can take $k(t_1) = k(t_0)$ for $0 < t_1 \le t_0$ by the proof of Lemma 4.2 and that $\exp[t\delta(Q(0)-P(0))] = e^{-\delta t}P(0) + e^{\delta t}Q(0)$. Therefore (4.8) yields

$$\|T_\Delta(s+t_0, s) - T(s+t_0, s)\| \le (1 + e^{\delta t_0})Nk(t_0)\beta(t_0)\gamma(t_0)\,\|\Delta(\cdot)\|_\infty$$

The assertion then follows from [25, Prop.2.3] (see also [4, Thm.5.23] and the references therein), where $\varepsilon_0 := (1 - e^{\delta t_0})^2\,((1 + e^{\delta t_0})8N^3)^{-1}$. □

We finally characterize the exponential stability of T from the perspective of control theory using the following notions. Recall the representation results Theorem 2.7 and Proposition 3.5 to see the relationsship with the concepts in e.g. [5], [6], [21], [26, §8.2].

Definition 5.7. *A non–autonomous control system (T, Φ) is called* stabilizable *if there exists an observation system (T_F, Ψ^F) with an exponentially stable evolution family T_F on X such that $T_F(t,s)x = T(t,s)x + \Phi_{t,s}\Psi_s^F x$ for all $x \in X$ and $t \ge s \ge 0$.*

Definition 5.8. *A non–autonomous observation system (T, Ψ) is called* detectable *if there exists a control system (T_K, Φ^K) with an exponentially stable evolution family T_K on X such that $T_K(t,s)x = T(t,s)x + \Phi_{t,s}^K\Psi_s x$ for all $x \in X$ and $t \ge s \ge 0$.*

The following theorem relates the exponential stability of T, i.e., *internal stability*, with the boundedness of $\mathbb{F} : L^p(\mathbb{R}_+, U) \to L^p(\mathbb{R}_+, Y)$, the so–called *input–output stability.* Versions of Theorem 5.9 for the autonomous Hilbert space setting are proved in [6, Thm.5.8] for the Pritchard–Salamon class and in [21, Cor.1.8] for regular systems. In that case the input–output stability can be replaced by the equivalent condition that the transfer function $H(\lambda) = CR(\lambda, A_{-1})B$ is bounded for $\operatorname{Re}\lambda > 0$, cf. [32, Thm.3.1]. In [5, Thm.4.3] our theorem was shown for bounded control and observation operators using the characterization of exponential stability given in [4, Thm.3.26]. Here we employ Datko's theorem, [8, Thm.1, Rem.3], in order to avoid some technical problems. But we remark that Datko's theorem can be deduced from [4, Thm.3.26], see [24, Thm.1.19]. A variant of the next result holds for admissible systems, [24, Thm.4.29].

Theorem 5.9. *Let $\Sigma = (T, \Phi, \Psi, \mathbb{F})$ be a regular non–autonomous system. Then the following assertions are equivalent.*

(a) *T is exponentially stable.*
(b) *(T, Φ) is stabilizable and $\Phi(\cdot, 0) \in \mathcal{L}(L^p(\mathbb{R}_+, U), L^p(\mathbb{R}_+, X))$.*
(c) *(T, Ψ) is detectable and $\|\Psi_s x\|_{L^p([s,\infty),Y)} \le c\,\|x\|$ for $s \ge 0$, $x \in X$, and a constant c.*
(d) *Σ is detectable and stabilizable and $\mathbb{F} \in \mathcal{L}(L^p(\mathbb{R}_+, U), L^p(\mathbb{R}_+, Y))$.*

Proof. Let (a) hold. Then Σ is always stabilizable (take $\Psi^F = 0$) and detectable (take $\Phi^K = 0$). The other assertions in (b)–(d) follow from Lemmas 2.3, 3.2, and 3.7.

Extending $u \in L^p([s,\infty), U)$ by 0 to $\mathbb{R}_+$ and using causality, we see that the norms of $\Phi(\cdot, s)$ and $\mathbb{F}_s$ decrease as s increases. The assumptions in (b) and Lemma 2.3 yield $\|T(\cdot,s)x\|_{L^p([s,\infty),Y)} \le c\,\|x\|$ for $s \ge 0$, $x \in X$, and a constant c. Thus (a) is a consequence of Datko's theorem, [8, Thm.1, Rem.3]. The implication '(c)$\Rightarrow$(a)' can be proved in the same way. If Σ is stabilizable, then Theorems 2.7 and 3.11 show that the operators $C(t)$ representing Ψ are also T_F–admissible and $\Psi_s x = C(\cdot)T_F(\cdot,s)x - \mathbb{F}_s\Psi^F_s x$ for $s \ge 0$ and $x \in X$. Hence, (d) implies (c) by Lemmas 2.3 and 2.5. $\square$

6. A parabolic problem with point control and observation

To illustrate Theorem 4.4, we give an easily accessible example which can be generalized in various directions. In [24, Exa.4.22] we considered a similar problem in one space dimension by a different method.

Let $\Omega \subset \mathbb{R}^n$, $n \ge 3$, be a bounded domain with C^2–boundary $\partial\Omega$ and $a_{kl}, a_k, a_0 \in C_b^\alpha(\mathbb{R}_+ \times \overline{\Omega})$, $k, l = 1, \cdots, n$, $\alpha \in (0,1)$, be real such that $\sum_{kl} a_{kl}(t,x)v_k v_l \ge \eta\,|v|^2$ for a constant $\eta > 0$ and $v \in \mathbb{R}^n$, $t \ge 0$, $x \in \overline{\Omega}$. Let $g \in C_0(\Omega)$, $s \ge 0$, $D_k = \frac{\partial}{\partial x_k}$, and

$$A(t,x,D) = \sum\nolimits_{kl} a_{kl}(t,x)D_k D_l + \sum\nolimits_{k} a_k(t,x)D_k + a_0.$$

The unique solution $w \in C([s,\infty) \times \overline{\Omega}) \cap C^{1,2}((s,\infty) \times \Omega)$ of the problem

$$\begin{aligned} w_t(t,x) &= A(t,x,D)w(t,x), \qquad x \in \Omega,\ t > s, \\ w(t,x) &= 0, \qquad x \in \partial\Omega,\ t \ge s, \\ w(s,x) &= g(x), \qquad x \in \Omega, \end{aligned} \tag{6.1}$$

is given by $w(t,x) = \int_\Omega k(t,s,x,y)g(y)\,dy$ for a continuous kernel $k(t,s,x,y)$, $t > s \ge 0$, $x, y \in \Omega$, satisfying the Gaussian estimate

$$|k(t,s,x,y)| \le M(t-s)^{-\frac{n}{2}} \exp(-w\,\tfrac{|x-y|^2}{t-s}) \tag{6.2}$$

for $0 < t - s \le t_0$ and constants $M, w > 0$ possibly depending on t_0, see e.g. [17, §IV.16]. By the uniqueness of solutions, we also have

$$k(t,s,x,y) = \int_\Omega k(t,r,x,z)k(r,s,z,y)\,dz, \quad t > r > s \ge 0,\ x, y \in \Omega. \tag{6.3}$$

We take $q, p \in (1,\infty)$ such that $\frac{2}{np} < \frac{1}{q} < 1 - \frac{2}{np'}$, where $\frac{1}{p} + \frac{1}{p'} = 1$. This is possible since $n \ge 3$. For instance, if $p = 2$ and $n = 3$, then $q \in (3/2, 3)$. Observe that

$$q > \frac{n}{n - \frac{2}{p'}} \qquad \text{and} \qquad q' > \frac{n}{n - \frac{2}{p}}\,. \tag{6.4}$$

Further, let $b, c : \mathbb{R}_+ \to \Omega$ be uniformly Lipschitz such that $|b(t) - c(t)| \ge \delta > 0$ for $t \ge 0$. We then define $X = L^q(\Omega)$, $U = Y = \mathbb{C}$, $T(s,s) = I$, $\Phi(s,s)u = 0$, and

$$T(t,s)f = \int_\Omega k(t,s,\cdot,y)f(y)\,dy, \qquad \Phi_{t,s}u = \int_s^t k(t,\tau,\cdot,b(\tau))u(\tau)\,d\tau,$$

$$(\Psi_s f)(t) = \int_\Omega k(t,s,c(t),y)f(y)\,dy, \qquad (\mathbb{F}_s u)(t) = \int_s^t k(t,\tau,c(t),b(\tau))u(\tau)\,d\tau$$

for $t > s \geq 0$, $f \in X$, and $u \in L^p_{loc}(\mathbb{R}_+)$. These maps correspond to the pde (6.1) complemented by the control $B(t)u(t) = \delta_{b(t)}u(t)$ and the output $y(t) = w(t, c(t))$.

The operators $T(t,s)$ yield an evolution family on X due to standard elliptic regularity and, e.g., [2, Thm.II.4.4.1]. Notice that

$$\exp(-w\,\tfrac{|x-b(s)|^2}{t-s}) \leq c_1 \exp(-w\,\tfrac{|x-b(t)|^2}{t-s}) \tag{6.5}$$

for $t > s \geq 0$, $x \in \Omega$ and a constant c_1 since b is Lipschitz. Hence, $|(\mathbb{F}_s u)(t)| \leq \varphi * |u|\,(t)$, $s \leq t \leq s + t_0$, where we have extended $u \in L^p_{loc}([s,\infty))$ by 0 and put $\varphi(t) = c_1 M\, t^{-\frac{n}{2}} \exp(-\frac{w\delta^2}{t})$ for $t > 0$ and $\varphi(t) = 0$ otherwise. Young's inequality then yields

$$\|\mathbb{F}_s u\|_{L^p[s,s+t_0]} \leq \|\varphi\|_{L^1[0,t_0]}\, \|u\|_{L^p[s,s+t_0]} \tag{6.6}$$

so that (3.11) holds. Using Hölder's inequality, (6.5), and a change of variables we compute

$$\begin{aligned}
|\Phi_{t,s}u(x)| &\leq c_1 M\, \|u\|_{L^p[s,t]} \Big(\int_s^t (t-\tau)^{-\frac{np'}{2}} \exp(-\tfrac{wp'\,|x-b(t)|^2}{t-\tau})\, d\tau \Big)^{\frac{1}{p'}} \\
&\leq c_2\, \|u\|_{L^p[s,t]} \Big(\int_0^\infty \tau^{\frac{np'}{2}-2}\, e^{-\tau}\, |x-b(t)|^{2-np'}\, d\tau \Big)^{\frac{1}{p'}} \\
&\leq c_3\, \|u\|_{L^p[s,t]}\, |x-b(t)|^{\frac{2}{p'}-n}
\end{aligned}$$

for constants c_k not depending on u, $x \in \Omega$, and $0 < t - s \leq t_0$, but possibly on $t_0 > 0$. This estimate implies (3.2) because of (6.4) and the boundedness of Ω. Similarly,

$$\begin{aligned}
\Big(\int_s^t |\Psi_s f(\tau)|^p\, d\tau \Big)^{\frac{1}{p}} &\leq c_4 \Big(\int_s^t \Big(\int_\Omega |\tau - s|^{-\frac{n}{2}} \exp(-\tfrac{w\,|c(s)-y|^2}{\tau - s})\, |f(y)|\, dy \Big)^p d\tau \Big)^{\frac{1}{p}} \\
&\leq c_4 \int_\Omega \Big(\int_s^t |\tau - s|^{-\frac{np}{2}} \exp(-\tfrac{wp\,|c(s)-y|^2}{\tau - s})\, |f(y)|^p\, d\tau \Big)^{\frac{1}{p}} dy \\
&\leq c_5 \int_\Omega |f(y)|\, |y - c(s)|^{\frac{2}{p}-n}\, dy,
\end{aligned}$$

where we have used the integral Minkowski inequality [10, §202]. Thus (2.2) holds due to Hölder's inequality and (6.4). Finally, (6.3) implies (2.1), (3.1), and (3.10). Altogether we have shown that $(T, \Phi, \Psi, \mathbb{F})$ is a well–posed non–autonomous system. In view of (6.6), this system is absolutely regular and every bounded feedback is admissible.

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R. Schnaubelt, FB Mathematik und Informatik, Martin–Luther–Universität, Theodor–Lieser–Str. 5, 06120 Halle, Germany.

E-mail address: `roland@euler.mathematik.uni-halle.de`