APPROXIMATION OF ABSTRACT DIFFERENTIAL EQUATIONS

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Abstract

This review paper is devoted to the numerical analysis of abstract differential equations in Banach spaces. The presentation is given on general approximation scheme and based on semigroup theory and functional analysis approach. The paper concerns mainly numerical analysis of differential equations of first order in time.

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1 General approximation scheme

Let $B(E)$ denote the Banach algebra of all linear bounded operators on a complex Banach space E. The set of all linear closed densely defined operators in E will be denoted by $\mathcal{C}(E)$. We denote by $\sigma(B)$ the spectrum of the operator B, by $\rho(B)$ the resolvent set of B, by $\mathcal{N}(B)$ the null space of B and by $\mathcal{R}(B)$ the range of B. Let us recall that $B \in B(E)$ is said to be a Fredholm operator if $\mathcal{R}(B)$ is closed, $dim\,\mathcal{N}(B) < \infty$ and $codim\,\mathcal{R}(B) < \infty$, the index of B is defined as $ind\,B = dim\,\mathcal{N}(B) - codim\,\mathcal{R}(B)$. The general approximation scheme, due to [79]-[81], [181], [204], [201] can be described in the following way. Let E_n and E be Banach spaces and $\{p_n\}$ be a sequence of linear bounded operators $p_n : E \to E_n, p_n \in B(E, E_n), n \in I\!N = \{1, 2, \cdots\}$, with the property:

$$\|p_n x\|_{E_n} \to \|x\|_E \text{ as } n \to \infty \text{ for any } x \in E.$$

Definition 1.1 *The sequence of elements $\{x_n\}, x_n \in E_n, n \in I\!N$, is said to be $\mathcal{P}$-convergent to $x \in E$ iff $\|x_n - p_n x\|_{E_n} \to 0$ as $n \to \infty$ and we write this $x_n \to x$.*

Definition 1.2 *The sequence of elements $\{x_n\}, x_n \in E_n, n \in I\!N$, is said to be $\mathcal{P}$-compact if for any $I\!N' \subseteq I\!N$ there exist $I\!N'' \subseteq I\!N'$ and $x \in E$ such that $x_n \to x$, as $n \to \infty$ in $I\!N''$.*

Definition 1.3 *The sequence of bounded linear operators $B_n \in B(E_n), n \in I\!N$, is said to be $\mathcal{PP}$-convergent to the bounded linear operator $B \in B(E)$ if for every $x \in E$ and for every sequence $\{x_n\}, x_n \in E_n, n \in I\!N$, such that $x_n \to x$ one has $B_n x_n \to Bx$. We write then $B_n \to B$.*

For general examples of notions of $\mathcal{P}-$convergence see [78], [181], [197], [205].

Remark 1.1 *If we put $E_n = E$ and $p_n = I$ for each $n \in I\!N$, where I is the identity operator on E, then Definition 1.1 leads to the traditional pointwise convergent bounded linear operators which we denote by $B_n \to B$.*

Let us denote by E^+ the positive cone in a Banach lattice E. The operator B is called positive if for any $x^+ \in E^+$ it follows $Bx^+ \in E^+$; we write $0 \preceq B$.

Definition 1.4 *The system $\{p_n\}$ is called discrete order preserving if for all sequences $\{x_n\}, x_n \in E_n$, and element $x \in E$ the following implication holds: $x_n \to x$ implies $x_n^+ \to x^+$.*

It is known [95] that $\{p_n\}$ preserves the order iff $\|p_n x^+ - (p_n x)^+\|_{E_n} \to 0$ as $n \to \infty$ for any $x \in E$.

In the case of unbounded operators, and we know in general infinitesimal generators are unbounded, we consider the notion of *compatibility.*

Definition 1.5 *The sequence of closed linear operators $\{A_n\}, A_n \in \mathcal{C}(E_n), n \in I\!N$, are said to be compatible with a closed linear operator $A \in \mathcal{C}(E)$ iff for each $x \in D(A)$ there is a sequence $\{x_n\}, x_n \in D(A_n) \subseteq E_n, n \in I\!N$, such that $x_n \to x$ and $A_n x_n \to Ax$. We write (A_n, A) are compatible.*

Usually in practice Banach spaces E_n are finite dimensional, although, in general, say for the case of a closed operator A, we have $\dim E_n \to \infty$ and $\|A_n\|_{B(E_n)} \to \infty$ as $n \to \infty$.

1.1 Approximation of spectrum of linear operators

The most important role in approximations of equation $Bx = y$ and approximations of spectra of operator B, is played by the notions of *stable* and *regular* convergence. These notions are used in different areas of numerical analysis (cf. [10], [15], [77], [82]-[85], [204], [206], [201], [216]).

Definition 1.6 *A sequence of operators $\{B_n\}, B_n \in B(E_n), n \in I\!N$, is stably convergence to an operator $B \in B(E)$ iff $B_n \to B$ and $\|B_n^{-1}\|_{B(E_n)} = O(1), n \to \infty$. We will write $B_n \to B$ stably.*

Definition 1.7 *A sequence of operators $\{B_n\}, B_n \in B(E_n)$, is called regularly convergent to the operator $B \in B(E)$ iff $B_n \to B$ and the following implication holds:*

$$\|x_n\|_{E_n} = O(1) \ \& \ \{B_n x_n\} \text{ is } \mathcal{P} - \text{compact} \Longrightarrow \{x_n\} \text{ is } \mathcal{P}\text{-compact}.$$

We write this as $B_n \to B$ regularly.

Theorem 1.1 *[204] For $B_n \in B(E_n)$ and $B \in B(E)$ the following conditions are equivalent:*
(i) $B_n \to B$ regularly, B_n are Fredholm operators of index 0 and $\mathcal{N}(B) = \{0\}$;
(ii) $B_n \to B$ stably and $\mathcal{R}(B) = E$;
(iii) $B_n \to B$ stably and regularly;
(iv) if one of the conditions (i)-(iii) is fulfilled, then there exist $B_n^{-1} \in B(E_n), B^{-1} \in B(E)$ and $B_n^{-1} \to B^{-1}$ regularly and stably.

This theorem admits an extension to the case of closed operators $B \in \mathcal{C}(E), B_n \in \mathcal{C}(E_n)$ [207].

Let $\Lambda \subseteq \mathbb{C}$ be some open connected set and let $B \in B(E)$. For an isolated point $\lambda \in \sigma(B)$, the corresponding maximal invariant space (or generalized eigenspace) will be denoted by $\mathcal{W}(\lambda; B) = P(\lambda)E$, where $P(\lambda) = \frac{1}{2\pi i}\int_{|\zeta-\lambda|=\delta}(\zeta I - B)^{-1}d\zeta$ and δ is small enough so that there are no points of $\sigma(B)$ in the disc $\{\zeta : |\zeta - \lambda| \le \delta\}$ other than λ. The isolated point $\lambda \in \sigma(B)$ is a *Riesz point* of B if $\lambda I - B$ is a Fredholm operator with index zero and $P(\lambda)$ is of finite rank. Denote by $\mathcal{W}(\lambda, \delta; B_n) = \bigcup_{|\lambda_n - \lambda| < \delta, \lambda_n \in \sigma(B_n)} \mathcal{W}(\lambda_n, B_n)$, where $\lambda_n \in \sigma(B_n)$ are taken from a δ-neighborhood of λ. It is clear that $\mathcal{W}(\lambda, \delta; B_n) = P_n(\lambda)E_n$, where $P_n(\lambda) = \frac{1}{2\pi i}\int_{|\zeta-\lambda|=\delta}(\zeta I - B_n)^{-1}d\zeta$. The following theorems state the complete picture of the approximation of the spectrum.

Theorem 1.2 *[78], [202]-[203] Assume that $L_n(\lambda) = \lambda I - B_n$ and $L(\lambda) = \lambda I - B$ are Fredholm operators with index zero for any $\lambda \in \Lambda$ and that $L_n(\lambda) \to L(\lambda)$ stably for any $\lambda \in \rho(B) \cap \Lambda \neq \emptyset$. Then*

(i) for any $\lambda_0 \in \sigma(B) \cap \Lambda$ there exists a sequence $\{\lambda_n\}$, $\lambda_n \in \sigma(B_n)$, $n \in I\!N$, such that $\lambda_n \to \lambda_0$ as $n \to \infty$;

(ii) if for some sequence $\{\lambda_n\}, \lambda_n \in \sigma(B_n)$, $n \in I\!N$, one has $\lambda_n \to \lambda_0 \in \Lambda$ as $n \to \infty$, then $\lambda_0 \in \sigma(B)$;

(iii) for any $x \in \mathcal{W}(\lambda_0, T)$ there exists a sequence $\{x_n\}$, $x_n \in \mathcal{W}(\lambda_0, \delta; B_n)$, $n \in I\!N$, such that $x_n \to x$ as $n \to \infty$;

(iv) there exists $n_0 \in I\!N$ such that $\dim \mathcal{W}(\lambda_0, \delta; B_n) \geq \dim \mathcal{W}(\lambda_0, B)$ for any $n \geq n_0$.

Remark 1.2 *The inequality in (iv) can be strict for all $n \in I\!N$ as is shown in [201].*

Theorem 1.3 *[204] Assume that $L_n(\lambda)$ and $L(\lambda)$ are Fredholm operators with index zero for all $\lambda \in \Lambda$. Suppose that $L_n(\lambda) \to L(\lambda)$ regularly for any $\lambda \in \Lambda$ and that $\rho(B) \cap \Lambda \neq \emptyset$. Then the statements (i)-(iii) of Theorem 1.2 hold and also*

(iv) there exists $n_0 \in I\!N$ such that $\dim \mathcal{W}(\lambda_0, \delta; B_n) = \dim \mathcal{W}(\lambda_0, B)$ for all $n \geq n_0$;

(v) any sequence $\{x_n\}, x_n \in \mathcal{W}(\lambda_0, \delta; B_n)$, $n \in I\!N$, with $\|x_n\|_{E_n} = 1$ is $\mathcal{P}$-compact and any limit point of this sequence belongs to $\mathcal{W}(\lambda_0, B)$.

Remark 1.3 *Estimates of $|\lambda_n - \lambda_0|$, $\mathrm{gap}\left(\mathcal{W}(\lambda_0, \delta; B_n), \mathcal{W}(\lambda_0, B)\right)$ and $|\hat{\lambda}_n - \lambda_0|$ are given in [204], where $\hat{\lambda}_n$ denotes the arithmetic mean (counting algebraic multiplicities) of the spectral values of B_n that contribute to $\mathcal{W}(\lambda_0, \delta; B_n)$. For the notion of gap and its properties, see [100].*

1.2 Regions of convergence

Theorems 1.2 and 1.3 have been generalized in [207] to the case of closed operators using the following notions introduced by Kato [100].

Definition 1.8 *The region of stability $\Delta_s = \Delta_s(\{A_n\})$, $A_n \in \mathcal{C}(B_n)$, is defined to be the set of all $\lambda \in \mathbb{C}$ such that $\lambda \in \rho(A_n)$ for almost all n and such that the sequence $\{\|(\lambda I - A_n)^{-1}\|\}_{n \in I\!N}$ is bounded. The region of convergence $\Delta_c = \Delta_c(\{A_n\})$, $A_n \in \mathcal{C}(E_n)$, is defined as the set of all $\lambda \in \mathbb{C}$ such that $\lambda \in \Delta_s(\{A_n\})$ and such that the sequence of operators $\{(\lambda I - A_n)^{-1}\}_{n \in I\!N}$ is $\mathcal{PP}$-convergent to some operator $S(\lambda) \in B(E)$.*

It is clear that $S(\cdot)$ is pseudo-resolvent and $S(\cdot)$ is a resolvent of some operator iff $\mathcal{N}(S(\lambda)) = \{0\}$ for some λ (cf. [100]).

Definition 1.9 *A sequence of operators $\{K_n\}$, $K_n \in \mathcal{C}(E_n)$, is called regularly compatible with the operator $K \in \mathcal{C}(E)$ if (K_n, K) are compatible and, for any bounded sequence $\|x_n\|_{E_n} = O(1)$ such that $x_n \in D(K_n)$ and such that $\{K_n x_n\}$ is $\mathcal{P}$-compact, it follows that $\{x_n\}$ is $\mathcal{P}$-compact and the $\mathcal{P}$-convergence of $\{x_n\}$ to some x and of $\{K_n x_n\}$ to some y for $n \to \infty$ in $I\!N' \subseteq I\!N$ imply that $x \in D(K)$ and $Kx = y$.*

Definition 1.10 *The region of regularity $\Delta_r = \Delta_r(\{A_n\}, A)$, is defined as the set of all $\lambda \in \mathbb{C}$ such that (K_n, K) are regularly compatible, where $K_n = \lambda I - A_n$ and $K = \lambda I - A$.*

The relationships between these regions are given by

Proposition 1.1 *[207] Suppose that $\Delta_c \neq \emptyset$ and $\mathcal{N}\big(S(\lambda)\big) = \{0\}$ at least for one point $\lambda \in \Delta_c$ so that $S(\lambda) = (I\lambda - A)^{-1}$. Then (A_n, A) are compatible and*

$$\Delta_c = \Delta_s \cap \rho(A) = \Delta_s \cap \Delta_r = \Delta_r \cap \rho(A).$$

It is shown in [207] that the following three conditions : *(A_n, A) are compatible, $\lambda I - A_n$ and $\lambda I - A$ are Fredholm operators with index zero for any $\lambda \in \Lambda$ and $\rho(A) \cap \Lambda \neq \emptyset$* imply (i)-(iv) of Theorem 1.2 when $\rho(A) \cap \Lambda \subseteq \Delta_s$ and imply (i)-(iii) of Theorem 1.2 and (iv)-(v) of Theorem 1.3, when $\Lambda \subseteq \Delta_r$.

Definition 1.11 *The Riesz point $\lambda_0 \in \sigma(A)$ is said to be strongly stable in Kato's sense if* $\dim \mathcal{W}(\lambda_0, \delta; B_n) \leq \dim \mathcal{W}(\lambda_0, B)$ *for all $n \geq n_0$.*

Theorem 1.4 *[207] The Riesz point $\lambda_0 \in \sigma(A)$ is strongly stable in Kato's sense iff $\lambda_0 \in \Lambda \cap \Delta_r \cap \sigma(A)$.*

Investigations of approximation of spectra and types of convergence, but not on general approximation scheme are given in [13], [36], [48], [123], [124], [135], [138].

1.3 Convergence in one space dimension

Throughout this subsection we assume that $E_n = E$ and $p_n = I$ for all $n \in I\!N$. Hence, the symbol $\mathcal{P}$ will be omitted in all notations of this subsection.

Let us recall that if $B_n \to B$ compactly (see Definition 1.12), then for any $\lambda \neq 0$ we have $\lambda I - B_n \to \lambda I - B$ regularly [201]. When $B_n \to B$ compactly and B is a compact operator, Anselone [10] has proved that

(1.1) $$\|(B_n - B)B_n\| \to 0,\ \|(B_n - B)B\| \to 0 \text{ as } n \to \infty.$$

Considering an approximation of a weakly singular compact integral operator Ahues [4] has proved that these convergence properties (1.1) are sufficient to state that a Riesz point is strongly stable in Kato's sense.

Theorem 1.5 *[7] Assume that $B \in B(E)$ is compact and that $B_n \to B$. If $\|(B_n - B)B_n\| \to 0$ as $n \to \infty$ then for any nonzero $\lambda_0 \in \sigma(B)$ the assertions (i)-(iii) of Theorem 1.2 and assertions (iv)-(v) of Theorem 1.3 hold.*

Theorem 1.6 *[7] Assume that $B_n \to B$ and (1.1) holds. Then for any non-zero Riesz point $\lambda_0 \in \sigma(B)$ the assertions (i)-(iii) of Theorem 1.2 and assertions (iv)-(v) of Theorem 1.5 hold.*

Corollary 1.1 *[5] Assume that $B_n \to B$, that $\lambda I - B_n$ are Fredholm operators with index zero for $\lambda \in \{z\ :\ |z - \lambda_0| \leq \delta\}$ and $\|(B_n - B)B_n^k\| \to 0$ as $n \to \infty$ for some $k \in I\!N$. Then for any non zero Riesz point $\lambda_0 \in \sigma(B)$ the assertions (i)-(iii) of Theorem 1.2 and assertions (iv)-(v) of Theorem 1.3 hold.*

Theorem 1.7 *[7] Assume that B is compact, $B_n \to B$ and $\|B_n(B_n - B)\| \to 0$. Then $\lambda_0 I - B_n \to \lambda_0 I - B$ regularly for any $\lambda_0 \neq 0$.*

Theorem 1.8 *[7] Assume that B is compact, $B_n \to B$ and $\|B_n^k(B_n - B)\| \to 0$ for some $k \in I\!N$. Then $\lambda_0 I - B_n \to \lambda_0 I - B$ regularly for any $\lambda_0 \neq 0$.*

Let $r(B)$ be a spectral radius of operator $B \in B(E)$.

Theorem 1.9 *[18] Let E be a Banach lattice. Let $0 \preceq B_n, B \in B(E)$ be such that $B_n \to B$ and $\|(B_n - B)^+\| \to 0$ as $n \to \infty$. Suppose that $r(B)$ is a Riesz point of $\sigma(B)$. Then $r(B_n)$ is a Riesz point of $\sigma(B_n)$ and $r(B_n) \to r(B)$ as $n \to \infty$.*

The conclusion on order of convergence of eigenvectors in Theorem 1.9 also is given in [17].

The application of Theorems 1.5 - 1.8 to the numerical solution of a mathematical model used in the jet printer industry is considered [6], [111].

1.4 Compact convergence of resolvents

We consider now the important class of operators which have compact resolvents. We will use this property of generator as assumption in section 5. It is natural to consider in this case approximate operators which "keep" this property.

Definition 1.12 *A sequence of operators $\{B_n\}$, $B_n \in B(E_n)$, $n \in I\!N$, converges compactly to an operator $B \in B(E)$, if $B_n \to B$ and the following compactness condition holds:*

$$\|x_n\|_{E_n} = O(1) \Longrightarrow \{B_n x_n\} \text{ is } \mathcal{P}\text{-compact.}$$

Definition 1.13 *The region of compact convergence of resolvents, $\Delta_{cc} = \Delta_{cc}(A_n, A)$, where $A_n \in \mathcal{C}(E_n)$ and $A \in \mathcal{C}(E)$ is defined as the set of all $\lambda \in \Delta_c \cap \rho(A)$ such that $(\lambda I - A_n)^{-1} \to (\lambda I - A)^{-1}$ compactly.*

Theorem 1.10 *Assume that $\Delta_{cc} \neq \emptyset$. Then for any $\zeta \in \Delta_s$ the following implication holds:*

$$\|x_n\|_{E_n} = O(1) \ \&\ \|(\zeta I - A_n)x_n\|_{E_n} = O(1) \Longrightarrow \{x_n\} \text{ is } \mathcal{P}\text{-compact.} \tag{1.2}$$

Conversely, if for some $\zeta \in \Delta_c \cap \rho(A)$ the implication (1.2) holds, then $\Delta_{cc} \neq \emptyset$.

Proof. Let $(\mu I - A_n)^{-1} \to (\mu I - A)^{-1}$compactly for some $\mu \in \Delta_{cc}$. Then for $\|x_n\|_{E_n} = O(1)$ and $\|(\zeta I - A_n)x_n\|_{E_n} = O(1)$, we get from Hilbert's identity

$$(\zeta I - A_n)^{-1} - (\mu I - A_n)^{-1} = (\mu - \zeta)(\zeta I - A_n)^{-1}(\mu I - A_n)^{-1}, \tag{1.3}$$

that $x_n = (\mu I - A_n)^{-1}(\zeta I - A_n)x_n - (\zeta - \mu)(\mu I - A_n)^{-1}x_n$ and it follows that $\{x_n\}$ is $\mathcal{P}$-compact. Conversely, let the implication (1.2) hold for some $\zeta_0 \in \Delta_c \cap \rho(A)$. We are going to show then that $\zeta_0 \in \Delta_{cc}$. Taking a bounded sequence $\{y_n\}$, $n \in I\!N$, one gets that the sequence $\|(\zeta_0 I - A_n)^{-1}y_n\|_{E_n} = O(1)$ for $n \in I\!N$. Let us apply the implication (1.2) to the sequence $x_n = (\zeta_0 I - A_n)^{-1}y_n$. It is easy to see that $\{x_n\}$ is $\mathcal{P}$-compact. Hence $\zeta_0 \in \Delta_{cc}$. □

Corollary 1.2 *Assume that $\Delta_{cc} \neq \emptyset$. Then $\Delta_{cc} = \Delta_c \cap \rho(A)$.*

Proof. It is clear that $\Delta_{cc} \subseteq \Delta_c \cap \rho(A)$. To prove that $\Delta_{cc} \supseteq \Delta_c \cap \rho(A)$ let us consider the Hilbert identity (1.3). Now let $\mu \in \Delta_{cc}$. Then $\mu \in \Delta_{cc} \cap \Delta_c \cap \rho(A)$. Hence for every $\zeta \in \Delta_c \cap \rho(A)$ and for any bounded sequence $\{x_n\}$, $n \in I\!N$, the sequence $\{(\zeta I - A_n)^{-1}x_n\}$ is $\mathcal{P}$-compact. □

Comparing Definitions 1.7, 1.8, 1.13 and implication (1.2) we see that $\Delta_{cc} \subseteq \Delta_r$.

Theorem 1.11 *Assume that $\Delta_{cc} \neq \emptyset$. Then $\Delta_r = \mathbb{C}$.*

Proof. Take any point $\lambda_1 \in \mathbb{C}$. We have to show that $(\lambda_1 I - A_n, \lambda_1 I - A)$ are regularly compatible. Assume that $\|x_n\|_{E_n} = O(1)$ and that $\{(\lambda_1 I - A_n)x_n\}$ is $\mathcal{P}$-compact. To show that $\{x_n\}$ is $\mathcal{P}$-compact take $\mu \in \Delta_{cc}$. Using (1.3) with $\zeta = \lambda_1$ one obtains $x_n = (\mu I - A_n)^{-1}(\lambda_1 I - A_n)x_n + (\lambda_1 - \mu)(\mu I - A_n)^{-1}x_n$ and therefore $\{x_n\}$ is $\mathcal{P}$-compact. Assume now that $x_n \to x$ and $(\lambda_1 I - A_n)^{-1}x_n \to y$, as $n \to \infty$ in $I\!N' \subseteq I\!N$. Then $x = (\mu I - A)^{-1}y - (\lambda_1 - \mu)(\mu I - A)^{-1}x$ and it follows that $x \in D(A)$ and $(\lambda_1 I - A)x = y$. □

2 Discretization of semigroups

Let us consider the well-posed Cauchy problem in the Banach space E with operator $A \in \mathcal{C}(E)$

$$(2.1)\qquad \begin{array}{l} u'(t) = Au(t),\ t \in [0,\infty), \\ u(0) = u^0, \end{array}$$

where operator A generates C_0-semigroup $\exp(\cdot A)$. It is well-known that the C_0-semigroup gives the solution of (2.1) by the formula $u(t) = \exp(tA)u^0$ for $t \geq 0$. The theory of well-posed problems and numerical analysis of these problems have been developed extensively, see for instance [71], [84], [100], [154], [156], [194], [210].

Let us consider on the general discretization scheme the semidiscrete approximation of the problem (2.1) in the Banach spaces E_n :

$$(2.2)\qquad \begin{array}{l} u_n'(t) = A_n u_n(t),\ t \in [0,\infty), \\ u_n(0) = u_n^0, \end{array} .$$

with the operators $A_n \in \mathcal{C}(E_n)$, such that they generate C_0-semigroups, which are consistent with the operator A and $u_n^0 \to u^0$.

2.1 The simplest discretization schemes

We have the following version of Trotter-Kato's Theorem on general approximation scheme:

Theorem 2.1 *[197] (Theorem ABC) The following conditions (A) and (B) are equivalent to condition (C).*

(A) Consistency. There exists $\lambda \in \rho(A) \cap \cap_n \rho(A_n)$ such that the resolvents converge $(\lambda I - A_n)^{-1} \to (\lambda - A)^{-1}$;

(B) Stability. There are some constants $M \geq 1$ and ω, which are not depending on n and such that $\|\exp(tA_n)\| \leq M\exp(\omega t)$ for $t \geq 0$ and any $n \in N$;

(C) Convergence. For any finite $T > 0$ one has $\max_{t\in[0,T]} \|\exp(tA_n)u_n^0 - p_n \exp(tA)u^0\| \to 0$ as $n \to \infty$, whenever $u_n^0 \to u^0$.

The analytic C_0-semigroup case is slightly different from the general case, but with the same property (A).

Theorem 2.2 *[154] Let operators A and A_n generate analytic C_0-semigroups. The following conditions (A) and (B_1) are equivalent to condition (C_1).*

(A) Consistency. There exists $\lambda \in \rho(A) \cap \cap_n \rho(A_n)$ such that the resolvents converge $(\lambda I - A_n)^{-1} \to (\lambda I - A)^{-1}$;

(B_1) Stability. There are some constants $M_2 \geq 1$ and ω_2 such that

$$\|(\lambda I - A_n)^{-1}\| \leq \frac{M_2}{|\lambda - \omega_2|},\ Re\lambda > \omega_2, n \in N;$$

(C_1) Convergence. For any finite $\mu > 0$ and some $0 < \theta < \frac{\pi}{2}$ we have

$$\max_{\eta\in\Sigma(\theta,\mu)} \|\exp(\eta A_n)u_n^0 - p_n \exp(\eta A)u^0\| \to 0$$

as $n \to \infty$ whenever $u_n^0 \to u^0$. Here $\Sigma(\theta,\mu) = \{z \in \Sigma(\theta) : |z| \leq \mu\}$, and $\Sigma(\theta) = \{z \in \mathbb{C} : |arg\, z| \leq \theta\}$.

Definition 2.1 *The linear operator* $A : D(A) \subseteq E \to E$ *is said to have the positive off-diagonal (POD) property if* $\langle Au, \phi \rangle \geq 0$ *whenever* $0 \preceq u \in D(A)$ *and* $0 \preceq \phi \in E^*$ *with* $\langle u, \phi \rangle = 0$.

Definition 2.2 *An element* $e \in E^+$ *is said to be an order unit in* E *if for every* $x \in E$ *there exists* $0 \leq \lambda \in R$ *such that* $-\lambda e \preceq x \preceq \lambda e$. *For* $e \in intE^+$ *we can define an order unit norm by*

$$\|x\|_e = \inf\{\lambda \geq 0 : -\lambda e \preceq x \preceq \lambda e\}.$$

The ordered Banach space E *is called an order unit space if there exists* $e \in intE^+$ *such that* $\|\cdot\|_E = \|\cdot\|_e$.

We can state now Trotter-Kato's version for positive semigroups.

Theorem 2.3 *[163] Let the operators* A_n *and* A *from (2.1) and (2.2) be consistent and let* E, E_n *be order unit spaces and* $e_n \in D(A_n) \cap intE_n^+$. *Assume that the operators* A_n *have the POD property and* $A_n e_n \preceq 0$ *for sufficiently large* n. *Then* $\exp(tA_n) \to \exp(tA)$ *uniformly in* $t \in [0, T]$.

We can assume that conditions (A) and (B) for the corresponding semigroup case are satisfied without any restriction of generality if any discretization processes in time are considered. If we denote by $T_n(\cdot)$ a family of discrete semigroups as in [100], i.e. $A_n = \frac{1}{\tau_n}(T_n(\tau_n) - I) \in B(E_n)$ and $T_n(t) = T_n(\tau_n)^{k_n}$, where $k_n = [\frac{t}{\tau_n}]$, as $\tau_n \to 0, n \to \infty$, then one gets

Theorem 2.4 *[197] (Theorem ABC-discr) The following conditions (A) and (B') are equivalent to condition (C').*

(A) Consistency. There exists $\lambda \in \rho(A) \cap \cap_n \rho(A_n)$ *such that the resolvents converge* $(\lambda I - A_n)^{-1} \to (\lambda I - A)^{-1}$;

(B') Stability. There are some constants $M_1 \geq 1$ *and* ω_1 *such that*

$$\|T_n(t)\| \leq M_1 \exp(\omega_1 t) \text{ for } t \in \overline{R}_+ = [0, \infty), n \in N;$$

(C') Convergence. For any finite $T > 0$ *one has* $\max_{t \in [0,T]} \|T_n(t)u_n^0 - p_n \exp(tA)u^0\| \to 0$ *as* $n \to \infty$, *whenever* $u_n^0 \to u^0$.

Theorem 2.5 *[197] Assume that conditions (A) and (B) of Theorem 2.1 hold. Then the implicit difference scheme*

$$\frac{\overline{U}_n(t + \tau_n) - \overline{U}_n(t)}{\tau_n} = A_n \overline{U}_n(t + \tau), \overline{U}_n(0) = u_n^0, \tag{2.3}$$

is stable, i.e. $\|(I_n - \tau_n A_n)^{-k_n}\| \leq M_1 e^{\omega_1 t}, t = k_n \tau_n \in \overline{I\!R}_+$, *and gives an approximation to the solution of the problem (2.1), i.e.* $\overline{U}_n(t) \equiv (I_n - \tau_n A_n)^{-k_n} u_n^0 \to \exp(tA)u_n^0$ $\mathcal{P}$*-converges uniformly with respect to* $t = k_n \tau_n \in [0, T]$ *as* $u_n^0 \to u^0, n \to \infty, k_n \to \infty, \tau_n \to 0$.

Theorem 2.6 *[197] Assume that conditions (A) and (B) of the Theorem 2.1 hold and condition*

$$\tau_n \|A_n^2\| = O(1) \tag{2.4}$$

is fulfilled. Then the difference scheme

$$\frac{U_n(t + \tau_n) - U_n(t)}{\tau_n} = A_n U_n(t), \; U_n(0) = u_n^0, \tag{2.5}$$

is stable, i.e. $\|(I_n + \tau_n A_n)^{k_n}\| \leq M e^{\omega t}, t = k_n \tau_n \in \overline{I\!R}_+$, *and gives an approximation to the solution of the problem (2.1), i.e.* $U_n(t) \equiv (I_n + \tau_n A_n)^{k_n} u_n^0 \to u(t)$ $\mathcal{P}$*-converges uniformly with respect to* $t = k_n \tau_n \in [0, T]$ *as* $n \to \infty, k_n \to \infty, \tau_n \to 0$.

Theorem 2.7 *[154] Assume that conditions (A) and (B_1) of Theorem 2.2 hold and condition*

$$\tau_n \|A_n\| \le 1/(M+2), n \in I\!N \tag{2.6}$$

is fulfilled. Then the difference scheme (2.5) is stable and gives an approximation to the solution of the problem (2.1), i.e. $U_n(t) \equiv (I_n + \tau_n A_n)^{k_n} u_n^0 \to u(t)$ *discretely $\mathcal{P}$-converge uniformly with respect to* $t = k_n \tau_n \in [0,T]$ *as* $u_n^0 \to u^0, n \to \infty, k_n \to \infty, \tau_n \to 0$.

Let us introduce the following conditions:
(B_1') Stability. There are constants M', ω' such that

$$\|\exp(tA_n)\| \le M' e^{\omega' t}, \ \|A_n \exp(tA_n)\| \le \frac{M'}{t} e^{\omega' t}, \ t \in I\!R_+.$$

(B_1'') Stability. There are constants M'', ω'' and $\tau^* > 0$ such that

$$\|(I - \tau_n A_n)^{-k}\| \le M'' e^{\omega'' k \tau_n}, \ \|k\tau_n A_n (I - \tau_n A_n)^{-1}\| \le M'' e^{\omega'' k \tau_n}, \ 0 < \tau_n < \tau^*, n, k \in I\!N.$$

Proposition 2.1 *[177] Conditions (B_1), (B_1'), (B_1'') are equivalent.*

Theorem 2.8 *The conditions (A) and (B_1'') are equivalent to the condition (C_1).*

Theorem 2.9 *[157] Let the assumptions of Theorem 2.7 and (2.4) be satisfied. Then*

$$tA_n (I + \tau_n A_n)^{k_n} \to tA \exp(tA) \ \textit{uniformly in} \ t = k_n \tau_n \in [0,T]. \tag{2.7}$$

Conversely, if $(I + \tau_n A_n)^{k_n} \to \exp(tA)$ *uniformly in* $t = k_n \tau_n \in [0,T]$ *and (2.7) is satisfied, then condition (C_1) holds.*

Theorem 2.10 *[157] Let condition (B_1) hold. Then*

$$\|\exp(tA_n) - (I - \tau_n A_n)^{-k_n}\| \le c \frac{\tau_n}{t} e^{\omega t}.$$

If, moreover, stability condition (2.6) holds, then

$$\begin{aligned} \|\exp(tA_n) - (I + \tau_n A_n)^{k_n}\| &\le c \frac{\tau_n}{t} e^{\omega t}, \\ \|(\exp(tA_n) - (I + \tau_n A_n)^{k_n}) x_n\| &\le c \tau_n e^{\omega t} \|A_n x_n\|, \\ \|A_n (\exp(tA_n) - (I + \tau_n A_n)^{k_n}) x_n\| &\le c \frac{\tau_n}{t} e^{\omega t} \|A_n x_n\|, \quad t = k_n \tau_n. \end{aligned}$$

In the case of analytic C_0-semigroups for the forward scheme, as we saw, the following stability condition: $\tau_n \|A_n\| < 1/(M+2)$, which can't be improved even in Hilbert spaces for self-adjoint operators. In the case of almost periodic C_0-semigroups and the forward scheme for differential equations of first order in time (2.1), one obtains necessary and sufficient stability condition $\tau_n \|A_n\| < 1$ [156]. It was discovered that the stability condition of the forward scheme like (2.5) for the positive C_0-semigroups also can be written in the form $\tau_n \|A_n\| < 1$, see [162].

Stability of difference schemes for differential equations in Hilbert spaces in the energy norm are investigated in [173], [174], where also schemes with weights were considered. Semidiscrete approximations are studied in [174].

2.2 Rational approximation

Let us denote by $P_p(z)$ the element of the set of all real polynomials of degree not greater than p and by $\pi_{p,q}$ the set of all rational functions $r_{p,q}(z) = \frac{P_p(z)}{P_q(z)}$ and $P_q(0) = 1$. Then a Padé (p,q)-approximation for e^{-z} is defined as an element $R_{p,q}(z) \in \pi_{p,q}$ such that

$$|e^{-z} - R_{p,q}(z)| = O(|z|^{p+q+1}) \text{ as } |z| \to 0.$$

It is well known that a Padé approximation for e^{-z} exists, is unique and is representable by the formula $R_{p,q}(z) = P_{p,q}(z)/Q_{p,q}(z)$, where

$$P_{p,q}(z) = \Sigma_{j=0}^{p} \frac{(p+q-j)!p!(-z)^j}{(p+q)!j!(p-j)!},\ Q_{p,q}(z) = \Sigma_{j=0}^{q} \frac{(p+q-j)!q!z^j}{(p+q)!j!(q-j)!}.$$

In the papers [168]-[169] details of the location of poles and order of convergence of rational approximations in different regions are given.

Definition 2.3 *A rational approximation $r_{p,q}(\cdot) \in \pi_{p,q}$ for e^{-z} is said to be*
a) A-acceptable if $|r_{p,q}(z)| < 1$ for $Re(z) > 0$;
b) $A(\theta)$-acceptable if $|r_{p,q}(z)| < 1$ for $z \in \Sigma(\theta) = \{z : -\theta < arg(z) < \theta, z \neq 0\}$.

It is well known that $R_{q,q}(z), R_{q-1,q}(z)$ and $R_{q-2,q}(z)$ are A-acceptable. But for $q \geq 3$ and $p = q - 3$ the Padé functions are not A-acceptable.

Theorem 2.11 *[169] For any $q \geq 2$ and $p \geq 0$ the Padé approximation to e^{-z} has no poles in the sector $S_{p,q} = \{z : |arg(z)| < cos^{-1}\left(\frac{q-p-2}{p+q}\right)\}$, in particular for $p \leq q \leq p+4$ all poles lies in the left-half plane.*

Since $r(\cdot) \in \pi_{p,q}$ is an approximation of e^{-z} it is natural to construct an operator-function $r(\tau_n A_n)^k$ which could be considered as an approximation of $\exp(tA_n)$ for $t = k\tau_n$. We assume in this section for simplicity that $\|\exp(tA_n)\| \leq M, t \in \overline{I\!R}_+$.

Theorem 2.12 *[40] Let condition (B) be satisfied. There is a constant C depending on r such that if r is A-acceptable, then*

$$\|r(\tau_n A_n)^k\| \leq CM\sqrt{k} \text{ for } \tau_n > 0, k \in I\!N.$$

Remark 2.1 *The term $\sqrt{k}$ in Theorem 2.12 can not be removed in general, moreover, there are examples [51], [93] which shows that the inequality $\|r(\tau_n A_n)^k\| \geq c\sqrt{k}, k \in I\!N$, hold.*

We say that $r(\cdot) \in \pi_{p,q}$ is accurate of order $1 \leq d \leq p+q$ if $|e^{-z} - r(z)| = O(|z|^{d+1})$ as $|z| \to 0$.

Theorem 2.13 *[40] Let condition (B) be satisfied. Then there is a constant C depending on r such that, for r which is A-acceptable and accurate of order d, one gets*

$$\|r(\tau_n A_n)^k u_n^0 - \exp(tA_n)u_n^0\| \leq CM\tau_n^d \|A_n^{d+1}u_n^0\| \text{ for } \tau_n > 0, k \in I\!N, u_n^0 \in D(A_n^{d+1}).$$

Theorem 2.14 *[40] Let condition (B_1) is satisfied. Then there is a constant C depending on r such that, for r which is A-acceptable and accurate of order d, one gets*

$$\|r(\tau_n A_n)^k u_n^0 - \exp(tA_n)u_n^0\| \leq CM\tau_n^d \|A_n^d u_n^0\| \text{ for } \tau_n > 0, k \in I\!N, u_n^0 \in D(A_n^d).$$

Theorem 2.15 *[179], [155] Let condition (B_1) be satisfied. Then there is a constant C depending on r such that for r which is A-acceptable and accurate of order d with $|r(\infty)| < 1$ or condition (2.6) is satisfied, then one gets*

$$\|r(\tau_n A_n)^k u_n^0 - \exp(tA_n)u_n^0\| \leq CM\frac{\tau_n^\gamma}{t^{d-\gamma}}\|A_n^\gamma u_n^0\| \text{ for } \tau_n > 0, 0 \leq \gamma \leq d, t = k\tau_n, k \in I\!N.$$

In [50] , [145], [147] the analogy of Theorems 2.13-2.15 was proved for multistep methods.

Let us recall that constant M_2 in condition (B_1) defines $0 < \alpha < \frac{\pi}{2}$ by $M_2 \sin\alpha < 1$ [105] such that

$$\|(\lambda I - A_n)^{-1}\| \leq \frac{M}{|\lambda - \omega|} \text{ for any } \lambda \in \Sigma(\pi/2 + \alpha). \tag{2.8}$$

Theorem 2.16 *[51], [143] Let condition (B_1) be satisfied. Then there is a constant C depending on r such that, for r which is $A(\theta)$-acceptable, accurate of order d and $\theta \in (\pi/2-\alpha, \pi/2]$ for α from condition (2.8), one gets*

$$\|r(\tau_n A_n)^k\| \leq CM \text{ for } \tau_n > 0, k \in I\!N,$$

and

$$\|r(\tau_n A_n)^k - \exp(tA_n) - \gamma^k \exp(-\tau_n^{-b} a k_n (-A_n)^{-b})\| \leq CM(k_n^{-d} + k_n^{-1/b}), \; t = k_n\tau_n,$$

where $\gamma = r(\infty)$ and a, b are some positive constants.

It is possible to show [144] that $\Pi_{j=1}^k r(\tau_{n,j}A_n)$ is a stable approximation for $\exp(\Sigma_{j=1}^k \tau_{n,j}A)$ with variable stepsize.

2.3 Richardson's Extrapolation method

Let us consider the schemes (2.3) and (2.5) which have an order of convergence $O(\tau_n)$ and denote $\mathcal{U}_n^{\tau_n}(k_n) = U_n(t)u_n^0$ and $\overline{\mathcal{U}}_n^{\tau_n}(k_n) = \overline{U}_n(t)u_n^0, t = k_n\tau_n$. The following deferred approach to the limit is valid:

Theorem 2.17 *[161] Assume that condition (B) is satisfied. Then for $\overline{V}_n(t) = 2\overline{\mathcal{U}}_n^{\tau_n}(k_n) - \overline{\mathcal{U}}_n^{\tau_n/2}(2k_n)$ one has*

$$\|\overline{V}_n(t) - u_n(t)\| \leq \tau_n^2 M e^{\omega t} t^2 \|A_n^3 u_n^0\|, \; t = k_n\tau_n.$$

If in addition the scheme (2.5) is stable, then for $V_n(t) = 2\mathcal{U}_n^{\tau_n}(k_n) - \mathcal{U}_n^{\tau_n/2}(2k_n), t = k_n\tau_n$,

$$\|V_n(t) - u_n(t)\| \leq \tau_n^2 M e^{\omega t} t^2 \|A_n^3 u_n^0\|, \; t = k_n\tau_n.$$

Let us consider the Crank-Nicolson scheme

$$\frac{\tilde{U}_n(k\tau_n + \tau_n) - \tilde{U}_n(k\tau_n)}{\tau_n} = A_n\frac{\tilde{U}_n(k\tau_n + \tau_n) + \tilde{U}_n(k\tau_n)}{2}, \; \tilde{U}_n(0) = I_n, \; k \in I\!N_0, \tag{2.9}$$

Theorem 2.18 *[161] Assume that condition (B) is satisfied and that scheme (2.9) is stable. Then $\psi_n(t) = \frac{4}{3}\tilde{\mathcal{U}}_n^{\tau_n/2}(2k_n) - \frac{1}{3}\tilde{\mathcal{U}}_n^{\tau_n}(k_n)$ satisfies*

$$\|\psi_n(t) - u_n(t)\| \leq c\tau_n^4 e^{\omega t} t^2 \|A_n^6 u_n^0\|, \; t = k_n\tau_n.$$

In general, let us put $\mathcal{V}_n^{\tau_n}(t) = R_{p,q}(\tau_n A_n)^{k_n} u_n^0, \; t = k_n\tau_n$.

Theorem 2.19 *[161] Assume that condition (B) is satisfied, $p = q$ and the scheme which corresponds to $\mathcal{V}_n^{\tau_n}$ is stable. Then for $\psi_n(t) = \frac{-1}{2^{2q}-1}\mathcal{V}_n^{\tau_n}(t) + \frac{2^{2q}}{2^{2q}-1}\mathcal{V}_n^{\tau_n/2}(t)$ one gets*

$$\|\psi_n(t) - u_n(t)\| \leq c\tau_n^{2q+2} e^{\omega t}\Big(\frac{t^{3/2}}{\sqrt{\tau_n}}\|A_n^{2q+3}u_n^0\| + t^3\tau_n^{2q-3}\|A_n^{4q+2}u_n^0\|\Big),\ t = k_n\tau_n.$$

Theorem 2.20 *[161] Assume that condition (B_1) is satisfied, $p = q$ and $\tau_n\|A_n\| \leq const.$ Then for $0 \leq \gamma \leq 2q$ and $\psi_n(t) = \frac{-1}{2^{2q}-1}\mathcal{V}_n^{\tau_n}(t) + \frac{2^{2q}}{2^{2q}-1}\mathcal{V}_n^{\tau_n/2}(t)$ one gets*

$$\|\psi_n(t) - u_n(t)\| \leq c\frac{\tau_n^{2q+2}}{t^{2q+2-\gamma}} e^{\omega t}\|A_n^{\gamma}u_n^0\|,\ t = k_n\tau_n.$$

2.4 Lax-type equivalence theorems with orders

The Lax equivalence Theorem on convergence of solution of approximation problem to the solution of the given well-posed Cauchy problem states that stability of the method is necessary and sufficient for convergence, provided it is consistent. More recently they furnish Lax's Theorem with orders, which make possible consider "not stable" approximations.

Definition 2.4 *The C_0-semigroups $\exp(tA_n)$ and $\exp(tA)$ are said to be consistent of order $O(\varphi(\tau_n))$ on a linear manifold $\mathcal{U} \subset E$ with respect to the semigroup $\exp(\cdot A)$ if $\exp(tA)\mathcal{U} \subseteq D(A)$ and there is a constant C such that*

$$\|(A_np_n - p_nA)\exp(tA)x\| \leq C\tau_n\varphi(\tau_n)e^{\omega t}|x|_{\mathcal{U}}\ \textit{for any}\ x \in \mathcal{U}, \tag{2.10}$$

where $|\cdot|$ denotes the seminorm on $\mathcal{U}$.

Definition 2.5 *The C_0-semigroups $\exp(tA_n)$ is said to be stable of order $O(M_ne^{\omega_nt})$ if there are constants M_n and ω_n such that*

$$\|\exp(tA_n)\| \leq M_ne^{\omega_nt}\ \textit{for any}\ t \in \overline{I\!R}_+. \tag{2.11}$$

The following is a slight modification of [43]-[46] and [62]-[64], which was proved in [157].

Theorem 2.21 *Let the C_0-semigroup $\exp(\cdot A_n)$ be consistent of order $O(\varphi(\tau_n))$ on a linear manifold $\mathcal{U} \subset E$ with respect to the semigroup $\exp(\cdot A)$, $\exp(tA)\mathcal{U} \subset \mathcal{U}$ and $|\exp(tA)x|_{\mathcal{U}} \leq M|x|_{\mathcal{U}}$. The following assertions are equivalent:*

(i) $\|\big(\exp(tA_n)p_n - p_n\exp(tA)\big)x\| \leq 2M_ne^{\omega_nt}K(\frac{C_n}{2}t\varphi(\tau_n), x; E, \mathcal{U})$;

(ii) $\|\big(\exp(tA_n)p_n - p_n\exp(tA)\big)x\| \leq M_ne^{\omega_nt}\begin{cases} M_x, & x \in E, \\ \frac{C_n}{2}t\varphi(\tau_n)|x|_{\mathcal{U}}, & t = k_n\tau_n \in [0,T], x \in \mathcal{U}; \end{cases}$

(iii) $\|\exp(tA_n)\| \leq M_ne^{\omega_nt}$, $\|(A_np_n - p_nA)\exp(tA)x\| \leq C_n\tau_n\varphi(\tau_n)e^{\omega t}|x|_{\mathcal{U}}$ *for any* $x \in \mathcal{U}, t \in \overline{I\!R}_+$,

where M_x is a constant depending only on x and $K(t, x; E, \mathcal{U}) = \inf_{y\in\mathcal{U}}\big\{\|x-y\|_E + t|y|_{\mathcal{U}}\big\}$ is Peetre functional.

Definition 2.6 *The family of discrete semigroups $\{U_n(k_n\tau_n)\}$ is said to be consistent of order $O(\varphi(\tau_n))$ on a linear manifold $\mathcal{U} \subset E$ with respect to the semigroup $\exp(\cdot A)$ provided $\overline{\mathcal{U}} = E$ and*

$$\|(U_n(\tau_n)p_n - p_n\exp(\tau_nA))\exp(tA)x\| \leq C\tau_n\varphi(\tau_n)|x|_{\mathcal{U}}\ \textit{for any}\ x \in \mathcal{U}. \tag{2.12}$$

Theorem 2.22 *Let* $\exp(tA_n)\mathcal{U}_n \subset \mathcal{U}_n$, *condition (B) holds and* $|\exp(tA_n)x|_{\mathcal{U}_n} \le Ce^{\omega t}|x_n|_{\mathcal{U}_n}$ *for any* $x_n \in \mathcal{U}_n$ *and* $t > 0$. *Then the following conditions are equivalent:*

(a) $\|(U_n(k_n\tau_n) - \exp(k_n\tau_n A_n))x_n\| \le M_n K(\frac{C_n k_n \tau_n}{2}\varphi(\tau_n), x_n, E_n, \mathcal{U}_n), n, k_n \in I\!N;$

(b) $\|(U_n(k_n\tau_n) - \exp(k_n\tau_n A_n))x_n\| \le M_n \begin{cases} M_{x_n}, & x_n \in E_n, \\ \frac{C_n}{2} t e^{t\omega}\varphi(\tau_n)|x_n|_{\mathcal{U}_n}, & x_n \in \mathcal{U}_n; \end{cases}$

(c) $\|U_n(k_n\tau_n)\|_{B(E_n)} \le M_n$, $\|(U_n(\tau_n) - \exp(\tau_n A))\exp(tA_n)x_n\| \le \frac{C_n M_n}{2}\tau_n e^{\omega t}\varphi(\tau_n)|x_n|_{\mathcal{U}_n}$, *where* $k_n\tau_n = t \in [0,T]$.

Definition 2.7 *The family of discrete semigroups* $\{U(k_n\tau_n)\}$ *is said to be stable of order* $O(1/\psi(n^{-1}))$ *if*

$$\|U_n(k_n\tau_n)\|_{B(E_n)} \le C/\psi(n^{-1}), \ \textit{for } n, k_n \in I\!N, 0 < \tau_n \le \tau^*, \tau_n k_n \in [0,T]. \tag{2.13}$$

Theorem 2.23 *Let the discrete semigroup* $\{U(k_n\tau_n)\}$ *be consistent of order* $O(\varphi(\tau_n))$ *on a linear manifold* $\mathcal{U} \subset E$ *with respect to the semigroup* $\exp(\cdot A)$. *The following assertions are equivalent:*

(i) $\|U_n(k_n\tau_n)\|_{B(E_n)} \le C/\psi(n^{-1})$;

(ii) $\|(U_n(k_n\tau_n)p_n - p_n\exp(k_n\tau_n A))x\| \le \frac{C}{\psi(n^{-1})}K(k_n\tau_n\varphi(\tau_n), x; E, \mathcal{U}), n, k_n \in I\!N;$

(iii) $\|(U_n(k_n\tau_n)p_n - p_n\exp(k_n\tau_n A))x\| \le \frac{C}{\psi(n^{-1})}\begin{cases} M_x, & x \in E, \\ k_n\tau_n\varphi(\tau_n)|x|_{\mathcal{U}}, & k_n\tau_n \in [0,T], x \in \mathcal{U}, \end{cases}$

where M_x *is a constant depending only on* x.

Theorem 2.24 *Let* $|\exp(tA)x|_{\mathcal{U}} \le C|x|_{\mathcal{U}}$ *for any* $x \in \mathcal{U}$ *and* $t \in [0,T]$. *Then the following conditions are equivalent:*

(i) The family of operators $\{U(k_n\tau_n)\}$ *is consistent of order* $O(\varphi(\tau_n))$ *on a linear manifold* $\mathcal{U} \subset E$ *with respect to the semigroup* $\exp(\cdot A)$ *and stable of order* $O(1/\phi(n^{-1}))$;

(ii) $\|(U_n(k_n\tau_n)p_n - p_n\exp(k_n\tau_n A))x\| \le \frac{C}{\psi(n^{-1})}K(k_n\tau_n\varphi(\tau_n), x; E, \mathcal{U}), n, k_n \in I\!N;$

(iii) $\|(U_n(k_n\tau_n)p_n - p_n\exp(tA))x\| \le \frac{C}{\psi(n^{-1})}\begin{cases} M_x, & x \in E, \\ k_n\tau_n\varphi(\tau_n)|x|_{\mathcal{U}}, & t = k_n\tau_n \in [0,T], \ x \in \mathcal{U}. \end{cases}$

On the extension of Lax-Richtmyer theory see [150], [175].

For the particular case, when $E = L^p(I\!R^d)$ and operator $A \equiv P(D) = \Sigma_{|\alpha|\le r}p_\alpha D^\alpha$ in E, one the can consider Cauchy problem

$$\frac{\partial u(x,t)}{\partial t} = P(D)u(x,t), \ u(x,0) = u^0(x), x \in \overline{I\!R}_+, \tag{2.14}$$

with $P(D)$ such that (2.14) is well-posed in the sense $\|u(\cdot,t)\|_{L^p(\boldsymbol{R})} \le c\,\|u^0(\cdot)\|_{L^p(\boldsymbol{R})}, t \in \overline{I\!R}_+$.

Let us denote $\hat{P}(\xi) = \Sigma_{|\alpha|\le r}p_\alpha(i\xi)^\alpha$. It is well-known that (2.14) well-posed iff $\|\exp(t\hat{P})\|_{M_p} \le C$, $t \in \overline{I\!R}_+$, where M_p is a space of Fourier multipliers.

Semidiscrete approximation of (2.14) is given by

$$\frac{\partial u_n(x,t)}{\partial t} = P_h(D)u_n(x,t), \ u_n(x,0) = u_n^0(x), x \in \overline{I\!R}_+, \tag{2.15}$$

where $P_h(D_h) = h^{-r}\Sigma_{|\alpha|\le r}p_\alpha(h)\Sigma_{\beta\in I_\alpha}b_\beta u_n(x+\beta h,t)$ and $\hat{P}_h(\xi) = h^{-r}\Sigma_{|\alpha|\le r}p_\alpha(h)\Sigma_{\beta\in I_\alpha}b_\beta e^{i\langle\xi,h\beta\rangle}$. Operator $P_h(D_h)$ is said to be consistent with operator $P(D)$ with order μ if $\hat{P}_h(\xi) - \hat{P}(\xi) = h^\mu|\xi|^{r+\mu}Q(h\xi), r = deg\hat{P}_h(\xi), Q$ is an infinitely differentiable function and $|Q(\eta)| \ge Q_0 > 0$ for $0 < |\eta| \le \epsilon_0$.

Theorem 2.25 *[39] Let $P(D)$ and $P_h(D_h)$ be consistent of order μ and (2.14) and (2.15) be well-posed. Then for every $T>0$ there exists $C>0$ such that $\|\big(e^{tP_h(D_h)}-\exp(tP(D))\big)u^0\|_{L^2(\boldsymbol{R}^d)} \le ch^{\mu}\|u^0\|_{W^{2,r+\mu}(\boldsymbol{R}^d)}$ and for $0<s<r+\mu$*

$$\begin{aligned}
\|\big(e^{tP_h(D_h)}-\exp(tP(D))\big)u^0\|_{L^2(\boldsymbol{R}^d)} &\le ch^{\frac{s\mu}{\mu+r}}\|u^0\|_{B_2^s},\\
\|\big(e^{tP_h(D_h)}-\exp(tP(D))\big)u_n^0\|_{L^\infty(\boldsymbol{R}^d)} &\le ch^{\frac{s\mu}{\mu+r}}\|u^0\|_{B_2^{d/2+s}},
\end{aligned}$$

where $B_p^\theta = B_{p,\infty}^\theta$ is a Besov space.

It is remarked in [28], that, for the quite general case $B_{p,q}^\theta = (L^p(I\!R), D(A))_{\theta,q}$.

If we consider full discretization scheme for (2.14) in the form $L_h U_n^{k+1} = B_h U_n^k, k=0,1,2,\dots$, where $L_h v = \Sigma_\beta a_\beta(h) v(x+\beta h), B_h v = \Sigma_\beta b_\beta(h) v(x+\beta h)$, then a discrete semigroup can be constructed as $U_n(k\tau_n)u_n^0 = \mathscr{F}^{-1}\big(\hat{U}_n^k(\xi)\hat{u}_n^0\big), \hat{U}_n(\xi) = \hat{B}_n(\xi)/\hat{L}_n(\xi), \hat{B}_n(\xi) = \Sigma_\beta a_\beta(h) e^{\langle \xi, \beta h\rangle}$ (the time step τ_n is tied to h by $\tau_n/h^r = constant$). Such a finite difference operator, $U_n(k\tau_n)$, approximates (2.14) with order μ if $\hat{U}_n(\xi) = e^{\tau_n \hat{P}(\xi)} + O(h^{r+\mu} + |\xi|^{r+\mu})$ as $\xi, h \to 0$.

Theorem 2.26 *[39] Let (2.14) be well-posed and $U_n(k\tau_n)$ be stable in $E = L^2(I\!R^d)$ and approximate (2.14) with order $\mu > 0$. Then for any $T>0$ there is a constant $c>0$ such that*

$$\|\big(U_n(t)-\exp(tP(D))\big)u_n^0\|_{L^2(\boldsymbol{R}^d)} \le ch^{\mu}\|u_n^0\|_{W^{2,r+\mu}(\boldsymbol{R}^d)}$$

and for $0<s<r+\mu$

$$\begin{aligned}
\|\big(U_n(t)-\exp(tP(D))\big)u^0\|_{L^2(\boldsymbol{R}^d)} &\le ch^{\frac{s\mu}{\mu+r}}\|u^0\|_{B_2^s},\\
\|\big(U_n(t)-\exp(tP(D))\big)u^0\|_{L^\infty(\boldsymbol{R}^d)} &\le ch^{\mu}\|u^0\|_{B_{2,1}^{d/2+\mu+r}},\\
\|\big(U_n(t)-\exp(tP(D))\big)u^0\|_{L^\infty(\boldsymbol{R}^d)} &\le ch^{\frac{s\mu}{\mu+r}}\|u^0\|_{B_2^{d/2+s}},\ t = k\tau_n \in [0,T].
\end{aligned}$$

Inverse results: order of convergence implies smoothness of u_n^0, see [28], [39].

Time discretization of parabolic problems with memory by backward Euler method was considered in [23]. Stability and error estimates take place in a Banach space framework, and the results are used to derive error estimates in the L_2 and maximum norms for piecewise linear finite-element discretizations in two space dimensions.

3 Backward Cauchy problem

Let us consider in the Banach space E the backward Cauchy problem:

$$\begin{aligned}
&v'(t) = Av(t),\ t\in[0,T],\\
&v(T) = v^T,
\end{aligned} \tag{3.1}$$

where we are going to find element $v(0)$. At least in two important cases it is not a well-posed problem, namely, if A is unbounded and generates an analytic C_0-semigroup or if the C_0-semigroup $\exp(\cdot A)$ is compact. Indeed, in these situations the problem $\exp(TA)x = v^T$ is ill-posed [47], [96], [190] in the sense that the operator $\exp(-TA)$ is not bounded on E and, moreover, in general $D(\exp(-TA)) \ne E$. This means that the Cauchy problem (3.1) has a solution in general only for some (but not every) initial data

v^T and the solution $v(0)$, if it is exists, does not depend continuously on the initial data. After changes of variables and putting $v(\eta) = u(T-\eta)$ one can rewrite the problem (3.1) in the form

$$
\begin{aligned}
&u'(t) = -Au(t),\ t \in [0,T], \\
&u(0) = u^0,
\end{aligned} \tag{3.2}
$$

where $u^0 = v^T$ is given and $u(T)$ is the element which we are going to find. In this section we are going consider the approximation of (3.2) with operator A, which generates analytic C_0-semigroup.

Definition 3.1 *A bounded linear operator $R_{\epsilon,T}$ on the space E is called a regularizator for the Cauchy problem (3.2) if for any $\delta > 0$ and any $u^0 \in E$, for which a solution of (3.2) exists, there exists $\epsilon = \epsilon(\delta) > 0$ such that $\epsilon(\delta) \to 0$ as $\delta \to 0$ and $\sup_{\|u^\delta - u^0\| \le \delta} \|R_{\epsilon(\delta),T} u^\delta - \exp(-TA)u^0\| \to 0$ as $\delta \to 0$.*

In [133] it is proved that for the existence of a linear regularizator of the problem (3.2), which commute with operator A, it is necessary and sufficient that $-A$ generates C_ϵ-semigroups $\mathcal{S}_\epsilon(t)$, $0 \le t \le T$, such that C_ϵ strongly converges to the identity operator I as $\epsilon \to 0$.

There are a lot of regularizators, which could be considered for problem (3.2). For example, in [158] it was shown that if $-A_n^2$ generates a cosine operator function, then the method of quasireversibility, which is given by the Cauchy problems

$$
u'_{n,\alpha}(t) = -A_n u_{n,\alpha}(t) - \alpha A_n^2 u_{n,\alpha}(t),\ u_{n,\alpha}(0) = u_n^0,
$$

is a regularization method for (3.2), and $\|u_{n,\alpha}(T) - p_n u(T)\| \le C\alpha\Big(\|u_n^0 - p_n u^0\|/\delta + \rho\Big)$, where $\alpha = \alpha(\delta) = 1/\Big(\ln(1/\delta) - \ln\ln(1/\delta) - o(\ln^{-1}(1/\delta))\Big)$. In this case $\mathcal{S}_\alpha(t) \equiv \exp(-tA)\exp(-\alpha TA^2)$ is a C_α-semigroup with $C_\alpha = \exp(-\alpha TA^2)$ and $C_\alpha \to I$ as $\alpha \to 0$. Moreover, the generator of this C_α-semigroup is $-A$.

It has been shown in [56] that the stochastic differential equation

$$
\begin{aligned}
&du_\alpha(t) = -Au_\alpha(t)dt - \alpha A u_\alpha(t)dw(t), \\
&u(0) = u^0,
\end{aligned} \tag{3.3}
$$

where $w(\cdot)$ is a standard one-dimensional Wiener process, yields a stochastic regularization of (3.2). Explicitly, the operator-function

$$
U_\alpha(t)u^0 = \frac{1}{2\pi i}\int_\Gamma e^{-t\lambda - \alpha\left(w(t)-w(0)\right)\lambda - \frac{1}{2}\alpha^2\lambda^2|t|}(\lambda - A)^{-1}u^0 d\lambda,\ t > 0,
$$

which represents a solution of (3.3) for any $u^0 \in \mathfrak{A}_c(A)$, possesses the following properties:

$$
\lim_{\alpha\to 0}\|U_\alpha(T)u^0 - \exp(-TA)u^0\| = 0, \tag{3.4}
$$

$$
\|U_\alpha(t)\| \le \frac{c_1}{\alpha\sqrt{|t|}}\exp\left(c_2\frac{\sqrt{|t|}}{\alpha} + c_3|t|^{-\mu})^2\right) + b(\alpha,|t|) \text{ for any } \alpha > 0. \tag{3.5}
$$

Here the function $b(\alpha,t)$ is bounded in the parameters α and t and $\mathfrak{A}_c(A)$ is the set of entire vectors of operator A. This means that by virtue of the inequality

$$
\|U_\alpha(T)u^\delta - \exp(-TA)u^0\| \le \|U_\alpha(T)\|\,\|u^\delta - u^0\| + \|U_\alpha(T)u^0 - \exp(-TA)u^0\|, \tag{3.6}
$$

there is a dependence on $\alpha = \alpha(\delta)$ in such a way that $U_\alpha(T)$ becomes a regularizator. The operator-function $t \mapsto \exp\Big((T-t)A\Big)U_\alpha(T)$, $0 \le t \le T$, is a C_α-semigroup with $C_\alpha = \exp(TA)U_\alpha(T)$. One can see that $C_\alpha \to I$ as $\alpha \to 0$ and that the generator of this C_α-semigroup is $-A$.

3.1 C-semigroups and ill-posed problems

Let C be a bounded linear operator on the Banach space E, i.e. $C \in B(E)$, and let $T > 0$ be some finite number.

Definition 3.2 *[185]. A family of bounded operators $\{\mathcal{S}(t) : 0 \le t < T\}$ is called a local C-semigroup on E if*

(i) $\mathcal{S}(t+s)C = \mathcal{S}(t)\mathcal{S}(s)$ for $t, s, t+s \in [0,T)$,

(ii) $\mathcal{S}(0) = C$,

(iii) $\mathcal{S}(\cdot)$ is strongly continuous on $[0,T)$.

Clearly, $\mathcal{S}(\cdot)$ is a commutative family. A local C-semigroup is called nondegenerate if condition $\mathcal{S}(t)x = 0$ for all $t \in (0,T)$ implies $x = 0$. It is seen from Definition 3.2 that a local C-semigroup is nondegenerate [59] if and only if C is injective, i.e., $\mathcal{N}(C) = \{0\}$. Starting from now on we will consider only the case that $C \in B(E)$ is an injective operator.

Definition 3.3 *The generator of $\{\mathcal{S}(t) : 0 \le t < T\}$ is defined as the limit $-Gx := C^{-1}\lim_{h\to 0+}\frac{1}{h}(\mathcal{S}(h)x - Cx)$, $x \in D(G)$, with a natural domain $D(G) := \Big\{x \in E : \exists \lim_{h\to 0+}\frac{1}{h}(\mathcal{S}(h)x - Cx) \in R(C)\Big\}$.*

Proposition 3.1 *[176] The operator G is closed, $R(C) \subseteq \overline{D(G)}$ and $C^{-1}GC = G$.*

We denote the C-semigroup $\mathcal{S}(\cdot)$ with generator $-G$ by $\mathcal{S}(\cdot G)$. Next, let $\tau \in (0,T)$. Put

$$L_\tau(\lambda)x := \int_0^\tau e^{-\lambda t}\mathcal{S}(tG)x\,dt,\ x \in E, \lambda > 0. \tag{3.7}$$

This is the so-called *local Laplace transform* of $\mathcal{S}(\cdot G)$.

Proposition 3.2 *Let $\mathcal{S}(\cdot G)$ be a local C-semigroup and $L_\tau(\cdot)$ be the local Laplace transform of $\mathcal{S}(\cdot G)$. Then for any $x \in E$ one has $L_\tau(\lambda)x \in D(G)$ and*

$$(\lambda + G)L_\tau(\lambda)x = Cx - e^{-\lambda\tau}\mathcal{S}(\tau G)x \text{ for all } \tau \in [0,T) \text{ and } \lambda > 0. \tag{3.8}$$

In the case of local C-semigroups spectrum $\sigma(-G)$ could be located on the line $[0,\infty)$. So in this case in general the Laplace transform of the local C-semigroup does not exist in principle and we follow the ideas from [26], [176], [185]. The function $L_\tau(\lambda)$ with the property (3.8) is said to be asymptotic resolvent.

Theorem 3.1 *[176] Let A be a closed linear operator in E and let $C \in B(E)$ be injective.*

(i) If operator A is the generator of a local C-semigroup $\{S(t); 0 \le t < T\}$ on E, then there exists an asymptotic C-resolvent $L_\tau(\lambda)$ of $-A$ such that

$$\|\frac{d^m}{d\lambda^m}L_\tau(\lambda)x\| \le M_\tau\frac{m!}{\lambda^{m+1}}\|x\|,\ x \in E, \tag{3.9}$$

with $0 \le m/\lambda \le \tau, \lambda > a, m \in I\!N \cup \{0\}$, and operator A satisfies $C^{-1}AC = A$.

(ii) If $-A$ has an asymptotic resolvent which satisfies (3.9), and $CD(A)$ is dense in $D(A)$, $D(C^{-1}AC) \subset D(A)$, i.e. $Cx \in D(A)$ and $ACx \in R(C)$ imply $x \in D(A)$, then the part A_0 of A in $E_0 := \overline{D(A)}$ generates a local C-semigroup on E_0, with C equal to $C_0 := C|_{E_0}$.

In particular, under the assumption that $\overline{CD(A)} = E$, the operator $-A$ generates a local C-semigroup on E if and only if $C^{-1}AC = A$ and there exists an asymptotic C-resolvent satisfying (3.9). In this case, A has dense domain.

Remark 3.1 *An asymptotic C-resolvent $L_\tau(\lambda)$ of operator $-A$ is compact for some $\lambda \in \mathbb{C}$ (and then for any λ large enough) if and only if $\mathcal{S}(\cdot A)$ is compact and uniformly continuous in t. Indeed, if $\mathcal{S}(\cdot A)$ is compact then by (3.7) and [213] it follows that $L_\tau(\lambda)$ is compact. Conversely, taking derivative of $L_\tau(\lambda)$ in τ and using the fact that $\mathcal{S}(\cdot A)$ is uniformly continuous in t we have that $\mathcal{S}(\cdot A)$ is compact as the uniform limit of compact operators. This fact could be used in the approximation of semilinear equations in case of the C-semigroups approach (cf. section 5).*

Let us consider the abstract Cauchy problem, which is given by (3.2).

Definition 3.4 *A function $u(\cdot)$ is called a solution to (ACP; T,y) if $u(\cdot)$ is continuously differentiable in $t \in [0,T)$, $u(t) \in D(A)$ for all $0 \le t < T$, and $u(\cdot)$ satisfies (3.2). We denote (ACP;T,y) with $y \in CD(A)$, also by (ACP;$T, CD(A)$).*

Definition 3.5 The Cauchy problem $(ACP;T,CD(A))$ is said to be *generalized well-posed* if for every $y \in CD(A)$, there is a unique solution $u(\cdot;y)$ to $(ACP;T,y)$ such that $\|u(t;y)\| \le M(t)\|C^{-1}y\|$ for $0 \le t < T$ and $y \in CD(A)$, where the function $M(t)$ is bounded on every compact subinterval of $[0,T)$.

We have to stress here that generalized well-posedness in the sense of Definition 3.5 is more general than in the case of the problem in (2.1). Moreover, we can state that this generalized well-posedness is a solvability condition of (3.2) for which a regularizator exists.

Theorem 3.2 *[176] Let C be a bounded linear injection on E and let A be a closed linear operator. Then the following assertions are equivalent:*

(I) The operator $-A$ is the generator of a local C-semigroup;

(II) $C^{-1}AC = A$, and the problem: $v'(t) = -Av(t) + Cx$, $t \in [0,T), v(0) = 0$, has a unique solution for every $x \in X$.

If either $\rho(A) \neq \emptyset$ or A has dense domain, (I) and (II) are also equivalent to

(III) $C^{-1}AC = A$, and the problem $\big(ACP;T,CD(A)\big)$ is generalized well-posed. Moreover, $u(t;y) = C^{-1}\mathcal{S}(tA)y$, $t \in [0,T)$, is a unique solution for every initial value $y \in CD(A)$.

Since the local C-semigroups are regularizators of the ill-posed problem (3.2) it is very important to present approximation theory of local C-semigroups.

3.2 Semidiscrete approximation theorem

Let us consider, within the general discretization scheme, the semidiscrete approximation of the problem (3.2) in the Banach spaces E_n :

$$\begin{aligned} &u_n'(t) = -A_n u_n(t), \quad t \in [0,T), \\ &u_n(0) = u_n^0, \end{aligned} \tag{3.10}$$

where the operators $-A_n$ are generators of local C_n-semigroups, which are compatible with the operator $-A$ and $u_n^0 \to u^0$. We understand compatibility in the sense of the general approximation scheme as the $\mathcal{PP}$-convergence of $C_n \to C$ and the $\mathcal{PP}$-convergence of resolvents $(\tilde\lambda - A_n)^{-1} \to (\tilde\lambda - A)^{-1}$ for some $\tilde\lambda \in \rho(A) \cap \rho(A_n)$. Recall that in our general case (3.1) such $\tilde\lambda$ exists, since the conditions (A) and (B) from Theorem 2.1 are naturally assumed to be satisfied.

Theorem 3.3 *[208] (*Theorem ABC-C*) Under the assumption $\overline{CD(A^2)} = E$, the following conditions (A_c) together with (B_c) are equivalent to condition (C_c).*

(A_c) Consistency. $C_n \to C$ and operators A_n, A are compatible;

(B_c) Stability. For any $0 < \tau < T$ there is some constant M_τ, not depending on n, such that

$$\|\mathcal{S}(tA_n)\| \leq M_\tau \text{ for } 0 \leq t \leq \tau \text{ and } n \in I\!N;$$

(C_c) Convergence. For any $0 < \tau < T$ we have $\max_{t\in[0,\tau]} \|\mathcal{S}(tA_n)x_n^0 - p_n\mathcal{S}(tA)x^0\| = 0$, as $n \to \infty$, whenever $x_n^0 \to x^0$.

Remark 3.2 *In the case of exponentially bounded C-semigroups [60], [61] we can trivially change the condition (A_c) to condition*

(A') $C_n \to C$ and $(\tilde{\lambda} - A_n)^{-1}C_n \to (\tilde{\lambda} - A)^{-1}C$ for some $\tilde{\lambda} \in \mathbb{C}$; see [218] for details. Since, the construction can be done just as with condition (A'), in this case we do not need to assume that $(\tilde{\lambda} - A_n)^{-1} \to (\tilde{\lambda} - A)^{-1}$ for some $\tilde{\lambda}$.

Remark 3.3 *We have put the condition $\overline{CD(A^2)} = E$ for simplicity. For the general case one gets convergence on the set $\overline{CD(A^2)}$. In the case of integrated semigroups such situations have been well investigated, see for instance [29], [31]. Actually the paper [29] is devoted to the following effect observed in the study of convergence of semigroups. Suppose we are given a sequence of uniformly bounded semigroups $\{\exp(tA_n), t \geq 0\}, n \geq 1$ ($\|\exp(tA_n)\| \leq M, t \in I\!R_+$) acting in the Banach space E. Assume furthermore that the limit $\lim_{n\to\infty}(\lambda I - A_n)^{-1}x = S(\lambda)x$ exists for any $x \in E$. If*

$$\overline{\mathcal{R}(S(\lambda))} = E \tag{3.11}$$

($\mathcal{R}(S(\lambda))$ is common for all $\lambda > 0$), the semigroups in question converge strongly, by the Trotter - Kato theorem. One may also show that if condition (3.11) is relaxed, the limit

$$\lim_{n\to\infty} \exp(tA_n)x \tag{3.12}$$

exists for all $x \in \overline{\mathcal{R}(S(\lambda))}$ (see e.g. [108], [66] p. 34, or [29], [31]). As observed by T.G. Kurtz [108] for any $x \in E$ there exists the limit

$$\lim_{n\to\infty} \int_0^t \exp(sA_n)x ds. \tag{3.13}$$

In general, however, one cannot expect that (3.12) holds for $x \notin \overline{\mathcal{R}(S(\lambda))}$. This effect is of course related to Arendt's theorem, or rather to generation theorem for "absolutely continuous integrated semigroups" presented in [33].

Let us consider a semi-discretization of problem (3.3) in Banach spaces E_n

$$\begin{aligned} du_{n,\alpha}(t) &= -A_n u_{n,\alpha}(t)dt - \alpha A_n u_{n,\alpha}(t)dw(t), \\ u_{n,\alpha}(0) &= u_n^0, \end{aligned} \tag{3.14}$$

where $u_n^0 \to u^0$, the operators A_n generate analytic semigroups and $\{(\Omega, \mathcal{F}, I\!P), w(t)\}$ is the standard one-dimensional Wiener process (Brownian motion). As usual the symbol $I\!E[\cdot]$ will denote mathematical expectation.

We emphasize that we consider the situation where $\sigma(A_n)$, $\sigma(A) \subset \mathbb{C} \setminus \Sigma(\frac{3}{4}\pi)$.

Theorem 3.4 *[208] Let the conditions (A) and (B_1) of Theorem 2.2 be satisfied and let $\delta_n > 0$ be a sequence which converges to 0 if $n \to \infty$. Then there exists a sequence α_n such that $u_{n,\alpha_n}(t) \to u(t)$ for every $t \in [0,T]$ as $n \to \infty$. Here $u_{n,\alpha_n}(\cdot)$ is a solution of (3.14) and $u(\cdot)$ is a solution of (3.2) with $u^0 \in \mathfrak{A}_c(A)$. Convergence is to be understood in the following sense:*

$$\sup_{\|u_n^0 - p_n u^0\| \leq \delta_n} \|u_{n,\alpha_n}(t) - p_n u(t)\| \to 0, \quad I\!P\text{-almost surely as } \delta_n \to 0.$$

3.3 Approximation by discrete C-semigroups

Following Section 2 we denote by $\{T_n(\cdot)\}$ a family of discrete semigroups, on E_n, respectively, i.e., $T_n(t) = T_n(\tau_n)^{k_n}$, where $k_n = [t/\tau_n]$. We define the generator of $T_n(\cdot)$ by the formula $-A_n = \frac{1}{\tau_n}(T_n(\tau_n) - I)$ and we are interested in the process $\tau_n \to 0, k_n, n \to \infty$. We assume that $C_n \in B(E_n)$ is an injective operator such that $T_n C_n = C_n T_n$. The discrete C_n-semigroup $U_n(\cdot)$ is defined as $U_n(t) = T_n(t)C_n$. In this section we also assume that the dimension of each space E_n is finite, but $\dim(E_n) \to \infty$ as $n \to \infty$.

Theorem 3.5 *[208] (Theorem ABC-C-discr) Under condition (A) in Theorem 2.1 and the assumption $\overline{CD(A^2)} = E$, the following conditions (A_{cd}) and (B_{cd}) together are equivalent to condition (C_{cd}).*

(A_{cd}) Consistency. $C_n \to C$, operators A_n, A are compatible and $A_n \in B(E_n)$, $n \in I\!N$;

(B_{cd}) Stability. For any $0 < \tau < T$ there is some constant M_τ, not depending on n, such that

$$\|U_n(t)\| \leq M_\tau \text{ for all } 0 \leq t \leq \tau < T \text{ and } n \in I\!N.$$

is satisfied uniformly for any choice of $\{\tau_n\}$ and $\{k_n\}$ as long as $\tau_n \to 0$, and $k_n = [t/\tau_n]$;

(C_{cd}) Convergence. For any $0 < \tau < T$ one gets $\max_{t\in[0,\tau]} \|U_n(t)x_n^0 - p_n\mathcal{S}(tA)x^0\| \to 0$ as $\tau_n \to 0, n \to \infty$, whenever $x_n^0 \to x^0$.

Theorem 3.6 *[208] Let the conditions (A_c) and (B_c) be satisfied. Assume that condition (A) in Theorem 2.1 and the assumption $\overline{CD(A^2)} = E$ are satisfied and also that $\tau_n\|A_n^2 C_n^{-1}\| \leq \frac{q}{M_\tau T}$ with $q < 1$. Then*

$$\|U_n(t)\| \leq M_\tau(1-q)^{-1} \text{ for } 0 \leq t \leq \tau < T \text{ and any } n \in I\!N.$$

is satisfied uniformly for any choice of $\{\tau_n\}$ and $\{k_n\}$ with $\tau_n \to 0$, as long as $k_n = \left[\frac{t}{\tau_n}\right]$. Moreover, for any $0 < \tau < T$ one gets $\max_{t\in[0,\tau]} \|U_n(t)x_n^0 - p_n\mathcal{S}(tA)x^0\| \to 0$ as $\tau_n \to 0$, $n \to \infty$, whenever $x_n^0 \to x^0$.

Remark 3.4 *As a matter of fact the scheme $\overline{U}_n(t) \equiv (I + \tau_n A_n)^{-k_n} C_n$ with $t = k_n\tau_n$ could be constructed even under condition (2.4). Indeed, $\tau_n A_n = \tau_n \lambda A_n(\lambda - A_n)^{-1} - \tau_n A_n^2(\lambda - A_n)^{-1}$, and by the choice of λ we can make the second term less than ϵ and then by choosing τ_n appropriately for a fixed λ, we obtain $\|\tau_n A_n\| \leq 2\epsilon$, so that the scheme $\overline{U}_n(\cdot)$ is well defined.*

Remark 3.5 *In contrast to the well-posed case, for ill-posed problems it looks that the implicit and explicit methods of discretization in time are not so different in the sense of stability advantages (compare with Theorems 2.5, 2.6). Moreover, under condition (2.4) from the identity*

$$(I - \tau_n A_n)^{k_n} C_n = (I - \tau_n^2 A_n^2)^{k_n}(I + \tau_n A_n)^{-k_n} C_n$$

and inequality $\|(I \pm \tau_n^2 A_n^2)^{k_n}\| \leq Ce^{t\tau_n\|A_n^2\|}, t = k_n\tau_n$, it follows that the stability properties of the implicit and explicit methods are the same.

There are a lot of stochastic finite-difference schemes which could be written for problem (3.14). For example, some of the simplest are

$$U_{n,\alpha}(t+\tau_n) - U_{n,\alpha}(t) = -\tau_n A_n U_{n,\alpha}(t) - \alpha\, \Delta w(t) A_n U_{n,\alpha}(t), \tag{3.15}$$

$$\bar{U}_{n,\alpha}(t+\tau_n) - \bar{U}_{n,\alpha}(t) = -\tau_n A_n \bar{U}_{n,\alpha}(t+\tau_n) - \alpha\, \Delta w(t) A_n \bar{U}_{n,\alpha}(t), \tag{3.16}$$

where $\Delta w(t) = \Big(w(t) - w(t-\tau_n)\Big)$, $t = k_n\tau_n$, and $U_{n,\alpha}(0) = \bar{U}_{n,\alpha}(0) = I_n$.

Theorem 3.7 *[208] Let the conditions (A) and (B_1) of Theorem 2.2 be satisfied. Assume that the stability conditions (2.4) and $\tau_n\|A_n^2\|e^{c\|A_n\|} = O(1)$ are fulfilled for some constant $c > 0$. Then for $\alpha_n = \sqrt{\tau_n}$ the scheme (3.15) behaves stably in the following sense:*

$$\eta_n := \sup_{n\in I\!N} \sup \left\{ I\!E \left[\left\| U_{n,\alpha_n}(t)u_n^0 - \exp\left(-tA_n + \alpha_n(w(t)-w(0))A_n - \frac{t}{2}\alpha_n^2 A_n^2\right) u_n^0 \right\| \right] : \|u_n^0\| \le 1 \right\} \to 0,$$

and converges in the following sense:

$$I\!E\left[\|U_{n,\alpha_n}(t)u_n^0 - p_n u(t)\|\right] \le C\sqrt{\tau_n}\|A\exp(-TA)u^0\| + \|u_{n,\alpha_n}(t) - p_n u_{\alpha_n}(t)\| + C\eta_n\|u_n^0\|, \quad 0 < t \le T.$$

For the scheme $\overline{U}_{n,\alpha_n}(\cdot)$ similar notions are employed.

We can also study the convergence behaviour of more sophisticated numerical methods. For example, in [41] in order to approximate (3.14) the following Runge-Kutta scheme was considered

$$\begin{aligned} Y_1 &= U_{n,\alpha}(t) + \sqrt{\tau_n}\alpha A_n U_{n,\alpha}(t), \\ U_{n,\alpha}(t+\tau_n) - U_{n,\alpha}(t) &= -\tau_n A_n U_{n,\alpha}(t) + \alpha\Delta w(t) A_n U_{n,\alpha}(t) + \\ &+ \frac{\sqrt{\tau_n}}{2}\Big(\frac{(\Delta w(t))^2}{\sqrt{\tau_n}} - 1\Big)\Big(\alpha_n A_n Y_1 - \alpha A_n U_{n,\alpha}(t)\Big). \end{aligned} \tag{3.17}$$

Thus the solution can be written in the form

$$\begin{aligned} &U_{n,\alpha}(t+\tau_n) = \\ &= (I_n - \tau_n A_n - \frac{\alpha^2\tau_n}{2}A_n^2)^{k_n} \Pi_{k=1}^{k_n}\Big(I_n + Z_n^{-1}\alpha\Delta w(t)A_n + (Z_n^{-1}/2)\alpha^2\Delta w(t)^2 A_n^2\Big)U_{n,\alpha}(0), \end{aligned} \tag{3.18}$$

where $Z_n = (I_n - \tau_n A_n - \frac{\alpha^2\tau_n}{2}A_n^2)$.

Theorem 3.8 *[41] Let the conditions (A) and (B) of Theorem 2.2 be satisfied. Assume that the stability conditions (2.4) and $\tau_n\|A_n^2\|e^{c\|A_n\|} = O(1)$ are fulfilled for some constant $c > 0$. Then for $\alpha_n = \sqrt{\tau_n}$ the scheme (3.18) behaves stably in the following sense:*

$$\eta_n := \sup \left\{ I\!E \left[\left\| U_{n,\alpha_n}(t)u_n^0 - e^{\left(-tA_n + \alpha_n(w(t)-w(0))A_n - \frac{t}{2}\alpha_n^2 A_n^2\right)} u_n^0 \right\| \right] : \|u_n^0\| \le 1 \right\} \to 0,$$

and converges in the following sense:

$$\begin{aligned} &I\!E\left[\|U_{n,\alpha_n}(t)u_n^0 - p_n u(t)\|\right] \le \\ &\le C\sqrt{\tau_n}\|A\exp(-TA)u^0\| + \|u_{n,\alpha_n}(t) - p_n u_{\alpha_n}(t)\| + C\eta_n\|u_n^0\|, \quad 0 < t \le T. \end{aligned}$$

In the case of the well-posed problem

$$\begin{cases} du_\alpha(t) = Au_\alpha(t)dt + \alpha Au_\alpha(t)dw(t), \\ u(0) = u^0, \end{cases} \tag{3.19}$$

where the operator A generates an analytic C_0-semigroup, the semidiscrete and full-discretisation schemes do not need additional stability assumptions and the order of convergence will be defined just by the consistency property of the scheme. More exactly the term $e^{\lambda t}$ under the integral leads to the absolute convergence of the integral independently of the behaviour of α on any compact set. For example we have the following

Theorem 3.9 *[41] Let the conditions (A) and (B") of Theorem 2.2 be satisfied. Assume that the stability conditions (2.4) is fulfilled for some constant $C > 0$. Then for any $\alpha_n \in [0, \alpha']$ the scheme like (3.18) behaves stably in the following sense:*

$$\sup \left\{ I\!E \left[\left\| U_{n,\alpha_n}(t) u_n^0 - e^{\left(tA_n + \alpha_n (w(t) - w(0)) A_n - \frac{t}{2} \alpha_n^2 A_n^2\right)} u_n^0 \right\| \right] : \|u_n^0\| \leq 1 \right\} \leq \tau_n,$$

and converges in the following sense:

$$I\!E \left[\|U_{n,\alpha_n}(t) u_n^0 - p_n \exp(tA) u^0\| \right] \leq C\alpha_n \|A \exp(tA) u^0\| + \|u_{n,\alpha_n}(t) - p_n u_{\alpha_n}(t)\| + C\tau_n \|u_n^0\|, \quad 0 < t \leq T.$$

4 Coercive inequalities

Let us consider in the Banach space E the nonhomogeneous Cauchy problem:

$$(4.1) \qquad \begin{aligned} u'(t) &= Au(t) + f(t), \ t \in [0, T], \\ u(0) &= u_0, \end{aligned}$$

with operator A which generates C_0-semigroup and $f(\cdot)$ be some function from $[0, T]$ to E. The problem (4.1) can be considered in different functional spaces. The most popular situations are the following settings: well-posedness in $C([0, T]; E), C^{\alpha,0}([0, T]; E)$ and $L^p([0, T]; E)$ spaces (see [9], [21], [132], [211]).

We say that problem (4.1) is well-posed, say in $C([0, T]; E)$, if for any $f(\cdot) \in C([0, T]; E)$ and any $u_0 \in D(A)$ the problem (4.1) is

i). uniquely solvable, i.e. $u(\cdot)$ satisfies the equation and boundary condition (4.1) , $u(\cdot)$ is continuously differentiable on $[0, T]$, $u(t) \in D(A)$ for any $t \in [0, T]$ and $Au(\cdot)$ is continuous on $[0, T]$;

ii). operator $(f(\cdot), u^0) \to u(\cdot)$, regarded as an operator from $C([0, T]; E) \times D(A)$ to $C([0, T]; E)$ is continuous.

In case $u^0 \equiv 0$, the coercive well-posedness in $C([0, T]; E)$ means that $\|Au(\cdot)\|_{C([0,T];E)} \leq c\|f(\cdot)\|_{C([0,T];E)}$. In general coercive well-posedness in the space $\Upsilon([0, T]; E)$ means for problem (4.1) that it is well-posed in the space $\Upsilon([0, T]; E)$ and one has

$$\|u'(\cdot)\|_{\Upsilon([0,T];E)} + \|Au(\cdot)\|_{\Upsilon([0,T];E)} \leq C\left(\|f(\cdot)\|_{\Upsilon([0,T];E)} + \|u^0\|_F\right),$$

where F is some subspace of E. For results of coercive well-posedness see [9], [21], [132].

The semidiscrete approximation of (4.1) are the Cauchy problems in Banach spaces E_n :

$$(4.2) \qquad \begin{aligned} u_n'(t) &= A_n u_n(t) + f_n(t), \ t \in [0, T], \\ u_n(0) &= u_n^0, \end{aligned}$$

with operators A_n which generate C_0-semigroups, A_n and A are compatible, $u_n^0 \to u^0$ and $f_n \to f$ in appropriate sense. Following section 2 it is natural to assume that conditions (A) and (B_1) are satisfied.

Here we are going to describe the discretization of (4.2) in time. The simplest difference scheme (Rothe scheme) is

$$(4.3) \qquad \begin{aligned} \frac{\overline{U}_n^k - \overline{U}_n^{k-1}}{\tau_n} &= A_n \overline{U}_n^k + \varphi_n^k, \ k \in \{1, ..., [\tfrac{T}{\tau_n}]\}, \\ \overline{U}_n^0 &= u_n^0, \end{aligned}$$

where, for example, in the case of $f_n(\cdot) \in C([0, T]; E_n)$ one can put $\varphi_n^k = f_n(k\tau_n), k \in \{1, ..., K\}, K = [\frac{T}{\tau_n}]$, and in the case $f_n \in L^1([0, T]; E_n)$ one can set $\varphi_n^k = \frac{1}{\tau_n} \int_{t_{k-1}}^{t_k} f_n(s) ds, \ t_k = k\tau_n, k \in \{1, ..., K\}$.

4.1 Coercive inequality in $C_{\tau_n}([0,T];E_n)$ spaces

Denote by $C_{\tau_n}([0,T];E_n)$ the space of the elements $\overline{\varphi}_n=\{\varphi_n^k\}_{k=0}^K$ such that $\varphi_n^k\in E_n, k\in\{0,...,K\}$, with the norm $\|\overline{\varphi}_n\|_{C_{\tau_n}([0,T];E_n)}=\max_{0\le k\le K}\|\varphi_n^k\|_{E_n}$.

We recall that coercive well-posedness in $C([0,T];E)$ implies [21] that A generates an analytic C_0-semigroup.

Theorem 4.1 *[21] Let condition (B_1) be satisfied. The problem (4.3) is stable in the space $C_{\tau_n}([0,T];E_n)$, i.e.*

$$\|\overline{U}_n\|_{C_{\tau_n}([0,T];E_n)}\le C\Big(\|\overline{\varphi}_n\|_{C_{\tau_n}([0,T];E_n)}+\|u_n^0\|\Big).$$

Theorem 4.2 *[21] Let condition (B_1) be satisfied. The problem (4.3) is almost coercive stable in the space $C_{\tau_n}([0,T];E_n)$, i.e.*

$$\|A_n\overline{U}_n\|_{C_{\tau_n}([0,T];E_n)}\le M\Big(\|A_nu_n^0\|_{E_n}+\min\Big(\ln(1/\tau_n),1+\Big|\ln\|A_n\|\Big|\Big)\|\overline{\varphi}_n\|_{C_{\tau_n}([0,T];E_n)}\Big).$$

We have note that if (4.1) is coercive well-posed in the space $C([0,T];E)$, then [65] operator A should be bounded or the space E contains a subspace isomorphic to c_0. It means that in general problem (4.3) is not coercive well-posed in $C_{\tau_n}([0,T];E_n)$ space.

For the explicit scheme

$$(4.4)\qquad \begin{aligned} \frac{U_n^k-U_n^{k-1}}{\tau} &= A_nU_n^{k-1}+\varphi_n^k,\ k\in\{1,...,K\},\\ U_n^0 &= u_n^0, \end{aligned}$$

Theorem 4.2 could be reconstructed, but under a stability condition.

Theorem 4.3 *[21] Let condition (B_1) is satisfied and $\tau_n\ln(\frac{1}{\tau_n})\|A_n\|\le\epsilon$ for sufficiently small $\epsilon>0$. Then the problem (4.4) is almost coercive stable in the space $C_{\tau_n}([0,T];E_n)$, i.e.*

$$\begin{aligned} &\|A_nU_n\|_{C_{\tau_n}([0,T];E_n)}+\|U_n\|_{C_{\tau_n}([0,T];E_{n,1-\frac{1}{\ln\frac{1}{\tau_n}}}}\le\\ \le\ & M\Big(\|A_nu_n^0\|_{E_{n,1-\frac{1}{\ln\frac{1}{\tau_n}}}}+\min\Big(\ln(1/\tau_n),1+\Big|\ln\|A_n\|\Big|\Big)\|\overline{\varphi}_n\|_{C^{\tau_n}([0,T];E_n)}\Big), \end{aligned}$$

where $\|u_n\|_{E_{n,\alpha}}=\Big(\int_0^\infty\|A_n\exp(tA_n)u_n\|_{E_n}^{\frac{1}{1-\alpha}}dt\Big)^{1-\alpha}$.

Remark 4.1 *The space $E_{n,\alpha}$ coincides with equivalent norm with the real interpolation space $(E_n,D(A_n))_{1-1/p,p}$ see [132].*

4.2 Coercive inequality in $C_{\tau_n}^{\alpha,0}([0,T];E_n)$ spaces

Denote by $C_{\tau_n}^{\alpha,0}([0,T];E_n)$ for $0<\alpha<1$ the space of the elements $\overline{\varphi}_n$ with the norm $\|\overline{\varphi}_n\|_{C_{\tau_n}^{\alpha,0}([0,T];E_n)}=\max_{0\le k\le K}\|\varphi_n^k\|_{E_n}+\max_{1\le k<k+l\le K}\|\varphi_n^{k+l}-\varphi_n^k\|_{E_n}(\tau_nk)^\alpha(l\tau_n)^{-\alpha}$.

Theorem 4.4 *[177] Let condition (B_1) hold. Then the scheme (4.3) is coercive well-posed in $C_{\tau_n}^{\alpha,0}([0,T];E_n)$ with $0<\alpha<1$, i.e.*

$$\|A_n\overline{U}_n\|_{C_{\tau_n}^{\alpha,0}([0,T];E_n)}\le\frac{M}{\alpha(1-\alpha)}\Big(\|A_nu_n^0\|_{E_n}+\|\overline{\varphi}_n\|_{C_{\tau_n}^{\alpha,0}([0,T];E_n)}\Big).$$

Roughly speaking, assumption (B_1) is necessary and sufficient for coercive well-posedness in $C_{\tau_n}^{\alpha,0}([0,T];E_n)$ space.

4.3 Coercive inequality in $L^p_{\tau_n}([0,T];E_n)$ spaces

Denote by $L^p_{\tau_n}([0,T];E_n)$ for $1 \leq p < \infty$ the space of the elements $\overline{\varphi}_n$ with the norm $\|\overline{\varphi}_n\|_{L^p_{\tau_n}([0,T];E_n)} = \left(\Sigma_{j=0}^K \|\varphi_n^k\|^p_{E_n}\tau_n\right)^{1/p}$.

Theorem 4.5 *[177] Let condition (B_1) hold. Let the difference scheme (4.3) be coercive well-posed in $L^{p_0}_{\tau_n}([0,T];E_n)$ for some $1 < p_0 < \infty$. Then it is coercive well-posed in $L^p_{\tau_n}([0,T];E_n)$ for any $1 < p < \infty$ and*

$$\|A_n\overline{U}_n\|_{L^p_{\tau_n}([0,T];E_n)} + \max_{0\leq k\leq K}\|\overline{U}^k_n\|_{E_{n,1-1/p}} \leq \frac{Mp^2}{p-1}\Big(\|\overline{\varphi}_n\|_{L^p_{\tau_n}([0,T];E_n)} + \|\overline{U}^0_n\|_{1-1/p}\Big).$$

We have to note that to the contrary to $C^{\alpha,0}$ space case the analyticity of semigroup $\exp(\cdot A)$ is not enough to get coersive well-posedness in L^p space [120], so to state coercive well-posedness in L^p we need some additional assumptions. The necessary and sufficient conditions for coercive well-posedness of the problem (4.1) in $L^p([0,T];E)$ was obtained in [215], [214], [97]. It is not clear yet whether these conditions provide, say, coercive well-posedness of the scheme (4.3) in $L^p_{\tau_n}([0,T];E_n)$.

Theorem 4.6 *[177] Let $1 < p,q < \infty, 0 < \alpha < 1$ and condition (B_1) hold. Then the difference scheme (4.3) is coercive well-posed in $L^p_{\tau_n}([0,T];E_{n,\alpha,q})$, i.e.*

$$\|A_n\overline{U}_n\|_{L^p_{\tau_n}([0,T];E_{n,\alpha,q})} + \max_{0\leq k\leq K}\|U^k_n\|_{E_{n,1-1/p}} \leq \frac{Mp^2}{(p-1)\alpha(1-\alpha)}\Big(\|\overline{\varphi}_n\|_{L^p_{\tau_n}([0,T];E_{n,\alpha,q})} + \|U^0_n\|_{1-1/p}\Big),$$

where $E_{n,\alpha,q}$ is the interpolation space $(E_n, D(A_n))_{\alpha,q}$ with the norm $\|u_n\|_{E_{n,\alpha,q}} = \Big(\int_0^\infty \|\lambda^\alpha A_n(\lambda - A_n)^{-1}\|^q_{E_n}\frac{d\lambda}{\lambda}\Big)^{1/q}$.

Of course for schemes like

$$(4.5)\qquad \begin{aligned} \frac{U_n^k - U_n^{k-1}}{\tau_n} &= A_n\Big(\frac{U_n^k + U_n^{k-1}}{2}\Big) + \varphi_n^k,\ n \in \{1,...,K\},\\ U_n^0 &= u_n^0. \end{aligned}$$

coercive well-posedness could be considered. Moreover, it is possible to consider a more general Padé difference scheme [21] for $p = q-1$ or $p = q-2$. In this case the difference scheme will be written in the form

$$(4.6)\qquad \frac{U_n^k - U_n^{k-1}}{\tau_n} = (\overline{A}_nU_n)^{k-1} + \varphi_n^{p,q,k},\ U_n^0 = u_n^0,\ 1 \leq k \leq K.$$

where $(\overline{A}_nU_n)^k = (\frac{R_{p,q}(\tau_nA_n)-I}{\tau_n}U_n)^{k-1}$ and $\|\varphi_n^k - \varphi_n^{p,q,k}\|_{E_n} \leq M\tau_n^{p+q}$. To formulate coercive statements of subsections 4.1-4.3 we just need to change operator A_n for $\overline{A}_n$. We know from Theorem 2.16 that Padé approximation under condition (B_1) with $p = q$ is stable, but in general it will be not coercive stable. To get coercive inequality we need condition (2.6). Spaces where the problem considered also could be very different [21].

4.4 Coercive inequality in $B_{\tau_n}([0,T];C^\theta_h(\Omega_h)) \cap C_h([0,T];C_h(\bar{\Omega}_h))$

From the numerical analysis point of view it would be very interesting to consider the Problem (4.1) in the space $\Upsilon([0,T];E)$ that E would be smoother than $C(\Omega)$ (elements of such a space could easily be well approximated) and $\Upsilon([0,T];E)$ be like $C([0,T];E)$ or bounded functions space. The interesting fact is that such a situation actually possible at least for the strongly elliptic operator of order 2 with coefficients of class $C^\theta(\overline{\Omega})$. Since in such space E operator $(p_nv)_i = v(ih)$ very concrete, i.e. taking values in the grid points, we consider the case omitting in this section the notion of p_n.

Theorem 4.7 *[35] Let Ω be an open bounded subset of $I\!R^d$, lying on one side of its topological boundary $\partial\Omega$, which is a submanifold of $I\!R^d$ of dimension $d-1$ and class $C^{2+\theta}$, for some $\theta \in (0,2)\setminus\{1\}$. Let $\mathcal{A} = \mathcal{A}(x, D_x) = \sum_{|\alpha|\le 2} a_\alpha(x) D_x^\alpha$ be a strongly elliptic operator of order two (thus, $Re\sum_{|\alpha|=2} a_\alpha(x)\xi^\alpha \ge \nu|\xi|^2$ for some $\nu > 0$ and for any $(x,\xi) \in \overline{\Omega}\times I\!R^d$) with coefficients of class $C^\theta(\overline{\Omega})$. Then, there exist $\mu \ge 0$, $\phi_0 \in (\frac{\pi}{2}, \pi)$ such that for any $\lambda \in \mathbb{C}$, with $|\lambda| \ge \mu$ and $|Arg\lambda| \le \phi_0$ the problem*

$$\begin{aligned} &\lambda v - \mathcal{A}v = y,\\ &\gamma_0 v = 0, \end{aligned}$$

has for any $y \in C^\theta(\overline{\Omega})$ a unique solution v belonging to $C^{2+\theta}(\overline{\Omega})$ and for a certain $M > 0$,

$$|\lambda|^{1+\frac{\theta}{2}}\|v\|_{C(\overline{\Omega})} + |\lambda|\,\|v\|_{C^\theta(\overline{\Omega})} + \|v\|_{C^{2+\theta}(\overline{\Omega})} \le M\Big(\|y\|_{C^\theta(\overline{\Omega})} + |\lambda|^{\frac{\theta}{2}}\|\gamma_0 y\|_{C(\partial\Omega)}\Big), \tag{4.7}$$

where γ_0 is the trace operator on $\partial\Omega$.

It is clear from (4.7) that operator $\mathcal{A}$ in general does not generate C_0-semigroup in $E = C^\theta(\overline{\Omega})$ space, but, following say [132], one can construct semigroup $\exp(t\mathcal{A}), t \ge 0$, which will be analytic.

Let $\mathcal{I} = Z\!\!Z$ and E be a Banach space with norm $\|\cdot\|$. For a grid function $U : \mathcal{I} \to E$, writing U_j or $(U)_j$ instead of $U(j)$ for any $j \in \mathcal{I}$, we set

$$B(\mathcal{I}; E) := \{U : \mathcal{I} \to E \,:\, \sup_{j\in\mathcal{I}}\|U_j\| < +\infty\},\ \|U\|_{B(\mathcal{I};E)} := \sup_{j\in\mathcal{I}}\|U_j\|.$$

It is easily seen that $B(\mathcal{I}; E)$ is a Banach space with the norm $\|\cdot\|_{B(\mathcal{I};E)}$. If the set $\mathcal{I}$ is some interval, say $\mathcal{I} = (a, \infty)$, we denote by $B(\mathcal{I}; E)$ the set of all bounded functions from $\mathcal{I}$ to E.

For the grid function $U : \mathcal{I} \to E$ and $h > 0$, we define operator ∂_h by formula

$$(\partial_h U)_j := h^{-1}(U_{j+1} - U_j).$$

For any $m \in Z\!\!Z$ we set $(\partial_h^m U)_j := h^{-m}\sum_{i=0}^m \binom{m}{i}(-1)^{m-i}U_{j+i}$. If $U \in B(\mathcal{I}; E)$, we set

$$\|U\|_{C_h^m(\mathcal{I};E)} := \max\Big\{\|\partial_h^r U\|_{B(\mathcal{I};E)} \,:\, 0 \le r \le m\Big\}.$$

Finally, let $\theta \in (0,1)$. We define

$$[U]_{C_h^\theta(\mathcal{I};E)} := \sup\Big\{\Big((k-j)h\Big)^{-\theta}\|U_k - U_j\| \,:\, j,k \in \mathcal{I}, j < k\Big\}$$

and, if $m \in I\!N_0$,

$$\|U\|_{C_h^{m+\theta}(\mathcal{I};E)} := \max\Big\{\|U\|_{C_h^m(\mathcal{I};E)}, [\partial_h^m U]_{C_h^\theta(\mathcal{I},E)}\Big\}.$$

In the same context we shall indicate $B(\mathcal{I}; E)$ with $C_h^0(\mathcal{I}; E)$. If $E = \mathbb{C}$ we write simply $B(\mathcal{I})$ or $C_h^0(\mathcal{I})$.

Let $f \in B(I\!N; E)$. We indicate with $\tilde{f}$, the extension of f to $I\!N_0$ such that $\tilde{f}_0 = 0$. For a nonnegative real number ω, we define

$$\|f\|_{C_{h,0}^\omega(I\!N;E)} := \|\tilde{f}\|_{C_h^\omega(I\!N_0;E)}. \tag{4.8}$$

Let now $L > 0, n \in I\!N, n \ge 3$, and $h = \frac{L}{n}$. For $j \in \mathcal{I} := \{1, ..., n-1\}$ we are given complex numbers a_j, b_j, b'_j, c_j satisfying the following conditions (ι):

$(\iota 1)$: there exists $\nu > 0$ such that $Re(a_j) \ge \nu$, for every $j \in \mathcal{I}$;

$(\iota 2)$: $\max\Big\{|a_j|, |b_j|, |b'_j|, |c_j|\Big\} \le Q$ for every $j \in \mathcal{I}$ with $Q > \nu$;

(ι3); there exists $\omega : [0, L] \to [0, +\infty)$ such that $\omega(0) = 0$ and ω is continuous in 0 such that for $j, k \in \mathcal{I}$ with $j \le k$

$$|a_k - a_j| \le \omega\Big((k-j)h\Big).$$

For $\lambda \in \mathbb{C}$ we are going to study the following problem:

$$\begin{array}{ll} \lambda U_j - a_j(\partial_h^2 U)_{j-1} - b_j(\partial_h U)_j - b'_j(\partial_h U)_{j-1} - c_j U_j = f_j & \text{for } j = 1, ..., n-1, \\ U_0 = U_n = 0. \end{array} \tag{4.9}$$

To this aim, we set $\overline{\mathcal{I}} := \{0, 1, ..., n-1, n\}$ and for $U \in B(\mathcal{I}; E)$, we define

$$\tilde{U}_j = \begin{cases} U_j & \text{if } j \in \mathcal{I}, \\ 0 & \text{if } j \in \{0, n\}. \end{cases}$$

We introduce operator A_h in $B(\mathcal{I}; E)$ defined as follows:

$$(A_h U)_j := a_j(\partial_h^2 \tilde{U})_{j-1} + b_j(\partial_h \tilde{U})_j + b'_j(\partial_h \tilde{U})_{j-1} + c_j \tilde{U}_j \ \text{ for } j \in \mathcal{I}. \tag{4.10}$$

Further, we assume that

(ι_θ1): there exists $\nu > 0$ such that $Re(a_j) \ge \nu$ for every $j \in \overline{\mathcal{I}}$;

(ι_θ2): $\max\Big\{\|a\|_{C_h^\theta(\overline{\mathcal{I}})}, \|b\|_{C_h^\theta(\overline{\mathcal{I}})}, \|b'\|_{C_h^\theta(\overline{\mathcal{I}})}, \|b\|_{C_h^\theta(\overline{\mathcal{I}})}\Big\} \le Q$ with $Q > \nu$.

Proposition 4.1 *[91] Assume that assumptions (ι_θ) are satisfied for some $\theta \in (0, 2) \setminus \{1\}$. Fix $\phi_0 \in [0, \pi - \arccos(\frac{\nu}{Q}))$. Then there exists $\mu_0 > 0$ such that $\{\lambda \in \mathbb{C} \ : \ |\lambda| \ge \mu_0, |Arg(\lambda)| \le \phi_0\} \subseteq \rho(A_h)$ where A_h is the operator defined in (4.10). Moreover, for every $r \in [0, 2]$ there exists $c > 0$ depending only on L, ν, Q, r such that for every $f \in B(\mathcal{I}; E)$, and any $F \in B(\overline{\mathcal{I}}; E)$ with $F|_{\mathcal{I}} = f$, one has*

$$\|(\lambda - A_h)^{-1} f\|_{C_{h,0}^{\theta+r}(\mathcal{I}; E)} \le c|\lambda|^{\frac{r}{2}-1}\Big(\|F\|_{C_h^\theta(\overline{\mathcal{I}}; E)} + |\lambda|^{\frac{\theta}{2}} \max\{\|F_0\|, \|F_n\|\}\Big).$$

Let us consider the following mixed Cauchy-Dirichlet parabolic problem:

$$\begin{array}{ll} \frac{\partial u}{\partial t}(t, x) = \mathcal{A}u(t, x) + f(t, x), & t \in [0, T], x \in [0, L], \\ u(t, x') = 0, & t \in [0, T], x' \in \{0, L\}, \\ u(0, x) = 0, & x \in [0, L], \end{array} \tag{4.11}$$

where $\mathcal{A}$ is a second order differential operator and $L > 0$. We say that problem (4.11) has a strict solution if there exists a continuous function $u(t, x)$ with the first derivative with respect to t and the derivatives of order less than or equal to 2 with respect to x which are continuous up to boundary of $[0, T] \times [0, L]$, i.e. $u \in C^1\Big([0, T]; C(\overline{\Omega})\Big) \cap C\Big([0, T]; C^2(\overline{\Omega})\Big)$, and the equations in (4.11) are satisfied.

Theorem 4.8 *[87] Consider problem (4.11) under the following assumptions:*

(I) T and L are positive real numbers;

(II) $\theta \in (0, 1) \setminus \{\frac{1}{2}\}$;

(III) $\mathcal{A}u(x) = a(x)\frac{\partial^2 u}{\partial x^2}(t, x) + b(x)\frac{\partial u}{\partial x}(t, x) + c(x)u(t, x)$, with $a, b, c \in C^{2\theta}([0, L])$;

(IV) a is real valued and $\min a = \nu > 0$;

(V) $f \in C([0, T] \times [0, L])$, $t \to f(t, .) \in B\Big([0, T]; C^{2\theta}([0, L])\Big)$, $t \to f(t, 0)$ and $t \to f(t, L)$ belong to $C^\theta([0, T])$, $f(0, 0) = f(0, L) = 0$.

Then there exists a unique strict solution $u(\cdot)$ of problem (4.11). Such solution belongs to $B\Big([0, T]; C^{2+2\theta}([0, L])\Big)$ and $\frac{\partial u}{\partial t} \in B\Big([0, T]; C^{2\theta}([0, L])\Big)$.

Now let $\mathcal{I}$ be a set which can depend on a positive parameter h and the Banach space $X_h = B(\mathcal{I})$. Next we introduce a linear operator A_h in X_h depending on h. In every case $\rho(A_h)$ contains $\{\lambda \in \mathbb{C} \setminus \{0\} : |\lambda| \geq R$ and $|Arg(\lambda)| \leq \phi_0\}$, where $R > 0$ and $\phi_0 \in (\frac{\pi}{2}, \pi)$, and there exists $M > 0$ such that for λ in the specified subset of $\mathbb{C}$, $\|(\lambda - A_h)^{-1}\|_{L(X_h)} \leq M|\lambda|^{-1}$. Here R, ϕ_0 and M are independent of h. Then we consider another set $\tilde{\mathcal{I}}$ such that $\mathcal{I} \subseteq \tilde{\mathcal{I}}$, we put $\tilde{X}_h := B(\tilde{\mathcal{I}})$. We define an extension operator $\mathscr{E}_h$ from X_h to $\tilde{X}_h$: in all our concrete cases this was the extension with zero. Next, for $\theta \in (0,1)$, we introduce the norms $\|\cdot\|_{2\theta,h}$ and $\|\cdot\|_{2+2\theta,h}$ in $\tilde{X}_h$. The first of these norms is connected to $\|\cdot\|_X$ and the operator A_h by the following property: there exist two positive constants c_1 and c_2 independent of h such that for every $U \in X_h$

$$c_1\|\mathscr{E}_h U\|_{2\theta,h} \leq \|U\|_{(X_h,D(A_h))_\theta} \leq c_2\|\mathscr{E}_h U\|_{2\theta,h}.$$

Then for every h we consider a restriction operator $\mathcal{R}_h \in L(\tilde{X}_h, X_h)$, such that $\mathcal{R}_h\mathscr{E}_h = I_{X_h}$. Let us also introduce a seminorm p_h in $\tilde{X}_h$: in the concrete cases we have $p_h(U) = \|U|_{\tilde{\mathcal{I}}\setminus\mathcal{I}}\|_{B(\tilde{\mathcal{I}}\setminus\mathcal{I})}$. We assume that, if $|\lambda| \geq R$ and $|Arg(\lambda)| \leq \phi_0$, for every $G \in \tilde{X}_h$, then

$$|\lambda|\,\|\mathscr{E}_h(\lambda - A_h)^{-1}\mathcal{R}_h G\|_{2\theta,h} + \|\mathscr{E}_h(\lambda - A_h)^{-1}\mathcal{R}_h G\|_{2+2\theta,h} \leq M\Big(\|G\|_{2\theta,h} + |\lambda|^\theta p_h(G)\Big).$$

Another inequality we impose is the following. If $|\lambda| \geq R$, $|Arg(\lambda)| \leq \phi_0$ and $G \in \tilde{X}_h$, then

$$\|A_h(\lambda - A_h)^{-1}\mathcal{R}_h G\|_{X_h} \leq M|\lambda|^{-\theta}\Big(\|G\|_{2\theta,h} + |\lambda|^\theta p_h(G)\Big).$$

Such an inequality can be easily deduced in each of our examples. In the formulation below we remove the parameter h. In the case of the backward Euler scheme (4.3) we have

Theorem 4.9 *[91] Let X and $\tilde{X}$ be Banach spaces with norms $\|\cdot\|_X$ and $\|\cdot\|_{\tilde{X}}$ respectively, $A \in B(X), \mathscr{E} \in B(X,\tilde{X}), \mathcal{R} \in B(\tilde{X}, X)$ such that $\mathcal{R}\mathscr{E} = I_X$. Assume, moreover, that $\theta \in (0,1)$ and $\|\cdot\|_{2\theta}$ and $\|\cdot\|_{2+2\theta}$ are norms in $\tilde{X}$, while p is a seminorm in the same space. Finally, assume that there exist $R > 0, \phi_0 \in (\frac{\pi}{2}, \pi), M > 0$, such that the following conditions are satisfied:*

(a) $\{\lambda \in \mathbb{C} : |\lambda| \geq R, |Arg(\lambda)| \leq \phi_0\} \subseteq \rho(A)$ and, for λ in this set,

$$\|(\lambda - A)^{-1}\|_{B(X)} \leq M(1 + |\lambda|)^{-1};$$

(b) for every $F \in X$

$$\|\mathscr{E}F\|_{2\theta} \leq M\|F\|_{(X,D(A))_\theta};$$

(c) for every $V \in \tilde{X}, \lambda \in \mathbb{C}$ with $|\lambda| \geq R$ and $|Arg(\lambda)| \leq \phi_0$

$$(1 + |\lambda|)^{-1}\|\mathscr{E}(\lambda - A)^{-1}\mathcal{R}V\|_{2\theta} + \|\mathscr{E}(\lambda - A)^{-1}\mathcal{R}V\|_{2+2\theta}$$

$$+(1 + |\lambda|)^\theta\|A(\lambda - A)^{-1}\mathcal{R}V\|_X \leq M\Big(\|V\|_{2\theta} + (1 + |\lambda|)^\theta p(V)\Big);$$

(d) $p(V) \leq \|V\|_{2\theta}$ for every $V \in \tilde{X}$ and $p(\mathscr{E}F) = 0$ for every $F \in X$;

(e) $\|\mathcal{R}V\|_X \leq \|V\|_{2\theta}$ for every $V \in \tilde{X}$.

Let $T > 0$, $K \in \mathbb{N}, K \geq 2$, $\tau = \frac{T}{K}$. Assume that $\tau R < 1$.

Let $\overline{G} \in B(\{0,1,...,K\};\tilde{X})$ be such that $G^0 = 0$ and consider problem (4.3) with $\varphi^k = \mathcal{R}G^k$ for $k = 1,...,K$ and $U^0 = 0$. Then for $\overline{U} \in B(\{0,1,...,K\};X)$ which solves (4.3) one has

$$\|\mathscr{E}\overline{U}^k\|_{2+2\theta} \leq c\Big(\max_{0\leq k\leq K}\|G^k\|_{2\theta} + \max_{0\leq k_1<k_2\leq K}((k_2 - k_1)\tau)^{-\theta}p(G^{k_2} - G^{k_1})\Big), \tag{4.12}$$

for $k = 0,1,...,K$, where c is a positive constant depending only on θ, R, ϕ_0, M, T and independent of τ_n and $\overline{G}$.

Now we consider the Crank-Nicolson scheme: we replace (4.3) with (4.5). Theorem 4.9 has the following correspondent:

Theorem 4.10 *[90] Assume that the assumptions of Theorem 4.9 are all satisfied and, moreover,*
(f) $\|\tau A\|_{B(X)} \le S$ *with some* $S > 0$;
(g) if $|\lambda| \ge 2S$, *then*

$$\|\mathscr{E}(\lambda - \tau A)^{-1}\mathcal{R}V\|_{2\theta} \le M\left(\|V\|_{2\theta} + \tau^{-\theta}p(V)\right) \textit{ for every } V \in \tilde{X};$$

(h) $\|\mathscr{E}\mathcal{R}V\|_{2\theta} \le M\Big(\|V\|_{2\theta} + \tau^{-\theta}p(V)\Big)$ *for every* $V \in \tilde{X}$;
(i) $2\tau R < 1$.
Let $\overline{G} \in B(\{0,1,...,K\};\tilde{X})$ *be such that* $G^0 = 0$ *and consider problem (4.5) with* $\varphi^k = \mathcal{R}G^k$ *for* $k = 1,...,K$ *and* $U^0 = 0$. *If* $U \in B\Big(\{0,1,...,K\};X\Big)$ *solves (4.5) for* $k = 0,1,...,K$, *then*

$$\|\mathscr{E}U^k\|_{2+2\theta} \le c\Big(\max_{0\le k\le K}\|G^k\|_{2\theta} + \max_{0\le k_1<k_2\le K}((k_2-k_1)\tau)^{-\theta}p(G^{k_2}-G^{k_1})\Big), \tag{4.13}$$

where c *is a positive constant depending only on* $\theta, R, \phi_0, M, S, T$ *and is independent of* τ_n *and* $\overline{G}$.

An application of Theorems 4.9 and 4.10 to the discretization of problem (4.11) is the following. Let $K, n \in I\!N$. We put $\tau := \frac{T}{K}, h := \frac{L}{n}$. We assume that $K \ge 2, n \ge 3$. For $j = 0,1,...,n$ we set $a_j := a(jh)$, $b_j = \frac{1}{2}b(jh), c_j := c(jh), \mathbf{N}_{n-1} := \{1,...,n-1\}$, $\overline{\mathbf{N}}_n := \{0,1,...,n-1,n\}$,

$$X := B(\mathbf{N}_{n-1}), \tilde{X} := B(\overline{\mathbf{N}}_n). \tag{4.14}$$

If $V \in X$, we put, as before for $i \in \mathbf{N}_{n-1}$,

$$(A_hV)_i := a_i\frac{\tilde{V}_{i+1} - 2\tilde{V}_i + \tilde{V}_{i+1}}{h^2} + b_i\frac{\tilde{V}_{i+1} - \tilde{V}_{i-1}}{2h} + c_i\tilde{V}_i, \tag{4.15}$$

where

$$\tilde{V}_i = (\mathscr{E}V)_i = \begin{cases} V_i & \text{if } 1 \le i \le n-1, \\ 0 & \text{if } i \in \{0,n\}. \end{cases}$$

Next, we define

$$\mathcal{R} \in B(\tilde{X}, X), \mathcal{R}V := V|_{\mathbf{N}_{n-1}} \tag{4.16}$$

for every $V \in \tilde{X}$. Then, again for $V \in \tilde{X}$, we set, for $\theta \in (0, \frac{1}{2})$,

$$\|V\|_{2\theta} := \max\{\|V\|_{\tilde{X}}, \max_{0\le i_1<i_2\le n}((i_2-i_1)h)^{-2\theta}|V_{i2} - V_{i_1}|\}, \tag{4.17}$$

$$\|V\|_{2+2\theta} := \max\{\|V\|_{\tilde{X}}, \max_{0\le i\le n-1}|(\partial_hV)_i|, \max_{0\le i\le n-2}|(\partial_h^2V)_i|,$$

$$\max_{0\le i_1<i_2\le n-2}((i_2-i_1)h)^{-2\theta}|(\partial_h^2V)_{i2} - (\partial_h^2V)_{i_1}|\}, \tag{4.18}$$

with

$$(\partial_hV)_i := \frac{V_{i+1} - V_i}{h} \text{ for } 0 \le i \le n-1, (\partial_h^2V)_i := \frac{V_{i+2} - 2V_{i+1} + V_i}{h} \text{ for } 0 \le i \le n-2,$$

$$p(V) := \max\{|V_0|, |V_n|\}. \tag{4.19}$$

One has the following result:

Theorem 4.11 *[90] With the positions (4.14),(4.15), (4.16) and (4.17) the assumptions* $(a)-(e)$ *of Theorem 4.9 are satisfied, with* R, ϕ_0, M *independent of* h. *If we impose the further condition*

$$\tau_n \le \alpha h^2, \tag{4.20}$$

the same holds true also for assumptions $(f)-(h)$ *of Theorem 4.10 (with even* S *independent of* n*).*

As a consequence, we have:

Theorem 4.12 *[91] Consider the problem (4.11) under the assumptions of Theorem 4.8. With the conventions (4.14),(4.15), (4.16), (4.17), set* $G_j^k := f(k\tau_n, jh)$ *for* $k \in \{1, ..., K\}, j = 0, ..., n$. *Put* $\varphi^k := \mathcal{R}G^k$ *and indicate with* G^0 *the zero in* $B(\overline{\mathbf{N}}_n)$.

Then if τ_n *is sufficiently small,the problem*

$$\begin{aligned} &\tfrac{\tilde{U}_j^k-\tilde{U}_j^{k-1}}{\tau_n} = a_i \tfrac{\tilde{U}_{i+1}^k - 2\tilde{U}_i^k + \tilde{U}_{i-1}^k}{h^2} + b_i \tfrac{\tilde{U}_{i+1}^k - \tilde{U}_{i-1}^k}{2h} + c_i \tilde{U}_i^k + \varphi_j^k, , \\ &\tilde{U}_j^0 = 0, \end{aligned} \tag{4.21}$$

for $j \in 1, \ldots, n-1$, $k \in \{1, ..., K\}$ *has a unique solution such that*

$$\|\tilde{U}^k\|_{C_h^{2+2\theta}(\overline{\mathbf{N}}_n)} \le c\Big(\|f\|_{B([0,T];C^{2\theta}([0,L]))} + \max\{\|f(\cdot,0)\|_{C^\theta([0,T])}, \|f(\cdot,L)\|_{C^\theta([0,T])}\}\Big), \tag{4.22}$$

with c *independent of* h *and* τ_n.

An analogous result holds true for the Crank-Nicholson scheme (4.5). Then we set $G_j^k := f((k-\frac{1}{2})\tau_n, jh)$, under the further condition (4.20).

Remark 4.2 *From Theorem 4.11 it follows that, for the scheme (4.5) with* $u_n^0 = 0$ *and under condition (4.20)*

$$\|A_h U\|_{C_h^{2\theta}(\mathbf{N}_{n-1})} \le c\Big(\|f\|_{B([0,T];C^{2\theta}([0,L]))} + \max\{\|f_n(\cdot,0)\|_{C^\theta([0,T])}, \|f(\cdot,L)\|_{C^\theta([0,T])}\}\Big), \tag{4.23}$$

with c *independent of* h *and* τ_n. *In the quoted papers Theorems 4.9 and 4.10 are applied also to the discretization of the heat equation in a square.*

A counter example in [90] and [89] shows that condition (4.20) cannot be removed in general. Finally estimates of order of convergence are given in [89].

5 Approximations of Semilinear Equations

Let us consider in a Banach space E the semilinear Cauchy problem

$$u'(t) = Au(t) + f(t, u(t)), \;\; u(0) = u^0, \tag{5.1}$$

with operator A, which generates analytic C_0-semigroup of type $\omega(A) < 0$ and function f is smooth enough. Existence and uniqueness of solution of problem (5.1) have been discussed in [9], [27], [92], [94], [132] for example.

5.1 Approximations of Cauchy problem

By a semidiscrete approximation of the problem (5.1) we mean the Cauchy problems in the Banach spaces E_n

(5.2) $$u_n'(t) = A_n u_n(t) + f_n(t, u_n(t)), \;\; u_n(0) = u_n^0,$$

where the operators A_n generate analytic semigroups in E_n, A_n and A are compatible, functions f_n approximate f and $u_n^0 \to u^0$.

Let Ω be an open set in a Banach space F and let $\mathcal{B} : \bar{\Omega} \to F$ be a compact operator having no fixed points on the boundary of Ω. Then for the vector field $\mathcal{F}(x) = x - \mathcal{B}x$ there is defined the rotation $\gamma(I - \mathcal{B}; \partial\Omega)$, being an integer-valued characteristic of this field. Let x^* be a unique isolated fixed point of the operator $\mathcal{B}$ in the ball S_{r_0} of radius r_0 with the centre at x^*. Then $\gamma(I - \mathcal{B}; \partial S_r) = \gamma(I - \mathcal{B}; \partial S_{r_0})$ for $0 < r \le r_0$, and this common value of the rotations is called the index of the fixed point x^* and it is denoted by $ind\ x^*$.

Theorem 5.1 *[159] Assume that conditions (A) and (B_1) hold and compact resolvents $(\lambda I - A)^{-1}, (\lambda I - A_n)^{-1}$ converge $(\lambda I - A_n)^{-1} \to (\lambda I - A)^{-1}$ compactly for some $\lambda \in \rho(A)$ and $u_n^0 \to u^0$. Assume that*

(i) the functions f_n, f are bounded and sufficiently smooth so that there exists a unique mild solution $u^*(\cdot)$ of the problem (5.1) on $[0,T]$ (in this situation $ind\ u^*(\cdot) = 1$);

(ii) $f_n(t, x_n) \to f(t,x)$ uniformly with respect to $t \in [0,T]$ for $x_n \to x$;

(iii) the space E is separable .

Then for almost all n the problems (5.2) have in the neighbourhood of $p_n u^*(\cdot)$ mild solutions $u_n^*(t), t \in [0,T]$. Each sequence $\{u_n^*(t)\}$ is $\mathcal{P}$-compact and $u_n^*(t) \to u^*(t)$ uniformly with respect to $t \in [0,T]$.

Let us consider time discretization with respect to the explicit difference scheme

(5.3) $$\frac{U_n(t+\tau_n) - U_n(t)}{\tau_n} = A_n U_n(t) + f_n(t, U_n(t)), \, U_n(0) = u_n^0, \, t = k\tau_n, k = \{0, \cdots, K\}.$$

Theorem 5.2 *[159] Assume that the conditions of Theorem 5.1 and condition (2.6) are satisfied. Then the functions $U_n(t)$ from (5.3) give an approximate mild solution $u^*(\cdot)$ of the problem (5.1) and, moreover, $U_n(t) \to u^*(t)$ uniformly with respect to $t \in [0,T]$.*

Let us define the operator $\Re(u_n)(t) = u_n(t) - \int_0^t \exp((t-s)A_n) f(s, u_n(s)) ds$.

Remark 5.1 *If we assume that the conditions of Theorem 5.1 hold and the functions $f(\cdot), f_n(\cdot)$ have Fréchet derivatives in some balls containing the solutions u^* and u_n^* and moreover, assume that $f'_{n_u}(t, p_n u^*(t))$ are uniformly continuous with respect to the first and second arguments and $f'_{n_{u_n}}(t, u_n(t)) \to f'_u(t, u^*(t))$ uniformly with respect to $t \in [0,T]$ for $u_n \to u^*$, then [159] for almost all n the problems (5.2) have in the neighbourhood of $p_n u^*(\cdot)$ mild solutions $u_n^*(t), t \in [0,T]$. Each sequence $\{u_n^*(\cdot)\}$ is $\mathcal{P}$-compact and $u_n^*(t) \to u^*(t)$ uniformly with respect to $t \in [0,T]$ and, moreover, for sufficiently large $n \ge n_0$ and some $T^* \le T$ we have*

$$c_1 \; \epsilon_n(u^*, u_n^0) \le \|u_n^* - p_n u^*\|_{F_n} \le c_2 \; \epsilon_n(u^*, u_n^0),$$

where the constants c_1, c_2 are not dependent on n, $F_n = C([0,T]; E_n)$, while

$$\epsilon_n(u^*, u_n^0) = \max_{t \in [0,T^*]} \|\Re(p_n u^*)(t) - \exp(tA_n) u_n^0\|_{E_n}.$$

Let $\mathcal{U}_n(t) = (I_n + \tau_n A_n)^k, \Im_n(u_n)(t) = u_n(t) - \Sigma_{l=1}^{k-1} \mathcal{U}_n((k-l)\tau_n) \; f_n(l\tau_n, u_n(l\tau_n)) \; \tau_n$.

Remark 5.2 *If we assume that the conditions of Theorem 5.2 hold and the functions $f(\cdot), f_n(\cdot)$ have Fréchet derivatives in some balls containing the solutions $u^*(\cdot)$ and $u_n^*(\cdot)$ and moreover, assume that $f'_{n u_n}(t, p_n u^*(t))$ are uniformly continuous with respect to the first and second arguments and $f'_{n u_n}(t, u_n(t)) \to f'_u(t, u^*(t))$ uniformly with respect to $t \in [0,T]$ for $u_n \to u^*$, and condition (2.6) holds, then [159] the functions $U_n(t)$ from (5.3) give an approximate mild solution of the problem (5.1) and $U_n^*(t) \to u^*(t)$ uniformly with respect to $t \in [0,T]$ and , moreover, for sufficiently large $n \geq n_0$ and some $T^* \leq T$ we have*

$$c_1 \ \epsilon_n(u^*, u_n^0) \leq \|U_n^* - p_n u^*\|_{F_n^{\tau_n}} \leq c_2 \ \epsilon_n(u^*, u_n^0),$$

where the constants c_1, c_2 do not depend on n, $F_n^{\tau_n} = \{u_n(k\tau_n) : \max_{0 \leq k\tau_n \leq T} \ \|u_n(k\tau_n)\|_{E_n} \ < \infty \ \}$, while $\epsilon_n(u^*, u_n^0) = \max_{t \in [0,T^*]} \|\Im_n(p_n u^*)(t) - \mathcal{U}_n(t) u_n^0\|_{E_n}$.

Schemes which have higher order of convergence than (5.3) are considered in [139], [160]. The Runge-Kutta methods for semilinear equations were considered in [75], [130], [129], [128], [139], [142].

5.2 Approximation of periodic problem

Let us consider in a Banach space E the semilinear T-periodic problem

$$v'(t) = Av(t) + f(t, v(t)), \ \ v(t) = v(T+t), t \in \overline{I\!R}_+, \tag{5.4}$$

with operator A, which generates an analytic compact C_0-semigroup and with the function f which is smooth enough and $f(t,x) = f(t+T, x)$ for any $x \in E$ and $t \in \overline{I\!R}_+$. Let $u(\cdot; u^0)$ be the solution of Cauchy problem (5.1) with initial data $u(0; u^0) = u^0$. This function $u(\cdot; u^0)$ is also a mild solution, i.e. it satisfies the integral equation

$$u(t) = \exp(tA) u^0 + \int_0^t \exp\left((t-s)A\right) f(s, u(s)) ds, \ t \in \overline{I\!R}_+. \tag{5.5}$$

Then the shift operator $\mathcal{K}(u^0) = u(T; u^0)$ can be defined and it maps E to E. If $u(\cdot; x^*)$ is a periodic solution of (5.1), then x^* is a zero of the compact vector field defined by $I - \mathcal{K}$, i.e. $\mathcal{K}(x^*) = x^*$.

Remark 5.3 *We assume here that operator $(I - \exp(TA))^{-1}$ exists and is bounded. In the mean time it is just enough to assume that $(I - \exp(tA))^{-1} \in B(E)$ holds for $t \geq t_0$ with some $t_0 > 0$. This assumption is not restrictive as without loss of generality we are able to change A to $A - \omega I$ and get $\|\exp\left(t(A-\omega)\right)\| \leq Me^{-\delta t}$ for $\delta > 0, t \geq 0$. It follows then [25] that $(I - \exp(tA))^{-1} \in B(E)$ for any $t > 0$.*

Remark 5.4 *We said that function f is smooth enough in the sense that it has to be at least continuous in both arguments, $\sup_{t \in [0,T], \|x\| \leq c_1} \|f(t,x)\| \leq C_2$ and such that there exists the global mild solution of the problem $u'(t) = Au(t) + f(t, u(t)), \ \ u(0) = u^0, t \in \overline{I\!R}_+$.*

Definition 5.1 *The solution $u(\cdot)$ of Cauchy problem (5.1) is said to be stable in the Lyapunov sense if for any $\epsilon > 0$ there is $\delta > 0$ such that inequality $\|u(0) - \tilde{u}(0)\| \leq \delta$ implies $\max_{0 \leq t < \infty} \|u(t) - \tilde{u}(t)\| \leq \epsilon$, where $\tilde{u}(\cdot)$ is a mild solution of (5.1) with the initial value $\tilde{u}(0)$.*

Definition 5.2 *The solution $u(\cdot)$ of Cauchy problem (5.1) is said to be uniformly asymptotically stable at the point $u(0)$ if it is stable in the Lyapunov sense and for any mild solution $\tilde{u}(\cdot)$ of (5.1) with $\|u(0) - \tilde{u}(0)\| \leq \delta$ it follows that $\lim_{t \to \infty} \|u(t) - \tilde{u}(t)\| = 0$ uniformly in $\tilde{u}(\cdot) \in B(u(0); \delta)$, i.e. there is a function $\phi_{u(0),\delta}(\cdot)$ such that $\|u(t; u(0)) - u(t; \tilde{u}(0))\| \leq \phi_{u(0),\delta}(t)$ with $\phi_{u(0),\delta}(t) \to 0$ as $t \to \infty$ and $\|u(0) - \tilde{u}(0)\| \leq \delta$.*

Constructive conditions on operator A and f to ensure that equation $u'(t) = Au(t) + f(u(t)), u(u) = u^0$ is asymptotically $k-$dimensional are given in [167], [166]. They concern the location of eigenvalues of A, i.e. $\lambda_{k+1} - \lambda_k > 2L, \lambda_{k+1} > L$.

Theorem 5.3 *[34] Assume that conditions (A) and (B") hold and compact resolvents $(\lambda I - A)^{-1}, (\lambda I - A_n)^{-1}$ converge $(\lambda I - A_n)^{-1} \to (\lambda I - A)^{-1}$ compactly for some $\lambda \in \rho(A)$. Assume that*

(i) the functions f, f_n are sufficiently smooth, so that there exists an isolated mild solution $v^*(\cdot)$ of periodic problem (5.4) with $v^*(0) = x^*$ such that the Cauchy problem (5.1) with $u(0) = x^*$ has an uniformly asymptotically stable isolated solution at the point x^* (in this case $ind\, v^*(\cdot) = 1$);

(ii) $f_n(t, x_n) \to f(t, x)$ for any $t \in [0, T]$ as $x_n \to x$;

(iii) the space E is separable .

Then for almost all n the problems

$$v_n'(t) = A_n v_n(t) + f_n(t, v_n(t)), v_n(t) = v_n(t + T), t \in \overline{I\!R}_+, \tag{5.6}$$

have periodic mild solutions $v_n^*(t), t \in [0, T]$, in the neighbourhood of $p_n v^*(\cdot)$, where $v^*(\cdot)$ is a mild periodic solution of (5.4) with $v^*(0) = x^*$. Each sequence $\{v_n^*(\cdot)\}$ is $\mathcal{P}$-compact and $v_n^*(t) \to v^*(t)$ uniformly with respect to $t \in [0, T]$.

We say that a fixed point x^* to operator $\mathcal{K}$ in Banach lattice E is stable from above [94] if given $\epsilon > 0$, there is a $\delta > 0$ such that $\|\mathcal{K}^k x - x^*\| \le \epsilon$ for all $k \in I\!N$ if $x^* \preceq x$ and $\|x - x^*\| \le \delta$. Using this notion Theorem 5.3 can be reformulated for positive semigroups due to result from [58].

Theorem 5.4 *Let the operators A_n and A from the problems (5.4) and (5.6) be consistent and let E, E_n be order unit spaces and $e_n \in D(A_n) \cap int E_n^+$. Assume that the operators A_n have the POD property and $A_n e_n \preceq 0$ for sufficiently large n and compact resolvents $(\lambda I - A)^{-1}, (\lambda I - A_n)^{-1}$ converge $(\lambda I - A_n)^{-1} \to (\lambda I - A)^{-1}$ compactly for some $\lambda \in \rho(A)$. Assume that*

(i) the functions f, f_n are sufficiently smooth, bounded and positive, so that there exists a mild solution $u^*(\cdot)$ of the Cauchy problem (5.3) such that elements $u^*(0) = x^*$ is a stable from above and fixed points of operator $\mathcal{K}$, with $x^* \prec y, \mathcal{K}y \preceq y$ (in this situation $ind\, x^* = 1$);

(ii) $f_n(t, x_n) \to f(t, x)$ uniformly with respect to $t \in [0, T]$ for $x_n \to x$;

(iii) the space E is separable .

Then for almost all n the problems (5.6) have periodic mild solutions $v_n^*(t), t \in [0, T]$ in the neighbourhood of $p_n v^*(\cdot)$, where $v^*(\cdot)$ is any stable from above mild periodic solution of (5.4). Each sequence $\{v_n^*(\cdot)\}$ is $\mathcal{P}$-compact and $v_n^*(t) \to v^*(t)$ uniformly with respect to $t \in [0, T]$.

Remark 5.5 *The technique which is used here could be applied to the case of condensing operators [3]. For example, the resolvent of Δ in $L^2(I\!R^d)$ is condensing, but it is not compact.*

In [113] they study the qualitative behavior of spatially semidiscrete finite element solutions of a semilinear parabolic problem near an unstable hyperbolic equilibrium.

The shadowing approach to study the long-time behavior of numerical approximations of semilinear parabolic equations was studied in [112].

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