THE SPECTRUM OF A LINEARIZED 2D EULER OPERATOR

Y. LATUSHKIN, Y. LI, AND M. STANISLAVOVA

Abstract. We study the spectral properties of the linearized Euler operator obtained by linearizing the equations of incompressible two dimensional fluid at a steady state with the vorticity that contains only two nonzero complex conjugate Fourier modes. We prove that the essential spectrum coincides with the imaginary axis, and give an estimate from above for the number of isolated nonimaginary eigenvalues. In addition, we prove that the spectral mapping theorem holds for the group generated by the linearized 2D Euler operator.

1. Introduction

In recent years a new interest has been drawn to understanding the *stability spectrum* of the Euler equations for the motion of inviscid fluid linearized about a steady state. We will not even attempt to review a vast literature on the subject, and refer the readers to the recent survey [Fr] and the bibliography therein, as well as to the closely related to this paper work in [BFY, FH, FVY].

In this paper, we consider 2D Euler equation under periodic boundary conditions. We apply Fourier transform and linearize the Euler equation about a steady state that contains only two nonzero complex conjugate Fourier modes. Two issues are addressed.

First, we give a full description of the spectrum of the linearization. Using quite different methods, some results in this direction were obtained in [L1, L2]. We show that the essential spectrum coincides with the imaginary axis, and that the discrete nonimaginary spectrum consists of finitely many points. Moreover, we give an estimate from above for the number of the nonimaginary isolated eigenvalues in terms of a transparent geometric quantity. Although the existence of nonimaginary eigenvalues is well-known, see, e.g. [F], we are not aware of any results in the literature that would give an estimate from above for the number of nonimaginary eigenvalues. The results in [BFY] indicate that this estimate is sharp.

Second, we show that the spectral mapping theorem holds for the group generated by the linearized Euler operator L, that is, we prove that $\sigma(e^{tL}) = e^{t\sigma(L)}$, $t \neq 0$, for the spectrum $\sigma(\cdot)$. Note that the validity of this spectral mapping property for (non-analytic) semigroups is a rather delicate issue. For instance, the spectral mapping property does not hold even for a group obtained by a first order perturbation of a two-dimensional wave equation, see [R] and many other examples and general discussion of this phenomenon in, e.g., [CL, vN].

Our strategy is to use some ideas from the theory of Jacobi matrices and operators on spaces with indefinite metrics as well as some general results from the theory of strongly continuous semigroups. It can be summarized as follows.

The steady state considered in this paper gives a flow on the torus parallel to a vector $\mathbf{p} \in \mathbb{Z}^2$. The linearized Euler operator L is a difference operator acting on a space of sequences on $\mathbb{Z}^2$, see (4). To study the spectrum of L we at first "slice" the grid $\mathbb{Z}^2$ using subsets $\Sigma_{\mathbf{q}}$, $\mathbf{q} \in \mathbb{Z}^2$, of lines parallel to $\mathbf{p}$, see (5). This gives us a way to represent L as a direct sum of operators $L_{\mathbf{q}}$ acting on the space of sequences on $\mathbb{Z}$. We show that $\sigma(L) = \bigcup_{\mathbf{q}} \sigma(L_{\mathbf{q}})$ and $R(\lambda, L) = \bigoplus_q R(\lambda, L_{\mathbf{q}})$ for the resolvent operators. Using an appropriate rescaling and "symmetrization", we replace the study of $\sigma(L_{\mathbf{q}})$ by that of $\sigma(B_{\mathbf{q}})$, where $B_{\mathbf{q}}$ is a certain two-diagonal infinite matrix. The essential spectrum $\sigma_{\text{ess}}(B_{\mathbf{q}})$ is described using Weyl's Theorem. We show that if $\|\mathbf{q}\| > \|\mathbf{p}\|$ then $B_{\mathbf{q}}$ is a selfadjoint operator and, therefore, $L_{\mathbf{q}}$ does not have nonimaginary spectrum. If $\|\mathbf{q}\| \leq \|\mathbf{p}\|$ then $B_{\mathbf{q}}$ is a finite rank perturbation of a selfadjoint operator. However, we give an appropriate choice of an indefinite metric that makes $B_{\mathbf{q}}$ a J-selfadjoint operator on a Pontryagin space with finitely many positive squares. Standard facts about J-selfadjoint operators give the estimate for the cardinality of the nonimaginary point spectrum of L in terms of the number of points in $\mathbb{Z}^2$ located inside of the open disc with the radius $\|\mathbf{p}\|$. Finally, the proof of the spectral mapping theorem is based on a general Gearhart-Prüss theorem, see, e.g., [vN], and an estimate for the norm of $R(\lambda, L)$, $\operatorname{Re}\lambda \neq 0$.

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2. Linearization

Consider the two dimensional Euler equations in the vorticity form:

$$\frac{\partial \Omega}{\partial t} = -u\frac{\partial \Omega}{\partial x} - v\frac{\partial \Omega}{\partial y}, \quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \tag{1}$$

Here $\Omega = \operatorname{curl}\mathbf{u} = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$ is the vorticity and $\mathbf{u} = (u, v)$ is the velocity. We assume that $u = u(x, y)$ and $v = v(x, y)$ are 2π-periodic, $x, y \in [0, 2\pi]$, and have zero spatial means. If ψ is the stream function, then $u = -\frac{\partial \psi}{\partial y}$, $v = \frac{\partial \psi}{\partial x}$, and $\Omega = \Delta\psi$. If $\Omega = \sum_{\mathbf{k}\in\mathbb{Z}^2\setminus\{0\}} \omega_{\mathbf{k}} e^{i\mathbf{k}\cdot(x,y)}$, where $\omega_{-\mathbf{k}} = \overline{\omega_{\mathbf{k}}}$, $\mathbf{k} \in \mathbb{Z}^2 \setminus \{0\}$, then, after a short calculation, see [L1, L2], we can rewrite (1) as follows:

$$\frac{d\omega_{\mathbf{k}}}{dt} = \sum_{\mathbf{q}\in\mathbb{Z}^2\setminus\{0\}} A(\mathbf{k}-\mathbf{q}, \mathbf{q})\omega_{\mathbf{k}-\mathbf{q}}\omega_{\mathbf{q}}, \quad \mathbf{k} \in \mathbb{Z}^2. \tag{2}$$

Here $A(\mathbf{p}, \mathbf{q})$, $\mathbf{p}, \mathbf{q} \in \mathbb{Z}^2$, are defined by the formula

$$A(\mathbf{p}, \mathbf{q}) = A(\mathbf{q}, \mathbf{p}) = \frac{1}{2}\left[\frac{1}{\|\mathbf{p}\|^2} - \frac{1}{\|\mathbf{q}\|^2}\right] \begin{vmatrix} p_1 & q_1 \\ p_2 & q_2 \end{vmatrix},$$

provided $\mathbf{p} \neq \pm\mathbf{q}$, $\mathbf{p} \neq 0$, $\mathbf{q} \neq 0$, and $A(\mathbf{p}, \mathbf{q}) = A(\mathbf{q}, \mathbf{p}) = 0$ otherwise. Here and below we denote $\mathbf{p} = (p_1, p_2)$, $\mathbf{q} = (q_1, q_2)$, $\mathbf{k} = (k_1, k_2)$; vertical bars denote the determinant, $\|\cdot\|$ is the Euclidean norm.

Fix $\mathbf{p} \in \mathbb{Z}^2\setminus\{0\}$ and $\Gamma \in \mathbb{C}$, and consider the steady state $\omega^0 = (\omega^0_{\mathbf{k}})_{\mathbf{k}\in\mathbb{Z}^2}$ for (2) defined as follows:

$$\omega^0_{\mathbf{k}} = \begin{cases} \frac{1}{2}\Gamma, & \mathbf{k} = \mathbf{p} \\ \frac{1}{2}\overline{\Gamma}, & \mathbf{k} = -\mathbf{p} \\ 0, & \mathbf{k} \neq \pm\mathbf{p} \end{cases}, \tag{3}$$

where bar denotes the complex conjugate.

If $\Gamma = a + ib$ and $\mathbf{p} = (p_1, p_2)$ then the vorticity Ω^0, corresponding to ω^0, is given by the formula

$$\Omega^0(x, y) = a\cos(p_1 x + p_2 y) - b\sin(p_1 x + p_2 y).$$

The corresponding velocity field has the following components:

$$u^0(x, y) = (-p_2 a \sin\mathbf{p}\cdot(x, y) - p_2 b\cos\mathbf{p}\cdot(x, y))\|\mathbf{p}\|^{-2},$$
$$v^0(x, y) = (p_1 a\sin\mathbf{p}\cdot(x, y) + p_1 b\cos\mathbf{p}\cdot(x, y))/\|\mathbf{p}\|^{-2}.$$

In particular, if $\mathbf{p} = (0, p_2)$, then the steady state (3) defines a parallel shear flow (see, e.g., [F, Fr]) with the vorticity $\Omega^0(x, y) = a\cos p_2 y - b\sin p_2 y$ and the profile $-ap_2\sin p_2 y - bp_2\cos p_2 y$. For $b = \operatorname{Im}\Gamma = 0$, therefore, we have the sinusoidal profile, studied in [MS, Y] for the case of a viscous shear flow and, recently, in [BFY, FSV].

The linearization of (2) around the steady state ω^0 as in (3) gives the following operator L:

$$L : (\omega_{\mathbf{k}})_{\mathbf{k}\in\mathbb{Z}^2} \mapsto \Big(A(\mathbf{p}, \mathbf{k}-\mathbf{p})\Gamma\omega_{\mathbf{k}-\mathbf{p}} + A(-\mathbf{p}, \mathbf{k}+\mathbf{p})\overline{\Gamma}\omega_{\mathbf{k}+\mathbf{p}}\Big)_{\mathbf{k}\in\mathbb{Z}^2}. \tag{4}$$

The choice of the space on which we want to consider the operator L is related to the choice of the space for vorticity in (1). Thus, if $\Omega \in H^r(\mathbb{T}^2)$, $r \geq 0$, the Sobolev space, then L should be considered on the space $\ell^2_r(\mathbb{Z}^2)$ of weighted $\ell^2(\mathbb{Z}^2)$-sequences with the weight $(1+\|\mathbf{k}\|^2)^{r/2}$, $\mathbf{k} \in \mathbb{Z}^2$. In what follows we will consider the operator L on $\ell^2(\mathbb{Z}^2)$, that is, for $r = 0$ and $\Omega \in L^2(\mathbb{T}^2)$.

Denote by W the shift operator $W : (\omega_{\mathbf{k}}) \mapsto (\omega_{\mathbf{k}-\mathbf{p}})_{\mathbf{k}\in\mathbb{Z}^2}$. This is a unitary operator on $\ell^2(\mathbb{Z}^2)$. Also, consider the following operator $D_{\mathbf{p}} : (\omega_{\mathbf{k}}) \mapsto (A(\mathbf{p}, \mathbf{k})\Gamma\omega_{\mathbf{k}})$, and remark that $D_{-\mathbf{p}} = -D_{\mathbf{p}}$. Thus, the linearized Euler operator L can be represented as follows:

$$L = WD_{\mathbf{p}} + W^*D^*_{-\mathbf{p}} = WD_{\mathbf{p}} - W^*D^*_{\mathbf{p}}.$$

Here and below $*$ denotes the adjoint operator, $W^* : (\omega_{\mathbf{k}}) \mapsto (\omega_{\mathbf{k}+\mathbf{p}})$ and $D^*_{\mathbf{p}} : (\omega_{\mathbf{k}}) \mapsto (A(\mathbf{p}, \mathbf{k})\overline{\Gamma}\omega_{\mathbf{k}})$.

Remark 1. The definition of $A(\mathbf{p}, \mathbf{q})$ imply that $D_{\mathbf{p}} = D^0_{\mathbf{p}} + D^1_{\mathbf{p}}$, where

$$D^0_{\mathbf{p}} : (\omega_{\mathbf{k}}) \mapsto \left(\frac{1}{2\|\mathbf{p}\|^2}\begin{vmatrix} p_1 & k_1 \\ p_2 & k_2 \end{vmatrix}\Gamma\omega_{\mathbf{k}}\right),$$

$$D^1_{\mathbf{p}} : (\omega_{\mathbf{k}}) \mapsto \left(-\frac{1}{2\|\mathbf{k}\|^2}\begin{vmatrix} p_1 & k_1 \\ p_2 & k_2 \end{vmatrix}\Gamma\omega_{\mathbf{k}}\right).$$

The operator $D^1_{\mathbf{p}}$ is a compact operator on $\ell^2(\mathbb{Z}^2)$. ◇

Remark 2. Let us define the operator $L^0 = WD^0_{\mathbf{p}} - W^*D^{0*}_{\mathbf{p}}$. Note that

$$(WD^0_{\mathbf{p}}\omega)_{\mathbf{k}} = 2^{-1}\|\mathbf{p}\|^{-2}\begin{vmatrix} p_1 & k_1 - p_1 \\ p_2 & k_2 - p_2 \end{vmatrix}\Gamma\omega_{(k_1-p_1, k_2-p_2)}$$

implies $(WD^0_{\mathbf{p}}\omega)_{\mathbf{k}} = (D^0_{\mathbf{p}}W\omega)_{\mathbf{k}}$. Therefore, $(L^0)^* = D^{0*}_{\mathbf{p}}W^* - D^0_{\mathbf{p}}W = -L^0$, and, as a result, $\sigma(L^0) \subset i\mathbb{R}$. As we will see below, Weyl's Theorem implies that $\sigma_{\mathrm{ess}}(L) = \sigma_{\mathrm{ess}}(L^0)$. ◇

3. Decomposition and symmetrization

In this section, we will first perform a "decomposition" of the operator L. The operator L will be represented as a direct sum of certain operators $L_{\mathbf{q}}$ acting on the space $\ell^2(\mathbb{Z})$. Next, we will perform a "symmetrization" of the operators $L_{\mathbf{q}}$. This procedure will allow us to study, instead of the operators $L_{\mathbf{q}}$, certain operators $B_{\mathbf{q}}$ that are finite rank perturbations of selfadjoint operators.

For $\mathbf{p} \in \mathbb{Z}^2 \backslash \{0\}$ as above, and any $\mathbf{q} = (q_1, q_2) \in \mathbb{Z}^2$ we denote

$$\Sigma_{\mathbf{q}} = \{\mathbf{q} + n\mathbf{p} : n \in \mathbb{Z}\}. \tag{5}$$

Note that $\Sigma_{\mathbf{q}}$ is a subset of the line $\{\mathbf{q} + t\mathbf{p} : t \in \mathbb{R}\}$, but, generally, $\Sigma_{\mathbf{q}}$ does not contain all points with integer coordinates that belong to this line. For each $\mathbf{q} \in \mathbb{Z}^2$ select $\widehat{\mathbf{q}} = \widehat{\mathbf{q}}(\mathbf{q})$ such that $\widehat{\mathbf{q}} \in \Sigma_{\mathbf{q}}$, $\|\widehat{\mathbf{q}}\| = \inf\{\|\mathbf{q} + n\mathbf{p}\| : n \in \mathbb{Z}\}$, and $\widehat{\mathbf{q}} = \mathbf{q} + \max\{n : \|\widehat{\mathbf{q}}\| = \|\mathbf{q} + n\mathbf{p}\|\}\mathbf{p}$. The last condition just takes one of the two possible points $\mathbf{q} + n\mathbf{p} \in \Sigma_{\mathbf{q}}$ such that $\|\mathbf{q} + n\mathbf{p}\| = \|\widehat{\mathbf{q}}\|$. Denote $Q = \{\widehat{\mathbf{q}}(\mathbf{q}) : \mathbf{q} \in \mathbb{Z}^2\}$. Clearly, $\Sigma_{\mathbf{q}} = \Sigma_{\widehat{\mathbf{q}}(\mathbf{q})}$ and $\bigcup_{\widehat{\mathbf{q}} \in Q} \Sigma_{\widehat{\mathbf{q}}} = \mathbb{Z}^2$.

Fix $\mathbf{q} \in Q$ and let $X_{\mathbf{q}} = \{(\omega_{\mathbf{k}}) \in \ell^2(\mathbb{Z}^2) : \omega_{\mathbf{k}} = 0 \text{ for } \mathbf{k} \notin \Sigma_{\mathbf{q}}\}$. Note that $\Sigma_{\mathbf{q}} \cap \Sigma_{\mathbf{q}'} = \emptyset$ provided $\mathbf{q} \neq \mathbf{q}'$; therefore, $\{X_{\mathbf{q}}\}_{\mathbf{q} \in Q}$ is a system of orthogonal subspaces in $\ell^2(\mathbb{Z}^2)$ such that $\bigoplus_{\mathbf{q} \in Q} X_{\mathbf{q}} = \ell^2(\mathbb{Z}^2)$.

Note that $J : X_{\mathbf{q}} \to \ell^2(\mathbb{Z}) : (\omega_{\mathbf{q}+n\mathbf{p}})_{n \in \mathbb{Z}} \mapsto (\omega_n)_{n \in \mathbb{Z}}$ is an isometric isomorphism. Also, the operator L leaves the subspace $X_{\mathbf{q}}$ invariant. Thus, $L = \bigoplus_{\mathbf{q} \in Q} L_{\mathbf{q}}$, where $L_{\mathbf{q}} = VD - V^*D^*$, and we denote

$$V : (\omega_n) \mapsto (\omega_{n-1}), \quad D = D(\mathbf{p}, \mathbf{q}) : (\omega_n) \mapsto (A(\mathbf{q} + n\mathbf{p}, \mathbf{p})\Gamma\omega_n).$$

Remark 3. If $\mathbf{q} = t\mathbf{p}$, $t \in \mathbb{R}$, then $D = 0$ by the definition of $A(\mathbf{p}, \mathbf{q})$. Thus, in what follows we assume that $\mathbf{q} \neq t\mathbf{p}$, that is, that $\Sigma_{\mathbf{q}}$ does not belong to the line $\{t\mathbf{p} : t \in \mathbb{R}\}$. In particular, $\mathbf{q} + n\mathbf{p} \neq 0$ and $\mathbf{q} + n\mathbf{p} \neq \pm\mathbf{p}$ for all $n \in \mathbb{Z}$. ◇

Our next step is to perform a symmetrization of the operator $L_{\mathbf{q}}$. Fix $\mathbf{p}$ as above, and $\mathbf{q} \in Q$ such that $\mathbf{q} \neq t\mathbf{p}$, $t \in \mathbb{R}$. Denote

$$\beta = -\frac{1}{2\|\mathbf{p}\|^2} \begin{vmatrix} q_1 & p_1 \\ q_2 & p_2 \end{vmatrix} \Gamma, \quad \gamma_n = -\frac{\|\mathbf{p}\|^2}{\|\mathbf{q} + n\mathbf{p}\|^2}, \quad n \in \mathbb{Z}. \tag{6}$$

Note that $\beta \neq 0$ due to $\mathbf{q} \neq t\mathbf{p}$. By the definition of $A(\mathbf{p}, \mathbf{q})$ we have

$$A(\mathbf{q} + n\mathbf{p}, \mathbf{p})\Gamma = \frac{1}{2}\left[\frac{1}{\|\mathbf{q} + n\mathbf{p}\|^2} - \frac{1}{\|\mathbf{p}\|^2}\right] \begin{vmatrix} q_1 & p_1 \\ q_2 & p_2 \end{vmatrix} \Gamma = \beta(1 + \gamma_n).$$

Therefore,

$$L_{\mathbf{q}} = (V\beta - V^*\overline{\beta}) \operatorname{diag}(1 + \gamma_n)_{n \in \mathbb{Z}}. \tag{7}$$

Define $\alpha = i\beta^{\frac{1}{2}}/\overline{\beta}^{\frac{1}{2}}$, $|\alpha| = 1$, and set $S = \alpha V$. Note that

$$V\beta - V^*\overline{\beta} = -i\overline{\beta}^{\frac{1}{2}}(S + S^*)\beta^{\frac{1}{2}} = -i|\beta|(S + S^*).$$

Denote $M^0_{\mathbf{q}} = S + S^*$, observe that $(M^0_{\mathbf{q}})^* = M^0_{\mathbf{q}}$, and remark that

$$M^0_{\mathbf{q}} = \begin{bmatrix} \ddots & & & & \\ & 0 & \alpha & 0 & \\ & \overline{\alpha} & \boxed{0} & \alpha & \\ & 0 & \overline{\alpha} & 0 & \\ & & & & \ddots \end{bmatrix}, \quad |\alpha| = 1. \tag{8}$$

Let $M_{\mathbf{q}} = M^0_{\mathbf{q}} \operatorname{diag}(1+\gamma_n)_{n\in\mathbb{Z}}$, and note that $L_{\mathbf{q}} = -i|\beta| M_{\mathbf{q}}$. Thus,

$$\sigma(L_{\mathbf{q}}) = -i|\beta|\sigma(M_{\mathbf{q}}) \quad \text{and} \quad L = \bigoplus_{\mathbf{q}\in Q} -i|\beta| M_{\mathbf{q}}. \tag{9}$$

If $\lambda \notin \sigma(L)$, then $\lambda \notin \sigma(L_{\mathbf{q}})$ and we have

$$R(\lambda, L) = \bigoplus_{\mathbf{q}\in Q} R(\lambda, -i|\beta| M_{\mathbf{q}}) = \bigoplus_{\mathbf{q}\in Q} \frac{i}{|\beta|} R\left(\frac{\lambda}{i|\beta|}, M_{\mathbf{q}}\right).$$

The operator $M_{\mathbf{q}}$ is a multiplicative perturbation of a selfadjoint operator $M^0_{\mathbf{q}}$. Instead of $M_{\mathbf{q}}$ we will consider a symmetric operator $B_{\mathbf{q}}$ with the same spectrum. To do this, for $n \in \mathbb{Z}$ we define:

$$\delta_n = |1+\gamma_n|^{\frac{1}{2}} \text{ if } 1+\gamma_n \geq 0 \text{ and } \delta_n = i|1+\gamma_n|^{\frac{1}{2}} \text{ if } 1+\gamma_n \leq 0. \tag{10}$$

Note that $\delta_n^2 = 1+\gamma_n$, and $\delta_n \in \mathbb{R}$ if and only if $\|\mathbf{q}+n\mathbf{p}\| \geq \|\mathbf{p}\|$, and $\delta_n \in i\mathbb{R}$ if and only if $\|\mathbf{q}+n\mathbf{p}\| < \|\mathbf{p}\|$. Denote

$$B_{\mathbf{q}} = \operatorname{diag}(\delta_n) M^0_{\mathbf{q}} \operatorname{diag}(\delta_n) = \begin{bmatrix} \ddots & & & & \\ & 0 & \alpha\delta_{-1}\delta_0 & 0 & \\ & \overline{\alpha}\delta_{-1}\delta_0 & \boxed{0} & \alpha\delta_0\delta_1 & \\ & 0 & \overline{\alpha}\delta_0\delta_1 & 0 & \ddots \end{bmatrix}, \tag{11}$$

and note that

$$M_{\mathbf{q}} = M^0_{\mathbf{q}} \operatorname{diag}(\delta_n) \cdot \operatorname{diag}(\delta_n), \quad B_{\mathbf{q}} = \operatorname{diag}(\delta_n) M^0_{\mathbf{q}} \operatorname{diag}(\delta_n). \tag{12}$$

Proposition 1. *The nonzero elements in $\sigma(M_{\mathbf{q}})$ and $\sigma(B_{\mathbf{q}})$ coincide.*

Proof. This is a consequence of the following elementary fact: If A and B are bounded operators, then $\sigma(AB)\backslash\{0\} = \sigma(BA)\backslash\{0\}$. $\square$

Remark 4. Assume $\|\mathbf{q}\| \geq \|\mathbf{p}\|$. By the choice of $\mathbf{q} \in Q$ we have $\|\mathbf{q}+n\mathbf{p}\| \geq \|\mathbf{q}\| \geq \|\mathbf{p}\|$. Therefore, $\delta_n \in \mathbb{R}$ for all $n \in \mathbb{Z}$ and $B_{\mathbf{q}} = B^*_{\mathbf{q}}$. Hence, $\sigma(B_{\mathbf{q}}) \subset \mathbb{R}$ or $\sigma(L_{\mathbf{q}}) \subset i\mathbb{R}$. Since $L_{\mathbf{q}}$ is a bounded operator for each $\mathbf{q} \in Q$, we have that $\bigcup_{\|\mathbf{q}\|\leq\|\mathbf{p}\|} \sigma(L_{\mathbf{q}})$ is a bounded set. Moreover, $\bigcup_{\mathbf{q}\in Q} \sigma(L_{\mathbf{q}})\backslash i\mathbb{R}$ is a bounded set. $\diamond$

Remark 5. Assume $\|\mathbf{q}\| < \|\mathbf{p}\|$. Then $\delta_n\delta_{n+1} \in i\mathbb{R}$ for exactly *two* values of $n \in \mathbb{Z}$. Each of these values corresponds to a pair of two consecutive points in $\Sigma_{\mathbf{q}}$ such that one of the points lies inside and another outside of the disc with the radius $\|\mathbf{p}\|$. For all other values of $n \in \mathbb{Z}$ we have $\delta_n\delta_{n+1} \in \mathbb{R}$. Thus, $B_{\mathbf{q}}$ is a perturbation of a selfadjoint operator by a rank four skew-selfadjoint operator. $\diamond$

It is quite simple to describe the spectrum of the "constant coefficients" infinite matrix $M^0_{\mathbf{q}}$. For $\mathbf{q} \in Q$ and $\beta = \beta(\mathbf{q})$, defined in (6), let

$$L^0_{\mathbf{q}} = V\beta - V^*\overline{\beta} = -i|\beta|M^0_{\mathbf{q}},$$

where $M^0_{\mathbf{q}}$ is as in (8). Note that $\|M^0_{\mathbf{q}}\| \le 2$ and, thus, $\sigma(M^0_{\mathbf{q}}) \subset \{|z| \le 2\}$ and $\sigma(L^0_{\mathbf{q}}) \subset \{|z| \le |\beta|\}$.

If $\mathbb{F} : \ell^2(\mathbb{Z}) \to L^2(\mathbb{T})$ is the Fourier transform, then

$$\mathbb{F}((\lambda + M^0_{\mathbf{q}})(\omega_n)) = (\alpha\overline{z} + \lambda + \overline{\alpha}z)\mathbb{F}[(\omega_n)](z), \quad |z| = 1,$$

for each $\lambda \in \mathbb{C}$, and $\alpha = i\beta^{\frac{1}{2}}/\overline{\beta}^{\frac{1}{2}}$. Therefore, $-\lambda \notin \sigma(M^0_{\mathbf{q}})$ if and only if the function $(\lambda + M^0_{\mathbf{q}})(\cdot)$ defined by the formula $(\lambda + M^0_{\mathbf{q}})(z) = \alpha\overline{z} + \lambda + \overline{\alpha}z$ is not equal to zero for all $|z| = 1$. If this is the case, then $(\lambda + M^0_{\mathbf{q}})^{-1} = \mathbb{F}^{-1}\frac{1}{(\lambda + M^0_{\mathbf{q}})(z)}\mathbb{F}$. Therefore, the following fact holds.

Proposition 2. (a) $\sigma(M^0_{\mathbf{q}}) = \sigma_{\text{ess}}(M^0_{\mathbf{q}}) = [-2, 2]$*;*
(b) $\sigma(L^0_{\mathbf{q}}) = \sigma_{\text{ess}}(L^0_{\mathbf{q}}) = i[-2|\beta|, 2|\beta|]$.

We note that very interesting and deep results on the spectrum of the "variable coefficients" infinite matrices of the same type as $M_{\mathbf{q}}$ could be found in [JN] and the literature on Jacobi matrices cited therein. However, the results that we need for the specific rate of decay of γ_n to zero for the problem in hand do not seem to be available.

4. Spectrum

In this section, we describe the spectrum of the linearized Euler operator L. Let $\sigma_p(\cdot)$ denote the point spectrum. For $\lambda = a + i\tau$, $a \neq 0$, and $\beta = \beta(\mathbf{q})$ defined in (6) we denote $z = \lambda/(i|\beta|)$.

Theorem 3. (a) *For each* $\mathbf{q} \in Q$ *we have:*

$$\sigma_{\text{ess}}(L_{\mathbf{q}}) = \sigma(V\beta - V^*\overline{\beta}) = i[-2|\beta|, 2|\beta|].$$

(b) $\sigma(L) = \bigcup_{\mathbf{q}\in Q}\sigma(L_{\mathbf{q}})$*;*

(c) $\sigma_{\text{ess}}(L) = i\mathbb{R}$ *and* $\sigma_p(L) \setminus i\mathbb{R} = \bigcup_{\|\mathbf{q}\|\le\|\mathbf{p}\|}\left(\sigma_p(L_{\mathbf{q}}) \setminus i\mathbb{R}\right)$ *is a bounded set with accumulation points only on* $i\mathbb{R}$.

Proof. **(a).** Split $L_{\mathbf{q}} = L^0_{\mathbf{q}} + L^{\text{comp}}_{\mathbf{q}}$, where $L^0_{\mathbf{q}} = V\beta - V^*\overline{\beta}$, and $L^{\text{comp}}_{\mathbf{q}} = L^0_{\mathbf{q}}\,\text{diag}(\gamma_n)_{n\in\mathbb{Z}}$ is a compact operator due to $\lim_{n\to\infty}|\gamma_n| = 0$. Apply Weyl's theorem (see Lemma XIII.4.3 in [RS]) for $A = L^0_{\mathbf{q}}$ and $B = L_{\mathbf{q}}$. Note that $\sigma(A) = i[-2|\beta|, 2|\beta|]$ has an empty interior in $\mathbb{C}$, and $\mathbb{C}\backslash\sigma(A)$ consists of only one component. Since $A - B$ is a compact operator, we have $\sigma_{\text{ess}}(L^0_{\mathbf{q}}) = \sigma_{\text{ess}}(L_{\mathbf{q}})$. Using Proposition 2, we have (a).

(b). Split L into the direct sum of two operators, L^s and L^b, that correspond to "small" and "big" values of $\|\mathbf{q}\|$, that is, write $L = L^s + L^b$, where $L^s = \bigoplus_{\|\mathbf{q}\|\le\|\mathbf{p}\|} L_{\mathbf{q}}$ and $L^b = \bigoplus_{\|\mathbf{q}\|>\|\mathbf{p}\|} L_{\mathbf{q}}$. Since L^s is a direct sum of finitely many operators, we have

$$\sigma(L) = \Big(\bigcup_{\|\mathbf{q}\|\le\|\mathbf{p}\|} \sigma(L_{\mathbf{q}})\Big)\bigcup\sigma(L^b),$$

and we need to see only that $\sigma(L^b) \subset \bigcup_{\|\mathbf{q}\|>\|\mathbf{p}\|}\sigma(L_{\mathbf{q}})$. But $|\beta| = |\beta(\mathbf{q})| \to \infty$ as $\|\mathbf{q}\| \to \infty$. Using (a) in the theorem, we have:

$$\bigcup_{\|\mathbf{q}\|>\|\mathbf{p}\|} \sigma(L_{\mathbf{q}}) \supset \bigcup_{\|\mathbf{q}\|>\|\mathbf{p}\|} i[-2|\beta|, 2|\beta|] = i\mathbb{R}.$$

Thus, it suffices to show that $\sigma(L^b) \subset i\mathbb{R}$. Since

$$L^b = \bigoplus_{\|\mathbf{q}\|>\|\mathbf{p}\|} L_{\mathbf{q}} = \bigoplus_{\|\mathbf{q}\|>\|\mathbf{p}\|} -i|\beta(\mathbf{q})|M_{\mathbf{q}},$$

it suffices to prove the following claim:

$$\text{if } \lambda \notin i\mathbb{R}, \text{ then } \sup_{\|\mathbf{q}\|>\|\mathbf{p}\|}\left\|\frac{1}{\beta(\mathbf{q})}\left(z + M_{\mathbf{q}}\right)^{-1}\right\| < \infty. \tag{13}$$

Indeed, assume that (13) is proved. Then

$$\|(\lambda - L^b)^{-1}\| = \left\|\bigoplus_{\|\mathbf{q}\|>\|\mathbf{p}\|} \frac{1}{i|\beta(\mathbf{q})|}\left(z + M_{\mathbf{q}}\right)^{-1}\right\| < \infty$$

and $\lambda \notin \sigma(L^b)$.

To prove (13), fix $\lambda = a + i\tau$, $a \neq 0$. Note that for any $\mathbf{q} \in Q$ we have (see (12)):

$$\begin{aligned} z + M_{\mathbf{q}} &= z + M^0_{\mathbf{q}} + M^0_{\mathbf{q}}\,\text{diag}(\gamma_n) \\ &= \left(z + M^0_{\mathbf{q}}\right)\left[I + \left(z + M^0_{\mathbf{q}}\right)^{-1} M^0_{\mathbf{q}}\,\text{diag}(\gamma_n)\right]. \end{aligned} \tag{14}$$

Note that the resolvent $(z+M^0_{\mathbf{q}})^{-1}$ in this identity exists since $z \notin \mathbb{R}$ and $M^0_{\mathbf{q}}$ is a self-adjoint operator; moreover,

$$\left\| \left(z+M^0_{\mathbf{q}}\right)^{-1} \right\| = \frac{1}{|\mathrm{Im}\ z|} = \frac{|\beta|}{|a|}. \tag{15}$$

Proposition 4. *There exists a constant $c(\mathbf{p})$ such that for all $\mathbf{q} \in Q$, $\mathbf{q} \neq 0$, we have:*

$$|\beta(\mathbf{q})| \| \operatorname{diag}(\gamma_n)_{n\in\mathbb{Z}} \| \leq c(\mathbf{p})/\|\mathbf{q}\|.$$

Proof. Using (6), we have:

$$\begin{aligned}
|\beta| \| \operatorname{diag}(\gamma_n)_{n\in\mathbb{Z}} \| = |\beta| \sup_{n\in\mathbb{Z}} |\gamma_n| &\leq c \sup_{n\in\mathbb{Z}} \left| \begin{vmatrix} q_1 & p_1 \\ q_2 & p_2 \end{vmatrix} \|\mathbf{q}+n\mathbf{p}\|^{-2} \right| \\
&= c \sup_{n\in\mathbb{Z}} \left| \begin{vmatrix} q_1+np_1 & p_1 \\ q_2+np_2 & p_2 \end{vmatrix} \|\mathbf{q}+n\mathbf{p}\|^{-2} \right| \\
&= c \sup_{n\in\mathbb{Z}} \left| (\mathbf{q}+n\mathbf{p}) \cdot \mathbf{p}^{\perp} \|\mathbf{q}+n\mathbf{p}\|^{-2} \right|,
\end{aligned}$$

where $\mathbf{p}^{\perp} = (p_2, -p_1)$ is the $\mathbb{Z}^2$-vector, perpendicular to $\mathbf{p}$. Using Cauchy-Schwartz inequality, we have that

$$|\beta| \| \operatorname{diag}(\gamma_n)_{n\in\mathbb{Z}} \| \leq c \sup_{n\in\mathbb{Z}} \frac{\|\mathbf{q}+n\mathbf{p}\| \|\mathbf{p}^{\perp}\|}{\|\mathbf{q}+n\mathbf{p}\|^2} \leq \frac{c_1}{\inf_{n\in\mathbb{Z}} \|\mathbf{q}+n\mathbf{p}\|} = \frac{c_1}{\|\mathbf{q}\|},$$

where the definition of $\mathbf{q} \in Q$ has been used. □

For $\lambda = a + i\tau$ as above, and $c(\mathbf{p})$ from Proposition 4, fix $\mathbf{q}_0 = \mathbf{q}_0(a)$ such that $\|\mathbf{q}_0\| > \|\mathbf{p}\|$ and if $\|\mathbf{q}\| \geq \|\mathbf{q}_0\|$ then the inequality

$$\frac{2c(\mathbf{p})}{|a|\|\mathbf{q}\|} \leq \frac{1}{2} \tag{16}$$

holds. Note that the set $Q_s := \{\mathbf{q} \in Q : \|\mathbf{q}\| \in [\|\mathbf{p}\|, \|\mathbf{q}_0\|]\}$ is finite, and let $Q_b = \{\mathbf{q} \in Q : \|\mathbf{q}\| > \|\mathbf{q}_0\|\}$.

Using Proposition 4, and the inequality $\|M^0_{\mathbf{q}}\| \leq 2$, we have (thanks to (15)) that if $\mathbf{q} \in Q_b$, then

$$\left\| \left(z+M^0_{\mathbf{q}}\right)^{-1} M^0_{\mathbf{q}} \operatorname{diag}(\gamma_n)_{n\in\mathbb{Z}} \right\| \leq \frac{2|\beta|}{|a|} \| \operatorname{diag}(\gamma_n) \| \leq \frac{2c(\mathbf{p})}{|a|\|\mathbf{q}\|} \leq \frac{1}{2}.$$

Thus, the operator $I + (z+M^0_{\mathbf{q}})^{-1} M^0_{\mathbf{q}} \operatorname{diag}(\gamma_n)$ is invertible and, for $\mathbf{q} \in Q_b$,

$$\left\| \left[I + \left(z+M^0_{\mathbf{q}}\right)^{-1} M^0_{\mathbf{q}} \operatorname{diag}(\gamma_n) \right]^{-1} \right\| \leq 2. \tag{17}$$

For each $\mathbf{q} \in Q_s$ we remark that $z \notin \sigma(B_{\mathbf{q}})$, see (12), and, hence, $z \notin \sigma(M_{\mathbf{q}})$. Since Q_s is a finite set, we have

$$\sup_{\mathbf{q}\in Q_s} \left\| \frac{1}{|\beta|} \left(z + M_{\mathbf{q}}\right)^{-1} \right\| < \infty.$$

Remark 6. Let $L_s^b = \bigoplus_{\mathbf{q}\in Q_s} L_{\mathbf{q}}$. Then L_s^b is a bounded operator. Hence, if $\lambda = a + i\tau$, $a \neq 0$, then $\|(\lambda - L_s^b)^{-1}\| = O\left(|\tau|^{-1}\right)$ as $|\tau| \to \infty$. $\diamond$

To finish the proof of (13), we use (14)-(15) and (17):

$$\begin{aligned} \sup_{\mathbf{q}\in Q_b} \left\||\beta|^{-1} \left(z + M_{\mathbf{q}}\right)^{-1}\right\| &\le \sup_{\mathbf{q}\in Q_b} \frac{2}{|\beta|} \left\|\left(z + M_{\mathbf{q}}^0\right)^{-1}\right\| \\ &\le \sup_{\mathbf{q}\in Q_b} 2|\beta|^{-1} \cdot \frac{|\beta|}{|a|} = \frac{2}{|a|}. \end{aligned} \tag{18}$$

This proves (13) and (b) in the theorem.

(c). Since

$$\sigma_{\text{ess}}(L) = \Big(\bigcup_{\|\mathbf{q}\|\le\|\mathbf{p}\|} \sigma_{\text{ess}}(L_{\mathbf{q}})\Big) \bigcup \sigma_{\text{ess}}(L^b),$$

the first statement follows from (a) in the theorem. The second statement follows from Remark 4 and (a) in the theorem. $\square$

Since L is the sum of a skew-adjoint operator L^0 and a compact operator, see Remarks 1 and 2, the operator L generates a strongly continuous group. The following spectral mapping theorem holds for the group $\{e^{tL}\}_{t\ge0}$. Its proof is similar to the proof of Theorem 1 in [GJLS].

Theorem 5. *If L is the linearized Euler operator, then*

$$\sigma(e^{tL}) = e^{t\sigma(L)}, t \neq 0.$$

Proof. Let $L^b = \bigoplus_{\mathbf{q}\in Q_b} L_{\mathbf{q}}$. Inequality (18) shows that if $\lambda = a + i\tau$, $a \neq 0$, then $\|(\lambda - L^b)^{-1}\| \le 2/|a|$. Using Remark 6 we have that

$$\|(\lambda - L)^{-1}\| = O(1) \quad \text{as} \quad |\tau| \to \infty. \tag{19}$$

Now the assertion follows from the resolvent estimate (19) and the following general Gearhart–Prüss spectral mapping theorem: On a Hilbert space the spectrum $\sigma(e^{tL})$, $t \neq 0$, is the set of the points $e^{\lambda t}$ such that either $\mu_n = \lambda + 2\pi n/t$ belongs to $\sigma(L)$ for some $n \in \mathbb{Z}$, or the sequence $\{\|R(\mu_n, L)\|\}_{n\in\mathbb{Z}}$ is unbounded, see, e.g. [CL]. Recall, that the spectral mapping property always holds for the point spectrum. Due to (19) we conclude that $\sigma_{\text{ess}}(e^{tL})$ belongs to the unit circle. $\square$

5. J-Theory

In this section, we obtain an estimate from above for the number of non-imaginary isolated eigenvalues of the operator L. Recall that $\sigma_p(L_{\mathbf{q}})\backslash i\mathbb{R}$ is empty as soon as $\|\mathbf{q}\| \geq \|\mathbf{p}\|$. Let $\varkappa$ denote the number of points $\mathbf{q} \in \mathbb{Z}^2$ that belong to the open disk of radius $\|\mathbf{p}\|$, and such that $\mathbf{q} \neq t\mathbf{p}$. Since for such $\mathbf{q}$ we have $(\pm q_1)^2 + (\pm q_2)^2 < \|\mathbf{p}\|^2$, we conclude that $\varkappa$ is even.

Theorem 6. *The number of nonimaginary eigenvalues of L (counting the multiplicities) does not exceed $2\varkappa$.*

Proof. Since $L_{\mathbf{q}} = 0$ for $\mathbf{q} = t\mathbf{p}$, only those $L_{\mathbf{q}}$ for which $\|\mathbf{q}\| < \|\mathbf{p}\|$ and $\mathbf{q} \neq t\mathbf{p}$ will contribute to the nonimaginary point spectrum of L. For each such $\mathbf{q}$ let $n'_{\mathbf{q}}$ denote the smallest and $n''_{\mathbf{q}}$ the largest integer such that $\delta_n \in i\mathbb{R}$ for $n = n'_{\mathbf{q}}, \dots, n''_{\mathbf{q}}$, see (10) for the definition of δ_n. Let $J_{\mathbf{q}} = \operatorname{diag}(j_n)_{n\in\mathbb{Z}}$, where $j_n = 1$ if $n = n'_{\mathbf{q}}, \dots, n''_{\mathbf{q}}$ and $j_n = -1$ otherwise. Note that $J_{\mathbf{q}} = J_{\mathbf{q}}^{-1} = J_{\mathbf{q}}^*$ and

$$J_{\mathbf{q}} \operatorname{diag}(\delta_n) = \operatorname{diag}(\delta_n) J_{\mathbf{q}} = (\operatorname{diag}(\delta_n))^*.$$

Using (11), we have that $J_{\mathbf{q}} B_{\mathbf{q}} J_{\mathbf{q}} = B_{\mathbf{q}}^*$, that is, that the operator $B_{\mathbf{q}}$ is $J_{\mathbf{q}}$-selfadjoint. Let $\langle \omega, \omega' \rangle$ be the standard scalar product in $\ell^2(\mathbb{Z})$. Note that the formula $[\omega, \omega'] = \langle J_{\mathbf{q}}\omega, \omega' \rangle$ defines an indefinite metric on $\ell^2(\mathbb{Z})$. Thus, $\ell^2(\mathbb{Z})$ is a Pontryagin space with $n''_{\mathbf{q}} - n'_{\mathbf{q}}$ positive squares. By a standard result of the theory of J-selfadjoint operators on Pontryagin spaces (see, e.g., [AI, Cor. II.3.15]), we have that the number of nonreal eigenvalues of $B_{\mathbf{q}}$ does not exceed $2(n''_{\mathbf{q}} - n'_{\mathbf{q}})$. By (12), (9), and $\varkappa = \sum_{\mathbf{q}} (n''_{\mathbf{q}} - n'_{\mathbf{q}})$ we have the result. $\square$

Proposition 7. *The nonimaginary eigenvalues of L are symmetric about the coordinate axes.*

Proof. Re-write (7) as $L_{\mathbf{q}} = N \operatorname{diag}(1 + \gamma_n)$, where $N = V\beta - V^*\bar{\beta} = -N^*$. Using the argument in the proof of Proposition 1, we have:

$$\begin{aligned}\sigma(L_{\mathbf{q}}^*)\backslash\{0\} &= \sigma(\operatorname{diag}(1+\gamma_n)N^*)\backslash\{0\} = -\sigma(\operatorname{diag}(1+\gamma_n)N)\backslash\{0\} \\ &= -\sigma(N \operatorname{diag}(1+\gamma_n))\backslash\{0\} = -\sigma(L_{\mathbf{q}})\backslash\{0\}.\end{aligned}$$

Thus, $\sigma(L_{\mathbf{q}})\backslash\{0\} = \overline{\sigma(L_{\mathbf{q}}^*)}\backslash\{0\} = -\overline{\sigma(L_{\mathbf{q}})}\backslash\{0\}$, and the nonimaginary eigenvalues of L are symmetric about $i\mathbb{R}$. Next, define $\hat{J}$ on $\ell^2(\mathbb{Z})$ by $\hat{J} : (\omega_n) \mapsto ((-1)^n \omega_n)$ and note that $V\hat{J} = -\hat{J}V$ and $\hat{J}V^* = -V^*\hat{J}$. Thus,

$$\hat{J} L_{\mathbf{q}} \hat{J} = \hat{J}(V \operatorname{diag}(1+\gamma_n)\beta - V^* \operatorname{diag}(1+\gamma_n)\bar{\beta})\hat{J} = -L_{\mathbf{q}},$$

and $\sigma(L_{\mathbf{q}}) = \sigma(\hat{J} L_{\mathbf{q}} \hat{J}) = -\sigma(L_{\mathbf{q}})$. $\square$

Note that the change of variables $\xi = p_1 x + p_2 y$, $\eta = -p_2 x + p_1 y$ in (1) converts Ω^0 to the vorticity $\widetilde{\Omega}^0 = \widetilde{\Omega}^0(\xi)$, that is, to a parallel shear flow. However, this does not simplify our analysis, since the new flow looses 2π-periodicity, and the results from, say, [F] cannot be applied directly.

Recently, a very interesting case of a steady state whose vorticity has four (symmetric) nonzero Fourier modes was considered in [FVY]. For general $\mathbf{p} = (p_1, p_2)$, the steady state considered in the current paper is in an "intermediate position" between the parallel shear flow as in [F] and the Kolmogorov flow as in [BFY], and the more sophisticated case studied in [FVY]. For the case in [FVY], the continuous spectrum of the linearization is unstable, while in our case it is always stable (that is, located on the imaginary axis). Note that the vorticity Ω^0 for the case considered in [FVY] has the following representation:

$$\Omega^0(x,y) = \mathrm{Re}(\Gamma_1 e^{i\mathbf{p}_1\cdot(x,y)} + \Gamma_2 e^{i\mathbf{p}_2\cdot(x,y)})), \quad \Gamma_{1,2} \in \mathbb{C}, \quad \mathbf{p}_{1,2} \in \mathbb{Z}^2.$$

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Department of Mathematics, University of Missouri, Columbia, MO 65211

E-mail address: `yuri@math.missouri.edu`

E-mail address: `cli@math.missouri.edu`

Department of Mathematics, University of Massachusetts, Amherst, MA 01003-4515

E-mail address: `stanis@math.umass.edu`