Marcinkiewicz and Mihlin multiplier theorems, and R-boundedness

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Abstract

We prove the R-boundedness of a family of operators in $L^p(\mathbb{T}, X)$ (where X is a UMD Banach space and $1 < p < \infty$) that are defined by scalar valued Fourier multipliers, under the assumption that the multipliers satisfy a uniform condition of Marcinkiewicz type. An analogous result is obtained for the Mihlin multiplier theorem in $L^p(\mathbb{R}, X)$.

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1 Introduction

Let $\mathcal{T}$ be a set of bounded linear operators on a Banach space X to a Banach space Y. $\mathcal{T}$ is said to be R-bounded if $\exists\, C > 0$ such that for any positive integer N and for arbitrary choices of $T_1, \dots, T_N \in \mathcal{T}$ and $x_1, \dots, x_N \in X$ one has

$$\sum_{\varepsilon \in \{-1,1\}^N} \Big\| \sum_{k=1}^{N} \varepsilon_k\, T_k\, x_k \Big\|_Y \le C \sum_{\varepsilon \in \{-1,1\}^N} \Big\| \sum_{k=1}^{N} \varepsilon_k\, x_k \Big\|_X . \tag{1}$$

If $j_1, \dots, j_N$ are positive integers such that $j_h \neq j_k$ for $h \neq k$, then the Rademacher functions $r_{j_1}, \dots, r_{j_N}$ are equidistributed, so that for any function f on $\{-1,1\}^N$ we have $\int_0^1 f(r_{j_1}(t), \dots, r_{j_N}(t))\, dt = 2^{-N} \sum_{\varepsilon \in \{-1,1\}^N} f(\varepsilon)$. Therefore (1) becomes

$$\int_0^1 \Big\| \sum_{k=1}^{N} r_{j_k}(t)\, T_k\, x_k \Big\|\, dt \le C \int_0^1 \Big\| \sum_{k=1}^{N} r_{j_k}(t)\, x_k \Big\|\, dt,$$

and now, because of the Khintchine-Kahane inequality (see [3, 11.1]), formula (1) can be written in an equivalent form as

$$\Big(\sum_{\varepsilon \in \{-1,1\}^N} \Big\| \sum_{k=1}^{N} \varepsilon_k\, T_k\, x_k \Big\|_Y^q \Big)^{1/q} \le C_q \Big(\sum_{\varepsilon \in \{-1,1\}^N} \Big\| \sum_{k=1}^{N} \varepsilon_k\, x_k \Big\|_X^q \Big)^{1/q}$$

where $q \in \mathbb{R}^+$. The best C_q will be called the R_q-bound of $\mathcal{T}$.

In the last years the notion of R-boundedness has proven to be very useful in connection with such problems as the maximal regularity for linear differential equations of parabolic type and the boundedness of the H^∞ functional calculus for sectorial operators. We refer to [2, 5] for further information on the subject.

In this note it is proved that if a family of Fourier multipliers on the real line satisfies a uniform bound of Marcinkiewicz type, then the corresponding set of bounded operators on $L^p(\mathbb{T}, X)$ to itself (where X is a UMD Banach space and $1 < p < \infty$) is R-bounded. An analogous result for a family of Mihlin multipliers in $L^p(\mathbb{R}, X)$ is proved.

2 The discrete case

Let X be a complex Banach space, and let $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ be the one-dimensional torus. Let $(\lambda_k)_{k\in\mathbb{Z}}$ be a family of complex numbers, and let Λ be the operator defined on the space of the X-valued trigonometric polynomials in the following way: $(\Lambda f)\widehat{\ }(k) = \lambda_k\, \hat{f}(k)$, where for an arbitrary function $g \in L^1(\mathbb{T}, X)$ $\hat{g}(k) := \int_0^1 e^{-2\pi i k t} g(t)\, dt$ is the k-th Fourier coefficient of g. One says that $(\lambda_k)_{k\in\mathbb{Z}}$ is a family of multipliers in $L^p(\mathbb{T}, X)$ (where $p \in [1, \infty[\,$) if the operator Λ has a continuous extension on $L^p(\mathbb{T}, X)$ to $L^p(\mathbb{T}, X)$. In this case, as the operator $g \mapsto \hat{g}(k)$ is continuous on $L^p(\mathbb{T}, X)$ to X, the equality $(\Lambda f)\widehat{\ }(k) = \lambda_k\, \hat{f}(k)$ also holds when Λ denotes the continuous extension to $L^p(\mathbb{T}, X)$ of the homonymous operator originally defined on the space of the trigonometric polynomials.

In theorem 2 we state an extension of the Marcinkiewicz multiplier theorem to the case of vector valued functions that was proved in [1, Theorem 4]. We agree that $(\Delta_k)_{k\in\mathbb{Z}}$ is the usual dyadic decomposition of $\mathbb{Z}$: precisely we set $\Delta_0 = \{0\}$, and for any positive integer k, $\Delta_k = \{n \in \mathbb{Z};\ 2^{k-1} \le n < 2^k\}$, $\Delta_{-k} = -\Delta_k$.

Theorem 2 *Let $(\lambda_k)_{k\in\mathbb{Z}}$ be a family of complex numbers. Assume that*

$$\max\Big\{\sup_{r\in\mathbb{Z}} |\lambda_r|,\ \sup_{k\in\mathbb{Z}} \sum_{r,r-1\in\Delta_k} |\lambda_r - \lambda_{r-1}|\Big\} = \gamma \in \mathbb{R}^+.$$

Then for each Banach space X of type UMD and $\forall p \in]1,\infty[$ $(\lambda_k)_{k\in\mathbb{Z}}$ is a family of multipliers in $L^p(\mathbb{T}, X)$, and the norm of the corresponding operator $\Lambda \in \mathcal{L}(L^p(\mathbb{T}, X))$ satisfies the inequality $\|\Lambda\|_{\mathcal{L}(L^p(\mathbb{T},X))} \le C(p, X)\,\gamma$.

Our first aim is to prove the following R-version of theorem 2.

Theorem 3 *Let Ξ be any set of indices, and $\forall \xi \in \Xi$ let $(\lambda_{k,\xi})_{k\in\mathbb{Z}}$ be a family of complex numbers. Assume that*

$$\sup_{\xi\in\Xi}\Big(\max\Big\{\sup_{r\in\mathbb{Z}} |\lambda_{r,\xi}|,\ \sup_{k\in\mathbb{Z}} \sum_{r,r-1\in\Delta_k} |\lambda_{r,\xi} - \lambda_{r-1,\xi}|\Big\}\Big) = \gamma \in \mathbb{R}^+.$$

Let X be a UMD Banach space, $p \in]1,\infty[$, and let Λ_ξ be the operator in $L^p(\mathbb{T}, X)$ associated to the family of multipliers $(\lambda_{k,\xi})_{k\in\mathbb{Z}}$. Then $\{\Lambda_\xi;\ \xi \in \Xi\}$ is a R-bounded set in $\mathcal{L}(L^p(\mathbb{T}, X))$, and its R_p-bound is $\le C(p, X)\,\gamma$.

Let us establish some notations and recall some results.

If X is a UMD Banach space, then $(-i \operatorname{sgn} k)_{k\in\mathbb{Z}}$ is a family of multipliers in $L^p(\mathbb{T}, X)$ for every $p \in]1,\infty[$: in fact that could be taken as a definition of the class UMD. The corresponding operator is the discrete Hilbert transform H. The Riesz projection operator $R = R_0$ is the operator defined by $Rf = \frac{1}{2}(f + iHf) + \frac{1}{2}\hat{f}(0)$. This is obviously a bounded operator on $L^p(\mathbb{T}, X)$ to itself ($p \in]1,\infty[$), whose norm only depends on the norm of H in $\mathcal{L}(L^p(\mathbb{T}, X))$, and it is the operator associated to the family $(\lambda_k)_{k\in\mathbb{Z}}$ of multipliers, where $\lambda_k = 1$ if $k \ge 0$ and $\lambda_k = 0$ if $k < 0$.

Let $m \in \mathbb{Z}$. The shifted Riesz operator R_m is defined by $R_m f = \varphi_m\, R(\varphi_{-m} f)$, where, $\forall k \in \mathbb{Z}$, $\varphi_k(t) = e^{2\pi i k t}$. This is obviously a bounded operator on $L^p(\mathbb{T}, X)$ to itself (always, as it is understood from now on, for $p \in]1,\infty[$), with $\|R_m\|_{\mathcal{L}(L^p(\mathbb{T},X))} = \|R\|_{\mathcal{L}(L^p(\mathbb{T},X))}$, and it is in fact the operator associated to the family of multipliers $(\lambda_k)_{k\in\mathbb{Z}}$, where $\lambda_k = 1$ if $k \ge m$ and $\lambda_k = 0$ if $k < m$. Therefore, if Δ is a "segment" of $\mathbb{Z}$, say $\Delta = \{k \in \mathbb{Z};\ m \le k \le n\}$, then the characteristic function of Δ is a family of multipliers, and the associated operator is $S_\Delta = R_m - R_{n+1}$, so that the operator norm of S_Δ is $\le 2\,\|R\|_{\mathcal{L}(L^p(\mathbb{T},X))}$.

In order to prove theorem 3, we need three lemmas.

Lemma 4 *Let X be a UMD Banach space, $1 < p < \infty$. Suppose that $T_1, \ldots, T_n$ belong to the linear subspace of $\mathcal{L}(L^p(\mathbb{T}, X))$ generated by $\{R_k; k \in \mathbb{Z}\}$, $T_k = \sum_{j\in\mathbb{Z}} \lambda_{kj} R_j$. If $\mu = \sup_{1\le k\le n} \sum_{j\in\mathbb{Z}} |\lambda_{kj}|$, then*

$$\Big(\sum_{\varepsilon\in\{-1,1\}^n} \|\sum_{k=1}^n \varepsilon_k\, T_k\, f_k\|^p_{L^p(\mathbb{T},X)}\Big)^{1/p} \le \mu\, C(p, X)\,\Big(\sum_{\varepsilon\in\{-1,1\}^n} \|\sum_{k=1}^n \varepsilon_k\, f_k\|^p_{L^p(\mathbb{T},X)}\Big)^{1/p}.$$

Lemma 5 *Let X be a UMD Banach space, $1 < p < \infty$. Then for $f \in L^p(\mathbb{T}, X)$ and $m, n \in \mathbb{Z}$ with $m \leq n$,*

$$\Big(2^{-(n-m+1)} \sum_{\varepsilon} \| \sum_{k=m}^{n} \varepsilon_k \, S_{\Delta_k} \, f \|^p_{L^p(\mathbb{T},X)}\Big)^{1/p} \leq C(p,X) \|f\|_{L^p(\mathbb{T},X)}.$$

Lemma 6 *Let X be a UMD Banach space, $1 < p < \infty$, $m, n \in \mathbb{Z}$ with $m \leq n$, $f_m, \ldots, f_n \in L^p(\mathbb{T}, X)$. Then*

$$\| \sum_{k=m}^{n} S_{\Delta_k} \, f_k \|_{L^p(\mathbb{T},X)} \leq C(p,X) \Big(2^{-(n-m+1)} \sum_{\varepsilon} \| \sum_{k=m}^{n} \varepsilon_k \, S_{\Delta_k} \, f_k \|^p_{L^p(\mathbb{T},X)}\Big)^{1/p}.$$

A proof of lemma 4 can be found in [1, Lemma 7] (see also [6, Lemma 1.2]) and relies on the so-called "contraction principle" (that is the R-boundedness of $\{\lambda I_X; |\lambda| \leq M\}$, see [3, 12.2]) plus a convexity argument. The Littlewood-Paley type lemma 5 above is Theorem 3 of [1]. In order to obtain lemma 6 it is enough to apply to $f := \sum_{k=m}^{n} S_{\Delta_k} f_k$ the inequality

$$\|f\|_{L^p(\mathbb{T},X)} \leq C(p,X) \Big(2^{-(n-m+1)} \sum_{\varepsilon} \| \sum_{j=m}^{n} \varepsilon_j \, S_{\Delta_j} \, f \|^p_{L^p(\mathbb{T},X)}\Big)^{1/p}$$

that can be obtained from lemma 5 by means of a duality argument (see [6, (1.9)]).

Proof of theorem 6.
We have to prove that for a positive integer n, $(\xi_1, \ldots, \xi_n) \in \Xi^n$ and $f_1, \ldots, f_n \in L^p(\mathbb{T}, X)$ the following inequality holds:

$$\Big(\sum_{\varepsilon \in \{-1,1\}^n} \Big\| \sum_{i=1}^{n} \varepsilon_i \, \Lambda_{\xi_i} f_i \Big\|^p_{L^p(\mathbb{T},X)} \Big)^{1/p} \leq C(p,X) \, \gamma \Big(\sum_{\varepsilon \in \{-1,1\}^n} \Big\| \sum_{i=1}^{n} \varepsilon_i f_i \Big\|^p_{L^p(\mathbb{T},X)} \Big)^{1/p}.$$

It is enough to prove this inequality when $f_1, \ldots, f_n$ are X-valued trigonometric polynomials, since the space of the trigonometric polynomials is dense in $L^p(\mathbb{T}, X)$. In this case there exist q, $r \in \mathbb{Z}$, with $q < r$, such that the support of every f_i is contained in $\bigcup_{k=q}^{r} \Delta_k$. Now we call $\Lambda_{i,k}$ the operator associated to the family of multipliers $(\chi_{\Delta_k}(m) \, \lambda_{m,\xi_i})_{m \in \mathbb{Z}}$ which vanishes outside of Δ_k and equals λ_{m,ξ_i} if $m \in \Delta_k$. We have obviously $\Lambda_{i,k} = \Lambda_{\xi_i} \, S_{\Delta_k} = S_{\Delta_k} \, \Lambda_{\xi_i}$, and we have also $\Lambda_{i,k} = \Lambda_{i,k} \, S_{\Delta_k} = S_{\Delta_k} \, \Lambda_{i,k}$. Moreover the support of $\Lambda_{\xi_i} f_i$ is contained in $\bigcup_{k=q}^{r} \Delta_k$, as well as the support of f_i; therefore

$$\Lambda_{\xi_i} f_i = \sum_{k=q}^{r} S_{\Delta_k} \, \Lambda_{\xi_i} \, f_i = \sum_{k=q}^{r} \Lambda_{i,k} \, f_i = \sum_{k=q}^{r} S_{\Delta_k} \, \Lambda_{i,k} \, f_i.$$

Hence we have

$$\sum_{\varepsilon \in \{-1,1\}^n} \Big\| \sum_{i=1}^{n} \varepsilon_i \, \Lambda_{\xi_i} \, f_i \Big\|^p_{L^p(\mathbb{T},X)} = \sum_{\varepsilon \in \{-1,1\}^n} \Big\| \sum_{k=q}^{r} S_{\Delta_k} \sum_{i=1}^{n} \varepsilon_i \, \Lambda_{i,k} \, f_i \Big\|^p_{L^p(\mathbb{T},X)} \leq$$

(by lemma 6)

$$\leq \frac{C(p,X)^p}{2^{r-q+1}} \sum_{\varepsilon} \sum_{\varepsilon'} \Big\| \sum_{k=q}^{r} \varepsilon'_k \, S_{\Delta_k} \sum_{i=1}^{n} \varepsilon_i \, \Lambda_{i,k} f_i \Big\|^p_{L^p(\mathbb{T},X)} =$$

$$\frac{C(p,X)^p}{2^{r-q+1}} \sum_{(\varepsilon,\,\varepsilon')} \Big\| \sum_{i,k} \varepsilon_i \, \varepsilon'_k \, \Lambda_{i,k} \, S_{\Delta_k} f_i \Big\|^p_{L^p(\mathbb{T},X)}.$$

Now we remark that the operators $\Lambda_{i,k}$ satisfy the assumptions of lemma 4, with $\mu \leq 3\gamma$. Indeed, if we set $\Delta_k = \{\alpha_k, \ldots, \beta_k\}$, then

$$\Lambda_{i,k} = \sum_{j=\alpha_k}^{\beta_k} \lambda_{i,j} \, (R_j - R_{j+1}) = \lambda_{i,\alpha_k} R_{\alpha_k} - \lambda_{i,\beta_k} R_{\beta_k+1} + \sum_{j,j-1 \in \Delta_k} (\lambda_{i,j} - \lambda_{i,j-1}) R_j.$$

Therefore, by lemma 4 we obtain

$$\sum_{\varepsilon \in \{-1,1\}^n} \Big\| \sum_{i=1}^{n} \varepsilon_i \, \Lambda_{\xi_i} \, f_i \Big\|^p_{L^p(\mathbb{T},X)} \leq \frac{(3\,\gamma\, C(p,X))^p}{2^{r-q+1}} \sum_{(\varepsilon,\,\varepsilon')} \Big\| \sum_{i,k} \varepsilon_i \, \varepsilon'_k \, S_{\Delta_k} \, f_i \Big\|^p_{L^p(\mathbb{T},X)} =$$

$$\frac{(3\,\gamma\, C(p,X))^p}{2^{r-q+1}} \sum_{\varepsilon} \sum_{\varepsilon'} \Big\| \sum_{k} \varepsilon'_k \, S_{\Delta_k} \Big(\sum_{i} \varepsilon_i \, f_i \Big) \Big\|^p_{L^p(\mathbb{T},X)} \leq$$

(by lemma 5)

$$\leq (3\,\gamma\, C(p,X))^p \sum_{\varepsilon} \Big\| \sum_{i} \varepsilon_i \, f_i \|^p_{L^p(\mathbb{T},X)}.$$

■

3 The continuous case

Now we want to obtain a result analogous to theorem 3 for multipliers of Mihlin type in $L^p(\mathbb{R}, X)$. This can be made by a transference trick based on lemmas 7 and 8 (see [6]).

$\forall f \in L^1(\mathbb{R}, X)$, the Fourier transform of f is the function $\mathcal{F}f : \mathbb{R} \to X$ defined by $(\mathcal{F}f)(\tau) = \int_{\mathbb{R}} e^{-2\pi i t\tau} f(t)\, dt$. We denote by J the operator acting on functions on $\mathbb{R}$ in the following way: $(Jf)(t) = f(-t)$: thus $(J\mathcal{F}f)(\tau) = \int_{\mathbb{R}} e^{2\pi i t\tau} f(t)\, dt = (\mathcal{F}Jf)(\tau)$.

The following results hold.

Lemma 7 (Poisson summation formula) *Let X be a Banach space and let $f : \mathbb{R} \to X$ be a continuous function. Suppose that for some $\varepsilon > 0$ we have, as $|t| \to +\infty$, $\|f(t)\| = \mathrm{O}(|t|^{-1-\varepsilon})$ and $\|(\mathcal{F}f)(t)\| = \mathrm{O}(|t|^{-1-\varepsilon})$. Then $\forall t \in \mathbb{R}$ and $\forall a > 0$*

$$\sum_{n \in \mathbb{Z}} f(t + na) = \frac{1}{a} \sum_{n \in \mathbb{Z}} e^{2\pi i n t/a} (\mathcal{F}f)(n/a).$$

Lemma 8 (see [6, Lemma 2.1]). *Let X be a Banach space, $f \in \mathcal{S}(\mathbb{R}, X)$, $p \in]1, \infty[$. $\forall k \in \mathbb{N}$ we define $g_k : \mathbb{R} \to X$, $g_k(t) = 2^{-k/p'} \sum_{n \in \mathbb{Z}} e^{2\pi i n t} (\mathcal{F}f)(2^{-k} n)$, where p' is the exponent conjugate to p. Then $g_k \in L^p(\mathbb{T}, X)$ and $\|f\|_{L^p(\mathbb{R},X)} = \lim_{k \to +\infty} \|g_k\|_{L^p(\mathbb{T},X)}$.*

A proof of lemma 7 can be given by comparing the Fourier coefficients of the 1-periodic functions $t \mapsto \sum_{n\in\mathbb{Z}} f(t+n)$ and $t \mapsto \sum_{n\in\mathbb{Z}} e^{2\pi int}(\mathcal{F}f)(n)$. A sketch of proof of lemma 8 works as follows: g_k is a C^∞ 1-periodic function on $\mathbb{R}$, and therefore can be considered as an element of $L^p(\mathbb{T}, X)$. Taking into account the Poisson summation formula, one obtains

$$\|g_k\|_{L^p(\mathbb{T},X)} = \Big(\int_{-2^{k-1}}^{2^{k-1}} \Big\|\sum_{n\in\mathbb{Z}} f(s+2^k\, n)\Big\|_X^p\, ds\Big)^{1/p} \leq$$

$$\Big(\int_{-2^{k-1}}^{2^{k-1}} \|f(s)\|_X^p\, ds\Big)^{1/p} + \sum_{0\neq n\in\mathbb{Z}} 2^{k/p} \sup_{|s|\geq 2^k(|n|-(1/2))} \|f(s)\|_X.$$

Here, as $k\to+\infty$, the first addendum tends to $\|f\|_{L^p(\mathbb{R},X)}$, while the following sum tends to 0 since f is rapidly decreasing at infinity.

The R-version of the Mihlin multipliers theorem for Banach valued functions is the following.

Theorem 9 *Let $(\varphi_i)_{i\in I}$ be a family of functions that belong to $L^\infty(\mathbb{R}) \cap W^{1,1}_{\mathrm{loc}}(\mathbb{R}\setminus\{0\})$, and assume that the functions $t\mapsto t\,\varphi_i'(t)$ (shortly denoted $t\,\varphi_i'$) belong to $L^\infty(\mathbb{R})$. Assume moreover that*

$$\sup_{i\in I} \max\Big\{\|\varphi_i\|_{L^\infty(\mathbb{R})},\ \|t\,\varphi_i'\|_{L^\infty(\mathbb{R})}\Big\} =: \eta < +\infty.$$

Let X be a UMD Banach space, and $p\in\,]1,\infty[$. Then $\exists\, C(p,X)\in\mathbb{R}^+$ such that for any positive integer n, $(i_1,\dots,i_n)\in I^n$ and $(f_1,\dots,f_n)\in(\mathcal{S}(\mathbb{R},X))^n$ we have

$$\Big(\sum_{\varepsilon\in\{-1,1\}^n} \Big\|\sum_{r=1}^n \varepsilon_r\, \mathcal{F}(\varphi_{i_r}\, \mathcal{F}f_r)\Big\|^p_{L^p(\mathbb{R},X)}\Big)^{1/p} \leq C(p,X)\,\eta\,\Big(\sum_{\varepsilon\in\{-1,1\}^n} \Big\|\sum_{r=1}^n \varepsilon_r\, f_r\Big\|^p_{L^p(\mathbb{R},X)}\Big)^{1/p}.$$

Proof. We perform the proof in several steps.

(I) Assume that $\forall i\in I$ there is a sequence $(\psi_{i,k})_{k\in\mathbb{N}}$ of functions satisfying the following conditions:

(a) $\varphi_i(t) = \lim_{k\to+\infty} \psi_{i,k}(t)$ almost everywhere on $\mathbb{R}$

(b) $\forall (i,k)\in I\times\mathbb{N}$ $\psi_{i,k}\in L^\infty(\mathbb{R})$ and $\forall i\in I$ $\exists\, M_i\in\mathbb{R}^+$ such that $\sup_{k\in\mathbb{N}} \|\psi_{i,k}\|_{L^\infty(\mathbb{R})} \leq M_i$

(c) $\exists\,\tilde M\in\mathbb{R}^+$ such that for any positive integer n, $\forall\, k\in\mathbb{N}$ and for arbitrary $(i_1,\dots,i_n)\in I^n$, $(f_1,\dots,f_n)\in(\mathcal{S}(\mathbb{R},X))^n$ it holds

$$\Big(\sum_{\varepsilon\in\{-1,1\}^n} \Big\|\sum_{r=1}^n \varepsilon_r\, \mathcal{F}(\psi_{i_r,k}\, \mathcal{F}f_r)\Big\|^p_{L^p(\mathbb{R},X)}\Big)^{1/p} \leq$$

$$\tilde M\,\Big(\sum_{\varepsilon\in\{-1,1\}^n} \Big\|\sum_{r=1}^n \varepsilon_r\, f_r\Big\|^p_{L^p(\mathbb{R},X)}\Big)^{1/p}.$$

If we now choose n, $(i_1,\dots,i_n)$ and $(f_1,\dots,f_n)$ as above, since $\mathcal{F}f_r\in\mathcal{S}(\mathbb{R},X)\subseteq L^1(\mathbb{R},X)$, from (a), (b) and the dominated convergence theorem we get

$$\|\psi_{i_r,k}\,\mathcal{F}f_r-\varphi_{i_r}\,\mathcal{F}f_r\|_{L^1(\mathbb{R},X)}\underset{k\to+\infty}{\to}0,$$

so that

$$\|\mathcal{F}(\psi_{i_r,k}\,\mathcal{F}f_r)-\mathcal{F}(\varphi_{i_r}\,\mathcal{F}f_r)\|_{L^\infty(\mathbb{R},X)}\underset{k\to+\infty}{\to}0$$

and also

$$\Big\|\sum_{r=1}^n\varepsilon_r\,\mathcal{F}(\psi_{i_r,k}\,\mathcal{F}f_r)-\sum_{r=1}^n\varepsilon_r\,\mathcal{F}(\varphi_{i_r}\,\mathcal{F}f_r)\Big\|_{L^\infty(\mathbb{R},X)}\underset{k\to+\infty}{\to}0.$$

Therefore, by Fatou's lemma and assumption (c) we get

$$\Big(\sum_{\varepsilon\in\{-1,1\}^n}\Big\|\sum_{r=1}^n\varepsilon_r\,\mathcal{F}(\varphi_{i_r}\,\mathcal{F}f_r)\Big\|^p_{L^p(\mathbb{R},X)}\Big)^{1/p}\le\tilde{M}\Big(\sum_{\varepsilon\in\{-1,1\}^n}\Big\|\sum_{r=1}^n\varepsilon_r\,f_r\Big\|^p_{L^p(\mathbb{R},X)}\Big)^{1/p}.$$

(II) Assume that $\forall i\in I$ $\varphi_i\in C^\infty(\mathbb{R})$. Then $\forall f\in\mathcal{S}(\mathbb{R},X)$ we have $\varphi_i\,\mathcal{F}f\in\mathcal{S}(\mathbb{R},X)$, and so $\varphi_i\,\mathcal{F}f=\mathcal{F}J\mathcal{F}(\varphi_i\,\mathcal{F}f)$. Now let us fix n, $(i_1,\dots,i_n)$ and $(f_1,\dots,f_n)$ as above. If $\varepsilon\in\{-1,1\}^n$, then $\sum_{r=1}^n\varepsilon_r\,\mathcal{F}(\varphi_{i_r}\,\mathcal{F}f_r)\in\mathcal{S}(\mathbb{R},X)$, and hence, by lemma 8,

$$\Big\|\sum_{r=1}^n\varepsilon_r\,\mathcal{F}(\varphi_{i_r}\,\mathcal{F}f_r)\Big\|_{L^p(\mathbb{R},X)}=\Big\|\sum_{r=1}^n\varepsilon_r\,J\mathcal{F}(\varphi_{i_r}\,\mathcal{F}f_r)\Big\|_{L^p(\mathbb{R},X)}=\lim_{k\to+\infty}\|h_{k,\varepsilon}\|_{L^p(\mathbb{T},X)}$$

where $h_{k,\varepsilon}(t)=2^{-k/p'}\sum_{m\in\mathbb{Z}}e^{2\pi imt}\sum_{r=1}^n\varepsilon_r\,\varphi_{i_r}(2^{-k}m)\,(\mathcal{F}f_r)(2^{-k}m)$, so that $h_{k,\varepsilon}$ is the 1-periodic function whose m-th Fourier coefficient is

$$2^{-k/p'}\sum_{r=1}^n\varepsilon_r\,\varphi_{i_r}(2^{-k}m)\,(\mathcal{F}f_r)(2^{-k}m).$$

Likewise

$$\Big\|\sum_{r=1}^n\varepsilon_r f_r\|_{L^p(\mathbb{R},X)}=\lim_{k\to+\infty}\|g_{k,\varepsilon}\|_{L^p(\mathbb{T},X)}$$

where $g_{k,\varepsilon}(t)=2^{-k/p'}\sum_{m\in\mathbb{Z}}e^{2\pi imt}\sum_{r=1}^n\varepsilon_r\,(\mathcal{F}f_r)(2^{-k}m)$, so that $g_{k,\varepsilon}$ is the 1-periodic function whose m-th Fourier coefficient is $2^{-k/p'}\sum_{r=1}^n\varepsilon_r\,(\mathcal{F}f_r)(2^{-k}m)$. Now we remark that $\sup_{m\in\mathbb{Z}}|\varphi_i(2^{-k}m)|\le\|\varphi_i\|_{L^\infty(\mathbb{R})}\le\eta$. Moreover, if the dyadic interval Δ_j contains two different elements, then $|j|\ge 2$; if $j\ge 2$, then

$$\sum_{m,m-1\in\Delta_j}|\varphi_i(2^{-k}m)-\varphi_i(2^{-k}(m-1))|=\sum_{m=2^{j-1}+1}^{2^j-1}\Big|\int_{2^{-k}(m-1)}^{2^{-k}m}\varphi_i'(s)\,ds\Big|\le$$

$$\sum_{m=2^{j-1}+1}^{2^j-1} \int_{m-1}^{m} |\varphi_i'(2^{-k}\sigma)|\, 2^{-k}\, d\sigma = \int_{2^{j-1}}^{2^j-1} |\varphi_i'(t)|\, dt \le$$

$$\|t\varphi_i'\|_{L^\infty(\mathbb{R})} \int_{2^{j-1}}^{2^j-1} t^{-1}\, dt \le \eta \,\log 2.$$

The same estimate is obtained when $j \le -2$, by means of a similar argument. Therefore if we apply theorem 3, we obtain

$$\Big(\sum_{\varepsilon\in\{-1,1\}^n} \|h_{k,\varepsilon}\|^p_{L^p(\mathbb{T},X)}\Big)^{1/p} \le C(p,X)\,(\log 2)\,\eta \Big(\sum_{\varepsilon\in\{-1,1\}^n} \|g_{k,\varepsilon}\|^p_{L^p(\mathbb{T},X)}\Big)^{1/p},$$

and as $k \to +\infty$ we get

$$\Big(\sum_{\varepsilon\in\{-1,1\}^n} \Big\| \sum_{r=1}^{n} \varepsilon_r\, \mathcal{F}(\varphi_{i_r}\, \mathcal{F} f_r) \Big\|^p_{L^p(\mathbb{R},X)}\Big)^{1/p} \le$$

$$C(p,X)\,(\log 2)\,\eta \Big(\sum_{\varepsilon\in\{-1,1\}^n} \Big\| \sum_{r=1}^{n} \varepsilon_r\, f_r \Big\|^p_{L^p(\mathbb{R},X)}\Big)^{1/p}.$$

(III) Finally we drop the additional assumption that $\varphi_i \in C^\infty$ $\forall i$. We fix $\omega \in C_0^\infty(\mathbb{R})$, with $\omega \ge 0$, $\displaystyle\int_{\mathbb{R}} \omega = 1$ and $\operatorname{supp}\omega \subseteq [-1,1]$. $\forall \rho \in \mathbb{R}^+$ we set $\omega_\rho(t) = \rho^{-1}\omega(\rho^{-1}t)$ and $\psi_{i,\rho} = \omega_\rho * \varphi_i$. Then $\psi_{i,\rho} \in C^\infty(\mathbb{R}) \cap L^\infty(\mathbb{R})$ and (by Young's inequality) $\|\psi_{i,\rho}\|_{L^\infty(\mathbb{R})} \le \|\varphi_i\|_{L^\infty(\mathbb{R})}$. Moreover

$$t\,\psi_{i,\rho}'(t) = t\,(\omega_\rho' * \varphi_i)(t) = t\int_{t-\rho}^{t+\rho} \omega_\rho'(t-s)\varphi_i(s)\, ds,$$

and if $0 < \rho < |t|$ partial integration also gives

$$t\,\psi_{i,\rho}'(t) = t\int_{t-\rho}^{t+\rho} \omega_\rho(t-s)\varphi_i'(s)\, ds.$$

Let $t \ne 0$ be fixed. If $\rho \ge \dfrac{|t|}{2}$, then

$$|t\,\psi_{i,\rho}'(t)| = |t\int_{\mathbb{R}} \omega_\rho'(t-s)\varphi_i(s)\, ds| \le$$

$$2\rho\, \|\varphi_i\|_\infty \int_{\mathbb{R}} \rho^{-2}\, |\omega'(\rho^{-1}(t-s))|\, ds = 2\|\omega'\|_{L^1(\mathbb{R})}\, \|\varphi_i\|_{L^\infty(\mathbb{R})}.$$

If, on the contrary, $\rho \le \dfrac{|t|}{2}$, then

$$|t\,\psi_{i,\rho}'(t)| = |t\int_{t-\rho}^{t+\rho} \omega_\rho(t-s)\varphi_i'(s)\, ds| \le$$

$$|t|\int_{\mathbb{R}} |\omega_\rho(\sigma)|\, d\sigma \sup_{|t-s|\le\rho} |\varphi_i'(s)| \le \|s\,\varphi_i'\|_\infty \sup_{|t-s|\le|t|/2} \frac{|t|}{|s|} = 2\,\|s\,\varphi_i'\|_{L^\infty(\mathbb{R})}.$$

Therefore

$$\max\{\|\psi_{i,\rho}\|_{L^\infty(\mathbb{R})},\, \|t\,\psi'_{i,\rho}\|_{L^\infty(\mathbb{R})}\} \leq$$
$$\max\{\|\varphi_i\|_{L^\infty(\mathbb{R})},\, 2\|\omega'\|_{L^1(\mathbb{R})}\,\|\varphi_i\|_{L^\infty(\mathbb{R})},\, 2\|t\,\varphi'_i\|_{L^\infty(\mathbb{R})}\} \leq 2\eta(1+\|\omega'\|_{L^1(\mathbb{R})}).$$

Next we remark that on $\mathbb{R}\setminus\{0\}$ φ_i is almost everywhere equal to a continuous function. Therefore for almost all $t \neq 0$ we have

$$|\psi_{i,\rho}(t) - \varphi_i(t)| = \Big|\int_{\mathbb{R}} \omega_\rho(s)\,(\varphi_i(t-s) - \varphi_i(t))\,ds\Big| \leq$$
$$\sup_{|s|\leq\rho} |\varphi_i(t-s) - \varphi_i(t)| \underset{\rho\to 0}{\to} 0.$$

Thus, combining steps (I) and (II), we get the desired inequality. ∎

Corollary 10 *Let $(\psi_i)_{i\in I}$ be a family of elements of $L^1(\mathbb{R})$ and assume that $\forall i \in I$ $\mathcal{F}f_i \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}\setminus\{0\})$, with $t\,(\mathcal{F}f_i)' \in L^\infty(\mathbb{R})$. Assume moreover that*

$$\sup_{i\in I} \max\Big\{\|\mathcal{F}f_i\|_{L^\infty(\mathbb{R})},\, \|t\,(\mathcal{F}f_i)'\|_{L^\infty(\mathbb{R})}\Big\} =: \eta < +\infty.$$

Let $p \in]1,\infty[$ and let X be a UMD Banach space. $\forall i \in I$ let $T_i : L^p(\mathbb{R},X) \to L^p(\mathbb{R},X)$ be the bounded linear operator defined by $T_i f = \psi_i * f$. Then $\{T_i;\, i \in I\}$ is a R-bounded set of operators. More precisely, for any positive integer n and for arbitrary $(i_1,\dots,i_n) \in I^n$, $(f_1,\dots,f_n) \in (L^p(\mathbb{R},X))^n$ the following inequality holds:

$$\Big(\sum_{\varepsilon\in\{-1,1\}^n} \Big\|\sum_{r=1}^n \varepsilon_r\, T_{i_r} f_r\Big\|^p_{L^p(\mathbb{R},X)}\Big)^{1/p} \leq C(p,X)\,\eta\,\Big(\sum_{\varepsilon\in\{-1,1\}^n} \Big\|\sum_{r=1}^n \varepsilon_r\, f_r\Big\|^p_{L^p(\mathbb{R},X)}\Big)^{1/p}.$$

Proof. It is sufficient to deal with the particular case of $f_1,\dots,f_n \in \mathcal{S}(\mathbb{R},X)$. In this case $\psi_{i_r} * f_r \in L^1(\mathbb{R},X)$ and $\mathcal{F}(\psi_{i_r} * f_r) = (\mathcal{F}\psi_{i_r})\,(\mathcal{F}f_r) \in L^1(\mathbb{R},X)$ (since $\mathcal{F}\psi_{i_r} \in L^\infty(\mathbb{R})$ and $\mathcal{F}f_r \in \mathcal{S}(\mathbb{R},X)$). Therefore $\psi_{i_r} * f_r = J\,\mathcal{F}((\mathcal{F}\psi_{i_r})\,(\mathcal{F}f_r))$, and so the result follows at once from theorem 9. ∎

Corollary 10 has been used in [4].

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