Fourier Multipliers in Stability and Control Theory

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Stability for strongly continuous semigroups on Banach spaces is described in terms of L^p–Fourier multiplier properties of the resolvent of the generator. A discrete version of this theory is developed, that lead to a description of the spectra of the semigroups in terms of discrete L^p–Fourier multipliers. Applications to stability of linear control systems on Banach spaces are given.

Key Words: Semigroups asymptotics, operator-valued Fourier multipliers, input-output stability

1. INTRODUCTION

A strongly continuous semigroup on a Hilbert space X is uniformly exponentially stable provided the resolvent of its generator A belongs to the Hardy class $H^\infty = H^\infty(\{\mathrm{Re}\,\lambda > 0\}; \mathcal{L}(X))$ of bounded analytic functions with values in the set $\mathcal{L}(X)$ of bounded linear operators, see, e.g., [9, Thm.V.1.11] or [7, Thm.5.1.5]. This theorem, generally, does not hold if X is a Banach space, see, e.g., [20].

In this paper we show, in particular, that on Banach spaces the H^∞-condition should be complemented by the condition that the resolvent re-

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stricted to the imaginary axis is a Fourier multiplier for $L^p(\mathbb{R};X)$ for some $p\in[1,\infty)$. Recall, that a function $\xi:\mathbb{R}\to\mathcal{L}(X)$ is called an $L^p(\mathbb{R};X)$–*Fourier multiplier* if the operator $M_\xi: f\mapsto \mathcal{F}^{-1}\xi(\cdot)\mathcal{F}f$ has a bounded extension from a dense in $L^p(\mathbb{R};X)$ set to all of $L^p(\mathbb{R};X)$; here $\mathcal{F}$ is the Fourier transform.

Various connections of stability and hyperbolicity for strongly continuous semigroups and Fourier multipliers were noticed in the work by M.A. Kaashoek and S. M. Verduyn Lunel [13], L. Weis [21], M. Hieber [12], W. Arendt and S. Bu [2], and in [4]. In this paper we continue this work in a different direction, and describe the uniform exponential stability and exponential stability for a semigroup in terms of $L^p(\mathbb{R};X)$–Fourier multiplier properties of the resolvent of its generator.

Using Datko–Pazy Theorem and a characterization of stability in terms of convolution with the semigroup, we prove for $m=0,1,2,\dots$ a formula for the m-th growth bound $\omega_m(A)$ in terms of the m-th spectral bound $s_m(A)$ and $L^p(\mathbb{R};X)$–Fourier multiplier properties of the function $\xi_m(s)=(w-A)^{-m}(is-A)^{-1}$, $s\in\mathbb{R}$, $w>\max\{\omega_0(A),0\}$. Recall, that $\omega_m(A)$ gives the best exponential rate of growth for the solutions of $\dot{x}=Ax$ that start in the domain $D(A^m)$, while $s_m(A)$ gives the minimal abscissa for the half-plane where the resolvent grows along vertical lines not faster than m-th power (see the definitions below and more information in [20, 23]). Although the case $m>0$ brings new effects, our proof works for all $m=0,1,2,\dots$, and is different from [4, 12] where the case $m=0$ was considered. Via Plancherel's Theorem, we recover the well known Hilbert space formula $\omega_m(A)=s_m(A)$ due to L. Gearhart and J. Prüss for $m=0$ and to G. Weiss for $m=1,2,\dots$, see [17, 20, 23], and also I. Herbst [10].

We also develop a discrete version of this theory, that lead to a description of the spectra of the semigroups in terms of discrete $L^p(\mathbb{Z};X)$–Fourier multipliers, cf. [14, Thm.2.3] and a recent interesting development in [2, Thm.3.6]. For $m=0,1,2,\dots$ the corresponding discrete multiplier operator gives a unique periodic mild solution for the inhomogeneous equations with inhomogeneity that takes values in the domain $D(A^m)$. In particular, for $m=0$ our result gives the spectral mapping theorem for evolution semigroups from [14] and the celebrated Prüss' Theorem [17]. However, for $m=1,2,\dots$ we also prove that the multiplier operator is bounded if and only if $D(A^m)\subset \mathrm{Rg}\,(e^{2\pi}-I)$, where $\mathrm{Rg}\,(\cdot)$ denotes the range of an operator. We do not know if this last condition ever appeared in the literature before. Using a connection to Kaashoek-Verduyn Lunel theory [13], the results of the current paper were recently generalized for the case of hyperbolicity of semigroups and $m\in\mathbb{R}_+$ in [15].

Finally, we use $L^p(\mathbb{R};X)$–Fourier multipliers to describe input–output stability for linear control systems on Banach spaces in terms of transfer function. Our main result here is that the control system is p-input-output stable (that is, the input-output operator is L^p-bounded) if and only if the corresponding transfer function is bounded on $\{\mathrm{Re}\,\lambda > 0\}$ and its restriction on $i\mathbb{R}$ is an L^p-Fourier multiplier. It is well-known that in the Hilbert space setting input-output stability is equivalent to external stability; the latter is defined as the boundedness of the transfer function in $\{\mathrm{Re}\,\lambda > 0\}$, see [6, 19, 25]. Thus, the lack of Plancherel's Theorem explains why this classical theorem cannot be, generally, extended to the Banach space setting. Finally, we prove that, under stabilizability and detectability assumptions, the exponential stability $\omega_1(A) < 0$ of the nominal system is equivalent to the fact that an appropriate modification of the transfer function is an L^p-Fourier multiplier.

2. SPECTRUM AND DISCRETE FOURIER MULTIPLIERS

The *discrete Fourier transform* $\mathcal{F}_d$ maps a Banach space valued function $f \in L^1([0,2\pi];X)$ to the sequence $(x_k)_{k\in\mathbb{Z}}$ of its Fourier coefficients. Conversely, if $\bar{x} = (x_k)_{k=-N}^N$ is a finitely supported sequence on the Banach space X, then its inverse Fourier transform $\mathcal{F}_d^{-1}\bar{x}$ is the trigonometric polynomial $(\mathcal{F}_d^{-1}\bar{x})(\tau) = \sum_{k=-N}^N e^{ik\tau}x_k, \tau \in [0,2\pi]$. By $\mathcal{P}$ we denote the set of all trigonometric polynomials $f(\tau) = \sum_{k=-N}^N x_k e^{ik\tau}$, $x_k \in X$, $\tau \in [0,2\pi]$, $N \in \mathbb{N}$, with coefficients in X. A function $\xi : \mathbb{Z} \to \mathcal{L}(X)$ is called a Fourier multiplier for $L^p([0,2\pi];X)$, $1 \le p < \infty$, if the operator $\mathcal{F}_d^{-1}\xi(\cdot)\mathcal{F}_d$ on $\mathcal{P}$ defined by

$$\mathcal{F}_d^{-1}\xi(\cdot)\mathcal{F}_d : \sum_{k=-N}^{N} e^{ik(\cdot)}x_k \mapsto \sum_{k=-N}^{N} e^{ik(\cdot)}\xi(k)x_k \tag{1}$$

can be extended to a bounded operator on $L^p([0,2\pi];X)$. Applying the operator $\mathcal{F}_d^{-1}\xi(\cdot)\mathcal{F}_d$ to the trigonometric monomials we have that a Fourier multiplier $\xi(\cdot)$ for $L^p([0,2\pi];X)$ always satisfies $\sup_{k\in\mathbb{Z}}\|\xi(k)\| < \infty$. If X is a Hilbert space, then by Parseval's identify the latter condition is necessary and sufficient for $\xi(\cdot)$ to be a Fourier multiplies for $L^2([0,2\pi];X)$.

Let A be the generator of a strongly continuous semigroup $\{e^{tA}\}_{t\ge 0}$ on the Banach space X. Recall that, for $f \in L^1([0,2\pi];X)$, the function $u \in C([0,2\pi];X)$ is a *mild solution* of the inhomogeneous equation

$$\dot{u}(t) = Au(t) + f(t), \quad 0 \le t \le 2\pi, \quad \text{if} \tag{$(E)_f$}$$

$$u(t) = e^{tA}u(0) + \int_0^t e^{(t-s)A}f(s)ds, \quad 0 \le t \le 2\pi. \tag{$(M)_{u,f}$}$$

In what follows we show that the existence of a unique periodic mild solution u of $(E)_f$ for f belonging to a certain class of functions can be expressed in terms of Fourier multipliers.

For $m \in \{1, 2, \dots\}$ we denote by X_m the domain $D(A^m)$ equipped with the norm $\|x\|_m = \sum_{k=0}^{n} \|A^k x\|$ and let $X_0 = X$. Denote by Rg B the range, by $D(B)$ the domain, by $\sigma(B)$ the spectrum, by $\rho(B)$ the resolvent set, and by $R(\lambda, B) = (\lambda - B)^{-1}$ the resolvent of an operator B. Let $\mathcal{E}([0, 2\pi]; X)$ be the space $L^p([0, 2\pi]; X)$, $1 \le p < \infty$, or the space $C_{\text{per}}([0, 2\pi]; X)$ of 2π-periodic continuous functions. If $m \in \{0, 1, 2, \dots\}$ and $i\mathbb{Z} \subset \rho(A)$, we let $\xi_m(k) = A^{-m}(A - ik)^{-1}$, $k \in \mathbb{Z}$, and consider the operator $L_m = \mathcal{F}_d^{-1} \xi_m(\cdot) \mathcal{F}_d$ defined in (1).

THEOREM 2.1. *Let $m \in \{0, 1, 2 \dots\}$. Then the following assertions are equivalent:*

(i) $i\mathbb{Z} \subseteq \rho(A)$ and $D(A^m) \subseteq Rg\ (e^{2\pi A} - I)$;

(ii) For each $f \in L^p([0, 2\pi]; X_m)$, $1 \le p < \infty$, there is a unique mild solution $u \in C_{per}([0, 2\pi]; X)$ of $(E)_f$;

(iii) For each $f \in C_{per}([0, 2\pi]; X_m)$ there is a unique mild solution $u \in C_{per}([0, 2\pi]; X)$ of $(E)_f$;

(iv) $i\mathbb{Z} \subseteq \rho(A)$ and the function $\xi_m(k) = A^{-m}(ik - A)^{-1}$, $k \in \mathbb{Z}$, is a Fourier multiplier for $L^p([0, 2\pi]; X)$, $1 \le p < \infty$.

(v) $i\mathbb{Z} \subseteq \rho(A)$ and $\xi_m(k) = A^{-m}(ik - A)^{-1}$, $k \in \mathbb{Z}$, is a Fourier multiplier for $C_{per}([0, 2\pi]; X)$.

Proof. **(iii)⇒(i)**: Define $R : C_{\text{per}}([0, 2\pi]; X_m) \to C_{\text{per}}([0, 2\pi]; X)$ by $Rf = u$ if $f \in C_{\text{per}}([0, 2\pi]; X_m)$ and $u \in C_{\text{per}}([0, 2\pi]; X)$ satisfy $(M)_{u,f}$. An elementary computation shows that the operator R is closed. By assumption and the closed Graph Theorem R is bounded.

We claim that for each $k \in \mathbb{Z}$ the operator $A - ik$ is uniformly injective on X. For that consider the part A_m of A in X_m with domain $D(A_m) = D(A^{m+1})$ and fix $x \in D(A_m)$. Define $f(k) = e^{ikt}(A - ik)x$ and $u(t) = e^{ikt}x$, $t \in [0, 2\pi]$. Then $f \in C_{\text{per}}([0, 2\pi]; X_m)$ and $u \in C_{\text{per}}([0, 2\pi]; X)$, and $(M)_{u,f}$ holds. Thus $Rf = u$, and hence $\|x\| = \|u(0)\| \le \|Rf\| \le \|R\|\|f\| = \|R\|\|(A - ik)x\|_m$. Therefore, there exists a constant $c > 0$ such that $c\|x\|_m \le \|(A - ik)x\|_m$ for all $x \in D(A_m)$. Since the operators A on X and A_m on X_m are similar the claim follows.

Next we show that $D(A^m) \subseteq \text{Rg}\ (e^{2\pi A} - I)$. Fix $x \in D(A^m)$. Choose $\gamma \in C[0, 2\pi]$ such that $\gamma(0) = 0$, $\gamma(2\pi) = 1$ and $\int_0^{2\pi} \gamma(s)ds = 1$. Define a function $f \in C_{\text{per}}([0, 2\pi]; X_m)$ by $f(t) = e^{tA}[(1 - \gamma(t))e^{2\pi A}x + \gamma(t)x]$, $t \in [0, 2\pi]$. By assumption, there exists $u \in C_{\text{per}}([0, 2\pi]; X)$ such that

$(M)_{u,f}$ holds. Then

$$\begin{aligned}(e^{2\pi A}-I)u(0) = e^{2\pi A}u(0)-u(2\pi) &= \int_0^{2\pi} e^{(2\pi-s)A}f(s)ds \\ &= e^{2\pi A}\int_0^{2\pi}[(1-\gamma(s))e^{2\pi A}x+\gamma(s)x]ds \\ &= e^{2\pi A}x = (e^{2\pi A}-I)x + x.\end{aligned}$$

Thus $x \in \mathrm{Rg}\ (e^{2\pi A}-I)$.

From $D(A^m) \subseteq \mathrm{Rg}\ (e^{2\pi A}-I)$ we obtain that $e^{2\pi A}-I$ has dense range. The spectral mapping theorem for the residual spectrum implies that $A-ik$ has dense range for all $k \in \mathbb{Z}$. Together with the uniform injectivity of $A-ik$ this shows $i\mathbb{Z} \subseteq \rho(A)$.

(i)⇒(iv),(v): Consider the convolution operator K on $\mathcal{E}([0,2\pi];X)$ defined by

$$\begin{aligned}(Kf)(\tau) &= \int_0^{2\pi} e^{sA}f(\tau - s[\mathrm{mod}\ 2\pi])ds \\ &= \int_0^{\tau} e^{sA}f(\tau-s)ds + \int_\tau^{2\pi} e^{sA}f(\tau-s+2\pi)ds, \quad \tau\in[0,2\pi].\end{aligned}$$

Clearly, the operator K is bounded. If $f=\sum_{k=-N}^{N} x_k e^{ik(\cdot)} \in \mathcal{P}$, then a simple computation shows

$$Kf = L_0[(e^{2\pi A}-I)f(\cdot)] = (e^{2\pi A}-I)[L_0 f](\cdot). \tag{2}$$

From $i\mathbb{Z}\subseteq\rho(A)$ and the spectral mapping theorem for the point spectrum it follows that $e^{2\pi A}-I$ is injective. Then $S=(e^{2\pi A}-I)^{-1}$ with domain $D(S) = \mathrm{Rg}\ (e^{2\pi A}-I)$ is a closed operator. Our assumption implies that SA^{-m} is defined on the whole of X. Since SA^{-m} is closed it is bounded. Thus, (2) yields $L_m f = SA^{-m}(Kf)(\cdot)$ for $f\in\mathcal{P}$, and the assertions follow.

(iv),(v)⇒(i): From (2) we obtain

$$K(A^{-m}f(\cdot)) = (e^{2\pi A}-I)[L_m f](\cdot) = L_m[(e^{2\pi A}-I)f(\cdot)] \tag{3}$$

for $f\in\mathcal{P}$. Since L_m has a bounded extension to $\mathcal{E}([0,2\pi];X)$, which we denote again by L_m, equation (3) holds for all $f\in\mathcal{E}([0,2\pi];X)$. Now let $y\in D(A^m)$. Choose $\beta\in C^1[0,2\pi]$ such that

$$0\le\beta(\cdot)\le 1, \quad \beta|_{[0,\frac{2\pi}{3}]}\equiv 1 \quad \text{and} \quad \beta|_{[\frac{4\pi}{3},2\pi]}\equiv 0. \tag{4}$$

Define $g \in \mathcal{E}([0,2\pi];X)$ by $g(\tau) = \beta'(\tau)e^{\tau A}A^m y$, $\tau \in [0,2\pi]$. Then (3) yields:

$$\begin{aligned}(e^{2\pi A} - I)(L_m g)(\tau) &= K(\beta'(\cdot)e^{(\cdot)A}y)(\tau)\\ &= \int_0^\tau e^{sA}\beta'(\tau - s)e^{(\tau-s)A}y ds + \int_\tau^{2\pi} e^{sA}\beta'(\tau - s + 2\pi)e^{(\tau-s+2\pi)A}y ds\\ &= (\beta(0) - \beta(\tau))e^{\tau A}y + (\beta(\tau) - \beta(2\pi))e^{(2\pi+\tau)A}y\\ &= (1 - \beta(\tau))e^{\tau A}y + \beta(\tau)e^{(2\pi+\tau)A}y.\end{aligned}$$

If $\mathcal{E} = C_{\text{per}}([0,2\pi];X)$ this implies

$$(e^{2\pi A} - I)(L_m g)(2\pi) = e^{2\pi A}y = (e^{2\pi A} - I)y + y, \tag{5}$$

and hence $y \in \text{Rg}\,(e^{2\pi A} - I)$. If $\mathcal{E} = L^p([0,2\pi];X)$ then the same argument works as soon as the following claim is proved: If (iv) holds, then $\text{Rg}\,(L_m) \subseteq C([0,2\pi];X)$. To prove the claim, let $g \in L^p([0,2\pi];X)$. Choose a sequence $\{g_k\}_{k=1}^\infty$ in $\mathcal{P}$ such that $g_k \to g$ in $L^p([0,2\pi];X)$. Then $u_k := L_m g_k \to L_m g$ in $L^p([0,2\pi];X)$, and $u_k \in \mathcal{P}$ for all $k \in \mathbb{N}$. Thus $\{u_k\}_{k=1}^\infty$ converges pointwise on a set $\Omega \subseteq [0,2\pi]$ of full measure. Since g_k and u_k are polynomials, a simple calculation shows that

$$u_k(t) = e^{tA}u_k(0) + \int_0^t A^{-m}e^{sA}g_k(t-s)ds, \quad t \in [0,2\pi], \quad \text{and therefore}$$

$$\int_0^t A^{-m}e^{sA}g_k(t-s)ds \to \int_0^t A^{-m}e^{sA}g(t-s)ds \text{ uniformly for } t \in [0,2\pi]. \tag{6}$$

Now fix $a \in \Omega \cap (0,2\pi)$. Then $\{u_k(a)\}_{k=1}^\infty$ converges in X, and we conclude from (6) that $\{e^{aA}u_k(0)\}_{k=1}^\infty$ converges in X as well. Let $x = \lim_k e^{aA}u_k(0)$. Then $e^{tA}u_k(0) = e^{(t-a)A}e^{aA}u_k(0) \to e^{(t-a)A}x$ uniformly for $t \in [a,2\pi]$. Thus, by (6), $h(t) := \lim_k u_k(t)$ exists uniformly for $t \in [a,2\pi]$ and $h \in C([a,2\pi];X)$. Since $u_k \to L_m g$ in $L^p([0,2\pi];X)$ we obtain $L_m g = h$ a.e. on $[a,2\pi]$. Since Ω has full measure we can choose $a \in \Omega$ arbitrarily close to 0 from which the claim follows.

(i)⇒(ii),(iii): Since $i\mathbb{Z} \subseteq \rho(A)$, the spectral mapping theorem for the point spectrum implies that $e^{2\pi A} - I$ is injective. Let $S = (e^{2\pi A} - I)^{-1}$ with domain $D(S) = \text{Rg}\,(e^{2\pi A} - I)$. By our assumption, $X_m \subseteq D(S)$. Now let $f \in \mathcal{E}([0,2\pi];X_m)$. Then $\int_0^{2\pi} e^{sA}f(2\pi - s)ds \in X_m$. Let

$$u(0) = -S\int_0^{2\pi} e^{sA}f(2\pi - s)ds \quad \text{and} \quad u(t) = e^{tA}u(0) + \int_0^t e^{sA}f(t-s)ds.$$

Note that $u(2\pi) = e^{2\pi A}u(0) - (e^{2\pi A} - I)u(0) = u(0)$. Hence, we have that $u \in C_{\text{per}}\ ([0, 2\pi]; X)$, and $(M)_{u,f}$ holds. In order to show that u is the only function from $C_{\text{per}}\ ([0, 2\pi]; X)$ satisfying $(M)_{u,f}$ we may assume that $f = 0$. Then $u(0) = u(2\pi) = e^{2\pi A}u(0)$ or, in other words, $(e^{2\pi A} - I)u(0) = 0$. The injectivity of $e^{2\pi A} - I$ then implies $u(0) = 0$, and hence $u \equiv 0$. The implication (ii)⇒(iii) follows from $C_{\text{per}}\ ([0, 2\pi]; X) \subset L^p([0, 2\pi]; X)$. ▌

Remark 2. 1. If $m = 0$ then (i) in Theorem 2.1 is equivalent to the fact that $1 \in \rho(e^{2\pi A})$. Therefore, the equivalence (i)⇔(iii) for $m = 0$ gives a theorem by J. Prüss [17]. Also, the equivalence (i)⇔(iv),(v) for $m = 0$ was proved in [14], see also [3, Thm.2.31]. Note, that though the term "Fourier multiplier" has never been mentioned in [14], the function β as in (4) is used in [14], see also [3, Lem. 2.29]. ◇

Remark 2. 2. For $m = 0$ the proof of the implication (iv), (v)⇒(i) leads to a representation of $(e^{2\pi A} - I)^{-1}$ in terms of the operator L_0. In fact, if one of the conditions in Theorem 2.1 holds, then formula (5) implies:

$$(e^{2\pi A} - I)^{-1}y = [L_0(\beta'(\cdot)e^{(\cdot)A}y)](2\pi) - y, \quad y \in X,$$

where β is defined in (4). ◇

Remark 2. 3. We can explicitly relate the operator L_0 to the operator F that appears in Greiner's Spectral Mapping Theorem (see for example [20, Thm. 2.2.1]). To define the operator F, we assume that $i\mathbb{Z} \subset \rho(A)$, set

$$F_n x = n^{-1} \sum_{N=0}^{n-1} \sum_{k=-N}^{N} (A - ik)^{-1}x,\ n \in \mathbb{N}, \ \text{ and let } \ Fx = \lim_{n\to\infty} F_n x$$

for those $x \in X$ where the limit exists. In other words, Fx is the Cesaro sum of the series $\sum (A - ik)^{-1}x$ for such x where the sequence $\{(A - ik)^{-1}x\}$ is Cesaro summable. Computing Fourier coefficients of the function $f(\tau) = e^{\tau A}$, $\tau \in [0, 2\pi]$, $x \in X$, and using Fejer's Theorem, we have the following formula:

$$\pi(e^{2\pi A} + I)x = F(e^{2\pi A} - I)x \text{ for each } x \in X. \tag{7}$$

It shows that F is defined on the range of $e^{2\pi A} - I$ which is dense in X due to $i\mathbb{Z} \subset \rho(A)$. Formula (2) for a polynomial $f \in \mathcal{P}$ shows that $(Kf)(\tau)$ is in the range of $e^{2\pi A} - I$ for each $\tau \in [0, 2\pi]$. Using (7) and (2), for every $f \in \mathcal{P}$ we have that $FKf = \pi Kf + 2\pi L_0 f$. Greiner's Spectral Mapping Theorem (see [20, Thm. 2.2.1]) says that for $m = 0$ assertion (i) in Theorem 2.1 is equivalent to the fact that $i\mathbb{Z} \subset \rho(A)$ and F has a bounded extension from the range of $e^{2\pi A} - I$ to X. Therefore, if one of the equivalent statements in Theorem 2.1 holds, then

$$Fy = \pi(e^{2\pi A} + I)[L_0(\beta'(\cdot)e^{(\cdot)A}y)(2\pi) - y] \quad \text{for} \quad y \in X,$$

where $\beta(\cdot)$ is the function defined in (4). ◇

Remark 2. 4. Finally, we note how the operator L_0 is related to the generator of the evolution semigroup, $\{E^t\}_{t\geq 0}$, on $L^p([0,2\pi];X)$, associated to the given semigroup $\{e^{tA}\}_{t\geq 0}$ on X. We define the *evolution semigroup* as follows, cf. [3]:

$$(E^t f)(\tau) = e^{tA} f(\tau - t[\text{mod } 2\pi]), \tau \in [0,2\pi], t \geq 0, f \in L^p([0,2\pi];X).$$

Let $\mathbf{d}$ denote the operator of differentiation (with the periodic boundary conditions) and let $\mathcal{A}$, $(\mathcal{A}f)(\tau) = Af(\tau)$, denote the multiplication operator by A (with the maximal domain) on $L^p([0,2\pi];X)$. Then $\{E^t\}_{t\geq 0}$ is the product of two commuting semigroups: the translation semigroup $\{e^{-t\mathbf{d}}\}_{t\geq 0}$, $(e^{-t\mathbf{d}} f)(\tau) = f(\tau - t[\text{mod } 2\pi])$, and the multiplication semigroup $\{e^{t\mathcal{A}}\}_{t\geq 0}$, $(e^{t\mathcal{A}}f)(\tau) = e^{tA}f(\tau)$. Therefore, the generator Γ of the semigroup $\{E^t\}_{t\geq 0}$ is the closure of the operator $-\mathbf{d} + \mathcal{A}$ with the domain $D(\mathbf{d}) \cap D(\mathcal{A})$. Note that the set $\mathcal{P}_A$ of all trigonometric polynomials f with coefficients in $D(A)$ is the core for Γ. Since

$$(\Gamma f)(\tau) = ([-\mathbf{d} + \mathcal{A}]f)(\tau) = \sum_{k=-N}^{N} (A - ik)x_k e^{ik\tau}, \quad \tau \in [0,2\pi],$$

for each polynomial $f(\tau) = \sum x_k e^{ik\tau}$, $x_k \in D(A)$, it is immediate that $\Gamma L f = f$ if $f \in \mathcal{P}$ and $L\Gamma f = f$ if $f \in \mathcal{P}_A$, and therefore $L_0 = \Gamma^{-1} \in \mathcal{L}(L^p([0,2\pi];X))$ if and only if for $m = 0$ assertion (i) in Theorem 2.1 is satisfied (cf. [3, Thm.2.30]). ◇

3. STABILITY AND FOURIER MULTIPLIERS

Let $m \in \{0,1,2,\ldots\}$ and recall the definitions of m-th *growth bound* $\omega_m(A)$ and m-th *spectral bound* $s_m(A)$ (see, e.g., [20, 23]):

$$\begin{aligned}\omega_m(A) = \inf \Big\{ \omega \in \mathbb{R} : \exists M = M(\omega) \text{ so that} \\ \|e^{tA}x\| \leq M e^{\omega t}\|x\|_m, x \in X_m, t \geq 0 \Big\},\end{aligned} \tag{8}$$

$$s_m(A) = \inf\Big\{\omega \in \mathbb{R} : \{\mathrm{Re}\ \lambda > \omega\} \subset \rho(A) \text{ and } \sup_{\mathrm{Re}\ \lambda>\omega} (1+|\lambda|)^{-m}\|R(\lambda, A)\| < \infty\Big\}.$$

Fix $w > \max\{\omega_0(A), 0\}$. Note that induction and the resolvent identity yield:

$$R^m(w, A)R(\lambda, A) = \sum_{k=0}^{m-1}(-1)^k(\lambda - w)^{-(k+1)}R^{m-k}(w, A) + (-1)^m(\lambda - w)^{-m}R(\lambda, A). \tag{9}$$

For $m \in \{0, 1, 2, \dots\}$ elementary estimates and (9) imply:

$$s_m(A) = \inf\Big\{\omega \in \mathbb{R} :\{\mathrm{Re}\ \lambda > \omega\} \subset \rho(A) \text{ and } \sup_{\mathrm{Re}\ \lambda>\omega} \|(w - A)^{-m}R(\lambda, A)\| < \infty\Big\}. \tag{10}$$

For $\lambda \in \mathbb{C}$ we denote $\xi_{m,\lambda}(s) = (w - A)^{-m}(is + \lambda - A)^{-1}$ and $\xi_m(s) = \xi_{m,0}(s)$, $s \in \mathbb{R}$, and define operators $\mathbb{G}_\lambda$ and $\mathbb{H}_{m,\lambda}$ as follows:

$$(\mathbb{G}_\lambda f)(t) = \int_0^t e^{(t-\tau)(A-\lambda)}f(\tau)d\tau,$$
$$(\mathbb{H}_{m,\lambda} f)(t) = (w - A)^{-m}\int_0^t e^{(t-\tau)(A-\lambda)}f(\tau)d\tau, \quad t \geq 0.$$

Let $\mathbb{G} = \mathbb{G}_0$ and $\mathbb{H}_m = \mathbb{H}_{m,0}$. Note that $\mathbb{G}_\lambda \in \mathcal{L}(L^p(\mathbb{R}_+; X_m); L^p(\mathbb{R}_+; X))$ if and only if $\mathbb{H}_{m,\lambda} \in \mathcal{L}(L^p(\mathbb{R}_+; X))$.

For $m = 0$, by a classical Datko–Pazy Theorem, $\omega_0(A) < 0$ if and only if the orbit $\{e^{tA}x : t \in \mathbb{R}_+\}$ belongs to $L^p(\mathbb{R}_+; X)$, $1 \leq p < \infty$, for each $x \in X$, see, e. g., [16, Thm.4.41]. In turn, by the Datko-van Neerven Theorem this happens if and only if $\mathbb{G} \in \mathcal{L}(L^p(\mathbb{R}_+; X))$, see [8, 20] or [3, Thm.1.44] and [12] for shorter proofs. A simple analytical continuation argument in Lemma 3.2 below shows that $\mathbb{G} \in \mathcal{L}(L^p(\mathbb{R}_+; X))$ if and only if $s_0(A) < 0$ and $f \mapsto \mathcal{F}^{-1}\xi_0(\cdot)\mathcal{F}f$ is a bounded operator on $L^p(\mathbb{R}; X)$. Note that for $p = 2$ and Hilbert space X this implies $s_0(A) = \omega_0(A)$.

For $m = 1, 2, \dots$ the situation is quite different. For instance, Example 3.1 below shows that condition $\mathbb{G} \in \mathcal{L}(L^p(\mathbb{R}_+; X_m); L^p(\mathbb{R}_+; X))$, generally, does not imply the strict inequality $\omega_m(A) < 0$. Nevertheless, in Theorem 3.2 we derive a formula for $\omega_m(A)$ in terms of $s_m(A)$ and Fourier multiplier properties of the function $\xi_m(\cdot)$.

LEMMA 3.1. *If* $\mathbb{G} \in \mathcal{L}(L^p(\mathbb{R}_+; X_m); L^p(\mathbb{R}_+; X))$, *then* $\omega_m(A) \le 0$.

Proof. Fix $x \in X_m$ and let $f_x(t) = e^{tA}x$ for $t \in [0,1]$ and $f_x(t) = 0$ for $t > 1$. Then $f_x \in L^p(\mathbb{R}_+; X_m)$ and for $t \ge 1$ we have:

$$(\mathbb{G}f_x)(t) = \int_0^1 e^{(t-\tau)A} e^{\tau A} x \, d\tau = e^{tA}x.$$

Thus the orbit $t \mapsto e^{tA}x$ belongs to $L^p(\mathbb{R}_+; X)$ and by [20, Lemma 3.1.9] this implies $\lim_{t\to\infty} \|e^{tA}x\| = 0$. Hence

$$\omega(x) := \inf\{\omega \in \mathbb{R} : \sup_{t \ge 0} \|e^{-\omega t} e^{tA} x\| < \infty\} \le 0.$$

Since $\omega_m(A) = \sup\{\omega(x) : x \in X_m\}$, we obtain $\omega_m(A) \le 0$. ▮

THEOREM 3.1. *The following assertions are equivalent:*

(i) $\omega_m(A) < 0$*;*
(ii) there exists $\epsilon > 0$ *such that* $\mathbb{G}_{-\epsilon} \in \mathcal{L}(L^p(\mathbb{R}_+; X_m); L^p(\mathbb{R}_+; X))$.

Proof. **(i)⇒(ii).** We prove that $\omega_m(A) < 0$ implies $\mathbb{G} \in \mathcal{L}(L^p(\mathbb{R}_+; X_m); L^p(\mathbb{R}_+; X))$. Using this for $A + \epsilon$ instead of A, we have that (i) implies (ii) for all $\epsilon \in (0, -\omega_m(A))$.

Assume $\omega_m(A) < 0$ and fix $\omega \in (\omega_m(A), 0)$. If $f \in L^p(\mathbb{R}_+; X_m)$ then let $\tilde{f}(t) = f(t)$ for $t \ge 0$ and $\tilde{f}(t) = 0$ for $t < 0$. By the definition of $\omega_m(A)$, we have for each $t \ge 0$:

$$\|(\mathbb{G}f)(t)\| \le \int_0^t \|e^{\tau A} f(t-\tau)\| ds \le M(\omega) \int_0^\infty e^{\omega\tau} \|\tilde{f}(t-\tau)\|_m d\tau.$$

Using Minkowski's inequality,

$$\begin{aligned}
\|\mathbb{G}f\|_{L^p(\mathbb{R}_+;X)} &= \left[\int_0^\infty \|(\mathbb{G}f)(t)\|^p dt\right]^{\frac{1}{p}} \\
&\le M(\omega) \left[\int_0^\infty \left(\int_0^\infty e^{\omega\tau} \|\tilde{f}(t-\tau)\|_m d\tau\right)^p dt\right]^{\frac{1}{p}} \\
&= M(\omega) \left\| \int_0^\infty e^{\omega\tau} \|\tilde{f}(\cdot - \tau)\|_m d\tau \right\|_{L^p(\mathbb{R}_+)} \\
&\le M(\omega) \int_0^\infty \left\| e^{\omega\tau} \|\tilde{f}(\cdot - \tau)\|_m \right\|_{L^p(\mathbb{R}_+)} d\tau \\
&= M(\omega) \int_0^\infty e^{\omega\tau} d\tau \|f\|_{L^p(\mathbb{R}_+;X_m)} = \frac{M(\omega)}{|\omega|} \|f\|_{L^p(\mathbb{R}_+;X_m)}.
\end{aligned}$$

(ii)⇒(i). This follows from Lemma 3.1 applied to $A+\epsilon$ instead of A. ▮

The following example shows that, unlike the case of $m=0$ in Datko-van Neerven Theorem [20, Thm.3.3.1], $\epsilon>0$ in condition (ii) of Theorem 3.1 cannot be dropped for $m=1$.

Example 3.1. On $X=\ell^1(\mathbb{Z}_+)$ we define the generator $A:(x_n)_{n\in\mathbb{Z}_+}\mapsto((in-n^{-1})x_n)_{n\in\mathbb{Z}_+}$, with $D(A)=\{(x_n)\in\ell^1(\mathbb{Z}_+):(nx_n)\in\ell^1(\mathbb{Z}_+)\}$, of the strongly continuous semigroup given by $e^{tA}:(x_n)\mapsto(e^{int}e^{-n^{-1}t}x_n)$. If $\overline{x}=(\delta_{kn})_{n\in\mathbb{Z}_+}$, the standard ort, then $e^{tA}\overline{x}=(e^{ikt}e^{-k^{-1}t}\delta_{kn})_{n\in\mathbb{Z}_+}$ and $\omega(\overline{x})=-k^{-1}$. Therefore, $\omega_1(A)=\sup\{\omega(x):x\in D(A)\}\geq 0$. To see that $\omega_1(A)\leq 0$, we check that the orbit $t\mapsto e^{tA}\overline{x}$ belongs to $L^1(\mathbb{R}_+;X)$ for each $\overline{x}\in D(A)$. Indeed, if $\overline{x}=(x_n)_{n\in\mathbb{Z}_+}\in D(A)$ then

$$\int_0^\infty \|e^{tA}\overline{x}\|_{\ell^1(\mathbb{Z}_+)}dt=\sum_{n\geq 0}\int_0^\infty e^{-n^{-1}t}|x_n|dt=\sum_{n\geq 0}n|x_n|<\infty.$$

Thus, we have $\omega_1(A)=0$.

However, the operator $\mathbb{H}_1=A^{-1}\mathbb{G}$ is bounded on $L^1(\mathbb{R}_+;X)$. Indeed, for $f(\cdot)=(f(\cdot,n))_{n\in\mathbb{Z}_+}\in L^1(\mathbb{R}_+;\ell^1(\mathbb{Z}_+))$ we have:

$$(\mathbb{H}_1 f)(t,n)=\int_0^t \frac{n}{in^2-1}e^{in\tau}e^{-n^{-1}\tau}f(t-\tau,n)d\tau,\quad n\in\mathbb{Z}_+,\quad t\in\mathbb{R}_+.$$

Therefore,

$$\begin{aligned}
\|\mathbb{H}_1 f\|_{L^1(\mathbb{R}_+;\ell^1(\mathbb{Z}_+))} &= \int_0^\infty \sum_{n\geq 0}|(\mathbb{H}_1 f)(t,n)|dt \\
&\leq \sum_{n\geq 0}\int_0^\infty\int_0^t \left|\frac{n}{in^2-1}\right| e^{-n^{-1}\tau}|f(t-\tau,n)|d\tau dt \\
&= \sum_{n\geq 0}\left|\frac{n}{in^2-1}\right|\int_0^\infty e^{-n^{-1}\tau}\int_\tau^\infty |f(t-\tau,n)|dtd\tau \\
&= \sum_{n\geq 0}\left|\frac{n}{in^2-1}\right|\int_0^\infty e^{-n^{-1}\tau}d\tau\|f(\cdot,n)\|_{L^1(\mathbb{R}_+)} \\
&\leq \sum_{n\geq 0}\|f(\cdot,n)\|_{L^1(\mathbb{R}_+)}=\|f\|_{L^1(\mathbb{R}_+;\ell^1(\mathbb{Z}_+))}.
\end{aligned}$$

Thus, $\mathbb{G}\in\mathcal{L}(L^p(\mathbb{R}_+;X_1);L^p(\mathbb{R}_+;X))$. ◇

Remark 3. 1. By a theorem of J. van Neerven [20, Thm.4.6.4], if $\{e^{tA}\}_{t\geq 0}$ is uniformly bounded and $\omega_1(A) \geq 0$, then there exists $\varepsilon > 0$ such that for each $t > 0$ there is an $x \in X$ with $\|x\|_X = 1$ so that

$$\text{meas}\left\{\tau \in \mathbb{R}_+ : \|e^{\tau A}x\| \geq \varepsilon\right\} \geq t. \tag{11}$$

An open question in [20] was whether this theorem could be improved to state that x in the assertion above could be taken in $D(A)$ with $\|x\|_{D(A)} = 1$. As is was noticed by J. van Neerven, our Example 3.1 shows that the answer to this question is negative. Indeed, suppose that $\mathbb{G} \in \mathcal{L}(L^p(\mathbb{R}_+; X_1); L^p(\mathbb{R}_+; X))$. A calculation shows that the $L^p(\mathbb{R}_+; X)$–norms of the orbits $e^{(\cdot)A}x$ with $x \in D(A)$ and $\|x\|_{D(A)} = 1$ are uniformly bounded. This contradicts the statement that for some $\varepsilon > 0$ and each $t > 0$ there is an $x \in D(A)$ with $\|x\|_{D(A)} = 1$ so that (11) holds. $\diamond$

Next, we will characterize $\omega_m(A)$ in terms of Fourier multiplier properties of ξ_m. To recall the definition of the operator-valued Fourier multiplier for $L^p(\mathbb{R}; X)$, see, e.g., [1, 11], we let $\mathcal{S}(\mathbb{R})$ denote the Schwartz class of rapidly decaying smooth scalar functions, and let $\mathcal{S}(\mathbb{R}; X)$ denote the Schwartz class of X-valued functions. If $x \in X$ and $\varphi \in \mathcal{S}(\mathbb{R})$, we let $x \otimes \varphi \in \mathcal{S}(\mathbb{R}; X)$ denote the function $(x \otimes \varphi)(t) = \varphi(t)x$, $t \in \mathbb{R}$. Note that the lineal $\mathcal{D}_{\mathcal{S}} = \text{lin}\,\{x \otimes \varphi : x \in X, \varphi \in \mathcal{S}(\mathbb{R})\}$ is dense in $L^p(\mathbb{R}; X)$. Let $\mathcal{F}$ and $\mathcal{F}^{-1}$ denote the Fourier transform and its inverse:

$$(\mathcal{F}f)(s) = \int_{-\infty}^{\infty} e^{-ist} f(t)\, dt, \quad (\mathcal{F}^{-1}f)(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ist} f(s)\, ds.$$

Definition 3.1. A function $\xi : \mathbb{R} \to \mathcal{L}(X)$ is called a Fourier multiplier for $L^p(\mathbb{R}; X)$ if the following assertions hold:

(a) $\xi \in L^\infty(\mathbb{R}; \mathcal{L}(X))$;
(b) there exists a constant $c > 0$ such that

$$\|\mathcal{F}^{-1}\xi(\cdot)\mathcal{F}f\|_{L^p(\mathbb{R};X)} \leq c\|f\|_{L^p(\mathbb{R};X)} \quad \text{for all } f \in \mathcal{D}_{\mathcal{S}}.$$

Note that if (a) holds and $f = x \otimes \varphi \in \mathcal{D}_{\mathcal{S}}$ then $\mathcal{F}f \in \mathcal{D}_{\mathcal{S}}$ and $\xi(\cdot)\mathcal{F}f(\cdot) \in L^1(\mathbb{R}; X)$. Therefore, $\mathcal{F}^{-1}\xi(\cdot)\mathcal{F}f \in C_0(\mathbb{R}; X)$. Define $M_\xi f := \mathcal{F}^{-1}\xi(\cdot)\mathcal{F}f$. If (b) holds, then $M_\xi f \in L^p(\mathbb{R}; X)$ and, moreover, $M_\xi : \mathcal{D}_{\mathcal{S}} \to L^p(\mathbb{R}; X)$ can be extended to a bounded operator on $L^p(\mathbb{R}; X)$. Thus, equivalently to Definition 3.1, a function $\xi \in L^\infty(\mathbb{R}; \mathcal{L}(X))$ is an Fourier multiplier for $L^p(\mathbb{R}; X)$ if and only if M_ξ maps $\mathcal{D}_{\mathcal{S}}$ to $L^p(\mathbb{R}; X)$ and can be extended from $\mathcal{D}_{\mathcal{S}}$ to $L^p(\mathbb{R}; X)$ as a bounded operator.

LEMMA 3.2. *Let $m \in \{0,1,2,\ldots\}$ and assume that*

$$\sup_{Re\ \lambda \geq 0} \|(w-A)^{-m} R(\lambda, A)\| < \infty. \tag{12}$$

The following assertions are equivalent:

(i) $\mathbb{G} \in \mathcal{L}(L^p(\mathbb{R}_+; X_m); L^p(\mathbb{R}_+; X))$;
(ii) ξ_m is a Fourier multiplier on $L^p(\mathbb{R}; X)$.

Moreover, if either (i) or (ii) holds, then $\mathbb{H}_m = M_{\xi_m}$.

Proof. Fix $x \in X$ and $\varphi \in \mathcal{S}(\mathbb{R})$. If Re $\lambda > \max\{\omega_0(A), 0\}$, then $\mathbb{G}_\lambda \in \mathcal{L}(L^p(\mathbb{R}_+; X_m); L^p(\mathbb{R}_+; X))$ and hence $\mathbb{H}_{m,\lambda} \in \mathcal{L}(L^p(\mathbb{R}_+; X))$. Moreover, an easy application of Fubini's theorem yields $\mathcal{F}\mathbb{H}_{m,\lambda}\varphi \otimes x = \xi_{m,\lambda}(\cdot)\mathcal{F}\varphi \otimes x$, and hence

$$\mathbb{H}_{m,\lambda}\varphi \otimes x = \mathcal{F}^{-1}\xi_{m,\lambda}(\cdot)\mathcal{F}\varphi \otimes x \text{ for Re } \lambda > \max\{\omega_0(A), 0\}. \tag{13}$$

Now fix $t \in \mathbb{R}$. For Re $\lambda \geq 0$ set

$$\begin{aligned} L(\lambda) &= \begin{cases} (w-A)^{-m} \int_0^t e^{-\lambda(t-\tau)} e^{(t-\tau)A} (\varphi \otimes x)(\tau) d\tau & \text{if } t \geq 0, \\ 0 & \text{if } t < 0, \end{cases} \\ &= (H_{m,\lambda}\varphi \otimes x)(t) \end{aligned}$$

and

$$R(\lambda) = (2\pi)^{-1} \int_{-\infty}^{\infty} e^{ist} \xi_{m,\lambda}(s) x (\mathcal{F}\varphi)(s) ds = (\mathcal{F}^{-1}\xi_{m,\lambda}(\cdot)\mathcal{F}\varphi \otimes x)(t).$$

Due to the assumption, the functions $L(\cdot)$ and $R(\cdot)$ are continuous on $\{\text{Re } \lambda \geq 0\}$ and an easy application of Morera's theorem shows that they are analytic on $\{\text{Re } \lambda > 0\}$. Hence, by (13) and the analyticity of L and R we have $L(0) = R(0)$, i.e.

$$(\mathbb{H}_m \varphi \otimes x)(t) = (\mathcal{F}^{-1}\xi_m \mathcal{F}\varphi \otimes x)(t).$$

From this and a density argument it follows easily that $\mathbb{H}_m \in \mathcal{L}(L^p(\mathbb{R}_+; X))$ if and only if ξ_m is a Fourier multiplier for $L^p(\mathbb{R}; X)$, and that $\mathbb{H}_m = M_{\xi_m}$. ∎

PROPOSITION 3.1. *If $\mathbb{G} \in \mathcal{L}(L^p(\mathbb{R}_+; X_m); L^p(\mathbb{R}_+; X))$, $m \in \{0,1,2,\ldots\}$, then the following holds:*

(a) $\sigma(A) \subseteq \{Re\ \lambda < 0\}$;

(b) $\sup_{s\in\mathbb{R}} \|(w-A)^{-m}(is-A)^{-1}\| < \infty$.

Proof. **(a).** Assume that the assertion is not true. Then there exists $\lambda \in \{\mathrm{Re}\ \lambda \geq 0\}$ belonging to the approximate point spectrum of A. Choose (x_n) in $D(A)$ such that $\|x_n\| = 1$ and $\lim_n (\lambda - A)x_n = 0$. Then

$$\lim_n \Big((w-\lambda)^{-m} - (w-A)^{-m}\Big)x_n = 0. \tag{14}$$

Moreover,

$$\lim_n \Big[e^{t(A-\lambda)}x_n - x_n\Big] = \lim_n \int_0^t e^{s(A-\lambda)}(A-\lambda)x_n ds = 0$$

uniformly for t in compact subsets in $\mathbb{R}_+$. By passing to a subsequence we may assume that

$$\lim_n \sup_{t\in[0,n]} \|(e^{tA} - e^{\lambda t})x_n\| = 0. \tag{15}$$

By our assumption $\mathbb{H}_m = \mathbb{G}(w-A)^{-m} \in \mathcal{L}(L^p(\mathbb{R}_+;X))$. Set $f_n = \chi_{[0,1]}e^{\lambda\cdot}x_n$, $n \in \mathbb{N}$. Note that $\sup_n \|f_n\|_p < \infty$ and for $t \geq 1$ we have

$$\begin{aligned}
(\mathbb{H}_m f_n)(t) &= (w-A)^{-m}\int_0^t e^{(t-s)A}f_n(s)ds \\
&= (w-A)^{-m}\int_0^1 e^{\lambda s}e^{(t-s)A}x_n ds \\
&= (w-A)^{-m}\int_0^1 e^{\lambda s}(e^{(t-s)A} - e^{\lambda(t-s)})x_n ds + e^{\lambda t}(w-A)^{-m}x_n.
\end{aligned}$$

By (15) we obtain $\lim\limits_n \sup\limits_{t\in[1,n]} \|(\mathbb{H}_m f_n)(t) - e^{\lambda t}(w-A)^{-m}x_n\| = 0$. Thus, by (14) we have

$$\begin{aligned}
\overline{\lim_n}\|\mathbb{H}_m f_n\|_p &\geq \overline{\lim_n}\left(\int_1^r \|e^{\lambda s}(w-A)^{-m}x_n\|^p ds\right)^{1/p} \\
&\geq \overline{\lim_n}\left(\int_1^r \left\|(w-\lambda)^{-m}e^{\lambda s}x_n\right\|^p ds\right)^{\frac{1}{p}} \\
&= \overline{\lim_n}\|x_n\|\left(\int_1^r |w-\lambda|^{-mp}e^{sp\mathrm{Re}\ \lambda}ds\right)^{\frac{1}{p}}
\end{aligned}$$

for $r \geq 1$. Since $\mathrm{Re}\ \lambda \geq 0$ this implies $\overline{\lim}_n\|\mathbb{H}_m f_n\|_p = \infty$ which is a contradiction.

(b). Assume that the assertion does not hold. There exist $s_n \in \mathbb{R}$ and $x_n \in D(A^{m+1})$, $n \in \mathbb{N}$, such that

$$\|x_n\| = 1 \quad \text{and} \quad \lim_n \|(w-A)^m(is_n - A)x_n\| = 0. \tag{16}$$

If $\varphi \in C_c^1(0,\infty)$, $x \in D(A)$, then integration by parts shows that

$$\mathbb{G}(\varphi' \otimes x - \varphi \otimes Ax) = \varphi \otimes x. \tag{17}$$

Now choose $0 \neq \psi \in C_c^1(0,\infty)$. Set $a_n = \max\{n\|(w-A)x_n\|, n\}$ and $\varphi_n(t) = e^{is_n t}\psi\left(\frac{t}{a_n}\right)$, $n \in \mathbb{N}$, $t \geq 0$. From (17) we obtain

$$\mathbb{G}((\varphi_n' - is_n\varphi_n) \otimes x_n - \varphi_n \otimes (is_n - A)x_n) = \varphi_n \otimes x_n.$$

Since $\mathbb{H}_m = \mathbb{G}(w-A)^{-m} \in \mathcal{L}(L^p(\mathbb{R}_+;X))$ we obtain

$$\begin{aligned}
\|\psi\|_{L^p(\mathbb{R}_+)} a_n^{\frac{1}{p}} &= \|\varphi_n \otimes x_n\|_{L^p(\mathbb{R}_+;X)} \\
&\leq \|\mathbb{H}_m\|(\|(\varphi_n' - is_n\varphi_n) \otimes (w-A)^m x_n\|_{L^p(\mathbb{R}_+;X)} \\
&\quad + \|\varphi_n \otimes (w-A)^m(is_n - A)x_n\|_{L^p(\mathbb{R}_+;X)}) \\
&= \|\mathbb{H}_m\|(\|\psi'\|_{L^p(\mathbb{R}_+)}(a_n)^{-1+\frac{1}{p}}\|(w-A)^m x_n\| \\
&\quad + \|\psi\|_{L^p(\mathbb{R}_+)}(a_n)^{\frac{1}{p}}\|(w-A)^m(is_n - A)x_n\|).
\end{aligned}$$

By the choice of a_n this implies

$$0 < \|\psi\|_{L^p(\mathbb{R}_+)} \leq \|\mathbb{H}_m\|(\|\psi'\|_{L^p(\mathbb{R}_+)}n^{-1} + \|\psi\|_{L^p(\mathbb{R}_+)}\|(w-A)^m(is_n - A)x_n\|).$$

Due to (16), the right hand side of the above inequality tends to 0 as $n \to \infty$ which is a contradiction. ∎

Remark 3. 2. If $m \in \{0,1,2,\ldots\}$ and $\mathbb{G} \in \mathcal{L}(L^p(\mathbb{R}_+;X_m);L^p(\mathbb{R}_+;X))$, then $s(A) \leq 0$, where $s(A) = \sup\{\text{Re } \lambda : \lambda \in \sigma(A)\}$ is the spectral bound for A. Unless $m = 0$, we cannot conclude that $s(A) < 0$ (see Example 3.1). Since $s_m(A) \leq \omega_m(A)$, from Lemma 3.1 we have that $s_m(A) \leq 0$ provided $\mathbb{G} \in \mathcal{L}(L^p(\mathbb{R}_+;X_m);L^p(\mathbb{R}_+;X))$. By the Datko-van Neerven Theorem [20, Thm.3.3.1] for $m = 0$ we have that $\mathbb{G} \in \mathcal{L}(L^p(\mathbb{R}_+;X))$ implies $s_0(A)(\leq \omega_0(A)) < 0$. ◇

For $m = 0$, using the resolvent identity, we have that if $s_0(A) \leq 0$ and $\sup_{s\in\mathbb{R}}\|(is-A)^{-1}\| < \infty$, then $s_0(A) < 0$. Thus, the resolvent $R(\cdot, A)$ satisfies assumption (12). Recall that if ξ_0 is a Fourier multiplier, then $\sup_{s\in\mathbb{R}}\|(is-A)^{-1}\| < \infty$.

COROLLARY 3.1. *$\omega_0(A) < 0$ if and only if $\sigma(A) \subset \{Re\ \lambda < 0\}$, $s_0(A) \leq 0$, and ξ_0 is a Fourier multiplier for $L^p(\mathbb{R}; X)$, $1 \leq p < \infty$.*

THEOREM 3.2. *If $m \in \{0, 1, 2, \dots\}$, then*

$$\omega_m(A) = \inf\{\omega > s_m(A) : \xi_{m,\omega} \text{ is a Fourier multiplier for } L^p(\mathbb{R}; X)\}. \tag{18}$$

Proof. Denote the right-hand side of (18) by ω_ξ and set $\omega_{\mathbb{G}} := \inf\{\omega \in \mathbb{R} : \mathbb{G}_\omega \in \mathcal{L}(L^p(\mathbb{R}_+; X_m); L^p(\mathbb{R}_+; X))\}$. We prove the following three inequalities.

$\boldsymbol{\omega_m(A) \leq \omega_{\mathbb{G}}}$. If $\omega > \omega_{\mathbb{G}}$ then $\mathbb{G}_\omega \in \mathcal{L}(L^p(\mathbb{R}_+; X_m); L^p(\mathbb{R}_+; X))$ and, by Lemma 3.1 for $A - \omega$ we have $\omega_m(A - \omega) = \omega_m(A) - \omega \leq 0$.

$\boldsymbol{\omega_{\mathbb{G}} \leq \omega_\xi}$. If $\omega > \omega_\xi$ then $s_m(A - \omega) < 0$. Using (10), we have:

$$\sup_{\text{Re } \lambda \geq 0} \|(w - A)^{-m} R(\lambda, A - \omega)\| < \infty. \tag{19}$$

Apply the implication (ii)$\Rightarrow$(i) of Lemma 3.2 to $A - \omega$ instead of A. Now $\mathbb{G}_\omega \in \mathcal{L}(L^p(\mathbb{R}_+; X_m); L^p(\mathbb{R}_+; X))$ since $\xi_{m,\omega}$ is a Fourier multiplier for $L^p(R; X)$. Thus, $\omega \geq \omega_{\mathbb{G}}$.

$\boldsymbol{\omega_\xi \leq \omega_m(A)}$. If $\omega > \omega_m(A)$, then $s_m(A - \omega) < 0$ and (19) holds. Also, $\mathbb{G}_\omega \in \mathcal{L}(L^p(\mathbb{R}_+; X_m), L^p(\mathbb{R}_+; X))$ by the proof of the implication (i)$\Rightarrow$(ii) in Theorem 3.1. Apply the implication (i)$\Rightarrow$(ii) of Lemma 3.2 to $A - \omega$ instead of A. Then $\xi_{m,\omega}$ is a Fourier multiplier for $L^p(\mathbb{R}; X)$, and $\omega \geq \omega_\xi$. ▌

Remark 3. 3. If X is a Hilbert space, then Plancherel's Theorem and Theorem 3.2 imply the well known results $\omega_0(A) = s_0(A)$ by Gearhart-Prüss and $\omega_m(A) = s_m(A)$ by G. Weiss (see [3, 10, 20, 24] and the literature therein). ◇

Remark 3. 4. For $\lambda \in \mathbb{C}$ and $m \in \{0, 1, 2, \dots\}$ we denote $\eta_{m,\lambda}(s) = (w - is)^{-m}(is + \lambda - A)^{-1}$ and abbreviate $\eta_m(s) = \eta_{m,0}(s)$, $s \in \mathbb{R}$. We claim that $\xi_{m,\omega}$ in Lemma 3.2 and Theorem 3.2 can be replaced by $\eta_{m,\omega}$. To prove the claim, it suffices to see that ξ_m and η_m are Fourier multipliers for $L^p(\mathbb{R}; X)$ only simultaneously. Passing to the functions $x \otimes \varphi$, $x \in X$, $\varphi \in \mathcal{S}(\mathbb{R})$, we have that the functions $s \mapsto (is - w)^{-(k+1)} R^{m-k}(w, A), k = 0, \dots, m - 1$, are Fourier multipliers on $L^p(\mathbb{R}; X)$. Using (9), the claim follows. ◇

There are many sufficient conditions that ensure that a function $\xi : \mathbb{R} \to \mathcal{L}(X)$ is a Fourier multiplier for $L^p(\mathbb{R}; X)$ (cf. Remark 4.1 below.)

Here we recall a convenient result called "Mikhlin+ε" Theorem (see [1, Cor.4.4] or [11, Prop.6.4]): Assume that $\xi \in W_2^\infty(\mathbb{R}; \mathcal{L}(X))$, the Sobolev space. If there exists an $\varepsilon > 0$ such that for the derivatives $\xi^{(n)}$ we have $\sup_{s\in\mathbb{R}}(1+|s|^{n+\varepsilon})\|\xi^{(n)}(s)\|_{\mathcal{L}(X)} < \infty$ for $n = 0, 1, 2$, then ξ is a Fourier multiplier for $L^p(\mathbb{R}; X)$. This result, Theorem 3.2, and Remark 3.4 imply the following.

COROLLARY 3.2. *Assume $s_0(A) < 0$. If there exists an $\varepsilon < 0$ such that*

$$\sup_{s\in\mathbb{R}}(1+|s|^{n+\varepsilon})\|(is-A)^{-(n+1)}\|_{\mathcal{L}(X)} < \infty \text{ for } n = 0, 1, 2,$$

then $\omega_0(A) < 0$. Assume $s_1(A) < 0$. If there exist $\epsilon > 0$ and $\varepsilon > 0$ such that

$$\sup_{s\in\mathbb{R}}(1+|s|^{n+\varepsilon})(1+|s|)^{-1}(is-\epsilon-A)^{-(n+1)}\|_{\mathcal{L}(X)} < \infty \text{ for } n = 0, 1, 2,$$

then $\omega_1(A) < 0$.

4. CONTROL THEORY AND FOURIER MULTIPLIERS

Let X, Y and U be Banach spaces, A be the generator of a strongly continuous semigroup on X, and assume that $B \in \mathcal{L}(U; X)$ and $C \in \mathcal{L}(X; Y)$. Consider the following control system (see [7] and the literature therein):

$$x(t) = e^{tA}x_0 + \int_0^t e^{(t-\tau)A}Bu(\tau)d\tau, \quad y(t) = Cx(t), \quad t \geq 0. \tag{20}$$

Let $H(z) = C(z-A)^{-1}B$, $z \in \rho(A)$, denote its transfer function, and let $\mathbb{L}$ denote the input-output operator, defined on $L^p_{\text{loc}}(\mathbb{R}_+; U)$ by the formula

$$(\mathbb{L}u)(t) = C\int_0^t e^{(t-\tau)A}Bu(\tau)d\tau, \quad t \geq 0.$$

Recall that the control system (20) is called *p-input-output stable* if $\mathbb{L}$ is a bounded operator from $L^p(\mathbb{R}_+; U)$ to $L^p(\mathbb{R}; Y)$, $1 \leq p < \infty$.

If $i\mathbb{R} \subset \rho(A)$, we denote $\zeta(s) = H(is) = C(is-A)^{-1}B$, $s \in \mathbb{R}$.

THEOREM 4.1. *Assume that $\sigma(A) \subset \{Re\ \lambda < 0\}$ and the transfer function H and its derivative H' belong to the Hardy class $H^\infty = H^\infty(\{Re\ \lambda > 0\}; \mathcal{L}(U; Y)))$. Then the following assertions are equivalent*

(i) Control system (20) is p-input-output stable;

(ii) ζ *is a Fourier multiplier from* $L^p(\mathbb{R};U)$ *to* $L^p(\mathbb{R};Y)$,

and $\|M_\zeta\|_{\mathcal{L}(L^p(\mathbb{R};U);L^p(\mathbb{R};Y))} = \|\mathbb{L}\|_{\mathcal{L}(L^p(\mathbb{R}_+;U);L^p(\mathbb{R}_+;Y))}$.

Proof. For $\lambda \in \mathbb{C}$ and $A_\lambda = A - \lambda$ we let $\zeta_\lambda(s) = C(is - A_\lambda)^{-1}B$, $s \in \mathbb{R}$, and let $\mathbb{L}_\lambda$ denote the input-output operator for A_λ. Fix $u_0 \in U$, $\varphi \in S(\mathbb{R})$ with supp $\varphi \subset (0,\infty)$, and define $u = u_0 \otimes \varphi$.

First, we claim that

$$(\mathcal{F}\mathbb{L}_\lambda u)(s) = \zeta_\lambda(s)(\mathcal{F}u)(s), \quad s \in \mathbb{R}, \quad \text{for Re } \lambda > \omega_0(A). \tag{21}$$

Indeed, since Re $\lambda > \omega_0(A)$ and B and C are bounded operators, for each $s \in \mathbb{R}$ and $u_0 \in U$ we have:

$$C(is - A_\lambda)^{-1}Bu_0 = C\int_0^\infty e^{(A_\lambda - is)\sigma}Bu_0 d\sigma.$$

Moreover, due to Re $\lambda > \omega_0(A)$ and $\varphi \in S(\mathbb{R})$, the integrals below converge absolutely and Fubini's theorem gives:

$$\begin{aligned}\int_0^\infty \zeta_\lambda(s)u_0\varphi(\tau)e^{-is\tau}d\tau &= C\int_0^\infty \int_0^\infty e^{(A_\lambda - is)\sigma}Bu_0\varphi(\tau)e^{-is\tau}d\sigma d\tau \\ &= C\int_0^\infty \int_0^t e^{(A_\lambda - is)(t-\tau)}Bu_0\varphi(\tau)e^{-is\tau}d\tau dt \\ &= \int_0^\infty e^{-ist}(\mathbb{L}_\lambda u)(t)dt.\end{aligned}$$

Thus, claim (21) is proved.

Second, recall that $\zeta_\lambda(s) = H(\lambda + is)$, $s \in \mathbb{R}$, Re $\lambda \geq 0$. Since $H(\cdot) \in H^\infty$, we have that $\zeta_\lambda(\cdot) = H(\lambda + i(\cdot)) \in L^\infty(\mathbb{R};\mathcal{L}(U;Y))$ for Re $\lambda \geq 0$. Since $\mathcal{F}u \in \mathcal{S}(\mathbb{R};U)$, we have $\zeta_\lambda(\cdot)(\mathcal{F}u)(\cdot) \in L^1(\mathbb{R};Y)$ and $\mathcal{F}^{-1}\zeta_\lambda\mathcal{F}u \in C_0(\mathbb{R};X)$ for Re $\lambda \geq 0$. Now (21) implies

$$(\mathbb{L}_\lambda u)(t) = (\mathcal{F}^{-1}\zeta_\lambda(\cdot)\mathcal{F}u)(t), \quad t \in \mathbb{R}, \quad \text{for Re } \lambda > \omega_0(A). \tag{22}$$

Third, we claim that for each $t \in \mathbb{R}$ the left- and right-hand sides of (22) define functions, analytic in $\{\text{Re } \lambda > 0\}$. Indeed, if

$$L(\lambda) := (\mathbb{L}_\lambda u)(t) = e^{-\lambda t}C\int_0^t e^{(t-\tau)A}Bu_0e^{\lambda\tau}\varphi(\tau)d\tau$$

then

$$\frac{d}{d\lambda}L(\lambda) = C\int_0^t (\tau - t)e^{-\lambda(t-\tau)}e^{A(t-\tau)}Bu_0\varphi(\tau)d\tau.$$

If $R(\lambda) := (\mathcal{F}^{-1}\zeta_\lambda(\cdot)u_0\mathcal{F}\varphi)(t)$ for Re $\lambda > 0$, then its derivative

$$\frac{dR}{d\lambda} = (\mathcal{F}^{-1}\zeta_\lambda'(\cdot)u_0\mathcal{F}\varphi)(t) = (\mathcal{F}^{-1}H'(\lambda + i(\cdot))u_0\mathcal{F}\varphi)(t). \tag{23}$$

By the assumption $H' \in H^\infty$ we conclude that the function $H'(\lambda + i(\cdot))$ belongs to $L^\infty(\mathbb{R}; \mathcal{L}(U;Y))$. Therefore, the improper integral in the inverse Fourier transform in the right-hand side of (23) converges, $\frac{dR}{d\lambda}$ exists, and $R(\cdot)$ is analytic in $\{\text{Re } \lambda > 0\}$.

Using (22), by analytic continuation we have:

$$(\mathbb{L}u)(t) = (\mathcal{F}^{-1}\zeta(\cdot)u_0\mathcal{F}\varphi)(t) = (M_\zeta u)(t), \quad t \in \mathbb{R}, \quad u = u_0 \otimes \varphi \in \mathcal{D}_\mathcal{S}.$$

From a density argument it follows easily that (i) and (ii) are equivalent, and the equality for the norms. ▮

Remark 4. 1. Recently, a significant progress has been made in finding sufficient conditions for operator valued functions to be L^p–Fourier multipliers, see [2, 5, 22] and literature therein. We formulate the following generalization of Bourgain's and Mikhlin's Theorems which is due to L. Weis [22]. Recall that a function $\zeta : \mathbb{R} \to \mathcal{L}(U, Y)$ is called *R–bounded* (randomized, or Rademacher, bounded) if there is a constant c such that for all $n \in \mathbb{N}$, $s_1, \ldots, s_n \in \mathbb{R} \setminus \{0\}$, and $u_1, \ldots, u_n \in U$ we have

$$\Big(\int_0^1 \Big\| \sum_{j=0}^n r_j(\tau)\zeta(s_j)u_j \Big\|_Y^p d\tau\Big)^{1/p} \le c\Big(\int_0^1 \Big\| \sum_{j=0}^n r_j(\tau)u_j \Big\|_U^p d\tau\Big)^{1/p},$$

where $r_j(\tau) = \text{sign } \sin(2^j\pi\tau)$, $\tau \in [0, 1]$, are the Rademacher functions, and $1 \le p < \infty$. Note that if this inequality holds for one p then it holds for all p. Also, if ζ is R–bounded, then it is bounded, and boundedness and R–boundedness are equivalent provided U and Y are Hilbert spaces. Assume that U and Y are UMD-spaces (for example, this holds for L^q– and many others function spaces). The result of L. Weis states that a differentiable function $\zeta : \mathbb{R} \setminus \{0\} \to \mathcal{L}(U; Y)$ is an L^p–Fourier multiplier provided $[s \mapsto \zeta(s)]$ and $[s \mapsto s\zeta'(s)]$ are R–bounded. By a direct application of this result in Theorem 4.1 for the transfer function $\zeta(s) = C(is - A)^{-1}B$ we have that the control system (20) is p–input-output stable provided $[s \mapsto \zeta(s)]$ and $[s \mapsto s\zeta'(s)]$ are R–bounded. Of course, if $X = Y = U$ is an UMD-space, and $B = C = I$, then we have that $\omega_0(A) < 0$ provided $[s \mapsto (is - A)^{-1}]$ and $s \mapsto s(is - A)^{-2}$ are R–bounded. ◇

Recall, that the control system (20) is called *externally stable* if $H(\cdot) \in H^\infty$. Recall that if ζ is an L^p-Fourier multiplier, then $\zeta \in L^\infty(\mathbb{R}; \mathcal{L}(U;Y))$

(see Definition 3.1). Also, $\|M_\zeta\|_{\mathcal{L}(L^p(\mathbb{R};\mathcal{L}(U;Y))} \geq \|\zeta\|_{L^\infty(\mathbb{R};\mathcal{L}(U;Y))}$. Therefore, Theorem 4.1 is related to a theorem of G. Weiss [25] stating that $\|\mathbb{L}\| \geq \|\zeta\|_\infty$. Using Plancherel's Theorem, one can also easily deduce from Theorem 4.1 the following classical result: if $p = 2$ and U and Y are Hilbert spaces, then input-output stability is equivalent to external stability.

Recall, that the system (20) is called *stabilizable* (respectively, *detectable*) if there is an operator $F \in \mathcal{L}(X;U)$ (respectively, $K \in \mathcal{L}(Y;X)$) such that $\omega_0(A + BF) < 0$ (respectively, $\omega_0(A + KC) < 0$). A well-known theorem due to R. Curtain [6] and R. Rebarber [19] says that if U and Y are Hilbert spaces then the control system (20) is *uniformly* exponentially stable (that is, $\omega_0(A) < 0$) if and only if it is stabilizable, detectable, and externally stable. As it was shown in [4], if U and Y are Banach spaces, then the external stability in this assertion must be replaced by input-output stability (that is, by the condition $\mathbb{L} \in \mathcal{L}(L^p(\mathbb{R}_+;U); L^p(\mathbb{R}_+;Y))$).

In the following result we relate the *exponential* stability ($\omega_1(A) < 0$) to input-output stability with respect to classical solutions; we define the latter as the boundedness of the operator $\mathbb{M}_A$ acting from $L^p(\mathbb{R}_+;U)$ to $L^p(\mathbb{R}_+;Y)$ by the rule

$$(\mathbb{M}_A u)(t) = C\int_0^t (w - A)^{-1} e^{(t-\tau)A} Bu(\tau)d\tau,\ t \geq 0,\ w > \max\{\omega_0(A), 0\}.$$

THEOREM 4.2. *Assume that control system* (20) *is stabilizable and detectable. The following assertions are equivalent.*

(a) $\omega_1(A) < 0$*;*
(b) there exists $\epsilon > 0$ *such that* $\mathbb{M}_{A_{-\epsilon}} \in \mathcal{L}(L^p(\mathbb{R}_+;U); L^p(\mathbb{R}_+;Y))$.

Proof. Consider the following operators:

$$(\mathbb{G}_{A+BF} f)(t) = \int_0^t e^{(t-\tau)(A+BF)} f(\tau)d\tau,$$
$$(\mathbb{G}_{A+KC} f)(t) = \int_0^t e^{(t-\tau)(A+KC)} f(\tau)d\tau.$$

Since $\omega_0(A + BF) < 0$ and $\omega_0(A + KC) < 0$ by assumptions, Datko-van Neerven Theorem implies that $\mathbb{G}_{A_{-\epsilon}+BF}$ and $\mathbb{G}_{A_{-\epsilon}+KC}$ belong to $\mathcal{L}(L^p(\mathbb{R}_+;X))$ for sufficiently small $\epsilon > 0$. Using variation of constant formulas for the semigroups $\{e^{t(A_{-\epsilon}+KC)}\}_{t\geq 0}$ and $\{e^{t(A_{-\epsilon}+BF)}\}_{t\geq 0}$, respectively, it is easy to obtain the following identities:

$$\mathbb{G}_{A_{-\epsilon}+KC} = \mathbb{G}_{-\epsilon} + \mathbb{G}_{A_{-\epsilon}+KC} KC \mathbb{G}_{-\epsilon}, \tag{24}$$

$$\mathbb{G}_{A_{-\epsilon}+BF} = \mathbb{G}_{-\epsilon} + \mathbb{G}_{-\epsilon} BF \mathbb{G}_{A_{-\epsilon}+BF}, \tag{25}$$

see [3, p.135] for more details. Note that $\mathbb{M}_{A_{-\epsilon}} = C\mathbb{H}_{1,-\epsilon}B$, where $\mathbb{H}_{1,-\epsilon} = (w-A)^{-1}\mathbb{G}_{-\epsilon}$. If (a) holds, then $\mathbb{H}_{1,-\epsilon} \in \mathcal{L}(L^p(\mathbb{R}_+;X))$ by Theorem 3.1. Since B and C are bounded operators, (b) holds. To see that (b) implies (a), we multiply (24) by $(w-A)^{-1}B$ and (25) by $(w-A)^{-1}$. Then

$$\mathbb{G}_{A_{-\epsilon}+KC}(w-A)^{-1}B = \mathbb{H}_{1,-\epsilon}B + \mathbb{G}_{A_{-\epsilon}+KC}K\mathbb{M}_{A_{-\epsilon}}, \tag{26}$$

$$(w-A)^{-1}\mathbb{G}_{A_{-\epsilon}+BF} = \mathbb{H}_{1,-\epsilon} + (\mathbb{H}_{1,-\epsilon}B)F\mathbb{G}_{A_{-\epsilon}+BF}. \tag{27}$$

If (b) holds, then $\mathbb{H}_{1,-\epsilon}B$ is a bounded operator due to (26). Indeed, $(w-A)^{-1}, B, K$ are bounded operators, $\mathbb{M}_{A_{-\epsilon}}$ is bounded by (b), and $\mathbb{G}_{A_{-\epsilon}+KC}$ is bounded due to the detectability assumption. As soon as we know that $\mathbb{H}_{1,-\epsilon}B$ is bounded, identity (27) shows that $\mathbb{H}_{1,-\epsilon}$ is bounded. Indeed, $(w-A)^{-1}$ and F are bounded, and $\mathbb{G}_{A_{-\epsilon}+BF}$ is bounded due to the stabilizability assumption. Now (a) holds by the implication (ii) $\Rightarrow$ (i) in Theorem 3.1. ▌

We remark that the condition $\mathbb{M}_A \in \mathcal{L}(L^p(\mathbb{R}_+;U);L^p(\mathbb{R}_+;Y))$ can be characterized in terms of the L^p–Fourier multiplier properties of the function $[s \mapsto (is-w)^{-1}C(is-A)^{-1}B]$, $s \in \mathbb{R}$, similarly to Theorem 3.2 and Remark 3.4.

Finally, we remark that the assumptions $B \in \mathcal{L}(U;X)$ and $C \in \mathcal{L}(X;Y)$ imposed in this section are rather restrictive. However, as it is recently shown in [18], the results of this section hold even if these assumptions are relaxed to allow that $B: U \to \overline{X}$ and $C: \underline{X} \to Y$ are bounded operators for a triple of Banach spaces $\underline{X} \hookrightarrow X \hookrightarrow \overline{X}$ such that certain conditions of "regularity" are satisfied to cover the Weiss-regular or Pritchard–Salamon classes.

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