Semilinear perturbations of Hille-Yosida operators

Aug 3, 2001

HORST R. THIEME [1] AND HAUKE VOSSELER [2]
Department of Mathematics
Arizona State University
Tempe, AZ 85287-1804, USA

Abstract

The semilinear Cauchy problem

$$u'(t) = Au(t) + G(u(t)), \qquad u(0) = x \in \overline{D(A)}, \tag{1}$$

with a Hille-Yosida operator A and a nonlinear operator $G : D(A) \to X$ is considered under the assumption that

$$\|G(x) - G(y)\| \leq \|B(x - y)\| \quad \forall x, y \in D(A)$$

with some linear $B : D(A) \to X$,

$$B(\lambda - A)^{-1}x = \lambda \int_0^\infty e^{-\lambda t} V(s) x ds,$$

where V is of suitable small strong bounded variation on some interval $[0, \varepsilon)$. We will prove the existence of a semiflow on $[0, \infty) \times \overline{D(A)}$ that provides Friedrichs solutions in L_1 for (1). If X is a Banach lattice, we replace the Lipschitz condition by

$$|G(x) - G(y)| \leq Bv \quad \text{whenever } x, y, v \in D(A), |x - y| \leq v,$$

with B being positive. We illustrate our result by an age-structured population model.

Key words and phrases. Hille-Yosida operators, semilinear perturbations, Cauchy problems, Friedrichs solutions, integral solutions, (integrated) operator semigroups, contraction mapping principle, cumulative outputs, Stieltjes convolutions, semiflows, dynamical systems, age-structured population dynamics

AMS Subject Classification. 34Gxx, 35F30, 47D06, 47D62, 47H20, 47J35, 58D25, 92D25

[1]Research partially supported by NSF Grant DMS-9706787
[2]Research partially supported by Studienstiftung des deutschen Volkes

1 Introduction

Let A be a Hille-Yosida operator in a Banach space X, i.e., after equivalent renormalization and an identity shift, an m-dissipative linear operator. Equivalently it is the generator of a locally Lipschitz continuous semigroup S. The part A_0 of A in $\overline{D(A)}$ generates a C_0-semigroup $\dot{S}$ which is the derivative of S. We will consider perturbations $A+G$ with nonlinear operators G that are Lipschitz continuous with respect to A in various ways which we will make precise later, and show that the solutions of the associated Cauchy problem induce a continuous semiflow (nonlinear semigroup) on $\overline{D(A)}$. This will generalize results that have obtained for the following three scenarios:

G satisfies a Lipschitz condition on $\overline{D(A)}$ [Thi90b, DvGLW95, MOT96],

A is densely defined and G a linear operator of Miyadera-Voigt type [Miy66, Voi77],

X is an abstract L space and G a positive linear operator on $D(A)$ [Voi89, Thi96].

Our approach does neither cover nor generalize the case that A is m-dissipative, $D(A)$ is dense, G is continuous, $D(G)=X$, and G continuous and dissipative [Web72].

For the sake of exposition let us assume that A is the generator of a C_0 semigroup T_0. The nonlinear nature of the perturbation does not allow the use of Dyson-Phillips series like in [Voi77] in constructing the semiflow Θ,

$$\Theta(t,x)=T_0(t)x+\int_0^t T_0(s)G(\Theta(t-s,x))ds.$$

Banach's fixed point theorem will be the basic tool of this paper, but it will be not be applied to the previous integral equation, but to

$$v(t)=G\left(T_0(t)x+\int_0^t T_0(s)v(t-s)ds\right), \tag{2}$$

and Θ is obtained via

$$\Theta(t,x)=T_0(t)x+\int_0^t T_0(s)v(t-s)ds.$$

Actually we do not solve (2), but something more complicated.
For $x\in\overline{D(A)}$, $u(t)=\Theta(t,x)$ will solve the Cauchy problem

$$u'=Au+G\circ u,\quad t>0,\qquad u(0)=x \tag{3}$$

in a generalized integral sense which we will explain later (Section 2) and in an L_1-*Friedrichs* sense:

(L_1-F) $u : [0, \infty) \to \overline{D(A)}$ is continuous, $u(0) = x$, and there exists a sequence of continuously differentiable functions (u_j) with values in $D(A)$ such that Au_j is continuous and

$$\left.\begin{array}{l} u_j \to u, \qquad t \to \infty, \text{ uniformly on } [0,T], \\ (u_j' - Au_j) \text{ and } (G \circ u_j) \text{ converge in } L_1([0,T], X), \\ \lim\limits_{j\to\infty} (u_j' - Au_j) = \lim\limits_{j\to\infty} G \circ u_j. \end{array}\right. \qquad \forall T > 0 \quad (4)$$

We are going to carry out this program in detail in two situations. In either scenario we will consider nonlinearities that are Lipschitz continuous relatively to A in a global sense. While this can be an unpleasant restriction in some applications, it makes the semiflow satisfy an exponential Lipschitz condition in the state variable:

$$\|\Theta(t,x) - \Theta(t,\tilde{x})\| \le N e^{\theta t} \|x - \tilde{x}\| \qquad \forall t \ge 0, x, \tilde{x} \in X, \tag{5}$$

with constants $\theta \in \mathbf{R}$, $N \ge 1$.
First, in the general case that X is a Banach space (Section 5), we consider a nonlinear operator $G : D(A) \to X$ such that

$$\|G(x) - G(y)\| \le \|B(x-y)\| \qquad \forall x, y \in D(A)$$

for some linear operator $B : D(A) \to Y$ into a Banach space Y such that

$$B \int_0^t T_0(s) x ds = V(t)x, \qquad x \in X,$$

where V is an operator family of suitable small strong bounded variation on some interval $[0, \varepsilon]$.
As a special case we obtain the following nonlinear Miyadera-Voigt type result (see Section 6 for more a more general and detailed result).

1.1 Theorem. *Let A be the generator of a C_0-semigroup T_0 on a Banach space X. Consider a Banach space Y, a Lipschitz continuous map $\hat{G} : Y \to X$,*

$$\|\hat{G}(y) - \hat{G}(\tilde{y})\| \le \Lambda \|y - \tilde{y}\| \qquad \forall y, \tilde{y} \in Y,$$

and a linear operator $B : D \to Y$ defined on a dense subspace D of X such that the following conditions are satisfied:

1. *$D \subseteq D(A)$, $T_0(t)D \subseteq D$, and the function $BT_0(\cdot)x$ is continuous for all $x \in D$.*

2. *There are α, $\lambda > 0$ and $\gamma > 0$ such that $\Lambda\gamma < 1$ and*

$$\int_0^\alpha e^{-\lambda t} \|BT_0(t)x\| \, dt \le \gamma \ \|x\| \qquad \forall x \in D.$$

Then there is an unique A-bounded extension $\hat{B}$ of B and a unique continuous semiflow Θ on X which satisfies the Lipschitz condition (5) such that, for each $x \in X$, $u(t) = \Theta(t,x)$ uniquely solves the Cauchy-problem (3) in the Friedrichs sense (L_1-F) with $G = \hat{G} \circ \hat{B}$.

In a second scenario, we concentrate on the Banach lattice case and a resolvent positive Hille-Yosida operator A. As illustration we present a result for the case that X is an abstract L space. For a general Banach lattice our assumptions become more involved (Section 7).

1.2 Theorem. *Let A be a resolvent positive Hille-Yosida operator in the abstract L-space X. Consider a nonlinear $G : D(A) \to X$ and a linear positive $B : D(A) \to X$ such that*

$$|G(x) - G(y)| \leq Bv \qquad \textit{whenever} \;\; x, y, v \in D(A), |x - y| \leq v,$$

and the spectral radius of $B(\lambda - A)^{-1}$ is smaller than 1 for some $\lambda > s(A)$. Then there is a unique continuous semiflow Θ on $\overline{D(A)}$ which satisfies the Lipschitz condition (5) such that, for each $x \in \overline{D(A)}$, $u(t) = \Theta(t,x)$ uniquely solves the Cauchy-problem (3) in the Friedrichs sense (L_1-F).
Moreover, if G is positive, so is Θ; more precisely: If G maps $D(A) \cap X_+$ into X_+, then $\Theta(t,\cdot)$ maps $\overline{D(A)} \cap X_+$ into itself.

We will not be able to write the nonlinear integral equation (2) in this strict form for the simple reason that, in general, the expression in $(\cdot)$ is not an element of $D(A)$. We will rather work with fixed points of the following operator

$$\begin{aligned} \Psi_x(v)(t) &= \lim_{h \to 0} G\Big(\frac{1}{h}[S(t+h)x - S(t)x] + \frac{1}{h}[(S * v)(t+h) - (S * v)(t)]\Big), \\ (S * v)(t) &= \int_0^t S(r)v(t-r)dr, \end{aligned} \tag{6}$$

where the convergence is required to hold in the L_1-sense and S is the integrated semigroup generated by A. In Section 2, we give abstract conditions under which the operator Ψ_x is well defined and has fixed points which we relate to Friedrichs solutions and integral solutions of the semilinear Cauchy problem (3). They will be connected to the results presented above by Stieltjes convolutions (Section 3) and cumulative outputs (Section 4).
Once it is known that the generalized solutions of the Cauchy problem induce a semiflow, the powerful theory of dynamical system is available to study their qualitative behavior [Hal88. e.g.].
In Section 8, we apply our results to an age-structured population model where per capita births are not described by an essentially bounded birth rate, but by a cumulative birth measure.
While, for the ease of exposition, we have restricted this paper to time-autonomous perturbations $A + G$, generalizations to non-autonomous semilinear perturbations $A + G(t)$ seem to be straightforward. Of course, the semiflow in (5)

will be non-autonomous as well, a nonlinear evolutionary system, and a suitable generalization of the exponential Lipschitz condition will only hold for all times if the Lipschitz conditions for $G(t)$ will be uniform for all $t \geq 0$.

2 Nonlinear perturbations in Banach spaces

Let X be a Banach space and A a Hille-Yosida operator in X, i.e., A is a linear closed operator such that the resolvent set $\rho(A)$ of A contains an infinite interval (ω, ∞) and there exists some $M \geq 1$ such that

$$\|(\lambda - A)^{-n}\| \leq M(\lambda - \omega)^{-n} \qquad \forall \lambda > \omega, n \in \mathbf{N}.$$

Equivalently [Are87b, KH89], A generates an integrated semigroup $S(t), t \geq 0$, such that

$$\|S(t) - S(r)\| \leq M \int_r^t e^{\omega s} ds.$$

In general, a (once) integrated semigroup is a strongly continuous operator family $(S(t))_{t \geq 0}$ that satisfies $S(0) = 0$ and the functional equation

$$S(t)S(r) = \int_0^{r+t} S(s)ds - \int_0^t S(s)ds - \int_0^r S(s)ds, \qquad t, r \geq 0,$$

strongly. The fact that A generates S can be either expressed as

$$x \in D(A), y = Ax \qquad \Longleftrightarrow \qquad S(t)x - tx = \int_0^t S(r)y dr \quad \forall t \geq 0, \tag{7}$$

[Thi90a] or, if S is exponentially bounded, in terms of Laplace transforms as

$$(\lambda - A)^{-1} = \lambda \int_0^\infty e^{-\lambda t} S(t) dt, \qquad \lambda > \omega, \tag{8}$$

[Are87b, KH89, Neu88]. If A is a Hille-Yosida operator, $S(t)x$ is strongly differentiable in $t \geq 0$ if and only if $x \in \overline{D(A)}$ and $\dot{S}(t)x := \frac{d}{dt} S(t)x$ is a C_0-semigroup on $\overline{D(A)}$ which is generated by the part of A in $\overline{D(A)}$.
The following result in [KH89] is one of the linear foundations for our nonlinear considerations.

2.1 Theorem. *Let $f \in L_1([0,T], X)$. Define*

$$(S * f)(t) = \int_0^t S(r) f(t-r) dr, \qquad t \geq 0.$$

Then $S * f$ is continuously differentiable, takes value in $D(A)$ with $A(S * f)(t)$ being a continuous function in $t \geq 0$, and $u = (S * f)'$ is the unique solution of

$$u(t) = A \int_0^t u(s) ds + \int_0^t f(s) ds, \qquad t \in [0, T]. \tag{9}$$

Finally

$$\|(S*f)'(t)\| \leq M \int_0^t e^{\omega r}\|f(t-r)\|dr \qquad \forall t \in [0,T].$$

The result that (9) has a unique solution which satisfies the estimate in Theorem 2.1 has already been obtained in [dPS87].

In the following we will write

$$S \diamond f := (S*f)'.$$

After this linear prelude, let Z be a closed subspace of X and

$$G : D(A) \to Z$$

be A-continuous, i.e., if X_A denotes $D(A)$ equipped with the graph norm, $\|x\|_A = \|x\| + \|Ax\|$, then $G : X_A \to Z$ is continuous. We consider the semilinear Cauchy problem

$$u'(t) = (A+G)u(t), \qquad u(0) = x \in \overline{D(A)}. \tag{10}$$

We will use two closely connected concepts of solving (10) in a generalized sense. Up to slight modifications, they can be found in [dPS87].

2.2 Definition. Let $T > 0$. A continuous function $u : [0,T] \to X$ with $u(0) = x$ is called *Friedrichs or F-solution in L_1* of (10) on $[0,T]$, if there are functions $u_j \in C^1([0,T],X) \cap C([0,T],X_A)$ such that $(Gu_j)_{j\in\mathbf{N}}$ converges in $L_1([0,T],X)$ and

$$\left.\begin{array}{l} u_j \to u, \qquad t \to \infty, \text{ uniformly on } [0,T], \\ (u_j' - Au_j) \text{ and } (G \circ u_j) \text{ converge in } L_1([0,T],X), \\ \lim\limits_{j\to\infty}(u_j' - Au_j) = \lim\limits_{j\to\infty} G \circ u_j. \end{array}\right. \qquad \forall T > 0$$

The second notion is a modification of the concept of integral solution.

2.3 Definition. A continuous function $u : [0,T] \to X$ is called *I-solution* of (10) if u can be continuously extended to some interval $[0, T+\epsilon]$, $\epsilon > 0$, such that $\int_0^t u(s)ds \in D(A)$ for all $t \in [0, T+\epsilon]$, and

$$u(t) = x + A\int_0^t u(s)ds + \lim_{h\to 0}\int_0^t G\left(\frac{1}{h}\int_s^{s+h} u(s)ds\right)ds \qquad \forall t \in [0,T], \tag{11}$$

with the limit existing in $L_1([0,T],X)$.

We establish a first relation between the two solution concepts.

2.4 Lemma. *If $0 < T_1 < T_2$, then any I-solution on $[0, T_2]$ is an F-solution in L_1 on $[0, T_1]$.*

Proof. Let u be an I-solution on $[0, T_2]$. Set

$$v(t) = \lim_{h\to\infty} G\left(\frac{1}{h}\int_t^{t+h} u(s)ds\right), \qquad t \in [0, T_2].$$

with convergence holding in $L_1([0, T_2], Z)$. Then

$$u(t) = x + A\int_0^t u(s)ds + v(t) \qquad \forall t \in [0, T_2].$$

Define

$$u_j(t) = j\int_t^{t+\frac{1}{j}} u(s)ds, \qquad v_j(t) = j\int_t^{t+\frac{1}{j}} v(s)ds, \qquad t \in [0, T_1].$$

Then $u_j \to u$ in $C([0, T_1], X)$ and $v_j \to v$ in $L_1([0, T_1], Z)$. Further $v = \lim_{j\to\infty} G \circ u_j$ in $L_1([0, T_1], X)$. Finally

$$u_j'(t) = j\left(u(t + \frac{1}{j}) - u(t)\right) = Au_j(t) + v_j(t),$$

so

$$u_j'(t) - Au_j(t) = v_j(t) \to v(t) \leftarrow G(u_j(t))$$

in $L_1([0, T_1], Z)$. □

Theorem 2.1 suggests to look for I-solutions of (10) as solutions of the integral equation

$$u(t) = \dot{S}(t)x + \lim_{h\to 0}\frac{d}{dt}\int_0^t S(s)G\left(\frac{1}{h}\int_{t-s}^{t-s+h} u(r)dr\right) ds,$$

where S is the integrated semigroup generated by A. Therefore for all $x \in \overline{D(A)}$ the following mapping Ψ from $\overline{D(A)} \times L_1([0, T], Z)$ to $L_1([0, T], Z)$ will play an important role whenever it is well-defined,

$$\Psi(x, v)(t) = \lim_{h\to 0} G\left(\frac{1}{h}\int_t^{t+h} \dot{S}(s)xds + \frac{1}{h}\int_t^{t+h} (S \diamond v)(s)ds\right). \qquad (12)$$

The convergence is required to be in $L_1([0, T], Z)$ and to be independent of how v is extended outside of $[0, T]$. Notice that this is the same map as in (6). We have the following fundamental result.

2.5 Proposition. *Assume the operator Ψ in (12) is well-defined and let $T > 0$, $x \in \overline{D(A)}$, $\Psi_x = \Psi(x, \cdot)$. Then there is a one-to-one correspondence between fixed points of Ψ_x and I-solutions. More precisely the following two relations holds:*

(1) If v_x is a fixed point of Ψ_x in $L_1([0,T],Z)$ and is extended by 0 outside of $[0,T]$, then $u = \dot{S}(\cdot)x + S \diamond v_x$ is an I-solution on $[0,T]$.

(2) If u is an I-solution of $[0,T]$, then

$$v(t) = L_1\text{-}\lim_{h\to 0} G\left(\frac{1}{h}\int_t^{t+h} u(s)ds\right)$$

is a fixed point of Ψ_x in $L_1([0,T],Z)$ and $u = \dot{S}(\cdot)x + S \diamond v$.

Proof. (1) Let v_x be a fixed point of Ψ_x and set $u = \dot{S}(\cdot)x + S \diamond v_x$. Extend v_x by 0 outside of $[0,T]$. By definition of Ψ, $v_x(t) = \lim_{h\to 0} G\left(\frac{1}{h}\int_t^{t+h} u(r)dr\right)$. u is an I-solution by Theorem 2.1.
(2) Let u be an I-solution of (10), i.e., satisfy (11) or rather all of Definition 2.3. Define v as stated in the theorem, then $u(t) = \dot{S}(t)x + (S \diamond v)(t)$ for $t \in [0,T]$ by Theorem 2.1. Since, by assumption, Ψ is well-defined and independent of how v is extended, v is a fixed point of Ψ_x.

□

Proposition 2.5 suggests to look for conditions which make the operator Ψ given by (12) well defined. The following subspace E of $\overline{D(A)} \times L_1([0,T];Z)$ will play an important role.
By definition, E consists of those pairs (x,v), with $x \in \overline{D(A)}$ and $v \in L_1([0,T],Z)$, such that

$$\begin{aligned} &\dot{S}(t)x + (S \diamond v)(t) \in D(A) \qquad \forall t \in [0,T], \\ &A\Big(\dot{S}(t)x + (S \diamond v)(t)\Big) \text{ is a continuous function of } t \in [0,T]. \end{aligned} \tag{13}$$

Since $G : D(A) \to Z$ is A-continuous,

$$\Psi_0(x,v)(t) = G\Big(\dot{S}(t)x + (S \diamond v)(t)\Big) \tag{14}$$

defines a map Ψ_0 from E to $C([0,T];Z) \subseteq L_1([0,T];Z)$.

2.6 Proposition. *Let $G : D(A) \to Z$ be A-continuous and E be the subspace defined in (13). Assume that Ψ_0 can be extended to a continuous map Ψ from $\overline{D(A)} \times L_1([0,T];Z)$ to $L_1([0,T];Z)$. Then Ψ satisfies (12).*

Notice in the last statement that the argument of G is in $D(A)$ and is a continuous function of t in the graph norm of A.

Proof. Set

$$w_h(t) = \frac{1}{h}(S(t+h) - S(t))x + \frac{1}{h}\int_t^{t+h} (S \diamond v)(s)ds.$$

Here we have extended v beyond T in an arbitrary way such that $v \in L_1([0,T+\epsilon],X)$ for some $\epsilon > 0$. By Theorem 2.1, $w_h(t) \in D(A)$, and

$$Aw_h(t) = \frac{1}{h}\big(\dot{S}(t+h) - \dot{S}(t)\big)x + \frac{1}{h}\Big((S \diamond v)(t+h) - (S \diamond v)(t)\Big) - \frac{1}{h}\int_t^{t+h} v(s)ds$$

is a continuous function of $t \in [0,T]$. Moreover

$$\begin{aligned}
& w_h(t) \\
= \; & \frac{1}{h}\dot{S}(t)S(h)x + \frac{1}{h}((S * v)(t+h) - (S * v)(t)) \\
= \; & \frac{1}{h}\dot{S}(t)S(h)x + \frac{1}{h}\int_0^t S(s)\big(v(t+h-s) - v(t-s)\big)ds \\
& + \frac{1}{h}\int_t^{t+h} S(s)v(t+h-s)ds \\
= \; & \frac{1}{h}\dot{S}(t)S(h)x + \frac{1}{h}\int_0^t S(s)\frac{d}{dt}\left(\int_{t-s}^{t+h-s} v(r)dr\right) ds \\
& + \frac{1}{h}\int_0^h S(t+s)v(h-s)ds \\
= \; & \frac{1}{h}\dot{S}(t)S(h)x + \frac{d}{dt}\int_0^t S(s)\left(\frac{1}{h}\int_{t-s}^{t+h-s} v(r)dr\right) \\
& + \frac{1}{h}\int_0^h (S(t+s) - S(t))v(h-s)ds \\
= \; & \dot{S}(t)x_h + (S \diamond v_h)(t)
\end{aligned}$$

with

$$x_h = \frac{1}{h}S(h)x + \frac{1}{h}\int_0^h S(s)v(h-s)ds, \quad v_h(t) = \frac{1}{h}\int_t^{t+h} v(r)dr.$$

Since $w_h(t) \in D(A)$ and $Aw_h(t)$ is continuous in t, by (13),

$$(x_h, v_h) \in E \quad \text{and} \quad G(w_h(t)) = \Psi_0\,(x_h, v_h)\,(t).$$

As $x_h \to x$ and $v_h \to v$ in $L_1([0,T],Z)$ as $h \to 0$ and Ψ is the continuous extension of Ψ_0 by assumption, we have

$$\Psi(x,v) = \lim_{h\to 0} G \circ w_h \quad \text{in } L_1([0,T];Z).$$

□

2.7 Proposition. *Suppose that $T > 0$ and the assumptions of Proposition 2.6 hold. If u is an F-solution of (9) on $[0,T]$, then $u = \dot{S}(\cdot)x + S \diamond v$ with a fixed point v of Ψ_x and u is an I-solution of (9).*

Proof. Let u be a F-solution in L_1. Then we find $u_j \in C^1([0,T],X) \cap C([0,T], X_A)$, $w_j \in L_1([0,T],X)$ such that $(Gu_j)_{j\in\mathbf{N}}$ converges in $L_1([0,T],X)$ and

$$\lim_{j\to\infty} \|u_j - u\|_\infty = 0$$
$$\lim_{j\to\infty} \|w_j\|_1 = 0$$

and

$$u_j' = Au_j + G \circ u_j + w_j.$$

We set $v_j := G \circ u_j$. By Theorem 2.1,

$$u_j = \dot{S}(\cdot)u_j(0) + S \diamond (G \circ u_j + w_j) = \dot{S}(\cdot)u_j(0) + S \diamond v_j + S \diamond w_j. \tag{16}$$

Since $u_j \in C([0,T], X_A)$, $(u_j(0), v_j + u_j)$ is an element of E by (13), and we have the following representation of v_j,

$$v_j = G \circ \left(\dot{S}(\cdot)u_j(0) + S \diamond v_j + S \diamond w_j\right) = \Psi_0(u_j(0), v_j + w_j).$$

Since Ψ is continuous and $u_j(0) \to u(0)$, $(v_j + w_j) \to v$ in L_1, we have

$$\Psi(u(0), v) = \lim_{j\to\infty} \Psi_0(u_j(0), v_j + w_j) = \lim_{j\to\infty} v_j = v$$

in $L_1([0,T],X)$ and v is a fixed point of Ψ_x, $x = u(0)$. Furthermore, by (16),

$$u(t) = \lim_{j\to\infty} u_j(t) = \lim_{j\to\infty} \dot{S}(\cdot)u_j(0) + (S \diamond v_j)(t) + (S \diamond w_j)(t) = \dot{S}(\cdot)u(0) + S \diamond v.$$

Then u is an I-solution by Proposition 2.5, Part 1. □

Once we have found the unique fixed point v_x of Ψ_x in $L_1([0,T],Z)$, motivated by Propositions 2.6 and 2.7, we define

$$\Theta(t,x) = \dot{S}(t)x + (S \diamond v_x)(t). \tag{18}$$

2.8 Theorem. *Let $T > 0$ and assume that $L_1([0,T],Z)$ can equivalently be renormalized such that*

$$\|\Psi_0(x,v) - \Psi_0(y,w)\| \le K_1\|x-y\| + K_2\|v-w\| \qquad \forall (x,v),(y,w) \in E$$

where $K_1 > 0$ and $K_2 \in (0,1)$.
Then there exists a continuous semiflow Θ on $[0,\infty) \times \overline{D(A)}$ such that the exponential Lipschitz condition (5) is satisfied and $u(t) = \Theta(t,x)$ are the unique I-solutions and the unique F-solutions of (10) on every finite interval $[0,\tau]$.

Proof. Since E contains the dense subspace $D(A_0) \times C_0^\infty((0,T);Z)$ of the Banach space $\overline{D(A)} \times L_1([0,T];Z)$, Ψ_0 has a Lipschitz extension Ψ to the latter space with the same Lipschitz constants.

Ψ is a uniform contraction on $\overline{D(A)} \times L_1([0,T];Z)$ and, by the parameter version of Banach's fixed point theorem, there are unique fixed points v_x of the mappings Ψ_x for all $x \in \overline{D(A)}$ and the dependence of the v_x on x is Lipschitz continuous on $[0,T]$ (see e.g. the proof of [CH87], Th. 2.2),

$$\|v_x - v_y\|_1 \leq C\|x-y\| \qquad \forall x,y \in \overline{D(A)},$$

with a suitable constant $C > 0$. We define Θ as in (18). By Theorem 2.1, for $0 < t \leq T$,

$$\begin{aligned}
&\|\Theta(t,x) - \Theta(t,y)\| \\
\leq\ & \sup_{s\in[0,T]} \left\|\dot{S}(s)\right\| \|x-y\| + \|S \diamond (v_x - v_y)\|_\infty \\
\leq\ & Me^{\omega_+ T}\left(\|x-y\| + \|v_x - v_y\|_1\right) \\
\leq\ & K\,\|x-y\|
\end{aligned}$$

where $K = Me^{\omega_+ T}(1+C)$ and $\omega_+ = \max\{\omega, 0\}$ is the positive part of ω. From Propositions 2.5 and 2.6 we know that $\Theta(\cdot,x)$ is the unique I-solution of (10). The semiflow property follows from the fact that $u = \Theta(\cdot,x)$ is also the unique F-solution of (10). See Lemma 2.4 and Lemma 2.7. Notice that $\tilde{u}(t) = \Theta(t+s,x)$ is the F solution of (10) with initial datum $\tilde{u}(0) = \Theta(s,x)$, so $\Theta(t+s,x) = \Theta(t,\Theta(s,x))$.
The semiflow Θ can be extended to $[0,2T]$ in the standard way: Let $t = s+r \in (T,2T]$ with $r,s \in [0,T]$. Set $\Theta(t,x) = \Theta(r,\Theta(s,x))$. The semiflow property on $[0,T]$ guarantees that this definition is independent of the choice of r and s and readily extends to $[0,2T]$. It also implies that $\Theta(\cdot,x)$ is an I-solution on $[0,2T]$,

$$\begin{aligned}
&\Theta(T+t,x) - x = \quad \Theta(t,\Theta(T,x)) - x \\
=\ & \Theta(T,x) - x + A\int_0^t \Theta(s,\Theta(T,x))ds \\
& + \lim_{h\searrow 0}\int_0^t G\left(\frac{1}{h}\int_s^{s+h}\Theta(r,\Theta(T,x))dr\right)ds \\
=\ & A\int_0^T \Theta(s,x)ds + \lim_{h\searrow 0}\int_0^T G\left(\frac{1}{h}\int_s^{s+h}\Theta(r,x)dr\right)ds \\
& + A\int_0^t \Theta(s+T,x))ds + \lim_{h\searrow 0}\int_0^t G\left(\frac{1}{h}\int_s^{s+h}\Theta(r+T,x))dr\right)ds \\
=\ & A\int_0^{T+t}\Theta(s,x))ds + \lim_{h\searrow 0}\int_0^{T+t} G\left(\frac{1}{h}\int_s^{s+h}\Theta(r,x))dr\right)
\end{aligned}$$

and the limit exists in L_1. The Lipschitz continuity of $\Theta(t,x)$ in x easily extends to $[T,2T]$ with the Lipschitz constant K^2. Continuing this way we can extend the semiflow to $[0,\infty)$ with

$$\|\Theta(t,x) - \Theta(t,y)\| \leq K^n\|x-y\| \qquad \forall t \in [(n-1)T, nT),\ n \in \mathbf{N}.$$

The exponential Lipschitz condition (5) is satisfied with $N = K$, $\theta = \ln K$.

$\square$

2.9 Corollary. *Let $T > 0$ and assume that $L_1([0,T],Z)$ can equivalently be renormalized such that*

$$\|\Psi_0(x,v) - \Psi_0(y,w)\| \le K_1\|x-y\| + K_2\|v-w\| \qquad \forall (x,v),(y,w) \in E, \tag{19}$$

where $K_1, K_2 > 0$. Then Ψ_0 has an extension Ψ satisfying the analogous inequality on $\overline{D(A)} \times L_1([0,T],X)$.

Further assume that there exist $n \in \mathbf{N}$ and $\varepsilon \in (0,1)$ such that for $\Psi_x = \Psi(x,\cdot)$,

$$\|\Psi_x^n(v) - \Psi_x^n(w)\| \le \varepsilon\|v-w\| \qquad \forall v,w \in L_1([0,T],Z), x \in \overline{D(A)}.$$

Then there is a unique continuous semiflow Θ on $[0,\infty) \times \overline{D(A)}$ such that the exponential Lipschitz condition (5) is satisfied and $u(t) = \Theta(t,x)$ are the unique I-solutions and the unique F-solutions of (10) on every finite interval $[0,\tau]$.

Proof. By assumption, with $\Psi_x := \Psi(x,\cdot)$, Ψ_x^n is a strict contraction on the Banach space $L_1([0,T],Z)$ and so has a unique fixed point v_x. Since $\Psi_x(v_x)$ is also a fixed point of Ψ_x^n, $\Psi_x(v_x) = v_x$. Since every fixed point of Ψ_x is also a fixed point of Ψ_x^n, v_x is the unique fixed point of Ψ_x. For all $x, y \in \overline{D(A)}$,

$$\begin{aligned}
\|v_x - v_y\| &= \|\Psi_x^n(v_x) - \Psi_y^n(v_y)\| \\
&\le \|\Psi_x^n(v_x) - \Psi_x^n(v_y)\| + \|\Psi_x^n(v_y) - \Psi_y^n(v_y)\| \\
&\le \|v_x - v_y\|\,\varepsilon + K_n\|x-y\|,
\end{aligned}$$

for some constant $K_n > 0$. Reorganizing this inequality,

$$\|v_x - v_y\| \le (1-\varepsilon)^{-1}K_n\|x-y\|.$$

The remaining part of the corollary follows as in Theorem 2.8. $\square$

For cases in which the estimate in Proposition 2.6 is difficult to check directly, we offer the following version which will be used in the Banach lattice case. Let $\phi : \mathbf{R} \to [0,\infty)$ be a continuous function with compact support and $\int_{\mathbf{R}} \phi(s)ds = 1$. Set

$$\phi_j(t) = j\phi(jt),$$

and for $v \in L_1([0,T);X)$ define

$$v_j^\phi(t) = \int_0^\infty \phi_j(s-t)v(s)ds,$$

where v has been extended outside of $[0,T]$ in an arbitrary measurable fashion. One readily checks that $v_j^\phi \to v$, $j \to \infty$, uniformly on compact subsets of $(0,T)$, if v is continuous on $(0,T)$, and $v_j^\phi \to v$ in $L_1([0,T];X)$. If $v : [0,T] \to D(A)$ and Av is continuous on $(0,T)$ then $v_j^\phi(t) \in D(A)$ and $Av_j^\phi \to Av$, $j \to \infty$, uniformly on compact subsets of $(0,T)$. ϕ with the properties above is called a *mollifier* because v^ϕ is typically smoother than v.

2.10 Corollary. *Let $G : D(A) \to Z$ be A-continuous and E be the subspace defined in (13). Let $K_1, K_2 \geq 0$ and ϕ a mollifier as above such that, for all (x,v), $(y,w) \in E$,*

$$\begin{aligned} &\varliminf_{j\to\infty} \int_0^T \left\| G\left(\left(\dot{S}(\cdot)x + S \diamond v\right)_j^{\phi}(t)\right) - G\left(\left(\dot{S}(\cdot)y + S \diamond w\right)_j^{\phi}(t)\right) \right\| dt \\ \leq\ & K_1\|x-y\| + K_2 \int_0^T \|v(t) - w(t)\| dt. \end{aligned}$$

Then the map Ψ_0 in (14) satisfies the inequalities (19) in Corollary 2.9, and its extension Ψ satisfies the analogous inequalities on $\overline{D(A)} \times L_1([0,T],X)$ and equation (12).

Proof. Since G is A-continuous, we have for all $t \in (0,T)$ and $(x,v) \in E$

$$\lim_{j\to\infty} G\left((\dot{S}(\cdot)x + S \diamond v)_j^{\phi}(t)\right) = G(\dot{S}(t)x + S \diamond v(t)).$$

By Fatou's lemma, the inequalities (19) in Corollary 2.9 holds, and Ψ_0 has an extension Ψ on $\overline{D(A)} \times L_1([0,T],X)$ which satisfies the analogous inequalities and, by Proposition 2.6, equation (12).

□

3 A convolution between operator families and vector-valued L_1-functions

The convolution we introduce in this section turns out to be useful in the proof of our perturbation theorems. The results we present are established in [TV99]. Let $I \subseteq \mathbf{R}$ be a interval which is bounded from below and

$$l(I) := \inf I \qquad \text{and} \qquad r(I) := \sup(I)$$

denote its left and right end-point. Unless explicitly stated otherwise, we always assume that $l(I) \in I$ and $r(I) \notin I$. Recall that, for every $f \in L_1(I,X)$, $\int_{l(I)}^{\cdot} f(t)dt$ is of bounded variation and its variation is given by

$$\mathbf{v}\left(\int_{l(I)}^{\cdot} f(t)dt; I \right) = \|f\|_1 = \int_{l(I)}^{r(I)} \|f(s)\|\, ds.$$

This motivates our next definition.

3.1 Definition. A function $f \in L_1(I,X)$ is of *bounded pseudo-variation* if

$$\mathbf{pv}(f;I) := \varlimsup_{h\searrow 0} \frac{1}{h} \int_{l(I)}^{r(I)-h} \|f(s+h) - f(s)\|\, ds < \infty.$$

If $r(I) = \infty$, $r(h) - h$ is to be interpreted as ∞. Recall that $f : I \to X$ is *of bounded variation*, if

$$\mathbf{v}(f; I) := \sup \left\{ \sum_{j=1}^{k} \|f(t_j) - (t_{j-1})\| \right\},$$

where the supremum is taken over all partitions $t_0 < \cdots < t_k$ of I where $k \in \mathbf{N}$ and $t_j \in I$. $f : I \to X$ is *Lipschitz continuous* if

$$\mathbf{pv}_\infty(f; I) := \sup \left\{ \frac{\|f(t) - f(s)\|}{|t - s|}, t, s \in I, t \neq s \right\} < \infty.$$

Every function of bounded pseudo-variation admits a version that is of bounded variation:

3.2 Lemma. *Let $f : I \to X$ be of bounded pseudo-variation. Then*

$$f^\oplus(t) := \lim_{h \searrow 0} \frac{1}{h} \int_t^{t+h} f(s) ds \tag{20}$$

exists for all $t \in I$, $f = f^\oplus$ a.e., and $f^\oplus$ is of bounded variation on I with $\mathbf{v}(f^\oplus, I) = \mathbf{pv}(f, I)$. Further $(f^\oplus)^\oplus = f^\oplus$ and $f^\oplus$ is right continuous.

We set

$$\mathbf{pv}^\oplus(f; I) := \mathbf{pv}(f; I) + \|f^\oplus(0)\|. \tag{21}$$

3.3 Proposition. *The vector space*

$$\mathcal{PV}(I; X) := \{f^\oplus | f : I \to X \text{ is of bounded pseudo-variation}\}$$

equipped with the norm $\mathbf{pv}^\oplus(f; I)$ is a Banach space.

The analogous result for the space of Lipschitz continuous functions, $\mathcal{PV}_\infty(I, X)$, is well known. We extend the previous concepts to operator families.

3.4 Definition. A family of bounded linear operators $(U(t))_{t \in I}$ from a Banach space X into a Banach space Y is called to be of *strong bounded pseudo-variation* if

$$\mathbf{pv}(U(\cdot)x; I) < \infty \qquad \forall x \in X.$$

It is easy to check that $x \mapsto U(\cdot)x$ is a closed linear operator from X into the Banach space $\mathcal{PV}(I, Y)$ and so a bounded linear operator; this implies that

$$\begin{aligned} \mathbf{PV}(U; I, X) &:= \sup_{\|x\| \leq 1,\ x \in X} \mathbf{pv}(U(\cdot)x; I) < \infty \\ \mathbf{PV}^\oplus(U; I, X) &:= \sup_{\|x\| \leq 1,\ x \in X} \mathbf{pv}^\oplus(U(\cdot)x; I) < \infty \end{aligned} \tag{22}$$

and

$$
\begin{aligned}
&\mathbf{pv}(U(\cdot)x; I, X) \leq \mathbf{PV}(U; I, X)\,\|x\|\,, \\
&\mathbf{pv}^{\oplus}(U(\cdot)x; I, X) \leq \mathbf{PV}^{\oplus}(U; I, X)\,\|x\|\,,
\end{aligned}
\qquad \forall x \in X. \tag{23}
$$

It is well known and follows in the same way that $U(\cdot)$ is Lipschitz continuous in operator norm if it is strongly Lipschitz continuous. We set

$$
\mathbf{PV}_{\infty}(U; I, X) = \sup\left\{ \frac{\|U(t) - U(t)\|}{|t-s|}; t, s \in I, t \neq s \right\}.
$$

We introduce

$$
U^{\oplus}(t)x = \lim_{h \downarrow 0} \frac{1}{h} \int_t^{t+h} U(s)x ds
$$

which exists according to Lemma 3.2. Notice that $U(t)x$ and $U^{\oplus}(t)x$ have the same pseudo-variation.
Before we define a Stieltjes type convolution between U and $f \in L_1(I, X)$, with $l(I) = 0$, we recall that the standard convolution

$$
(U * f)(t) := \int_0^t U(s) f(t-s) ds, \qquad t \in I,
$$

defines a function in $L_1(I, Y)$, if $f \in L_1(I, X)$ and $U(t)$ is strongly integrable on I. If U is of strong bounded pseudo-variation, we actually have the following result.

3.5 Proposition. *Let I be a bounded interval, U be of strong bounded pseudo-variation on I and $f \in L_1([0, T], X)$. Then $U * f \in L_{\infty}(I, Y)$ and*

$$
\|U * f\|_{\infty} \;\leq\; \sup_{\|x\| \leq 1} \|U(\cdot)x\|_{\infty}\, \|f\|_1.
$$

In fact, much more holds.

3.6 Theorem. *Let U be of strong pseudo-variation on I, $f \in L_1(I, X)$. Then $U * f$ is continuous on I. Moreover the limit*

$$
U \star f = \lim_{h \searrow 0} \frac{1}{h} \int_0^{\cdot} (U(s+h) - U(s)) f(\cdot - s) ds,
$$

exists in $L_1(I, Y)$,

$$
\|U \star f\|_1 \leq \mathbf{PV}(U; I, X) \|f\|_1,
$$

and

$$
(U * f)(t) = \int_0^t (U \star f)(s) ds + \int_0^t U^{\oplus}(0) f(s) ds.
$$

Finally

$$
U^{\oplus} \star f = U \star f.
$$

We define

$$(U \diamond f)(t) = (U \star f)(t) + U^{\oplus}(0)f(t). \tag{24}$$

Our Stieltjes type convolution relates to the standard convolution as follows.

3.7 Corollary. *Let $(U(t))_{t\in I}$ be a family of bounded linear operators from X to Y that is of strong bounded pseudo-variation.*

(a) *Then $(U * f) \in W_1^1(I, X)$ for all $f \in L_1(I, Y)$,*

$$(U * f)' = U \diamond f \ \text{ a.e. }, \qquad \|U \diamond f\|_1 \le \mathbf{PV}^{\oplus}(U; [0,T), X)\|f\|_1.$$

(b) *If $f : [0,T) \to X$ is of bounded pseudo-variation, then $U \diamond f$ is of bounded pseudo-variation on $[0,T)$,*

$$\mathbf{pv}^{\oplus}(U \diamond f; I) \le \mathbf{PV}^{\oplus}(U; I, X)\mathbf{pv}^{\oplus}(f; I).$$

Further $U^{\oplus} \diamond f^{\oplus} = U \diamond f^{\oplus} = U^{\oplus} \diamond f = U \diamond f$ a.e. on I,

$$(U \diamond f)^{\oplus}(0) = U^{\oplus}(0)f^{\oplus}(0).$$

(c) *If U is Lipschitz continuous and $f \in L_1(I, X)$, then $U \star f$ is continuous with*

$$\|U \star f\|_{\infty} \le \mathbf{PV}_{\infty}(U; I, X)\, \|f\|_1 .$$

If $U(t)$ and $V(t)$, $t \in I$, are families of bounded linear operators which are of strong bounded pseudo-variation, then, for each x, $U \diamond (V(\cdot)x)$ is defined and a function of bounded pseudo-variation, but difficult to turn into a family of bounded linear operators $W(t)x$, because it is uniquely determined up to a set of Lebesgue measure 0 which depends on x. This situation can be remedied by setting

$$(U \diamond V)(t)x = (U \diamond [V(\cdot)x])^{\oplus}.$$

By Lemma 3.2 and Corollary 3.7 (b), $U \diamond V$ is a family of bounded linear operators that is of locally strong bounded pseudo-variation and strongly right-continuous.

3.8 Proposition. *Let $f : [0,T) \to X$ be of bounded pseudo-variation and $U(t)$, $V(t)$, $t \in [0,T)$, be of strong bounded pseudo-variation. Then*

$$(U \diamond V) \diamond f = U \diamond (V \diamond f) \quad \text{a.e.}$$

We define recursively

$$U^{\diamond 1} = U^{\oplus}, \qquad U^{\diamond(n+1)} = U \diamond U^{\diamond n}.$$

3.9 Corollary. *Let $(U(t))_{t\in I}$ be a family of bounded operators of locally strong bounded variation. Then*

$$\mathbf{PV}^{\oplus}(U^{\diamond n}; I, X) \le [\mathbf{PV}^{\oplus}(U; I, X)]^n \qquad \forall n \in \mathbf{N}.$$

If $U(t)$ maps X into a closed subspace Z, then

$$\mathbf{PV}^{\oplus}(U^{\diamond(n+1)}; I, X) \le [\mathbf{PV}^{\oplus}(U; I, Z)]^n\, \mathbf{PV}^{\oplus}(U; I, X) \qquad \forall n \in \mathbf{N}.$$

Application to integrated semigroups. If we apply our results to integrated semigroups we obtain the following generalization of Theorem 2.1. Notice that $S \diamond f = S \star f$ because $S^{\oplus}(0) = 0$.

3.10 Lemma. *Let S be an integrated semigroup which is of strong bounded pseudo variation on every finite interval. Then $(S \diamond f)(t) \in \overline{D(A)}$ and $(S * f)(t) \in D(A)$ for all $f \in L_1(I, X)$ and almost all $t \geq 0$. Furthermore we have*

$$A(S * f)(t) = (S \diamond f)(t) - \int_0^t f(s)ds \text{ for a.a. } t \geq 0$$

and

$$(S * f)(t) = \int_0^t (S \diamond f)(s)ds \quad \forall t \in I.$$

4 Cumulative outputs of strong bounded variation

Recall that X, Y are Banach spaces and A is a Hille-Yosida operator on X. The locally Lipschitz continuous once integrated semigroup generated by A is denoted by $(S(t))_{t \geq 0}$ and the C_0-semigroup on $\overline{D(A)}$ generated by A_0 is denoted by $(\dot{S}(t))_{t \geq 0}$.

Assumption H4. Let B be an A-bounded operator with values in Y, $T \in [0, \infty]$, such that for every $x \in X$,

$$B \int_0^t S(s)xds = \int_0^t V(s)xds,$$

with a family $(V(t))_{t \geq 0}$ of bounded linear operators from X to Y that is of strong bounded pseudo-variation on $I = [0, T)$.

4.1 Remark. Replacing V by $V^{\oplus}$ we see that $B \int_0^t S(s)dx$ is right differentiable with derivative $V^{\oplus}(t)x$. So without restriction, we assume this from the very beginning, with $V(t)$ being strongly right continuous.

4.2 Lemma. *Let Assumption H4 be satisfied. Then the following holds:*

1. *For all $x \in \overline{D(A)}$ and all $t \in I$:* $\qquad BS(t)x = V(t)x.$

   For all $x \in D(A)$ and all $t \in I$, $V(t)x$ is right differentiable and

   $$Bx + V(t)Ax = V'(t)x.$$

   For all $x \in D(A_0)$ and all $t \in I$, $V(t)x$ is continuously differentiable and $B\dot{S}(t)x = V'(t)x$.

2. *For all $f \in L_1(I,X)$ and $t \in I$*

$$(V * f)(t) = B(S * f)(t).$$

3. *If $(x,v) \in E$, E as in (13), then $V(\cdot)x + V * f$ is continuously differentiable on $[0,T]$ and*

$$B\big(\dot S(t)x + (S \diamond f)(t)\big) = \frac{d}{dt}\big(V(t)x + (V * f)(t)\big).$$

In particular $V(\cdot)x \in W^1_{1,loc}[0,T]$ and

$$B\big(\dot S(t)x + (S \diamond f)(t)\big) = V'(t)x + (V * f)'(t).$$

Proof. The first statement in 1. is obvious because $A\int_0^t S(s)xds$ is differentiable for $x \in \overline{D(A)}$ with derivative $AS(t)x - x$. Similarly, for $x \in D(A_0)$, $AS(t)x$ is differentiable with derivative $A\dot S(t)x = \dot S(t)A_0x$ and so is $BS(t)x = V(t)x$ with derivative $B\dot S(t)x$. For $x \in D(A)$, by part 1 and (7),

$$V(t)x = Btx + B\int_0^t S(s)Axds = tBx + \int_0^t V(s)Axds.$$

We see that $V(t)x$ is right differentiable and

$$V'(t)x = Bx + V(t)Ax, \quad x \in D(A).$$

As for 2. we first recall that $A(S * f)(t)$ is continuous in t (Theorem 2.1). Since B is A-continuous, $B(S * f)(t)$ is continuous. Changing the order of integration shows that

$$\int_0^t (S * f)(r)dr = \int_0^t \left(\int_0^{t-s} S(r)f(s)dr\right) ds.$$

So

$$\begin{aligned}\int_0^t B(S * f)(r)dr &= B\int_0^t (S * f)(r)dr \\ &= \int_0^t B\left(\int_0^{t-s} S(r)f(s)dr\right) ds \\ &= \int_0^t \left(\int_0^{t-s} V(r)f(s)dr\right) ds = \int_0^t (V * f)(r)dr.\end{aligned}$$

Since $V * f$ is continuous, the assertion follows.
3. As B is A-bounded and $A(\dot S(t)x + (S \diamond f)(t))$ is continuous,

$$\begin{aligned}&B\Big(\dot S(t)x + (S \diamond f)(t)\Big) \\ &= \lim_{h\to 0}\frac{1}{h}B\Big((S(t+h)x - S(t)x) + (S * f)(t+h) - (S * f)(t)\Big) \\ &= \lim_{h\to 0}\frac{1}{h}\Big(V(t+h)x - V(t)x + (V * f)(t+h) - (V * f)(t)\Big)\end{aligned}$$

uniformly for t in bounded intervals. So $V(t)x + (V * f)(t)$ is continuously differentiable in t with derivative $B\Big(\dot{S}(t)x + (S \diamond f)(t)\Big)$.

$\square$

It follows from Lemma 4.2 (1) and the fact that $S(t)$ maps X into $\overline{D(A)}$ that V has the properties of a cumulative output defined below. In some circumstances it is convenient to start from the operator family $V(t)$ rather than the operator B.

4.3 Definition. A family $(V(t))_{t\geq 0}$ of bounded operators from X to Y is a *cumulative output* for $(S(t))_{t\geq 0}$ if the following holds:

$V(t)$ is strongly right continuous at any $t > 0$, the right limit $V(0+)$ exists strongly, and

$$V(t)S(r) = \int_0^{t+r} V(s)ds - \int_0^t V(s)ds - \int_0^r V(s)ds \qquad \forall t, r \geq 0. \tag{25}$$

It follows from the uniform boundedness theorem and Lemma 4.14 in [Thi96] that $V(t)$ is exponentially bounded.

4.4 Remark. Since $S(\cdot)x$ is continuously differentiable for all $x \in \overline{D(A)}$ we obtain

$$V(t)S'(r)x = V(t+r)x - V(r)x, \qquad t, r > 0.$$

If $w(V)$ denotes the exponentially bound for $(V(t))_{t\geq 0}$, we define

$$F(\lambda)x = \lambda \int_0^\infty e^{-\lambda t} V(t)x dt, \qquad \lambda > w(V),\ x \in X.$$

Since V is a cumulative output for S, $F(\lambda)$ is a resolvent output of A, i.e.,

$$F(\lambda)(\mu - A)^{-1} = \frac{1}{\mu - \lambda}(F(\lambda) - F(\mu)), \qquad \lambda, \mu > w(V), \lambda \neq \mu.$$

So we can define $\hat{B} = F(\lambda)(\lambda - A)$ and show that this definition is independent of $\lambda > w(V)$. A straightforward Laplace transform argument shows that Assumption H4 is satisfied with $\hat{B}$ replacing B.

5 Application to the semilinear Cauchy problem

We state the additional assumptions on G and A.

Assumption H5. There exists a linear operator $B : D(A) \to Y$ into a Banach space Y, $T > 0$ such that

$$\|G(x) - G(v)\| \leq \|B(x - y)\| \qquad \forall x, y \in D(A),$$

and

$$B\int_0^t S(r)xdr = \int_0^t V(r)xdr, \qquad t \in [0,T), x \in X,$$

with an operator family $(V(t))_{t\in[0,T)}$ from X to Y which is of strong bounded pseudo-variation and $\mathbf{PV}^{\oplus}(V;[0,T),Z) < 1$.

5.1 Theorem. *Under the assumption H5 there is a unique continuous semiflow Θ on $\overline{D(A)}$ such that the exponential Lipschitz condition (5) is satisfied and the orbits of Θ uniquely solve the Cauchy problem (3) in the L_1-Friedrichs sense (L_1-F).*

Proof. Let $T > 0$. We show that, under the assumption (AI), the assumptions of Theorem 2.8 are fulfilled with

$$K_1 = \mathbf{PV}(V;[0,T),X), \qquad K_2 = \mathbf{PV}^{\oplus}(V;[0,T),Z).$$

Let $(x,v),(y,w) \in E$, E as in (13). By Lemma 4.2 and H5,

$$\begin{aligned} & \left\| G\left(\dot S(t)x + (S\diamond v)(t)\right) - G\left(\dot S(t)x + (S\diamond w)(t)\right)\right\| \\ \leq & \left\| B\left(\dot S(t)x + (S\diamond v)(t)\right) - B\left(\dot S(t)x + (S\diamond w)(t)\right)\right\| \\ = & \|V'(t)(x-y) + (V\diamond(v-w))(t)\| \end{aligned}$$

for a.a. $t \geq 0$. By Corollary 3.7 (a),

$$\begin{aligned} &\int_0^T \left\| G\left(\dot S(t)x + (S\diamond v)(t)\right) - G\left(\dot S(t)x + (S\diamond w)(t)\right)\right\| dt \\ \leq & \mathbf{PV}(V;[0,T),X)\,\|x-y\| + \ \mathbf{PV}^{\oplus}(V;[0,T),Z)\ \|v-w\|. \end{aligned}$$

□

This theorem will work well if we have enough information about the integrated semigroup S generated by A. Esthetically it may be more pleasing if all assumptions are in terms of A (or its resolvent) and G.

5.2 Corollary. *There exists a linear operator $B : D(A) \to Y$ into a Banach space Y such that*

$$\|G(x) - G(v)\| \leq \|B(x-y)\| \qquad \forall x,y \in D(A),$$

$B(\lambda - A)^{-1}x$ converges as $\lambda \to \infty$ for every $x \in X$, and there exist $\mu > 0$ and $\gamma \in (0,1)$ and $c > 0$ such that

$$\begin{aligned} &\int_0^\infty \|B(\lambda+\mu-A)^{-(k+1)}x\|k\lambda^{k-1}d\lambda + \varliminf_{\lambda\to\infty} \|B(\lambda-A)^{-1}x\| \leq \gamma\|x\| \\ &\qquad \forall x \in Z, k \in \mathbf{N}. \\ &\int_0^\infty \|B(\lambda+\mu-A)^{-(k+1)}x\|k\lambda^{k-1}d\lambda \leq c\|x\| \qquad \forall x \in \overline{D(A)}, k \in \mathbf{N}. \end{aligned}$$

Then there is a unique continuous semiflow Θ on $\overline{D(A)}$ such that the orbits of Θ uniquely solve the Cauchy problem (3) in the Friedrichs sense (L_1-F) for all $x \in \overline{D(A)}$.

Proof. By the vector valued inversion of the Laplace transform for functions of bounded variation [Wei93, Are94]), there is a family of bounded operators $(V(t))_{t\geq 0}$ and some $w > 0$ such that

$$B(\lambda - A)^{-1}x = \lambda \int_0^\infty e^{-\lambda t} V(t)x dt \qquad \forall \lambda > w, x \in X,$$

and V is of strong bounded (pseudo)-variation on every finite interval, $V(0) = 0$. Moreover the strong right limit $V(0+) = V^{\oplus}(0)$ exists and

$$V^{\oplus}(0)x = \lim_{\lambda\to\infty} B(\lambda - A)^{-1}x.$$

For sufficiently large $\lambda > 0$,

$$\|G(x) - G(y)\| \leq \left\|B(\lambda - A)^{-1}\right\| \|(\lambda - A)(x - y)\| \qquad \forall x, y \in D(A)$$

and therefore G is A-continuous. A straightforward Laplace transform argument shows that Assumption H4 is satisfied on every finite interval provided we find some $T > 0$ with $\mathbf{PV}^{\oplus}(V; [0,T), Z) < 1$. To this end, let

$$V^{\mu}(t) = e^{-\mu t}V(t) + \mu \int_0^t e^{-\mu t}V(s)ds.$$

Then

$$B(\lambda + \mu - A)^{-1} = \lambda \int_0^\infty e^{-\lambda t}V^{\mu}(t)dt,$$

V^{μ} is of bounded pseudo-variation on $[0, \infty)$ and $\mathbf{PV}^{\oplus}(V^{\mu}; [0, \infty), Z) \leq \gamma$. See [Wei93] or [Are94]. It follows (Proposition 3.2 in [TV99]) that

$$\overline{\lim_{h\to 0}} \frac{1}{h} \int_0^\infty e^{-\mu t}\|V(t + h)x - V(h)x\|dt + \|V^{\oplus}(0)x\| \leq \gamma\|x\| \qquad \forall x \in Z.$$

So

$$\mathbf{PV}^{\oplus}(V; [0,T); Z) < \gamma e^{\mu T}.$$

By choosing $T > 0$ small enough we achieve that $\gamma e^{\mu T} < 1$ and our assertion follows from Theorem 5.1.

□

6 A semilinear Miyadera-Voigt theorem for C_0-semigroups

For A densely defined, $X = Z$ and $G = \mathrm{Id}$ we can use the proof of Theorem 5.1 to obtain a new proof of the well known Miyadera-Voigt perturbation theorem for C_0-semigroups [Voi77] that makes no use of the Dyson-Phillips series and works for semi-linear perturbations of Miyadera-Voigt type.

6.1 Theorem. *Let A be the generator of a C_0-semigroup $\dot{S}$ on a Banach space X, $Z \subseteq X$ be a closed subspace of X and Y be another Banach space. Let $\hat{G} : Y \to Z$ be Lipschitz continuous,*

$$\|\hat{G}(y) - \hat{G}(\tilde{y})\| \leq \Lambda \|y - \tilde{y}\| \qquad \forall y, \tilde{y} \in Y,$$

$B : D \to Y$ a linear operator defined on a dense subspace D of X such that $D \cap Z$ is dense in Z and the following conditions are satisfied:

1. *$D \subseteq D(A)$, $\dot{S}(t)D \subseteq D$, and $B\dot{S}(\cdot)x$ continuous for all $x \in D$.*
2. *There are α, λ, $K > 0$ and $\gamma > 0$ such that $\Lambda\gamma < 1$ and*
   (a) *$\int_0^\alpha e^{-\lambda t} \left\| B\dot{S}(t)x \right\| dt \leq K \|x\|$ for all $x \in D$.*
   (b) *$\int_0^\alpha e^{-\lambda t} \left\| B\dot{S}(t)x \right\| dt \leq \gamma \ \|x\|$ for all $x \in D \cap Z$.*

Then there is an unique A-bounded extension $\hat{B}$ of $B|_D$ and a unique continuous semiflow Θ on X such that the exponential Lipschitz condition (5) and the following are satisfied:

1. *For all $x \in X$, $\Theta(\cdot, x)$ uniquely solves the Cauchy problem (3) in the Friedrichs sense (L_1-F) with $G = \hat{G} \circ \hat{B}$.*
2. *For all $x \in X$ and $t \geq 0$ we have*

$$\Theta(t,x) = \dot{S}(t)x + \lim_{h \to 0} \int_0^t \dot{S}(t-s)(G \circ \hat{B}) \left(\frac{1}{h} \int_s^{s+h} \Theta(u,x) du \right) ds \tag{26}$$

Proof. Just as in [Voi77], p.168, one can show that for every $\gamma < c < \frac{1}{\Lambda}$ there is a $\mu_c > 0$ such that for all $\mu > \mu_c$

$$\int_0^\infty e^{-\mu s} \left\| B\dot{S}(s)x \right\| ds \leq c \|x\| \qquad \forall x \in D \cap Z, \tag{27}$$

$$\int_0^\infty e^{-\mu s} \left\| B\dot{S}(s)x \right\| ds \leq K \|x\| \qquad \forall x \in D. \tag{28}$$

Define

$$V(t)x = \int_0^t B\dot{S}(s)x ds, \qquad t \geq 0, x \in D,$$

and, for $\mu > \mu_c$,

$$F(\mu)x = \int_0^\infty e^{-\mu t} B\dot{S}(t)x dt.$$

By (27) and (28) all operators have a unique extension in $L(X,Y)$. We now check that $(F(\lambda))_{\lambda>\mu_c}$ satisfies the resolvent output identity, i.e.,

$$F(\lambda)(\mu-A)^{-1} = \frac{1}{\mu-\lambda}(F(\lambda)-F(\mu)), \qquad \mu,\lambda>\mu_c, \lambda\neq\mu.$$

Consider $x\in D$ and $\lambda,\eta>\mu_c$, $\lambda\neq\mu$. We have

$$F(\lambda)(\eta-A)^{-1}x = \int_0^\infty e^{-\eta s}F(\lambda)\dot{S}(s)xds.$$

Since $\dot{S}(s)x\in D$,

$$F(\lambda)(\eta-A)^{-1}x = \int_0^\infty e^{-\eta s}\left(\int_0^\infty e^{-\lambda t}B\dot{S}(t)\dot{S}(s)xdt\right)ds$$
$$= \int_0^\infty\left(\int_0^t e^{-\eta s}e^{-\lambda(t-s)}ds\right)B\dot{S}(t)xdt = \frac{1}{\lambda-\eta}(F(\eta)x-F(\lambda)x).$$

Since D is dense we have verified the resolvent output identity.
Thus the definition $\hat{B}:=F(\lambda)(\lambda-A)$ is independent of $\lambda>\mu_c$, provides an A-bounded operator $\hat{B}$, and

$$V(t)=\hat{B}\int_0^t\dot{S}(s)ds.$$

Notice that, for each $x\in X$, $B\dot{S}(s)$ can be extended to an function $\dot{V}(\cdot)x$ in $L_1([0,\alpha],X)$ with

$$V(t)x=\int_0^t\dot{V}(s)xds$$

and

$$\int_0^\alpha e^{-\lambda t}\|\dot{V}(t)x\|dt\leq K\|x\| \qquad \forall x\in X,$$
$$\int_0^\alpha e^{-\lambda t}\left\|\dot{V}(t)x\right\|dt\leq c\ \|x\| \qquad \forall x\in Z.$$

Let $\tau\in(0,\alpha]$. Then

$$\mathbf{pv}(V(\cdot)x;[0,\tau))=\int_0^\tau\|\dot{V}(t)x\|dt\leq Ke^{\lambda\tau}\|x\| \qquad \forall x\in X$$
$$\mathbf{pv}(V(\cdot)x;[0,\tau))\leq ce^{\lambda\tau}\|x\| \qquad \forall x\in Z.$$

Choosing $\tau>0$ small enough we can achieve that $ce^{\lambda\tau}\Lambda<1$.
$G=\hat{G}\circ\hat{B}$ and $\tilde{B}=\Lambda\hat{B}$ satisfy

$$\|G(x) - G(\tilde{x})\| \leq \|\tilde{B}(x - \tilde{x})\| \qquad \forall x, \tilde{x} \in D(A).$$

Further $\tilde{V}(t) = \Lambda V(t)$ satisfies Assumption H5 with $\tilde{V}$ replacing V. The existence and the properties of the semiflow Θ now follow in the same way as in the proof of Theorem 5.1.

It only remains to show that $\hat{B}$ is the unique A-bounded extension of B. Since $BT(\cdot)x$ is continuous, we have for $x \in D$ that

$$Bx = \lim_{\lambda\to\infty} \int_0^\infty e^{-t} BT\left(\frac{t}{\lambda}\right) x dt = \lim_{\lambda\to\infty} \lambda \int_0^\infty e^{-\lambda t} B\dot{S}(t) x dt. \qquad (29)$$

By the resolvent output identity and $\lim_{\lambda\to\infty} F(\lambda)x = 0$ for all $x \in X$, we have for $x \in D$ and $z > \mu_c$

$$\begin{aligned} Bx &= \lim_{\lambda\to\infty} \lambda F(\lambda)x = \lim_{\lambda\to\infty} \lambda F(\lambda)(z-A)^{-1}(z-A)x \\ &= \lim_{\lambda\to\infty} \frac{\lambda}{\lambda - z}(F(z)(z-A)x - F(\lambda)(z-A)x) = F(z)(z-A)x = \hat{B}x. \end{aligned}$$

Suppose $\tilde{B}$ is a A-bounded extension of B to $D(A)$. Let $\nu \in \rho(A)$. We have for $\lambda > \mu_c$ and $x \in D$

$$\begin{aligned} F(\lambda)x &= \int_0^\infty e^{-\lambda t} \tilde{B}\dot{S}(t) x dt = \int_0^\infty e^{-\lambda t} \tilde{B}\dot{S}(t)(\nu - A)^{-1}(\nu - A) x dt \\ &= \int_0^\infty \tilde{B}(\nu - A)^{-1} e^{-\lambda t} \dot{S}(t)(\nu - A) x dt \\ &= \tilde{B}(\nu - A)^{-1} \int_0^\infty e^{-\lambda t} \dot{S}(t)(\nu - A) x dt \\ &= \tilde{B}(\nu - A)^{-1}(\lambda - A)^{-1}(\nu - A)x = \tilde{B}(\lambda - A)^{-1}x. \end{aligned}$$

Since D is dense, we conclude

$$\hat{B}(\lambda - A)^{-1} = F(\lambda) = \tilde{B}(\lambda - A)^{-1},$$

and $\hat{B} = \tilde{B}$. □

7 Semilinear perturbations in Banach lattices

We consider a Banach lattice X with positive convex cone X_+. Following Arendt ([Are87a]), we call a linear operator A in X resolvent positive, if $(a, \infty) \subseteq \rho(A)$ for some $a \in \mathbf{R}$ and $(\lambda - A)^{-1} \geq 0$ for all $\lambda > a$. Here $\rho(A)$ denotes the resolvent set of A. We refer to [Sch74] for more details on ordered Banach spaces and positive operators.

We say that a linear operator A in X is *absolutely resolvent majorized* by a resolvent positive operator $\bar{A}$ if $(a, \infty) \in \rho(A) \cap \rho(\bar{A})$ for some $a \in \mathbf{R}$ with

$$\left|(\lambda - A)^{-1}x\right| \le (\lambda - \bar{A})^{-1}\,|x| \qquad \text{for all } x \in X \text{ and } \lambda > a.$$

Given a family of bounded operators $(V(t))_{t\ge 0}$ of locally strong bounded variation we obtain iterative convolutions $V^{\diamond n}$ for $n \in \mathbf{N}$ by

$$V^{\diamond(n+1)} := (V^{\diamond n} \diamond V)\,.$$

$V^{\diamond n}$ is again a family of bounded operators that is of locally strong bounded variation (see Corollary 3.9). We make the following assumption. Recall that an operator in a Banach lattice is called *positive* if it maps positive elements in its domain into X_+.

Assumption H7a. Let X be a Banach lattice and A be resolvent majorized by a resolvent positive Hille-Yosida operator $\bar{A}$. Let $G : D(A) \to Z$ map into a closed subspace Z of X and assume that there is a linear positive operator $B : D(\bar{A}) \to Z$ such that

$$|G(x) - G(y)| \le Bv \quad \text{whenever} \;\; x, y \in D(A), v \in D(\bar{A}), |x - y| \le v,$$

and there exists some $w \in \mathbf{R}$ such that

$$B(\lambda - \bar{A})^{-1}x = \lambda \int_0^\infty e^{-\lambda t} V(t)x dt \qquad \forall \lambda > w,\; x \in X,$$

where V is a family of strong bounded pseudo-variation on some interval $[0, T+\epsilon)$ with $T, \epsilon > 0$ and

$$\mathbf{PV}^{\oplus}(V^{\diamond n}; [0,T), Z) < 1$$

for some $n \in \mathbf{N}$.

7.1 Remark. (a) Since $(\lambda - \bar{A})^{-1}$ and B are positive we have

$$|G(x) - G(y)| \le B(\lambda - \bar{A})^{-1}\,|(\lambda - A)(x - y)|$$

and therefore G is A-continuous.
(b) Inductively one realizes that

$$|(\lambda - A)^{-n}x| \le (\lambda - \bar{A})^{-n}|x| \qquad \forall x \in X, n \in \mathbf{N},$$

which implies that A is a Hille-Yosida operator as well.

We are going to proof a theorem similar to Theorem 5.1 and proceed as in the previous section by showing first that Ψ_x is well defined and then that it admits a unique fixed point. Denote by

$$(\bar{S}(t))_{\ge 0}$$

the semigroup generated by $\bar{A}$. We start with

7.2 Lemma. *Let $T > 0$. Under Assumptions H7a, Ψ_0 satisfies the inequalities (19) in Corollary 2.9 with*

$$K_1 = \mathbf{PV}(V;[0,T+),X), \qquad K_2 = \mathbf{PV}^{\oplus}(V;[0,T+),Z),$$

where

$$\mathbf{PV}(V;[0,T+),X) := \overline{\lim_{h\downarrow 0}}\, \mathbf{PV}(V;[0,T+h),X).$$

Further the extension Ψ of Ψ_0 satisfies

$$|\Psi(x,v) - \Psi(x,w)| \le V \diamond |v-w| \qquad \forall v,w \in L_1([0,T];Z).$$

Proof. We start by showing for $v \in L_1([0,T],X)$ that

$$|(S \diamond v)(t)| \quad \le \quad (\bar{S} \diamond |v|)(t). \tag{30}$$

We have for locally Lipschitz continuous integrated semigroups (see [HP57] p.224)

$$S(t+h)x - S(t)x = \lim_{k\to\infty} \int_t^{t+h} \left(\frac{k}{s}\right)\left(\frac{k}{s} - A\right)^{-(k+1)} x ds.$$

Since A is resolvent dominated by $\bar{A}$,

$$|(S(t+h) - S(t))x| \le \lim_{k\to\infty} \int_t^{t+h} \left(\frac{k}{s}\right) \left|\left(\frac{k}{s} - A\right)^{-(k+1)} x\right| ds$$

$$\le \quad \lim_{k\to\infty} \int_t^{t+h} \left(\frac{k}{s}\right)\left(\frac{k}{s} - \bar{A}\right)^{-(k+1)} |x|\, ds = (\bar{S}(t+h) - \bar{S}(t))\, |x|\,,$$

and the inequality (30) follows a.e. from Theorem 3.6 and $S \diamond v = S \star v$. As a by-product, we see that $\{\bar{S}(t)\}$ is an increasing family. Since $S \diamond v$ and $\bar{S} \diamond |v|$ are continuous by Theorem 2.1, the inequality holds everywhere.
We apply Corollary 2.10. Let $\phi : \mathbf{R} \to [0,\infty)$ be infinitely often differentiable with support in $(0,1)$ and $\int_0^1 \phi(s)ds = 1$. Then, for $x, y \in \overline{D(A)}$,

$$\begin{aligned}
&\left|(\dot{S}(\cdot)x)_j^{\phi}(t) - (\dot{S}(\cdot)y)_j^{\phi}(t)\right| = \left|\int_0^\infty \phi_j(s-t)\dot{S}(s)(x-y)ds\right| \\
= \quad & \lim_{\lambda\to\infty} \left|\int_0^\infty \phi_j(s-t)\dot{S}(s)\lambda(\lambda - A)^{-1}(x-y)ds\right| \\
= \quad & \lim_{\lambda\to\infty} \lim_{h\searrow 0} \left|\int_0^\infty \phi_j(s-t)\frac{1}{h}[S(s+h) - S(s)]\lambda(\lambda - A)^{-1}(x-y)ds\right| \\
\le \quad & \lim_{\lambda\to\infty} \lim_{h\searrow 0} \int_0^\infty \phi_j(s-t)\frac{1}{h}[\bar{S}(s+h) - \bar{S}(s)]\lambda(\lambda - \bar{A})^{-1}|x-y|ds \\
= \quad & \lim_{\lambda\to\infty} \int_0^\infty \phi_j(s-t)\dot{\bar{S}}(s)\lambda(\lambda - \bar{A})^{-1}|x-y|ds.
\end{aligned}$$

Since ϕ is zero on $(-\infty, 0]$,

$$
\begin{aligned}
& \left|(\dot{S}(\cdot)x)_j^{\phi}(t) - (\dot{S}(\cdot)y)_j^{\phi}(t)\right| \leq - \lim_{\lambda \to \infty} \int_0^{\infty} \phi_j'(s-t)\bar{S}(s)\lambda(\lambda - \bar{A})^{-1}|x-y|ds \\
= \quad & - \int_0^{\infty} \phi_j'(s-t)\bar{S}(s)|x-y|ds = \int_0^{\infty} \phi_j''(s-t) \left(\int_0^s \bar{S}(r)|x-y|dr \right) ds \\
& \in D(\bar{A}).
\end{aligned}
$$

Let $v, w \in L_1([0,T], Z)$. We set $v(s) = w(s) = 0$ for $s > T$. Let $t \in [0,T]$. Then

$$
\begin{aligned}
& \left|(S \star v)_j^{\phi}(t) - (S \star w)_j^{\phi}(t)\right| \leq \int_0^{\infty} \phi_j(s-t)(\bar{S} \star |v-w|)(s)ds \\
= \quad & - \int_0^{\infty} \phi_j'(s-t)(\bar{S} * |v-w|)(s)ds \quad \in \ D(\bar{A}).
\end{aligned}
$$

By Assumption H7a,

$$
\begin{aligned}
& \left|G\left((\dot{S}(\cdot)x + S \star v)_j^{\phi}(t)\right) - G\left((\dot{S}(\cdot)y + S \star w)_j^{\phi}(t)\right)\right| \\
\leq \quad & B \int_0^{\infty} \phi_j''(s-t) \left(\int_0^s \bar{S}(r)|x-y|dr \right) ds \\
& - B \int_0^{\infty} \phi_j'(s-t)(\bar{S} * |v-w|)(s)ds \\
= \quad & \int_0^{\infty} \phi_j''(s-t) \left(\int_0^s V(r)|x-y|dr \right) ds - \int_0^{\infty} \phi_j'(s-t)(V * |v-w|)(s)ds \\
= \quad & \int_0^{\infty} \phi_j(s-t)V^{\oplus}(ds)|x-y| + \int_0^{\infty} \phi_j(s-t)(V \diamond |v-w|)(s)ds.
\end{aligned}
$$

Recall that $V^{\oplus}(t)x = V(t)x$ for a.a. t and that $V^{\oplus}(\cdot)$ is of strong bounded variation which equals the strong bounded pseudo-variation of $V(\cdot)$. So the Stieltjes integral in the last equation makes sense. Since the support of ϕ_j is contained in $(0, 1/j)$,

$$
\begin{aligned}
& \int_0^T \left\| G\left((\dot{S}(\cdot)x + S\star v)_j^\phi(t)\right) - G\left((\dot{S}(\cdot)y + S\star w)_j^\phi(t)\right) \right\| dt \\
\leq \; & \int_0^T \left\| \int_0^\infty \phi_j(s-t) V^\oplus(ds)|x-y| \right\| dt \\
& + \int_0^T \left(\int_0^\infty \phi_j(s-t) \|(V \diamond |v-w|)(s)\| ds \right) dt \\
\leq \; & \mathbf{PV}\left(V; \left[0, T+\frac{1}{j}\right), X\right) \; \|x-y\| \sup_{0\leq s\leq T} \int_0^\infty \phi_j(s-t)dt \\
& + \int_0^{T+\frac{1}{j}} \left(\int_0^\infty \phi(s-t)dt \right) \|V \diamond |v-w|)(s)\| ds \\
\leq \; & \mathbf{PV}\left(V; \left[0, T+\frac{1}{j}\right), X\right) \|x-y\| \\
& + \mathbf{PV}^\oplus\left(V; \left[0, T+\frac{1}{j}\right), Z\right) \int_0^{T+\frac{1}{j}} \|v(s)-w(s)\| ds.
\end{aligned}
$$

In the last inequality we have used Corollary 3.7 (a). Recalling that we have set $v(t) = 0 = w(t)$ for $t > T$, this implies the assumption in Corollary 2.10. Now

$$
\begin{aligned}
& \left| \frac{1}{h} \int_t^{t+h} (S\star v)(s)ds - \frac{1}{h} \int_t^{t+h} (S\star w)(s)ds \right| \\
\leq \; & \frac{1}{h} \int_t^{t+h} |(S\star (v-w))(s)| ds \leq \frac{1}{h} \int_t^{t+h} (\bar{S}\star |v-w|)(s)ds \\
= \; & \frac{1}{h}\big((\bar{S} * |v-w|)(t+h) - (\bar{S} * |v-w|)(t) \big) \in D(\bar{A}).
\end{aligned}
$$

By Assumption H7a and Lemma 3.4 (2.),

$$
\begin{aligned}
& \Bigg| \; G\left(\frac{1}{h} \int_t^{t+h} \dot{S}(s)x ds + \frac{1}{h} \int_t^{t+h} (S\star v)(s)ds \right) \\
& -G\left(\frac{1}{h} \int_t^{t+h} \dot{S}(s)x ds + \frac{1}{h} \int_t^{t+h} (S\star w)(s)ds \right) \Bigg| \\
\leq \; & B\frac{1}{h}\big((\bar{S} * |v-w|)(t+h) - (\bar{S} * |v-w|)(t) \big) \\
= \; & \frac{1}{h}\big((V * |v-w|)(t+h) - (V * |v-w|)(t) \big).
\end{aligned}
$$

The Corollaries 2.10 and 3.7 imply the second assertion. $\square$

We are now ready to prove a result similar to Theorem 5.1 for Banach lattices.

7.3 Theorem. *Under the assumption H7a there is a unique continuous semiflow Θ on $\overline{D(A)}$ satisfying (5) such that the orbits of Θ are the unique F-solutions in $L_1([0,T], X)$ of (10) for all $x \in X$ and $T > 0$.*

Proof. We want to apply Corollary 2.9. By Lemma 7.2, the first assumption holds and Ψ_0 has an extension Ψ. Therefore it only remains to show that there exists $n \in \mathbf{N}$ and $\varepsilon \in (0,1)$ such that

$$\|\Psi_x^n(v) - \Psi_x^n(w)\|_1 \leq \varepsilon \|v - w\|_1 \qquad \forall v, w \in L_1([0,T], Z), x \in \overline{D(A)}.$$

By Lemma 7.2,

$$|\Psi_x(v) - \Psi_x(w)| \leq V \diamond |v - w| \, .$$

Thus by induction

$$|\Psi_x^n(v) - \Psi_x^n(w)| \leq V^{\diamond n} \diamond |v - w| \, .$$

Hence

$$\|\Psi_x^n(v) - \Psi_x^n(w)\|_1 \leq \mathbf{PV}^{\oplus}(V^{\diamond n}; [0,T], X) \|v - w\|_1 .$$

and we have reached our goal with $\varepsilon = \mathbf{PV}^{\oplus}(V^{\diamond n}; [0,T], X)$.

□

We call a semiflow Θ on $\overline{D(A)}$ *positive* if $\Theta(t, \cdot)$ maps $\overline{D(A)} \cap X_+$ into itself for all $t \geq 0$, where X_+ denotes the positive cone of the Banach lattice X.

Corollary. *Let X be a Banach lattice and A be a resolvent positive Hille-Yosida operator. Let $G : D(A) \to Z$ map into a closed subspace Z of X and be positive, i.e., G maps $D(A) \cap X_+$ into X_+.*
Assume that there is a linear positive operator $B : D(A) \to Z$ such that

$$|G(x) - G(y)| \leq Bv \quad \text{ whenever } \; x, y, v \in D(A), |x - y| \leq v,$$

and there exists some $w \in \mathbf{R}$ such that

$$B(\lambda - A)^{-1} x = \lambda \int_0^\infty e^{-\lambda t} V(t) x dt \qquad \forall \lambda > w, \; x \in X.$$

where V is a family of strong bounded pseudo-variation on some interval $[0, T+\epsilon)$ with $T, \epsilon > 0$ and

$$\mathbf{PV}^{\oplus}(V^{\diamond n}; [0,T), Z) < 1 \qquad \text{for some } n \in \mathbf{N}.$$

Then there exists a unique continuous positive semiflow Θ on $\overline{D(A)}$ which satisfies (5) and whose orbits uniquely solve (3) in the Friedrichs sense (L_1-F).

Proof. By Theorem 7.3 we only need to show the positivity of the semiflow. Since G is positive and the integrated semigroup S is increasing, $S \star v$ takes values in X_+ if $v \in L_1([0,T], Z \cap X_+)$ by Theorem 3.6 and the map

$$\Psi(x, v) = \lim_{h \downarrow 0} G\left(\frac{1}{h}[S(t+h) - S(t)]x + \frac{1}{h}\int_t^{t+h} (S \star v)(s) ds\right)$$

maps $(\overline{D(A)} \cap X_+) \times L_1([0,T], Z \cap X_+)$ into itself. Hence, if $x \in \overline{D(A)} \cap X_+$, the fixed point v_x is in $L_1([0,T], Z \cap X_+)$ and

$$\Theta(t,x) = \dot{S}(t)x + (S \star v_x)(t) \in X_+$$

by Theorem 3.6.

$\square$

The Banach lattice X is called an abstract L-space, if in addition the norm on X is additive on the positive cone X_+ (in particular X may be a L_1-space or a space of measures).

Assumption H7b. Let X be an abstract L-space, A a linear operator in X, and

$$G : D(A) \to X.$$

Suppose there are a resolvent positive Hille-Yosida operator $\bar{A}$ and a positive $\bar{A}$-bounded operator $B : D(\bar{A}) \to Z$ such that A is absolutely resolvent majorized by $\bar{A}$,

$$\text{spr } B(\mu - \bar{A})^{-1} < 1 \qquad \text{for some } \mu \in \mathbf{R} \cap \rho(\bar{A})$$

and

$$|G(x) - G(v)| \leq Bw \quad \text{for all } x, y \in D(A), w \in D(\bar{A}) \text{ with } |x - y| \leq w.$$

7.4 Theorem. *The statement of Theorem 5.3 also holds if Assumption H7a is replaced by Assumption H7b.*

Proof. Since $\bar{A}$ is resolvent positive on (w, ∞) for some $w > 0$ and B is positive, we have

$$(-1)^n \frac{d^n}{d\lambda^n} BR(\lambda, \bar{A}) = n! BR(\lambda, \bar{A})^{n+1} \geq 0, \qquad \lambda > w,$$

i.e., $(BR(\lambda, \bar{A}))_{\lambda > w}$ is completely monotonic. By the vector valued version of Bernstein's theorem (see [Are87a], section 5) we find an exponentially bounded increasing family $(V(t))_{t \geq 0}$, $V(0) = 0$, such that

$$B(\lambda - \bar{A})^{-1} x = \int_0^\infty e^{-\lambda t} dV(t)x = \lambda \int_0^\infty e^{-\lambda t} V(t)x dt, \qquad \forall x \in X,\ \lambda > w.$$

Since every increasing family of bounded operators is of strong bounded variation on every finite interval, we know from (24), and Theorem 3.6 that $(V^{\diamond n}(t))_{t \geq 0}$ is an increasing family of bounded operators that is of bounded pseudo-variation on every finite interval. Since X is an abstract L-space and $V^{\diamond n}$ is increasing,

$$\int_0^{\epsilon-h} \|V^{\diamond n}(s+h)x - V^{\diamond n}(s)x\| ds = \int_0^{\epsilon-h} \Big\| |V^{\diamond n}(s+h)x - V^{\diamond n}(s)x| \Big\| ds$$
$$= \left\| \int_0^{\epsilon-h} |V^{\diamond n}(s+h)x - V^{\diamond n}(s)x| ds \right\|$$
$$\leq \left\| \int_0^{\epsilon-h} (V^{\diamond n}(s+h)|x| - V^{\diamond n}(s)|x|) ds \right\|$$
$$= \left\| \int_{\epsilon-h}^{\epsilon} V^{\diamond n}(s)|x| ds - \int_0^h V^{\diamond n}(s)|x| ds \right\| \leq h \| V^{\diamond n}(\epsilon)|x| - V^{\diamond n}(0)|x| \|.$$

By Definition 3.1 and (21), for $n \geq 2$,

$$\mathbf{pv}^{\oplus}(V^{\diamond n}(\cdot)x; [0,\epsilon]) = \mathbf{pv}(V^{\diamond n}(\cdot)x; [0,\epsilon]) + \|V^{\diamond n}(0)x\| \leq \|V^{\diamond n}(\epsilon)|x|\|$$
$$\leq e^{\lambda\epsilon} \left\| \int_0^{\epsilon} e^{-\lambda t} V^{\diamond n}(dt)|x| \right\| \leq e^{\lambda\epsilon} \left\| \left(B(\lambda - \bar{A})^{-1}\right)^n |x| \right\|$$
$$\leq e^{\lambda\epsilon} \left\| \left(B(\lambda - \bar{A})^{-1}\right)^n \right\| \|x\|.$$

By (22),
$$\mathbf{PV}^{\oplus}(V^{\diamond n}; [0,\epsilon], X) \leq e^{\lambda\epsilon} \| \left(B(\lambda - \bar{A})^{-1}\right)^n \|.$$
Choosing λ and n large and $\epsilon > 0$ small enough, we have $\mathbf{PV}^{\oplus}(V^{\diamond n}; [0,\epsilon], X) < 1$ and our statement follows from Theorem 7.3.

□

8 An application to age-structured population dynamics

The development of an age-structured population can be described as a partial differential equation for the age-density $u(t,\cdot)$ at time t as

$$\begin{aligned} \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial a}\right) \frac{u(t,a)}{\mathcal{F}(a)} &= 0; \quad t > 0, 0 < a < c, \\ u(t,0) &= f(u(t,\cdot)); \quad t > 0, \\ u(0,a) &= u_0(a); \quad 0 \leq a < c. \end{aligned} \tag{31}$$

The number $c \in (0,\infty]$ denotes the maximum age that an individual can reach. $\mathcal{F}(a)$ is the probability to survive till age a and is a decreasing function of a with $\mathcal{F}(0) = 1$, $\mathcal{F}(a) > 0$ for $a \in [0,c)$ and $\mathcal{F}(a) = 0$ for $a > c$ with the last only making sense if $c < \infty$. $f(u(t,\cdot))$ is the population birth rate as a function of the population age density. We make the following assumption:

H8a: There exist a non-negative not necessarily finite Borel measure m on $[0,c)$ such that $f : \mathcal{L}_1([0,c); m+\ell) \to [0,\infty)$, $f(0) = 0$, and

$$|f(u) - f(v)| \le \int_{[0,c)} |u(a) - v(a)| m(da).$$

ℓ represents the Lebesgue measure and $\mathcal{L}_1([0,c); m+\ell)$ the space of real-valued Borel measurable functions x such that

$$\int_{[0,c)} |x(a)| m(da) + \int_{[0,c)} |x(a)| da < \infty.$$

The assumption $f(0) = 0$ means that the birth rate is 0 if the population size is 0. As state space we choose the abstract L space $X = \mathbf{R} \times L_1([0,c); \mathbf{R})$. However, the solutions u of the abstract Cauchy problem in which we will recast (31) will take values in the subspace $\{0\} \times L_1([0,\infty); \mathbf{R})$ which can be identified with $L_1([0,\infty); \mathbf{R})$. We define the operator

$$A(0,x) = \left(x(0), -\mathcal{F}\left(\frac{x}{\mathcal{F}}\right)' \right), \qquad x \in L_1([0,c)),$$

with $D(A)$ being the set of functions $x \in L_1([0,c))$ such that $x/\mathcal{F}$ is absolutely continuous and $\mathcal{F}(x/\mathcal{F})'$ is integrable on $[0,c)$. (Cf. [Thi98], Section 5). We find that

$$(\lambda - A)^{-1}(\xi, x) = (0, v), \qquad \xi \in \mathbf{R}, x \in L_1[0,c),$$

with

$$v(a) = \xi e^{-\lambda a} \mathcal{F}(a) + \int_0^a e^{-\lambda(a-s)} \frac{\mathcal{F}(a)}{\mathcal{F}(s)} x(s) ds, \qquad a \in [0,c).$$

We see that $(0,\infty)$ is contained in $\rho(A)$, $\|(\lambda - A)^{-1}\| \le 1/\lambda$ for $\lambda > 0$ and that $(\lambda - A)^{-1}$ maps the positive cone into itself, i.e., A is resolvent-positive. We make the following assumption

H8b: There is some $w > 0$ such that

$$\int_{[s,c)} e^{-w(a-s)} \frac{\mathcal{F}(a)}{\mathcal{F}(s)} m(da) \quad \text{is a bounded function of } s \in [0,c).$$

With this assumption, $D(A)$ is a subspace of $\mathcal{L}_1([0,c), m+\ell)$ and the following operator B is defined on $D(A)$,

$$B(0,x) = \left(\int_{[0,c)} x(a) m(da), 0 \right), \qquad x \in D(A).$$

Obviously B is a positive operator from $D(A)$ to X. Further $G(x) = (f(x), 0)$ is an operator from $D(A)$ to X satisfying the assumptions of Theorem 1.2, provided we can verify that the spectral radius of $F(\lambda) = B(\lambda - A)^{-1}$ is smaller than 1. But B maps X into $Z = \mathbf{R} \times \{0\}$, so the spectral radius of $B(\lambda - A)^{-1}$ is smaller than the norm of $B(\lambda - A)^{-1}$ taken on this closed subspace, which is

$$\|B(\lambda - A)^{-1}\|_Z = \int_{[0,c)} e^{-\lambda a} \mathcal{F}(a) m(da).$$

This expression converges to $m(\{0\})$ for $\lambda \to \infty$. So it is sufficient to assume that

$$m(\{0\}) < 1.$$

This is particularly satisfied if $m(\{0\}) = 0$, which means that an individual that has just been born does not reproduce.

8.1 Theorem. *Let the Assumptions* **H8a** *and* **H8b** *be satisfied and* $m(\{0\}) < 1$. *For each* $u_0 \in L_1([0,c), \mathbf{R}) = L_1[0,c)$, *there exists a unique Friedrichs solution* u *which is defined for all times. More precisely* $u(t, \cdot)$ *is continuous in* $t \geq 0$ *as a function with values in* $L_1[0,c)$, $u(0, \cdot) = u_0$, *and there exists a sequence of functions* $u_n : [0, \infty) \times [0, c) \to \mathbf{R}$ *with the following properties:*

(a) *For all* $t \geq 0$, $u_n(t, \cdot) \in L_1[0,c)$ *and*

$$\int_0^c |u_n(t,a) - u(t,a)| da \to 0 \qquad \textit{locally uniformly in } t \geq 0,$$

(b) *For all* $t \geq 0$, $u_n(t, \cdot)/\mathcal{F}$ *is absolutely continuous with* $\mathcal{F} \frac{\partial}{\partial a} \frac{u_n(t,\cdot)}{\mathcal{F}} \in L_1[0,c)$, $u_n(t, \cdot)$ *is differentiable in* t *as a function with values in* $L_1[0, \infty)$, *and*

$$\int_0^T \int_0^c \left| \frac{\partial}{\partial t} u_n(t,a) + \mathcal{F}(a) \frac{\partial}{\partial a} \frac{u_n(t,a)}{\mathcal{F}(a)} \right| da dt \to 0, \qquad n \to \infty$$

where $\frac{\partial}{\partial t} u_n(t,a)$ *stands for the derivative of* $u_n(t, \cdot)$ *with respect to* t,

(c) $u_n(t, \cdot) \in L_1([0,c); m + \ell)$ *and, for all* $T > 0$,

$$\int_0^T |u_n(t,0) - f(u_n(t, \cdot))| dt \to 0, \qquad n \to \infty.$$

Moreover $\Theta(t, u_0) = u(t, \cdot)$ *defines a semiflow on* $L_1[0,c]$. *If* f *maps non-negative functions in* $\mathcal{L}([0,c), m + \ell)$ *onto non-negative functions, then* Θ *is a positive semiflow, i.e., if the initial datum* u_0 *is non-negative, so is the solution* u.

Literatur

[Are87a] W. Arendt. Resolvent positive operators. *Proc. London Math. Soc. 54*, pages 321–359, 1987.

[Are87b] W. Arendt. Vector valued Laplace transforms and Cauchy problems. *Israel J. Math. 59*, pages 327–352, 1987.

[Are94] W. Arendt. Vector-valued versions of some representation theorems in real analysis. *Functional Analysis (Bierstedt, Pietsch, Ruess, Vogt; eds.), Marcel Dekker*, pages 33–50, 1994.

[CH87] S-N. Chow and Jack K. Hale. *Dynamics of Infinite Dimensional Systems.* Springer, Berlin, Heidelberg, New York, 1987.

[dPS87] G. da Prato and E. Sinestrari. Differential operators with nondense domain. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. 14 (2)*, pages 285–344, 1987.

[DvGLW95] O. Diekmann, S.A. van Gils, S.M. Verduyn Lunel, and H.-0. Walther. *Delay Equations. Functional-, Complex, and Nonlinear Analysis.* Springer, New York, Berlin, Heidelberg, London, Paris, Tokyo, Hongkong, Barcelona, Budapest, 1995.

[HP57] E. Hille and R. Phillips. *Functional Analysis and Semigroups.* AMS, 1957.

[KH89] H. Kellermann and M. Hieber. Integrated semigroups. *J. Func. Anal. 84*, pages 160–180, 1989.

[Miy66] I. Miyadera. On perturbation theory for semi-groups of operators. *Tohoku Math. J. 18*, pages 299–310, 1966.

[MOT96] T. Matsumoto, S. Oharu, and H.R. Thieme. Nonlinear perturbations of a class of integrated semigroups. *Hiroshima Math. J. 26*, pages 433–473, 1996.

[Neu88] F. Neubrander. Integrated semigroups and their applications to the abstract Cauchy problem. *Pac. J. Math. 135*, pages 111–155, 1988.

[Sch74] H.H. Schaefer. *Banach Lattices and Positive Operators.* Springer Verlag, Berlin-Heidelberg-New York, 1974.

[Thi90a] H. R. Thieme. Integrated semigroups and integral solutions to abstract Cauchy problems. *J. Math. Anal. Appl. 152*, pages 416–447, 1990.

[Thi90b] H. R. Thieme. Semiflows generated by the Lipschitz perturbations of non-densely defined operators. *Differential and Integral Equations 3*, pages 1035–1066, 1990.

[Thi96] H. R. Thieme. Positive perturbations of dual and integrated semigroups. *Advances in Mathematical Sciences and Applications 2*, pages 445–507, 1996.

[Thi98] H. R. Thieme. Positive perturbations of operator semigroups: Growth bounds, essential compactness and asynchronous exponential growth. *Discrete and Continuous Dynamical Systems 4*, pages 735–764, 1998.

[TV99] H. Thieme and H. Vosseler. A Stieltjes type convolution for integrated semigroups of strong bounded variation and L_p-solutions to the abstract Cauchy problem. preprint, 1999.

[Voi77] J. Voigt. On the perturbation theory for strongly continuous semigroups. *Math. Annalen 229*, pages 163–171, 1977.

[Voi89] J. Voigt. On resolvent positive operators and positive C_0-semigroups on AL-spaces. *Semigroup Forum 38*, pages 263–266, 1989.

[Web72] G.F. Webb. Continuous nonlinear perturbations of linear accretive operators in Banach spaces. *J. Funct. Anal. 10*, pages 191–203, 1972.

[Wei93] L. Weis. Inversion of the vector valued Laplace transform in $L_p(X)$-spaces. *Proceedings of the conference on Differential Equations in Banach spaces, Bologna 1991, Marcel Dekker*, pages 235–253, 1993.