BANACH SPACES WITH THE RIEMANN-LEBESGUE OR THE ANALYTIC RIEMANN-LEBESGUE PROPERTY

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Abstract. In this note we propose and discuss two geometric properties of Banach spaces related to the validity of the Lemma of Riemann-Lebesgue in spaces of weakly integrable functions. We show that the class of Banach spaces with the analytic Riemann-Lebesgue property is precisely the class of Banach spaces for which a weak form of a theorem of Ingham holds.

1. Introduction

The vector-valued tauberian theory of Fourier and Laplace transforms and its applications to asymptotics of abstract evolution equations in Banach spaces has been intensively studied since the emergence of the articles [2], [28]; we refer the reader to [3] for an account of the present state of the art in this field.

One key step in the proof of most of the important results, namely in complex tauberian theorems, is an application of the classical Lemma of Riemann-Lebesgue on Fourier transforms of integrable functions; see [24, Proof of Theorems I, II, p.468-469], [26, p.114], [4, Proof of Theorem 3.5], [12, Proof of Proposition 1.3], [3, Theorem 4.9.5] for examples. The Lemma of Riemann-Lebesgue is there applied to boundary values of analytic functions (Laplace transforms, power series) with suitable growth near the boundary. However, it is well known that in general Banach spaces these boundary values need not exist, or only in a weak sense, and so their existence is usually assumed. On the other hand, one could ask when the Lemma of Riemann-Lebesgue still holds for the boundary values in the weak sense; the answer whether this is true or not depends then on the geometry of the underlying Banach space. Motivated by this example we are led to the definition of the (analytic) Riemann-Lebesgue property (see Definitions 3.1 and 4.1 below).

In this note we present a few geometric results on Banach spaces with the (analytic) Riemann-Lebesgue property and relations to other well known properties. As an application we show that the Banach spaces with the analytic Riemann-Lebesgue property is precisely the class of Banach spaces for which a vector-valued version of Ingham's Theorem holds (see [24, Theorem II], [4, Theorem 3.5], [12, Theorem 3.4]). By the importance of Ingham's Theorem in tauberian theory, it

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is thus the class of Banach spaces in which weak tauberian conditions imply certain types of asymptotic behaviour (e.g. stability, asymptotic almost periodicity). Another application of the (analytic) Riemann-Lebesgue property in the harmonic analysis of operators on spaces of continuous functions is described at the end.

2. Preliminaries

Throughout this article, we denote by X a (complex) Banach space, and by X' its dual. For every $1 \leq p \leq \infty$ and every closed interval $J \subset \mathbb{R}$ we define the space $L_p^*(J; X')$ of all (equivalence classes of) functions $f : J \to X'$ which are weak* measurable and for which the order supremum

$$f^* := \sup\{|\langle f, x\rangle| : x \in X, \|x\| \leq 1\} \tag{2.1}$$

exists in the space $L_p(J)$. Equipped with the norm $\|f\|_{L_p^*} := \|f^*\|_{L_p}$, this space is a Banach space.

For every $f \in L_p^*(0, 2\pi; X')$ and every $g \in C[0, 2\pi]$ the integral

$$\int_0^{2\pi} g(\theta) f(\theta)\, d\theta \in X'$$

exists in the weak* sense. We can thus define, for every norming subspace Y of X',

$$L_{p,X}^*(0, 2\pi; Y') := \{f \in L_p^*(0, 2\pi; Y') : \mathcal{F}f(n) \in X \text{ for all } n \in \mathbb{Z}\},$$

where $\mathcal{F}f$ is the *Fourier transform* of $f \in L_p^*(0, 2\pi; Y')$ defined by

$$\mathcal{F}f(n) := \int_0^{2\pi} e^{-in\theta} f(\theta)\, \frac{d\theta}{2\pi}, \quad n \in \mathbb{Z}. \tag{2.2}$$

By Fejer's Theorem (or Weierstrass' Theorem) and Lebesgue's Dominated Convergence Theorem, for every $f \in L_{p,X}^*(0, 2\pi; Y')$ and every $g \in L_\infty(0, 2\pi)$, the integral $\int_0^{2\pi} f(t)g(t)\, dt$ belongs to X. We point out that, by [10, Corollary 2.8], the space $L_{p,X}^*(0, 2\pi; Y')$ does not depend on the choice of the norming subspace $Y \subset X'$.

It is clear that $L_p(0, 2\pi; X)$ is a closed subspace of $L_{p,X}^*(0, 2\pi; Y')$. A Banach space is said to have the *Radon-Nikodym property* if both spaces coincide (see [14, Definition 3, p.61], [1, Theorem 1.5]).

Let $\mathbb{D}$ be the unit disc in $\mathbb{C}$, and define for $1 \leq p < \infty$ the space

$$h_p(\mathbb{D}; X) := \{f : \mathbb{D} \to X : f \text{ is harmonic and } \sup_{0<r<1} \int_0^{2\pi} \|f(re^{i\theta})\|^p\, d\theta < \infty\}$$

and define

$$h_\infty(\mathbb{D}; X) := \{f : \mathbb{D} \to X : f \text{ is harmonic and } \sup_{z\in\mathbb{D}} \|f(z)\| < \infty\}$$

with obvious norm. The *Poisson operator*

$$\Psi : L_{p,X}^*(0, 2\pi; X'') \to h_p(\mathbb{D}; X)$$

which maps every function $f \in L_{p,X}^*(0, 2\pi; X'')$ to the Poisson integral defined by

$$(\Psi f)(re^{i\theta}) := \int_0^{2\pi} P(r, \theta - \alpha) f(\alpha)\, d\alpha,$$

is isometric for every $1 \le p \le \infty$, and isomorphic if $p > 1$. Here P is the *Poisson kernel*:

$$P(r,\theta) := \frac{1-r^2}{1-2r\cos\theta + r^2}, \quad r \in [0,1), \theta \in [0,2\pi]. \tag{2.3}$$

Via this isomorphism, the Fourier transform $\mathcal{F}f$ of an element $f \in h_p(\mathcal{D};X)$ is just the sequence of Laurent coefficients of f.

We finally remark that the space $\mathcal{L}(L_q(0,2\pi),X)$ is isometrically isomorphic to $L^*_{p,X}(0,2\pi;X'')$ for every $1 \le q < \infty$, and $\frac{1}{p}+\frac{1}{q}=1$ (see [15, Theorem 8, p.279], [14]).

Moreover, the space $V_p(0,2\pi;X)$ of all absolutely continuous functions $F : [0,2\pi] \to X$ of bounded p-variation (see [14]) such that $F(0)=0$ is also isomorphic to $L^*_{p,X}(0,2\pi;X'')$ via integration.

3. The Riemann-Lebesgue property

Definition 3.1. We say that the Banach space X has the *Riemann-Lebesgue property* (and we write $X \in (RL)$) if the Lemma of Riemann-Lebesgue holds in the space $L^*_{1,X}(0,2\pi;X'')$, i.e. for every $f \in L^*_{1,X}(0,2\pi,X'')$ one has $\lim_{|n|\to\infty} \|\mathcal{F}f(n)\| = 0$.

Remark 3.2. Clearly, the Riemann-Lebesgue property is inherited by subspaces. Moreover, a Banach space X has the Riemann-Lebesgue property if and only if each separable subspace has it.

Proposition 3.3. *A Banach space X has the Riemann-Lebesgue property if and only if there exists $1 \le p \le \infty$ such that for every $f \in L^*_{p,X}(0,2\pi;X'')$ one has $\lim_{|n|\to\infty} \|\mathcal{F}f(n)\| = 0$.*

Proof. By continuity of the Fourier transform, it will suffice to show that the space $L^*_{\infty,X}(0,2\pi;X'')$ is dense in $L^*_{1,X}(0,2\pi;X'')$. So let $f \in L^*_{1,X}(0,2\pi;X'')$ be fixed and let f^* be as in (2.1).

First suppose that there exists a constant $C > 0$ such that $f^* \ge C$ almost everywhere. In this case we have $\ln C \le \ln f^* \le 1 + f^*$, so that the outer function φ associated with f^* exists and is in $H_1(\mathbb{D})$. Define $h(\theta) = 1/\varphi(e^{i\theta})$ and $h_m(\theta) = 1/\varphi(r_m e^{i\theta})$, where $(r_m)_{m\in\mathbb{N}} \uparrow 1$ is a fixed sequence in $(0,1)$.

We will show that $fh \in L^*_{\infty,X}(0,2\pi;X'')$. For this it suffices to show that $\mathcal{F}(fh)(n) \in X$ for $n \in \mathbb{Z}$. For every $m \in \mathbb{N}$, $n \in \mathbb{Z}$, one has $\mathcal{F}(fh_m)(n) = \sum_{k\in\mathbb{Z}} r_m^{|k|}\mathcal{F}h(k)\mathcal{F}f\ (n-k) \in X$ and it is not hard to show that $(\mathcal{F}(fh_m)(n))_{m\ge 0}$ is a Cauchy sequence in X. Denote by a_n its limit. For every $n \in \mathbb{Z}$ and $m \in \mathbb{N}$

$$\|\mathcal{F}(fh)(n) - \mathcal{F}(fh_m)(n)\| \le \int_0^{2\pi} f^*(\theta)|h(\theta) - h_m(\theta)| \,\frac{d\theta}{2\pi}$$

By Lebesgue's Dominated Convergence Theorem, $\mathcal{F}(fh)(n) = \lim_{m\to\infty} \mathcal{F}(fh_m)(n) = a_n \in X$, and so $fh \in L^*_{\infty,X}(0,2\pi;X'')$. To show that f is the limit of elements in $L^*_{\infty,X}(0,2\pi;X'')$, it will suffice to use the identity $f = (fh)/h$ and approximate $1/h \in H^1(\mathbb{D})$ by the sequence of functions $\phi_m(z) = 1/h(r_m z)$ and repeate the same argument as above.

For the general case, we consider the Banach space $Y = X \oplus \mathbb{C}$ with the norm $\|(x,\lambda)\|_Y = \|x\| + |\lambda|$, and the function $G(\theta) = (f(\theta), 2+\cos\theta)$. It is easy to see that $G \in L^*_{1,Y}(0,2\pi;Y'')$ and $\|G(\theta)\| \ge 1$ for all $\theta \in [0,2\pi]$. We deduce from the first part of the proof that there exists $G_n = (f_n,h_n) \in L^*_{\infty,Y}(0,2\pi;Y'')$ such that

$\lim_{n\to\infty} \|G_n - G\|_1 = 0$. Hence the function f is the limit of f_n in $L^*_{1,X}(0, 2\pi; X'')$. This finishes the proof. $\square$

Recall that a Banach space X has the *complete continuity property* if every operator $T : L^1(0, 2\pi) \to X$ is Dunford-Pettis, i.e. it maps weakly compact sets into norm compact sets (see [31, Definition II.3], [30, Definition 3.7.6]).

Proposition 3.4. *Let X be a Banach space, and consider the following statements:*
(a) The space X is reflexive.
(b) The space X has the Radon-Nikodym property.
(c) The space X has the complete continuity property.
(d) The space X has the Riemann-Lebesgue property.
Then (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (d).

Proof. The implication (a) $\Rightarrow$ (b) is proved in [14, Corollary 13, p.76], and (b) $\Rightarrow$ (c) follows from the respective definitions (see [31], or [14, Lemma 11, p.74] in combination with [14, Theorem 15, p.76], [30, Theorem 2.5.4]).

In order to show (c) $\Rightarrow$ (d), let X be a Banach space with the complete continuity property. By definition, every operator $L_1(0, 2\pi) \to X$ is Dunford-Pettis. In particular, the weak null sequence $(e_n)_{n\in\mathbb{Z}}$, defined by $e_n(t) := e^{-int}$, is mapped into a norm null sequence. Identifying the space $\mathcal{L}(L_1(0, 2\pi), X)$ with $L^*_{\infty,X}(0, 2\pi; X'')$, this means that the Fourier transform of every function in $L^*_{\infty,X}(0, 2\pi; X'')$ converges to 0 in norm. The claim follows from Proposition 3.3. $\square$

Remarks 3.5. (a) The proof of Proposition 3.4 (b) $\Rightarrow$ (d) is just a simple application of the classical Lemma of Riemann-Lebesgue.
(b) By Proposition 3.4, every reflexive Banach space and every separable dual space (for example l^1, see [14, Theorem 1, p.79]) have the Riemann-Lebesgue property. By [31, Theorem 4] and [31, Theorem 13], the dual of the James tree space ([25], [27, p.103]) and the Bourgain-Rosenthal space ([9]) provide nontrivial examples of Banach spaces with the Riemann-Lebesgue property and without the Radon-Nikodym property (see also [14, p.214]). In fact, these Banach spaces have the complete continuity property.

Problem 3.6. Is the converse of Proposition 3.4 (c) $\Rightarrow$ (d) true?

For the definition of *B-convexity* we refer to [21], [22].

Proposition 3.7. *If X is B-convex, then X has the Riemann-Lebesgue property.*

Proof. A Banach space is B-convex if and only if it has nontrivial Fourier type (see [7], [29]). Hence, if X is B-convex, then there exist $p > 1$ and $c \geq 0$ such that

$$\|(\mathcal{F}f(n))_{n\in\mathbb{Z}}\|_{l_q} \leq c\|f\|_{L_p}$$

for every $f \in L_p(0, 2\pi; X)$ ($\frac{1}{p} + \frac{1}{q} = 1$). Hence, for every $f \in h_p(0, 2\pi; X)$ we obtain

$$\begin{aligned} \|(\mathcal{F}f(n))_{n\in\mathbb{Z}}\|_{l_q} &= \sup_{0<r<1} \|(r^{|n|}\mathcal{F}f(n))_{n\in\mathbb{Z}}\|_{l_q} \\ &\leq c \cdot \sup_{0<r<1} \|f(re^{i\cdot})\|_{L_p} \\ &= c \cdot \|f\|_{h_p} \quad < \infty. \end{aligned}$$

This implies in particular $\lim_{|n|\to\infty} \|\mathcal{F}f(n)\| = 0$. The claim follows from Proposition 3.3. $\square$

Remark 3.8. By Proposition 3.7, the example of a Banach space X which is B-convex and which does not have the (analytic) Radon-Nikodym property (see [23, Example 2.4], [33], [34]) provides another nontrivial example of a Banach space with the Riemann-Lebesgue property. Clearly, the converse of Proposition 3.7 is not true; the space l^1 has the Radon-Nikdoym property, but it is not B-convex.

Proposition 3.9. *Let X be such that there exists a norming subspace Y of X' with $l^1 \not\subset Y$. Then X has the Riemann-Lebesgue property.*

Proof. Let $f \in L^*_{1,X}(0, 2\pi; Y')$, and define an operator $T \in \mathcal{L}(Y, c_0(\mathbb{Z}))$ which is the composition of the operator $T_1 \in \mathcal{L}(Y, L^1(0, 2\pi))$ given by $T_1 x' := \langle f, x'\rangle$, and the Fourier transform $\mathcal{F} : L^1(0, 2\pi) \to c_0(\mathbb{Z})$. We claim that under the assumption $l^1 \not\subset Y$ the operator T_1 is compact.

In fact, by Rosenthal's characterization of Banach spaces not containing l^1 (see [36], [17]), every bounded sequence $(x'_n)_{n\in\mathbb{N}} \subset Y$ has a subsequence which is a weak Cauchy sequence. Hence, by Lebesgue's Dominated Convergence Theorem and the definition of the space $L^*_1(0, 2\pi; Y')$, for every bounded sequence $(x'_n)_{n\in\mathbb{N}}$ there exists a subsequence $(x'_{n_k})_{k\in\mathbb{N}}$ such that $T_1 x'_{n_k} = \langle f, x'_{n_k}\rangle$ converges in $L^1(0, 2\pi)$. Thus the operator T_1 is compact.

Hence, the operator T is compact. Since every compact subset of $c_0(\mathbb{Z})$ is contained in some order interval, there exists a sequence $a = (a_n)_{n\in\mathbb{Z}} \in c_0(\mathbb{Z})$ such that $TB_Y \subset [-a, a]$. This implies

$$(\|\mathcal{F}f(n)\|)_{n\in\mathbb{Z}} = \sup_{\substack{x'\in Y \\ \|x'\|\leq 1}} (|\langle x', \mathcal{F}f(n)\rangle|)_{n\in\mathbb{Z}} \leq (a_n)_{n\in\mathbb{Z}},$$

i.e. $\lim_{|n|\to\infty} \|\mathcal{F}f(n)\| = 0$. ☐

Remark 3.10. Compare Proposition 3.9 with [32] and [31, Theorem II.4]. Proposition 3.9 provides a second proof of Proposition 3.7. If X is B-convex, then X' is B-convex by [21, Corollary II.4]. Hence, by the definition of B-convexity, the dual X' does not contain l^1 [22], and the Proposition 3.7 follows from Proposition 3.9.

Proposition 3.11. *If X has the Riemann-Lebesgue property, then, for every measure space (Ω, Σ, μ) and every $1 < p < \infty$, the space $L_p(\Omega; X)$ has also the Riemann-Lebesgue property.*

Proof. By Proposition 3.3, it will suffice to show that for all $f \in h_p(\mathbb{D}; L_p(\Omega; X))$, we have $\lim_{|n|\to\infty} \mathcal{F}f(n) = 0$ in $L_p(\Omega; X)$. For this purpose let

$$f(z) = \sum_{n\geq 0} a_n z^n + \sum_{n<0} a_n \overline{z}^{-n}$$

for $z \in \mathbb{D}$. We have $a_n \in L_p(\Omega; X)$ and using the subharmonicity of the function $\|f(\cdot)(\omega)\|^p$, one shows that

$$\int_\Omega \sup_{0\leq r<1} \int_0^{2\pi} \|f(re^{i\theta})(\omega)\|^p \, d\theta \, d\mu(\omega) \leq \|f\|^p_{h_p} < \infty.$$

This implies that for almost all $\omega \in \Omega$, the function $z \to f(z)(\omega)$ is in $h_p(\mathbb{D}; X)$. Since X has the Riemann-Lebesgue property, we have $\lim_{|n|\to\infty} a_n(\omega) = 0$ in X.

For $n \in \mathbb{N}$ and $\omega \in \Omega$,

$$\begin{aligned} \|a_n(\omega)\|^p &\leq \sup_{0<r<1} \Big(\int_0^{2\pi} \|f(re^{i\theta})(\omega)\| \frac{d\theta}{2\pi}\Big)^p \\ &\leq \sup_{0<r<1} \int_0^{2\pi} \|f(re^{i\theta})(\omega)\|^p \frac{d\theta}{2\pi} \\ &\leq \lim_{r\uparrow 1} \int_0^{2\pi} \|f(re^{i\theta})(\omega)\|^p \frac{d\theta}{2\pi}. \end{aligned}$$

By Lebesgue's Dominated Convergence Theorem, $\lim_{n\to\infty} a_n = 0$ in $L_p(\Omega; X)$. The same argument shows that $\lim_{n\to-\infty} a_n = 0$ in $L^p(\Omega; X)$. □

4. The analytic Riemann-Lebesgue property

Let X be a Banach space. For every $1 \leq p \leq \infty$ and every norming subspace Y of X' we define

$$L^{*,+}_{p,X}(0, 2\pi; Y') := \{f \in L^*_{p,X}(0, 2\pi; Y') : \mathcal{F}f(n) = 0 \text{ for all } n < 0\}.$$

Then the space $L^{*,+}_{p,X}(0, 2\pi; Y')$ is a closed subspace of $L^*_{p,X}(0, 2\pi; Y')$, and for every $1 \leq p \leq \infty$ the restriction of the Poisson operator Ψ to $L^{*,+}_{p,X}(0, 2\pi; Y')$ is an isometric isomorphism onto the Hardy space $H_p(\mathbb{D}; X) := \{g \in h_p(\mathbb{D}; X) : g \text{ is analytic}\}$.

Definition 4.1. We say that the Banach space X has the *analytic Riemann-Lebesgue property* ($X \in (ARL)$ for short) if the Lemma of Riemann-Lebesgue holds in the space $L^{*,+}_{1,X}(0, 2\pi; X'')$, i.e. for every $f \in L^{*,+}_{1,X}(0, 2\pi, X'')$ one has $\lim_{n\to\infty} \|\mathcal{F}f(n)\| = 0$.

Remark 4.2. Trivially, the Riemann-Lebesgue property implies the analytic Riemann-Lebesgue property. We will show below that the converse is not true (see Proposition 4.7). Moreover, the analytic Riemann-Lebesgue property is also inherited by subspaces, and a Banach space X has the analytic Riemann-Lebesgue property if and only if each separable subspace has it.

Proposition 4.3. *A Banach space X has the analytic Riemann-Lebesgue property if and only if there exists $1 \leq p \leq \infty$ such that for every function $f \in L^{*,+}_{p,X}(0, 2\pi; X)$ one has $\lim_{n\to\infty} \|\mathcal{F}f(n)\| = 0$.*

Proof. By continuity of the Fourier transform, it suffices to show that $L^{*,+}_{\infty,X}(0, 2\pi; X'')$ is dense in $L^{*,+}_{1,X}(0, 2\pi; X'')$. By isomorphy, this is equivalent to showing that $H_\infty(\mathbb{D}; X)$ is dense in $H_1(\mathbb{D}; X)$. So let $F \in H_1(\mathbb{D}; X)$. Then there exists a function $G \in H_\infty(\mathbb{D}; X)$ and a function $f \in H_1(\mathbb{D})$ such that $F = G \cdot f$. Next, the functions $f_r \in H_\infty(\mathbb{D})$ defined by $f_r(z) := f(rz)$ $(0 < r < 1)$ converge to f in $H_1(\mathbb{D})$ as $r \uparrow 1$. This implies that $G \cdot f_r \in H_\infty(\mathbb{D}; X)$ converges to F in $H_1(\mathbb{D}; X)$ as $r \uparrow 1$, and the claim is proved. □

Recall that a Banach space X has the *analytic Radon-Nikodym property* if every function $f \in L^{*,+}_{1,X}(0, 2\pi; X'')$ already belongs to $L_1(0, 2\pi; X)$ (see [10], [11]). A Banach space X has the *analytic complete continuity property* if every function $f \in L^{*,+}_{\infty,X}(0, 2\pi; X'')$ induces a Dunford-Pettis operator from $L^1(0, 2\pi)$ into X ([35]).

Proposition 4.4. *Let X be a Banach space, and consider the following statements:*
(a) The space X has the analytic Radon-Nikodym property.
(b) The space X has the analytic complete continuity property.
(c) The space X has the analytic Riemann-Lebesgue property.
Then (a) $\Rightarrow$ (b) $\Rightarrow$ (c).

Proof. The implication (a) $\Rightarrow$ (b) follows from the respective definitions (see [35], and [14, Lemma 11, p.74] in combination with [14, Theorem 15, p.76], [30, Theorem 2.5.4]).

The proof of the implication (b) $\Rightarrow$ (c) is similar to the proof of Proposition 3.4 (c) $\Rightarrow$ (d), using [35, Theorem 2.8] which says that every analytic operator $L^1(0,2\pi) \to X$ is Dunford-Pettis if X has the analytic complete continuity property. □

Remarks 4.5. (a) The proof of Proposition 4.4 (a) $\Rightarrow$ (c) is a simple application of the classical Lemma of Riemann-Lebesgue.
(b) It follows from results in [18] and [19] that the Davies-Figiel-Johnson interpolation space Z from [35, p.977] has the complete continuity property, but not the analytic Radon-Nikodym property. That space, the space $L_1(\Omega; Z)$ (see Proposition 4.11 below) and the space from Remark 3.8 provide thus nontrivial examples of Banach spaces with the analytic Riemann-Lebesgue property.
(c) By [35, Theorem 3.1], every Banach space in the Ghoussub-Rosenthal class (see [20]) has the analytic complete continuity property, and thus the analytic Riemann-Lebesgue property.

Problem 4.6. Is the converse of Proposition 4.4 (b) $\Rightarrow$ (c) true?

Proposition 4.7. *The space $L_1(0,2\pi)$ has the analytic Riemann-Lebesgue property, but it fails the Riemann-Lebesgue property.*

Proof. The first part of the statement follows from Proposition 4.4 and the fact that $L_1(0,2\pi)$ has the analytic Radon-Nikodym property (see [11, Theorem 1] and [30]).

In order to show that $L_1(0,2\pi)$ fails the Riemann-Lebesgue property, we consider the following harmonic function f defined on $\mathbb{D}$ with values in $L_1(0,2\pi)$:

$$f(z)(\theta) = f(re^{i\sigma})(\theta) := P(r, \sigma + \theta),$$

where P is the Poisson kernel (see (2.3)). Since $P(r,\cdot) \in L_1(0,2\pi)$ and $\|P(r,\cdot)\|_{L_1} = 2\pi$ for every $0 \leq r < 1$, the function f is bounded from $\mathbb{D}$ with values in $L_1(0,2\pi)$. In fact, $f \in h_\infty(\mathbb{D}; L_1(0,2\pi)) = L^*_{\infty, L_1(0,2\pi)}(0,2\pi; L_1(0,2\pi)'')$. We have $\mathcal{F}f(n)(\theta) = e^{in\theta}$, in particular $\|\mathcal{F}f(n)\| = 2\pi$, so $L_1(0,2\pi)$ fails the Riemann-Lebesgue property. □

A Banach lattice X has the analytic Radon-Nikodym property if and only if it does not contain c_0 ([11, Theorem 1]). Since c_0 does not have the analytic Riemann-Lebesgue property, this implies that a Banach lattice has the analytic Riemann-Lebesgue property if and only if it does not contain c_0. However, for general Banach spaces X the property not to contain c_0 does not characterize the analytic Riemann-Lebesgue property.

Proposition 4.8. *The quotient space L^1/H^1_0 does not contain c_0 and it does not have the analytic Riemann-Lebesgue property.*

Proof. By [8] the space L^1/H_0^1 has cotype 2, and thus it can not contain c_0. Another argument is given in [11, Proposition 4]: the space L^1/H_0^1 is weakly sequentially complete while c_0 is not. This proves the first part of the statement.

For the second part we consider the functions $a_n(\theta) := e^{in\theta}$ ($n \in \mathbb{N}_0$), viewed as elements in L^1/H_0^1, and we define $f(z) := \sum_{n=0}^{\infty} a_n z^n$ ($z \in \mathbb{D}$).

One checks that $\|a_n\|_{L^1/H_0^1} = 2\pi$ for every $n \in \mathbb{N}_0$. Moreover, similarly to Proposition 4.7,

$$\begin{aligned} \|f(re^{i\phi})\|_{L^1/H_0^1} &\leq \left\| \sum_{n=0}^{\infty} e^{in\cdot} r^n e^{in\phi} + \sum_{n=1}^{\infty} e^{-in\cdot} r^n e^{-in\phi} \right\|_{L^1} \\ &= \left\| \sum_{n=-\infty}^{\infty} r^{|n|} e^{in(\cdot+\phi)} \right\|_{L^1} \\ &= \int_0^{2\pi} P(r, \theta + \phi)\, d\theta \quad = 2\pi. \end{aligned}$$

This implies $f \in H^\infty(\mathbb{D}; L^1/H_0^1)$, but $\mathcal{F}f(n) = a_n$ does not converge to 0, and thus L^1/H_0^1 does not have the analytic Riemann-Lebesgue property. □

Remark 4.9. Proposition 4.8 generalizes [11, Proposition 4] (resp. [35, Proposition 3.3]) saying that L^1/H_0^1 does not have the analytic Radon-Nikodym property (resp. the analytic complete continuity property).

Proposition 4.10. *If a Banach space X has the analytic Riemann-Lebesgue property, then every weak l^1-sequence $(x_n)_{n\in\mathbb{N}_0} \subset X$ is a norm null sequence.*

Proof. Let $(x_n)_{n\in\mathbb{N}_0}$ be a weak l^1-sequence in X, and assume that X has the analytic Riemann-Lebesgue property. Define the analytic function $f(z) := \sum_{n=0}^{\infty} x_n z^n$ ($z \in \mathbb{D}$). Then f is weakly bounded by the assumption on $(x_n)_{n\in\mathbb{N}_0}$, and thus f is bounded by Banach-Steinhaus. The claim follows from the definition of the analytic Riemann-Lebesgue property. □

The same proof as in Proposition 3.11 shows

Proposition 4.11. *If X has the analytic Riemann-Lebesgue property, then, for every measure space (Ω, Σ, μ) and every $1 \leq p < \infty$, the space $L_p(\Omega; X)$ has also the analytic Riemann-Lebesgue property.*

5. Applications

5.1. **Fourier transforms on $\mathbb{R}$.** Let $L^*_{p,X}(\mathbb{R}; X'')$ ($1 \leq p \leq \infty$) be the space of all functions $f \in L^*_p(\mathbb{R}; X'')$ such that $\int_{\mathbb{R}} g(t) f(t)\, dt \in X$ for all $g \in C_c(\mathbb{R})$.

If $f \in L^*_{1,X}(\mathbb{R}; X'')$, then, by Lebesgue's Dominated Convergence Theorem, for every bounded continuous function g, the integral $\int_{\mathbb{R}} g(t) f(t)\, dt$ exists and belongs to X. We can thus define for every $f \in L^*_{1,X}(\mathbb{R}; X'')$ the *Fourier transform* $\mathcal{F}f$ by

$$\mathcal{F}f(\eta) := \int_{\mathbb{R}} e^{-i\eta t} f(t)\, dt, \quad \eta \in \mathbb{R}, \tag{5.1}$$

and the *inverse Fourier transform* $\mathcal{F}^{-1}f$ by

$$\mathcal{F}^{-1}f(\eta) := \frac{1}{2\pi} \int_{\mathbb{R}} e^{i\eta t} f(t)\, dt, \quad \eta \in \mathbb{R}. \tag{5.2}$$

A simple truncation and density argument shows the following lemma.

Lemma 5.1. *A Banach space X has the Riemann-Lebesgue property if and only if for every $f \in L^*_{1,X}(\mathbb{R}; X'')$ the Fourier transform belongs to $C_0(\mathbb{R}; X)$.*

Let $L^{*,+}_{1,X}(\mathbb{R}; X'')$ be the subspace of all functions $f \in L^*_{1,X}(\mathbb{R}; X'')$ such that the inverse Fourier transform $\mathcal{F}^{-1}f$ is supported in $\mathbb{R}_+$. Via the Poisson integral on $\mathbb{R}$ (see e.g. [5, p.70 and Theorem 3.9]) the space $L^{*,+}_{1,X}(\mathbb{R}; X'')$ is isometrically isomorphic to the Hardy space $H_1(\mathbb{C}_+; X)$ of all holomorphic functions $f : \mathbb{C}_+ \to X$ such that

$$\|f\|_1 := \sup_{\xi>0} \int_{\mathbb{R}} \|f(\xi + i\eta)\| \, d\eta < \infty.$$

Here, $\mathbb{C}_+ := \{z \in \mathbb{C} : \operatorname{Re} z > 0\}$.

Proposition 5.2. *A Banach space X has the analytic Riemann-Lebesgue property if and only if for every $f \in L^{*,+}_{1,X}(\mathbb{R}; X'')$ the Fourier transform belongs to $C_0(\mathbb{R}; X)$.*

Proof. Let $f \in L^*_{1,X}(\mathbb{R}; X'')$, and assume that $\mathcal{F}^{-1}f(\eta) = 0$ for $\eta < 0$. Since $\mathcal{F}f(\eta) = \mathcal{F}^{-1}f(-\eta)$ for $\eta \in \mathbb{R}$, we need to show that $\lim_{\eta\to\infty} \mathcal{F}^{-1}f(\eta) = 0$.

For $n \in \mathbb{N}$, let $\phi_n(t) = \sum_{j=-n}^{n} f(2\pi j - t)$ for $t \in [0, 2\pi]$. It is clear that $\phi_n \in L^*_1(0, 2\pi; X'')$ and the sequence ϕ_n is a Cauchy sequence in $L^*_1(0, 2\pi; X'')$. Since $L^*_1(0, 2\pi; X'')$ is a Banach space, ϕ_n converges to some element $\phi \in L^*_1(0, 2\pi; X'')$. We claim that $\phi \in L^*_{1,X}(0, 2\pi; X'')$. Indeed, for $k \in \mathbb{Z}$ and $x' \in X'$, one easily shows that $\langle x', \mathcal{F}\phi(k)\rangle = \langle x', \mathcal{F}^{-1}f(k)\rangle$, and so $\mathcal{F}\phi(k) = \mathcal{F}^{-1}f(k) \in X$ for every $k \in \mathbb{Z}$ and $\mathcal{F}\phi(k) = 0$ for $k \in \mathbb{Z} \setminus \mathbb{N}$. By assumption X has the analytic Riemann-Lebesgue property, so $\lim_{k\to\infty} \mathcal{F}^{-1}f(k) = \lim_{k\to\infty} \mathcal{F}\phi(k) = 0$. Finally, for $\lambda > 0$, let $f_\lambda \in L^*_{1,X}(\mathbb{R}; X'')$ be defined by $f_\lambda(t) = \lambda f(\lambda t)$ and denote by ϕ_λ the corresponding function in $L^*_{1,X}(0, 2\pi; X'')$. It is clear that $f_\lambda \in L^{*,+}_{1,X}(\mathbb{R}; X'')$ and it is not hard to show that $\mathcal{F}\phi_\lambda(k) = \mathcal{F}^{-1}f(k/\lambda)$ for $k \in \mathbb{Z}$ and so $\lim_{k\to\infty} \mathcal{F}^{-1}f(k/\lambda) = 0$ for $\lambda > 0$. This implies that $\lim_{\eta\to\infty} \mathcal{F}^{-1}f(\eta) = 0$.

In order to show the converse, assume that X does not have the analytic Riemann-Lebesgue property. Then, by Proposition 4.3, there exists a function $f \in H_\infty(\mathbb{D}; X)$ such that $\limsup_{n\to\infty} \|\hat{f}(n)\| > 0$. Define a function $g : \mathbb{R}_+ \to X$ by

$$g(t) := \mathcal{F}f(n) \quad \text{if} \quad t \in [n, n+1).$$

Then $\hat{g}(\lambda) = \frac{1-e^{-\lambda}}{\lambda} f(e^{-\lambda})$ for every $\lambda \in \mathbb{C}_+$. Thus, the Laplace transform $\hat{g}$ is bounded on $\mathbb{C}_+$. Let $\varphi \in \mathcal{D}(\mathbb{R}_+)$ be a test function, and define $g_\varphi := g * \varphi \in BUC(\mathbb{R}_+; X)$. Then $\widehat{g_\varphi}(\lambda) = \hat{g}(\lambda)\hat{\varphi}(\lambda)$ for every $\lambda \in \mathbb{C}_+$, so that $\widehat{g_\varphi} \in H_1(\mathbb{C}_+; X)$. Choosing $\varphi \in \mathcal{D}(\mathbb{R}_+)$ such that $\int_{\mathbb{R}_+} \varphi = 1$ and $\operatorname{supp} \varphi \subset [0, 1]$, one easily verifies that $\limsup_{t\to\infty} \|g_\varphi(t)\| > 0$. □

Remark 5.3. For $\lambda > 0$, define $L^{*,+}_{1,X,\lambda}(\mathbb{R}; X'') = \{f \in L^*_{1,X}(\mathbb{R}; X'') : \quad \mathcal{F}^{-1}f(\lambda k) = 0 \text{ for every } k \in \mathbb{Z} \setminus \mathbb{N}\}$. Then the argument used in the proof of Proposition 5.2 shows that X has the analytic Riemann-Lebesgue property if and only if for every $f \in L^{*,+}_{1,X,\lambda}(\mathbb{R}; X'')$, one has $\lim_{k\to\infty} \mathcal{F}^{-1}f(\lambda k) = 0$.

5.2. **Ingham's Theorem.**

Proposition 5.4. *Let $f \in L_\infty(\mathbb{R}_+; X)$, $\eta_0 \in \mathbb{R}$ and $\varepsilon > 0$ be such that*

$$\limsup_{\xi\to 0} \int_{\eta_0-\varepsilon}^{\eta_0+\varepsilon} \|\hat{f}(\xi + i\eta)\| \, d\eta < \infty,$$

where $\hat{f}$ denotes the Laplace transform of f. Let $\varphi \in \mathcal{S}(\mathbb{R})$ be such that $\operatorname{supp} \mathcal{F}\varphi \subset (\eta_0 - \varepsilon, \eta_0 + \varepsilon)$. *If X has the analytic Riemann-Lebesgue property, then $f * \varphi \in C_0(\mathbb{R}; X)$.*

Proof. Let f and φ be as in the assumptions. Put $g := f * \varphi \in BUC(\mathbb{R}; X)$, $g_+(t) := g(t)$ for $t \geq 0$, $g_+(t) := 0$ for $t < 0$, and $g_- := g - g_+$.

Similarly to the proof of [12, Proposition 3.1], i.e. using conformal mappings and Hardy spaces on Jordan domains, one shows that there exists $\tilde{f} \in L^*_{1,X}(\eta_0 - \varepsilon, \eta_0 + \varepsilon; X'')$ such that for every $x' \in X'$

$$\lim_{\xi \to 0} \langle x', \hat{f}(\xi + i\eta)\rangle = \langle \tilde{f}(\eta), x'\rangle \quad a.e.\ \eta.$$

Hence, for every $\psi \in \mathcal{S}(\mathbb{R})$, by Plancherel's Theorem,

$$\begin{aligned}
\langle g, \mathcal{F}\psi\rangle &= \int_{\mathbb{R}} \int_0^\infty f(s)\varphi(t-s)\, ds \mathcal{F}\psi(t)\, dt \\
&= \int_0^\infty f(s) \int_{\mathbb{R}} \varphi(t-s)\mathcal{F}\psi(t)\, dt\, ds \\
&= w - \lim_{\xi\to 0} \int_0^\infty e^{-\xi s} f(s) \int_{\mathbb{R}} \varphi(t-s)\mathcal{F}\psi(t)\, dt\, ds \\
&= w - \lim_{\xi\to 0} \int_{\mathbb{R}} \hat{f}(\xi + i\eta)\mathcal{F}\varphi(\eta)\psi(\eta)\, d\eta \\
&= \int_{\eta_0-\varepsilon}^{\eta_0+\varepsilon} \tilde{f}(\eta)\mathcal{F}\varphi(\eta)\psi(\eta)\, d\eta,
\end{aligned}$$

which implies $\mathcal{F}g = \tilde{f}\cdot\mathcal{F}\varphi =: \tilde{g} \in L^*_{1,X}(\mathbb{R}; X'')$ in the sense of distributions.

For every $t \in \mathbb{R}$

$$\|g(t)\| \leq \|f\|_\infty \cdot \int_{-\infty}^t \varphi(s)\, ds,$$

so that, since φ is rapidly decreasing, the function g_- belongs to $L^1(\mathbb{R}; X)$ and $\lim_{t\to-\infty} g_-(t) = 0$. By the Lemma of Riemann-Lebesgue, $\mathcal{F}g_- \in C_0(\mathbb{R}; X)$. By linearity, $\mathcal{F}g_+ = \tilde{g} - \mathcal{F}g_-$ in the sense of distributions.

Let $\psi \in \mathcal{D}(0,\infty)$. Then $g_+ * \psi \in BUC(\mathbb{R}; X)$ has support in $\mathbb{R}_+$, and $\mathcal{F}(g_+ * \psi) = (\tilde{g} - \mathcal{F}g_-)\mathcal{F}\psi$. Hence, the Fourier transform $\mathcal{F}(g_+ * \psi)$ belongs to $L^{*,+}_{1,X}(\mathbb{R}; X)$. Since X has the analytic Riemann-Lebesgue property, the function $g_+ * \psi$ belongs to $C_0(\mathbb{R}; X)$ by Proposition 5.2. Choosing an approximate identity $(\psi_n)_{n\in\mathbb{N}} \subset \mathcal{D}(0,\infty) \subset L^1(\mathbb{R}_+)$, and using the uniform continuity of g_+, we thus deduce $g_+ \in C_0(\mathbb{R}_+; X)$. Hence, $g = f * \varphi \in C_0(\mathbb{R}; X)$, and the claim is proved. □

Let X be a complex Banach space, and define

$$H^1_{loc}(\mathbb{C}_+; X) := \{g : \mathbb{C}_+ \to X \quad : \quad g \text{ is analytic, and for every } R > 0 \quad \limsup_{\xi\to 0^+} \int_{-R}^R \|g(\xi + i\eta)\|\, d\eta < \infty\}.$$

Theorem 5.5 (Ingham). *Let X be a Banach space with the analytic Riemann-Lebesgue property, and let $f \in BUC(\mathbb{R}_+; X)$. Assume that the Laplace transform $\hat{f}$ belongs to $H^1_{loc}(\mathbb{C}_+; X)$. Then $f \in C_0(\mathbb{R}_+; X)$.*

Proof. Let $\varphi \in \mathcal{S}(\mathbb{R})$ be such that $\mathcal{F}\varphi \in \mathcal{D}(\mathbb{R})$. Then, by the assumption on $\hat{f}$ and by Proposition 5.4, the convolution $f * \varphi$ belongs to $C_0(\mathbb{R}_+; X)$. Choosing an approximate identity $(\varphi_n)_{n\in\mathbb{N}} \subset L^1(\mathbb{R}_+)$ consisting of test functions in $\mathcal{S}(\mathbb{R})$ such that their Fourier transforms belong to $\mathcal{D}(\mathbb{R})$, and using the uniform continuity of f, we deduce $f \in C_0(\mathbb{R}_+; X)$. □

Remarks 5.6. (a) By Proposition 5.2, Theorem 5.5 is optimal in the sense that in every Banach space which fails the analytic Riemann-Lebesgue property, there exists a function $f \in BUC(\mathbb{R}_+; X)$ such that $\hat{f} \in H_1(\mathbb{C}_+; X)$ and $\limsup_{t\to\infty} \|f(t)\| > 0$.
(b) Theorem 5.5 generalizes Ingham's Theorem in [24] to the vector-valued case. It has been proved in a more general form in [12, Theorem 3.4], under the assumption that X has the analytic Radon-Nikodym property. Using Proposition 5.4 and the techniques from [12], one will actually show that [12, Theorem 3.4] remains true for Banach spaces with the analytic Riemann-Lebesgue property.

The following immediate corollary generalizes results from [23] in two directions: in [23] the underlying Banach space is either B-convex or it has the analytic Radon-Nikodym property (both subclasses of (ARL)), and the extension F of the local resolvent has to be locally bounded near the imaginary axis.

Corollary 5.7. *Let $(T(t))_{t\geq 0}$ be a C_0-semigroup with generator A on a Banach space X which has the analytic Riemann-Lebesgue property. Let $x \in X$ be such that the orbit $T(\cdot)x$ is bounded and uniformly continuous, and such that the analytic extension F of the local resolvent $R(\cdot, A)x$ to $\mathbb{C}_+$ belongs to $H^1_{loc}(\mathbb{C}_+; X)$. Then* $\lim_{t\to\infty} \|T(t)x\| = 0$.

Proof. Put $f(t) := T(t)x$ $(t \geq 0)$ and apply Theorem 5.5. □

5.3. **Operators $C_0(\mathbb{R}; Y) \to X$ and convolutions.** Let X, Y and Z be Banach spaces. According to the classical Riesz Representation Theorem we may regard the space $\mathcal{L}(C_0(\mathbb{R}; Y), X)$ as a space of generalized $\mathcal{L}(Y, X)$-valued measures on $\mathbb{R}$ (see for example [15], [16]), and thus we may also define convolutions of such operators or measures.

We say that an operator $\Phi \in \mathcal{L}(C_0(\mathbb{R}; Y), X)$ is *right convolution admissible* if the following operator

$$\begin{array}{rcl} L_\Phi : C_0(\mathbb{R}; Y) & \to & BUC(\mathbb{R}; X) \\ f & \mapsto & (t \mapsto \Phi(f(t + \cdot))) \end{array}$$

actually maps into $C_0(\mathbb{R}; X)$. If $\Phi \in \mathcal{L}(C_0(\mathbb{R}; Y), X)$ is right convolution admissible, and if $\Psi \in \mathcal{L}(C_0(\mathbb{R}; X), Z)$, then we define the *convolution* $\Psi * \Phi \in \mathcal{L}(C_0(\mathbb{R}; Y), Z)$ by $\Psi * \Phi := \Psi \circ L_\Phi$. See [13] for these definitions, a more detailed study of operators $\mathcal{L}(C_0(\mathbb{R}; Y), X)$, and applications to abstract delay equations.

Proposition 5.8. *If X has the Riemann-Lebesgue property, then every operator $\Phi \in \mathcal{L}(C_0(\mathbb{R}; Y), X)$ is right convolution admissible.*

Proof. Let $\mathcal{F}\Phi \in \mathcal{L}(L^1(\mathbb{R}; Y), X)$ be the Fourier transform of Φ defined by

$$\mathcal{F}\Phi(g) := \Phi(\mathcal{F}g), \quad g \in L^1(\mathbb{R}; Y).$$

We refer to [13, Section 6.2] for this definition. By [14], [1], for every $y \in Y$ there exists a function $f_y \in L^*_{\infty,X}(\mathbb{R}; X'')$ such that

$$\mathcal{F}\Phi(g \otimes y) = \int_{\mathbb{R}} g(\eta) f_y(\eta)\, d\eta$$

for every $g \in C_c(\mathbb{R})$. Hence, for every $g \in C_c(\mathbb{R})$ and every $y \in Y$,

$$\begin{aligned} L_\Phi(\mathcal{F}g \otimes y)(t) &= \Phi(\mathcal{F}g \otimes y(t+\cdot)) \\ &= \mathcal{F}\Phi(e^{-it\cdot} g \otimes y) \\ &= \int_{\mathbb{R}} g(\eta) e^{-i\eta t} f_y(\eta)\, d\eta \end{aligned}$$

tends to 0, as $|t| \to \infty$, by the assumption on X and by Lemma 5.1 (note that $gf'_y \in L^*_{1,X}(\mathbb{R}; X''))$. Since the functions of the form $g \otimes y$ are total in $L^1(\mathbb{R}; Y)$, this implies $L_\Phi(\mathcal{F}g) \in C_0(\mathbb{R}; X)$ for every $g \in L^1(\mathbb{R}; Y)$. Since the range of $L^1(\mathbb{R}; Y)$ under Fourier transform is dense in $C_0(\mathbb{R}; Y)$, the operator L_Φ maps into $C_0(\mathbb{R}; X)$. The proof is complete. □

Remarks 5.9. (a) Proposition 5.8 and [13, Lemma 6.5] imply that the operator space $\mathcal{L}(C_0(\mathbb{R}; X), X)$ is a Banach algebra for the convolution defined above, if the space X has the Riemann-Lebesgue property.
(b) Similarly to Proposition 5.8, using Ingham's Theorem (Theorem 5.5) instead of Lemma 5.1, one will prove that if X has the analytic Riemann-Lebesgue property, then every operator $\Phi \in \mathcal{L}(C_0(\mathbb{R}; Y), X)$ with support in $\mathbb{R}_+$ is right convolution admissible. In this case, the function f_y appears to be the boundary of the Laplace transform of Φ on the imaginary axis (see [13, Lemma 6.3]).

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