THE DOMAIN OF THE ORNSTEIN–UHLENBECK OPERATOR ON AN L^p–SPACE WITH INVARIANT MEASURE

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ABSTRACT. We show that the domain of the Ornstein–Uhlenbeck operator on $L^p(\mathbb{R}^N, \mu dx)$ equals the weighted Sobolev space $W^{2,p}(\mathbb{R}^N, \mu dx)$, where $\mu\, dx$ is the corresponding invariant measure. Our approach relies on the operator sum method, namely the commutative and the non commutative Dore–Venni theorems.

1. INTRODUCTION

In recent years the Ornstein-Uhlenbeck operator

$$Lu(x) = \frac{1}{2}\sum_{i,j=1}^{N} q_{ij}D_{ij}u(x) + \sum_{i,j=1}^{N} a_{ij}x_iD_ju(x) = \frac{1}{2}\operatorname{tr} QD^2u(x) + \langle Ax, Du(x)\rangle, \quad x \in \mathbb{R}^N,$$

and its associated semigroup $T(\cdot)$ on, say, $C_b(\mathbb{R}^N)$ given by

$$(T(t)\varphi)(x) = (2\pi)^{-\frac{N}{2}} (\det Q_t)^{-\frac{1}{2}} \int_{\mathbb{R}^N} \varphi(e^{tA}x - y)\, e^{-\frac{1}{2}<Q_t^{-1}y,y>}\, dy, \quad x \in \mathbb{R}^N,\ t > 0, \quad (1.1)$$

have attracted a lot of interest. These activities are in particular motivated by the fact that $T(\cdot)$ is the transition semigroup of the Ornstein–Uhlenbeck process (see [11])

$$X(t,x) = e^{tA}x + \int_0^t e^{(t-s)A}\, dW(s)$$

on $\mathbb{R}^N$, where W is an N–dimensional Brownian motion with covariance matrix Q, i.e.,

$$(T(t)\varphi)(x) = \mathbb{E}[\varphi(X(t,x))].$$

The main purpose of this paper is to determine the domain of the realization L_p of L in a certain weighted Lebesgue space $L^p_\mu = L^p(\mathbb{R}^N, \mu dx)$ assuming that $Q = (q_{ij})$ is a real, symmetric, positive definite $N \times N$–matrix and that $A = (a_{ij})$ is a real $N \times N$–matrix whose eigenvalues are contained in the open left half plane. These hypotheses,

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kept throughout Sections 1–3, ensure that the matrices

$$Q_t = \int_0^t e^{sA} Q e^{sA^*}\, ds, \qquad t \in (0, \infty], \tag{1.2}$$

are well defined, symmetric, and positive definite. (If $t \in (0, \infty)$, then one can allow for an arbitrary real A.) Moreover, the Gaussian measure $\mu\, dx$ given by the weight

$$\mu(x) = (2\pi)^{-\frac{N}{2}} (\det Q_\infty)^{-\frac{1}{2}}\, e^{-\frac{1}{2} < Q_\infty^{-1} x, x>}, \qquad x \in \mathbb{R}^N,$$

is the unique invariant measure for the semigroup $T(\cdot)$, i.e., μ is the only probability measure for which

$$\int_{\mathbb{R}^N} (T(t)\varphi)(x)\mu(x)\, dx = \int_{\mathbb{R}^N} \varphi(x)\mu(x)\, dx, \qquad \varphi \in C_b(\mathbb{R}^N),\ t \geq 0, \tag{1.3}$$

see [11, Theorems 11.7, 11.11]. As a result, $T(\cdot)$ extends to a semigroup of positive contractions on L^p_μ for $1 \leq p \leq \infty$ and it is not difficult to see that $T(t)f(x)$ is still defined by (1.1) for $f \in L^p_\mu$, $x \in R^N$, and $t > 0$. The semigroup $T(\cdot)$ is strongly continuous on L^p_μ if $1 \leq p < \infty$, analytic for $1 < p < \infty$ (but not for $p = 1$), and its generator L_p is the closure of L defined on the Schwartz class $\mathcal{S}(\mathbb{R}^N)$. We refer, e.g., to [16] for the proofs of these properties.

The equality $D(L_2) = W^{2,2}_\mu$ has been first proved by A. Lunardi in [14] making heavy use of interpolation theory. A simpler proof of the same result can be found in [7]. Recently, this result was extended to $p \in (1, \infty)$ in the *symmetric* case by A. Chojnowska–Michalik and B. Goldys in [3, Theorem 3.3], who studied (as in [7]) the infinite dimensional version of L where $\mathbb{R}^N$ is replaced by a separable real Hilbert space, and by G. Da Prato and V. Vespri in [10, Theorem 2.2], who allowed for more general drift terms of gradient type on $\mathbb{R}^N$. Both approaches are based on maximal regularity results from [2] and [1], respectively. We also refer to the previous papers [6], [9], [12], [20].

In our main Theorem 3.4 we establish the equality $D(L_p) = W^{2,p}_\mu$ for $1 < p < \infty$. In Section 2 we first diagonalize Q and Q_∞ simultaneously and describe the resulting drift matrix A_1. This allows to decompose $L = L^0 + B$, where L^0 is a symmetric, diagonal Ornstein–Uhlenbeck operator and B generates an isometric group on L^p_μ. Then we determine $D(L_p)$ in three steps. The one–dimensional case is first settled in Lemma 3.1 by rather elementary calculations. In Proposition 3.2 we then establish that $D(L^0_p) = W^{2,p}_\mu$ using the Dore–Venni theorem [13], Lemma 3.1, and elliptic regularity in $L^p(\mathbb{R}^N)$. In a final step we deduce that $D(L_p) = D(L^0_p) = W^{2,p}_\mu$ employing a perturbation argument based on a non commutative Dore–Venni type theorem, see [17].

In the last section of this paper we characterize the domain of the Ornstein-Uhlenbeck operator L in $L^p(\mathbb{R}^N)$, $1 < p < \infty$, for an arbitrary real drift matrix A applying again the results in [17]. As a byproduct, we can prove L^p estimates for L. We remark that Schauder estimates for L have been already obtained in [8].

Notation. The space of continuous functions f having continuous (resp. continuous and bounded) partial derivatives $D_\alpha f$ up to order k is denoted by $C^k(\mathbb{R}^N)$ (resp. $C^k_b(\mathbb{R}^N)$) and the corresponding weighted Sobolev space by $W^{k,p}_\mu = W^{k,p}(\mathbb{R}^N, \mu dx) = \{f \in L^p_\mu :$

$D_\alpha f \in L^p_\mu,\ |\alpha| \le k\}$, where $k \in \mathbb{N}_0$, $1 \le p < \infty$, $C_b^0(\mathbb{R}^N) = C_b(\mathbb{R}^N)$, $W_\mu^{0,p} = L^p_\mu = L^p(\mathbb{R}^N, \mu dx)$, and α is a multi index. The Schwartz class is designated by $\mathcal{S}(\mathbb{R}^N)$ and the space of test functions by $C_0^\infty(\mathbb{R}^N)$. We write $L^p(\mathbb{R}^N)$ if the underlying measure is the Lebesgue measure. By a slight abuse of notation we write xf or $x_j f$ for the functions $x \mapsto xf(x)$ or $x \mapsto x_j f(x)$, where $x = (x_1, \cdots, x_N) \in \mathbb{R}^N$. The symbol c denotes a generic constant.

2. Preparations

In this section we collect some results needed in the next section. For reader's convenience we give complete proofs of known facts.

The Ornstein–Uhlenbeck operator L is called *symmetric* if the semigroup $T(\cdot)$ is symmetric in L^2_μ. The next lemma is a slight modification of [3, Theorem 2.2].

Lemma 2.1. *The equality $AQ_\infty + Q_\infty A^* = -Q$ holds. Moreover, the following properties are equivalent.*

(a) *L is symmetric.*
(b) *$Q_\infty A^* = AQ_\infty$.*
(c) *$Q_t A^* = AQ_t$ for all $t \in (0, \infty)$.*
(d) *$QA^* = AQ$.*

Proof. Observe that (1.2) yields the formula

$$Q_t + e^{tA} Q_\infty e^{tA^*} = Q_\infty. \tag{2.1}$$

The first assertion can be verified by taking in (2.1) the derivative at $t = 0$. The equivalence of (a) and (d) was shown in [3, Theorem 2.2]. The implication (d)⇒(b) is an immediate consequence of (1.2). Assertion (c) follows from (b), due to (2.1). Finally, (c) implies (d) by differentiation. □

Given an invertible real $N \times N$–matrix M, we introduce the similarity transformation

$$\Phi_M : C(\mathbb{R}^N) \to C(\mathbb{R}^N); \quad (\Phi_M u)(y) = u(M^{-1}y).$$

For $u \in \mathcal{S}(\mathbb{R}^N)$ and $v = \Phi_M u \in \mathcal{S}(\mathbb{R}^N)$, one easily calculates that $Lu(x) = \tilde{L}v(Mx)$, $x \in \mathbb{R}^N$, where

$$\tilde{L}v = \tfrac{1}{2} \operatorname{tr} \tilde{Q} D^2 v + \langle \tilde{A}y, Dv \rangle, \quad \tilde{Q} = MQM^*, \quad \tilde{A} = MAM^{-1}.$$

This means that $L = \Phi_M^{-1} \tilde{L} \Phi_M$ on $\mathcal{S}(\mathbb{R}^N)$ and $\tilde{Q}_\infty = MQ_\infty M^*$. The corresponding Gaussian measure for $\tilde{L}$ is given by

$$\tilde{\mu}(x) = (2\pi)^{-\frac{N}{2}} (\det \tilde{Q}_\infty)^{-\frac{1}{2}} e^{-\frac{1}{2}\langle \tilde{Q}_\infty^{-1} x, x\rangle} = \frac{1}{|\det M|}\, \mu(M^{-1}x), \quad x \in \mathbb{R}^N.$$

As a result, Φ_M induces an isometry from L^p_μ onto $L^p_{\tilde{\mu}}$ and an isomorphism from $W^{k,p}_\mu$ onto $W^{k,p}_{\tilde{\mu}}$, $1 \le p \le \infty$, $k \in \mathbb{N}$. Recalling that the induced generators L_p and $\tilde{L}_p$ are the closures of L and $\tilde{L}$ defined on $\mathcal{S}(\mathbb{R}^N)$, respectively, we arrive at

$$L_p = \Phi_M^{-1} \tilde{L}_p \Phi_M \quad \text{with} \quad D(L_p) = \Phi_M^{-1} D(\tilde{L}_p).$$

There is an invertible real matrix M_1 such that $M_1QM_1^* = I$ and an orthogonal real matrix M_2 such that $M_2(M_1Q_\infty M_1^*)M_2^* = \operatorname{diag}(\lambda_1, \cdots, \lambda_N) =: D_\lambda$ for certain $\lambda_j > 0$. Taking $M = M_2M_1$, we have transformed L into the more convenient form described in the next lemma.

Lemma 2.2. *(a) There exists a real invertible $N \times N$–matrix M such that $L_p = \Phi_M^{-1}\tilde{L}_p\Phi_M$ and $D(L_p) = \Phi_M^{-1}D(\tilde{L}_p)$, where $\tilde{L}u = \frac{1}{2}\Delta u + \langle \tilde{A}x, Du\rangle$ and $\tilde{A} = MAM^{-1}$. Moreover, $\tilde{Q}_\infty = D_\lambda$ for certain $\lambda_j > 0$,*

$$\tilde{\mu}(x) = (2\pi)^{-\frac{N}{2}}\,(\lambda_1 \cdots \lambda_N)^{-\frac{1}{2}}\,\exp(-\textstyle\sum_j \frac{x_j^2}{2\lambda_j}), \tag{2.2}$$

and $\Phi_M : W_\mu^{k,p} \to W_{\tilde{\mu}}^{k,p}$, $1 \le p \le \infty$, $k \in \mathbb{N}_0$, is an isomorphism.

(b) Setting $\tilde{L}^0u = \frac{1}{2}\Delta u - \langle \frac{1}{2}D_{\frac{1}{\lambda}}x, Du\rangle$, $Bu = \langle A_1x, Du\rangle$, and $A_1 = \tilde{A} + \frac{1}{2}D_{\frac{1}{\lambda}}$, we can write $\tilde{L} = \tilde{L}^0 + B$. Moreover, $A_1D_\lambda = -D_\lambda A_1^*$ and hence the diagonal elements of A_1 equal zero. Finally, $\tilde{\mu}$ (defined in (2.2)) is the invariant measure of the Ornstein–Uhlenbeck semigroup generated by $\tilde{L}^0$, and $\tilde{L}^0$ is symmetric.

Proof. (a) holds in view of the discussion above. As regards (b), by Lemma 2.1 we have $\tilde{A}D_\lambda + D_\lambda\tilde{A}^* = -I$, and hence $A_1D_\lambda = -I - D_\lambda\tilde{A}^* + \frac{1}{2}I = -D_\lambda A_1^*$. Finally, $\tilde{\mu}$ is the invariant measure for $\tilde{L}^0$ by the explicit computation of the integral in (1.2) and $\tilde{L}^0$ is symmetric (in $L^2_{\tilde{\mu}}$) by Lemma 2.1. $\square$

In order to determine $D(L_p)$, we may thus assume that L is given by

$$L = L^0 + B, \quad L^0u = \tfrac{1}{2}\Delta u - \tfrac{1}{2}\langle D_{\frac{1}{\lambda}}x, Du\rangle, \quad Bu = \langle A_1x, Du\rangle, \quad \text{where} \tag{2.3}$$

$$A_1D_\lambda = -D_\lambda A_1^*, \quad Q_\infty = D_\lambda, \quad \mu(x) = (2\pi)^{-\frac{N}{2}}\,(\lambda_1 \cdots \lambda_N)^{-\frac{1}{2}}\,\exp(-\textstyle\sum_j \frac{x_j^2}{2\lambda_j}) \tag{2.4}$$

for $x \in \mathbb{R}^N$ and certain $\lambda_j > 0$. We recall that μ is the invariant measure for L^0 and that L^0 is symmetric.

We further need the following property of the space $W_\mu^{1,p}$ which was essentially proved in [16, Lemma 2.3], see also [14, Lemma 2.1] for the case $p = 2$.

Lemma 2.3. *Let $1 < p < \infty$. If $\varphi, D_j\varphi \in L^p_\mu$, then the function $x_j\varphi$ belongs to L^p_μ and $\|x_j\varphi\|_{L^p_\mu} \le C_p\,(\|\varphi\|_{L^p_\mu} + \|D_j\varphi\|_{L^p_\mu})$ for a constant $C_p > 0$ depending only on p and λ_j.*

Proof. It suffices to prove the lemma for $\varphi \in C_0^\infty(\mathbb{R}^N)$ and we may assume that μ is in the diagonal form (2.4). Integrating by parts, we obtain

$$\begin{aligned}\int_{\mathbb{R}^N} |x_j\varphi(x)|^p\mu(x)\,dx &= -\lambda_j\int_{\mathbb{R}^N} |x_j|^{p-1}\,|\varphi(x)|^p\,\operatorname{sign} x_j\,(D_j\mu)(x)\,dx \qquad (2.5)\\ &= \lambda_j\int_{\mathbb{R}^N}\big[(p-1)|x_j|^{p-2}|\varphi(x)|^p + p\,|x_j|^{p-1}(D_j\varphi)(x)\,|\varphi(x)|^{p-1}\operatorname{sign} x_j\big]\,\mu(x)\,dx.\end{aligned}$$

We set $I_1 = \int_{\mathbb{R}^N} |x_j|^{p-2}|\varphi(x)|^p\mu(x)\,dx$ and $I_2 = \int_{\mathbb{R}^N} |x_j|^{p-1}|D_j\varphi(x)|\,|\varphi(x)|^{p-1}\mu(x)\,dx$. Hölder's inequality yields

$$I_2 \le \Big[\int_{\mathbb{R}^N} |x_j\varphi(x)|^p\mu(x)\,dx\Big]^{\frac{p-1}{p}}\,\Big[\int_{\mathbb{R}^N} |D_j\varphi(x)|^p\mu(x)\,dx\Big]^{\frac{1}{p}}. \tag{2.6}$$

In order to deal with I_1, we first consider $p \geq 2$. Then, for each $\varepsilon > 0$ there is $M_\varepsilon > 0$ such that $|x_j|^{p-2} \leq \varepsilon\, |x_j|^p + M_\varepsilon$, and hence

$$I_1 \leq M_\varepsilon \|\varphi\|^p_{L^p_\mu} + \varepsilon \int_{\mathbb{R}^N} |x_j\varphi(x)|^p \mu(x)\, dx. \tag{2.7}$$

Combining (2.5), (2.6), (2.7) with Young's inequality, we arrive at

$$\begin{aligned}\|x_j\varphi\|^p_{L^p_\mu} &\leq \lambda_j(p-1)M_\varepsilon\, \|\varphi\|^p_{L^p_\mu} + \lambda_j(p-1)\varepsilon\, \|x_j\varphi\|^p_{L^p_\mu} + \lambda_j p\, \|x_j\varphi\|^{p-1}_{L^p_\mu}\, \|D_j\varphi\|_{L^p_\mu} \\ &\leq \lambda_j(p-1)M_\varepsilon\, \|\varphi\|^p_{L^p_\mu} + \lambda_j(p-1)\varepsilon\, \|x_j\varphi\|^p_{L^p_\mu} + \varepsilon^{\frac{p}{p-1}}\lambda_j(p-1)\, \|x_j\varphi\|^p_{L^p_\mu} \\ &\qquad + \varepsilon^{-p}\lambda_j\, \|D_j\varphi\|^p_{L^p_\mu}\,,\end{aligned}$$

and hence

$$[1-\lambda_j(p-1)(\varepsilon+\varepsilon^{\frac{p}{p-1}})]\, \|x_j\varphi\|^p_{L^p_\mu} \leq \lambda_j(p-1)M_\varepsilon\, \|\varphi\|^p_{L^p_\mu} + \lambda_j\varepsilon^{-p}\, \|D_j\varphi\|^p_{L^p_\mu}. \tag{2.8}$$

Therefore the lemma is proved for $p \geq 2$ taking a sufficiently small $\varepsilon > 0$. If $1 < p < 2$, we write $x = (x_j, \hat{x})$ and estimate

$$\begin{aligned} I_1 &\leq \int_{\mathbb{R}^{N-1}} \Big[\int_{-1}^1 |x_j|^{p-2}\, |\varphi(x)|^p\, \mu(x)\, dx_j + \int_{\mathbb{R}\setminus[-1,1]} |\varphi(x)|^p\, \mu(x)\, dx_j\Big] d\hat{x} \\ &\leq \int_{-1}^1 |x_j|^{p-2}\, dx_j \int_{\mathbb{R}^{N-1}} [\sup_{x_j\in[-1,1]} |\varphi(x_j,\hat{x})|]^p\, \mu(0,\hat{x})\, d\hat{x} \;+\; \|\varphi\|_{L^p_\mu} \\ &\leq c\,(\|D_j\varphi\|^p_{L^p_\mu} + \|\varphi\|^p_{L^p_\mu}) \end{aligned} \tag{2.9}$$

using the embedding $W^{1,p}(-1,1) \hookrightarrow L^\infty(-1,1)$. As in (2.8), the assertion follows from (2.5), (2.6), and (2.9). ☐

From the above lemma we deduce that $W^{2,p}_\mu$ is a core for L_p if $1 < p < \infty$.

Proposition 2.4. *For $1 < p < \infty$, the operator L_p is the closure of the differential operator L defined on $W^{2,p}_\mu$.*

Proof. As shown in [16, Lemma 2.1], L_p is the closure of the operator L defined on $\mathcal{S}(\mathbb{R}^N)$. Let $u \in W^{2,p}_\mu$ and $(u_n) \subset \mathcal{S}(\mathbb{R}^N)$ converge to u in the norm of $W^{2,p}_\mu$. Then $u_n \in D(L_p)$ and $L_pu_n = Lu_n$ converges to Lu in L^p_μ, by Lemma 2.3. Since L_p is closed, $u \in D(L_p)$ and $L_pu = Lu$. ☐

3. Main results

We first compute the domain of the one-dimensional Ornstein-Uhlenbeck operator L_p on L^p_μ for $1 < p < \infty$. In view of Lemma 2.2 we may assume that $Lu = \frac{1}{2}\, u'' - \frac{1}{2\lambda}\, xu'$ and $\mu(x) = (2\pi\lambda)^{-1/2}\exp\{-x^2/2\lambda\}$.

Lemma 3.1. *If $N = 1$ and $1 < p < \infty$, then $D(L_p) = W^{2,p}_\mu$.*

Proof. Thanks to Proposition 2.4 and Lemma 2.3, it remains to show that

$$\|u\|_{W^{2,p}_\mu} \leq c\,(\|Lu\|_{L^p_\mu} + \|u\|_{L^p_\mu}) \tag{3.1}$$

for $u \in W^{2,p}_\mu$. For a given $u \in W^{2,p}_\mu$, we write $f = (I - L)u \in L^p_\mu$ and $v = u'$. Hence,

$$-\tfrac{1}{2}v'(x) + \tfrac{1}{2\lambda}xv(x) = f(x) - u(x), \quad x \in \mathbb{R}.$$

Integrating we obtain

$$e^{-\frac{1}{2\lambda}\xi^2}v(\xi) - e^{-\frac{1}{2\lambda}x^2}v(x) = 2\int_x^\xi (u(y) - f(y))e^{-\frac{1}{2\lambda}y^2}\,dy, \quad \xi \in \mathbb{R}.$$

Since $(u - f)e^{-\frac{1}{2\lambda}y^2} \in L^1(\mathbb{R})$ and $v \in L^p_\mu$, the function $e^{-\frac{1}{2\lambda}\xi^2}v(\xi)$ tends to 0 as $\xi \to \pm\infty$ and therefore

$$e^{-\frac{1}{2\lambda}x^2}v(x) = -2\int_x^{\pm\infty} (u(y) - f(y))e^{-\frac{1}{2\lambda}y^2}\,dy.$$

Setting $\varphi(x) = e^{-\frac{1}{2p\lambda}x^2}v(x)$ and $g(x) = 2e^{-\frac{1}{2p\lambda}x^2}(f(x) - u(x))$, we arrive at

$$\varphi(x) = \begin{cases} \int_x^\infty \exp(\frac{1}{2p'\lambda}(x^2 - y^2))\,g(y)\,dy, & x \geq 0, \\ -\int_{-\infty}^x \exp(\frac{1}{2p'\lambda}(x^2 - y^2))\,g(y)\,dy, & x \leq 0. \end{cases}$$

Note that $\|xu'\|_{L^p_\mu} = \|x\varphi\|_{L^p(\mathbb{R})}$ and $2\|f - u\|_{L^p_\mu} = \|g\|_{L^p(\mathbb{R})}$. We thus have to control the $L^p(\mathbb{R})$–norm of $x\varphi$. It suffices to estimate $\|x\varphi\|_{L^p(\mathbb{R}_+)}$. Hölder's inequality and Fubini's theorem yield

$$\begin{aligned}
\|x\varphi\|^p_{L^p(\mathbb{R}_+)} &= \int_0^\infty \left|\int_x^\infty xe^{-\frac{1}{2p'\lambda}(y+x)(y-x)}g(y)\,dy\right|^p dx \\
&\leq \int_0^\infty \left(\int_x^\infty xe^{-\frac{1}{p'\lambda}x(y-x)}|g(y)|^p\,dy\right)\left(\int_x^\infty xe^{-\frac{1}{p'\lambda}x(y-x)}\,dy\right)^{p-1} dx \\
&= (p'\lambda)^{p-1}\int_0^\infty\int_x^\infty xe^{-\frac{1}{p'\lambda}x(y-x)}|g(y)|^p\,dy\,dx \\
&= (p'\lambda)^{p-1}\int_0^\infty |g(y)|^p\int_0^y xe^{-\frac{1}{p'\lambda}x(y-x)}\,dx\,dy.
\end{aligned}$$

We further compute

$$\begin{aligned}
\int_0^y xe^{-\frac{1}{p'\lambda}x(y-x)}\,dx &= y^2\int_0^1 te^{-\frac{1}{p'\lambda}y^2t(1-t)}\,dt \leq 2y^2\int_0^{\frac{1}{2}} e^{-\frac{1}{p'\lambda}y^2t(1-t)}\,dt \\
&\leq 2y^2\int_0^{\frac{1}{2}} e^{-\frac{1}{2p'\lambda}y^2t}\,dt \leq 4p'\lambda.
\end{aligned}$$

As a result,

$$\|xu'\|_{L^p_\mu} = \|x\varphi\|_{L^p(\mathbb{R})} \leq c\,\|g\|_{L^p(\mathbb{R})} \leq c\,(\|Lu\|_{L^p_\mu} + \|u\|_{L^p_\mu}).$$

It follows that $\|u''\|_{L^p_\mu} \leq c\,(\|Lu\|_{L^p_\mu} + \|u\|_{L^p_\mu})$, and the analogous estimate for u' easily follows from those for u'' and xu'. We have therefore established (3.1). □

We now treat the N-dimensional, symmetric case. The next result was recently shown in [3, Theorem 3.3] and [10, Theorem 2.2] in more general situations, but with completely different arguments.

Proposition 3.2. *If the Ornstein–Uhlenbeck operator L is symmetric, then $D(L_p) = W^{2,p}_\mu$ for $1 < p < \infty$.*

Proof. In view of Lemma 2.2 we may assume that

$$L = L^{(1)} + \cdots + L^{(N)} \quad \text{with} \quad L^{(j)}u = \frac{1}{2}D_{jj} - \frac{x_j}{2\lambda_j}\,D_j \qquad \text{and}$$

$$\mu(x) = (2\pi)^{-\frac{N}{2}}\,(\lambda_1\cdots\lambda_N)^{-\frac{1}{2}}\,\exp(-\textstyle\sum_j \frac{x_j^2}{2\lambda_j}).$$

We consider the semigroup of positive and self-adjoint contractions

$$T^{(j)}(t)\varphi(x) = \big(2\pi\lambda_j(1-e^{-t/\lambda_j})\big)^{-\frac{1}{2}}\int_{\mathbb{R}} \varphi(e^{-\frac{t}{2\lambda_j}}x_j - y, \hat{x})\,\exp(-\tfrac{y^2}{2\lambda_j(1-e^{-t/\lambda_j})})\,dy$$

on L^p_μ, where we write $x = (x_j, \hat{x}) \in \mathbb{R}^N$. Its generator in L^p_μ is the operator $L^{(j)}_p = L^{(j)}$ with domain $D(L^{(j)}_p) = \{u \in L^p_\mu : D_j u, D_{jj}u \in L^p_\mu\}$, see the next lemma.

Since $T(\cdot)$ is the product of the commuting semigroups $T^{(j)}(\cdot)$, the generator L_p is the closure of the sum $L^{(1)}_p + \cdots + L^{(N)}_p$ defined on $D(L^{(1)}_p) \cap \cdots \cap D(L^{(N)}_p)$. The operator $(I - L^{(j)}_p)$ admits bounded imaginary powers on L^p_μ, $1 < p < \infty$, with power angle

$$\theta(L^{(j)}_p) := \lim_{|s|\to\infty} \tfrac{1}{|s|}\log\|(I - L^{(j)})^{is}\| \le \tfrac{\pi}{2}$$

due to the transference principle [5, §4], see [4, Theorem 5.8]. The semigroup $T^{(j)}(\cdot)$ is symmetric on L^2_μ so that $\theta(L^{(j)}_2) = 0$ by the functional calculus for selfadjoint operators. Hence, $\theta(L^{(j)}_p) < \frac{\pi}{2}$ by the Riesz–Thorin interpolation theorem. Since the resolvents of $L^{(j)}_p$ commute, we can apply the Dore–Venni theorem [13] in the version of [18, Corollary 4]. As a consequence, $L^{(1)}_p + \cdots + L^{(N)}_p$ is closed on the intersection of the domains and so

$$D(L_p) = \bigcap_{j=1}^N D(L^{(j)}_p) = \{u \in L^p_\mu : D_j u, D_{jj}u \in L^p_\mu \;\text{ for } j = 1, \cdots, N\}.$$

Let $u \in D(L_p)$. In order to check $D_{ij}u \in L^p_\mu$, we set $v(x) = u(x)\exp(-\frac{1}{2p}\langle D_{\frac{1}{\lambda}}x, x\rangle)$, $x \in \mathbb{R}^N$. Notice that $v \in L^p(\mathbb{R}^N)$ and

$$D_{jj}v(x) = \Big[D_{jj}u(x) - \frac{2x_j}{p\lambda_j}\,D_j u(x) - \frac{1}{p\lambda_j}\,u(x) + \frac{x_j^2}{(p\lambda_j)^2}\,u(x)\Big]\,\exp(-\frac{1}{2p}\langle D_{\frac{1}{\lambda}}x, x\rangle)$$

for $j = 1, \cdots, N$ and $x \in \mathbb{R}^N$. Lemma 2.3 shows that $x_j D_j u, x_j^2 u \in L^p_\mu$, hence $|x|^2 u \in L^p_\mu$. This implies that $D_{jj}v, |x|^2 v \in L^p(\mathbb{R}^N)$. From standard regularity properties of the Laplacian it follows that $D_{ij}v \in L^p(\mathbb{R}^N)$ for $i, j = 1, \cdots, N$. On the other hand,

$$D_{ij}u(x) = \Big[D_{ij}v(x) + \frac{x_i}{p\lambda_i}\,D_j v(x) + \frac{x_j}{p\lambda_j}\,D_i v(x) + \frac{x_i x_j}{p^2\lambda_i\lambda_j}\,v(x)\Big]\exp(\frac{1}{2p}\langle D_{\frac{1}{\lambda}}x, x\rangle) \tag{3.2}$$

for $i, j = 1, \cdots, N$ and $x \in \mathbb{R}^N$. For $i \ne j$ we have, writing $x = (x_i, \hat{x})$,

$$\begin{aligned}\|x_j D_i v\|^p_{L^p(\mathbb{R}^N)} &= \int_{\mathbb{R}^{N-1}} |x_j|^p \int_{\mathbb{R}} |D_i v(x)|^p\,dx_i\,d\hat{x} \\ &\le c\int_{\mathbb{R}^{N-1}} \left(\int_{\mathbb{R}} |D_{ii}v(x)|^p\,dx_i\right)^{\frac{1}{2}} |x_j|^p \left(\int_{\mathbb{R}} |v(x)|^p\,dx_i\right)^{\frac{1}{2}} d\hat{x} \\ &\le c\,\|D_{ii}v\|^{\frac{p}{2}}_{L^p(\mathbb{R}^N)}\,\|x_j^2 v\|^{\frac{p}{2}}_{L^p(\mathbb{R}^N)}\end{aligned}$$

so that $x_j D_i v \in L^p(\mathbb{R}^N)$. Hence, $D_{ij}u \in L^p_\mu$ by (3.2). This means that $D(L_p) = W^{2,p}_\mu$. $\square$

The following lemma has been used in the proof of the preceding result.

Lemma 3.3. *The generator of the semigroup $T^{(j)}(\cdot)$ in L^p_μ is the operator $L^{(j)}_p = L^{(j)}$ with domain $D(L^{(j)}_p) = \{u \in L^p_\mu : D_j u, D_{jj}u \in L^p_\mu\}$*

Proof. Let $W^{2,p,j}_\mu = \{u \in L^p_\mu : D_j u, D_{jj} u \in L^p_\mu\}$. As in Proposition 2.4 one verifies that $L^{(j)}_p$ is the closure of $L^{(j)}$ defined on $W^{2,p,j}_\mu$ and therefore the equality $D(L^{(j)}_p) = W^{2,p,j}_\mu$ will follow from the estimate

$$\|D_j u\|_{L^p_\mu} + \|D_{jj}u\|_{L^p_\mu} \le c\,(\|L^{(j)}u\|_{L^p_\mu} + \|u\|_{L^p_\mu}) \tag{3.3}$$

for $u \in W^{2,p,j}_\mu$. By density, it suffices to prove (3.3) for $u \in C_0^\infty(\mathbb{R}^N)$. By Lemma 3.1 and writing $x = (x_j, \hat{x})$, $\mu(x) = \mu_j(x_j)\hat{\mu}(\hat{x})$, we have

$$\int_{\mathbb{R}} \big(|D_j u(x_j,\hat{x})|^p + |D_{jj}u(x_j,\hat{x})|^p\big)\mu_j(x_j)\,dx_j \le c\int_{\mathbb{R}} \big(|L^{(j)}u(x_j,\hat{x})|^p + |u(x_j,\hat{x})|^p\big)\mu(x_j)\,dx_j\,.$$

Now (3.3) follows multiplying by $\hat{\mu}(\hat{x})$ and integrating with respect to $\hat{x}$. $\square$

We now come to the main result of our paper.

Theorem 3.4. *Assume that Q is real, symmetric and positive definite, that A is real with eigenvalues in the open left halfplane, and that $1 < p < \infty$. Then the generator L_p of the Ornstein–Uhlenbeck semigroup $T(\cdot)$ on L^p_μ is the Ornstein–Uhlenbeck operator L defined on $W^{2,p}_\mu$.*

Proof. Taking into account Proposition 2.4 and Lemma 2.2, it suffices to show that $L = L^0 + B$ is closed on the domain $W^{2,p}_\mu$ (see (2.3)).

(a) Consider the group $S(t)\varphi(x) = \varphi(e^{tA_1}x)$ for $\varphi \in L^p_\mu$. Due to (2.4) we have $-A_1 = D_\lambda A_1^* D_{\frac{1}{\lambda}}$ and $\operatorname{tr} A_1 = 0$. This yields

$$\begin{aligned}
\|S(t)\varphi\|^p_{L^p_\mu} &= (2\pi)^{-\frac{N}{2}}(\lambda_1\cdots\lambda_N)^{-\frac12}\int_{\mathbb{R}^N} |\varphi(e^{tA_1}x)|^p \exp(-\tfrac12\langle D_{\frac{1}{\lambda}}x, x\rangle)\,dx \\
&= (2\pi)^{-\frac{N}{2}}(\lambda_1\cdots\lambda_N)^{-\frac12}\int_{\mathbb{R}^N} |\varphi(y)|^p \exp(-\tfrac12\langle D_{\frac{1}{\lambda}}e^{-tA_1}y, e^{-tA_1}y\rangle)\,dy \\
&= (2\pi)^{-\frac{N}{2}}(\lambda_1\cdots\lambda_N)^{-\frac12}\int_{\mathbb{R}^N} |\varphi(y)|^p \exp(-\tfrac12\langle e^{tA_1^*}D_{\frac{1}{\lambda}}y, e^{-tA_1}y\rangle)\,dy \\
&= \|\varphi\|^p_{L^p_\mu}
\end{aligned}$$

Hence, $S(\cdot)$ is a group of isometries on L^p_μ. It is then easy to see that $S(\cdot)$ is strongly continuous on L^p_μ and that its generator B_p coincides with the operator B on $C_0^\infty(\mathbb{R}^N)$. Since $C_0^\infty(\mathbb{R}^N)$ is dense in L^p_μ and $S(\cdot)$–invariant, it is a core for B_p. Using Lemma 2.3 and the density of $C_0^\infty(\mathbb{R}^N)$ in $W^{2,p}_\mu$, we deduce that the domain $D(B_p)$ contains $W^{2,p}_\mu$ and that $B_p u = Bu$ for $u \in W^{2,p}_\mu$. In particular, $D(L^0_p) \cap D(B_p) = W^{2,p}_\mu$.

Since B_p generates a positive contraction semigroup on L^p_μ, $w - B_p$ has bounded imaginary powers with power angle $\theta(B_p) \le \pi/2$ on L^p_μ for every $w > 0$ thanks to the transference principle [5, §4], see [4, Theorem 5.8]. By the same argument $I - L^0_p$ has bounded imaginary powers with power angle $\theta(L_p^0) \le \pi/2$. Moreover, L_2^0 is self adjoint on L_μ^2 and thus has power angle 0 on L_μ^2. By interpolation we obtain that $\theta(B_p) + \theta(L_p^0) < \pi$ for $1 < p < \infty$.

(b) We next compute the commutator $[B_p, L_p^0]$. If $u \in C_b^4(\mathbb{R}^N)$, then $u \in D(L_p^0) \cap D(B_p)$, $L_p^0 u = L^0 u \in D(B_p)$, $B_p u = Bu \in D(L_p^0)$ and $[B_p, L_p^0]u = BL^0 u - L^0 Bu$. Denoting the coefficients of A_1 by b_{ij}, we obtain

$$\begin{aligned}
2BL^0 u &= \sum_{klj} b_{kl} x_l\, D_k\,(D_{jj} u - \frac{x_j}{\lambda_j}\, D_j u) \\
&= \sum_{klj} b_{kl} x_l\, D_{kjj} u - \sum_{kl} b_{kl}\, \frac{x_l}{\lambda_k}\, D_k u - \sum_{klj} b_{kl}\, \frac{x_l x_j}{\lambda_j}\, D_{kj} u, \\
2L^0 Bu &= \sum_{klj} (D_{jj} - \frac{x_j}{\lambda_j}\, D_j)\,(b_{kl} x_l\, D_k u) \\
&= \sum_{klj} b_{kl} x_l\, D_{kjj} u + 2 \sum_{kj} b_{kj}\, D_{kj} u - \sum_{klj} b_{kl}\, \frac{x_j x_l}{\lambda_j}\, D_{kj} u - \sum_{kj} b_{kj}\, \frac{x_j}{\lambda_j}\, D_k u, \\
L^0 Bu - BL^0 u &= \sum_{kj} b_{kj}\, D_{kj} u - \frac{1}{2} \sum_{kl} \Big(\frac{b_{kl}}{\lambda_l} - \frac{b_{kl}}{\lambda_k}\Big)\, x_l D_k u \\
&= \operatorname{tr} A_1 D^2 u - \frac{1}{2} \sum_{kl} (b_{kl} + b_{lk})\, \frac{x_l}{\lambda_l}\, D_k u.
\end{aligned}$$

In the last line we have used (2.4). Due to Proposition 3.2 and Lemma 2.3, the operator $[B_p, L_p^0] R(1, L_p^0)$ is bounded on L_μ^p.

(c) It is clear that $S(\cdot)$ is exponentially bounded on $C_b^4(\mathbb{R}^N)$, hence $R(\mu, B_p) C_b^4(\mathbb{R}^N) \subseteq C_b^4(\mathbb{R}^N)$ for $\operatorname{Re} \mu > w_0$ and a suitable $w_0 \ge 0$. Moreover, using (1.1) it is easy to see that the Ornstein–Uhlenbeck semigroup associated to L^0 is contractive in $C_b^k(\mathbb{R}^N)$ for every $k \in \mathbb{N}$. Consequently, $R(\mu, L_p^0) C_b^4(\mathbb{R}^N) \subseteq C_b^4(\mathbb{R}^N)$ for $\operatorname{Re} \mu > 0$. Let $G = I - L_p^0$ and $B_w = w - B_p$ for $w \ge w_0$. We then compute

$$\begin{aligned}
C_w(\mu, \nu) u &:= G(\nu + G)^{-1}\, \{G^{-1}(\mu + B_w)^{-1} - (\mu + B_w)^{-1} G^{-1}\} u \\
&= -G(\nu + G)^{-1}\, (\mu + B_w)^{-1} G^{-1}\, [B_p, L_p^0] G^{-1} (\mu + B_w)^{-1} u \\
&= R(\nu + 1, L_p^0)(L_p^0 - I) R(1, L_p^0) R(\mu + w, B_p)\, [B_p, L_p^0] R(1, L_p^0) R(\mu + w, B_p) u \\
&\quad + C_w(\mu, \nu) [B_p, L_p^0] R(1, L_p^0) R(\mu + w, B_p) u
\end{aligned}$$

for $u \in C_b^4(\mathbb{R}^N)$. Since $\theta(B_p) + \theta(L_p^0) < \pi$, we can fix $\phi_B < \min\{\pi/2, \pi - \theta(B_p)\}$, $\pi/2 < \phi_{L^0} < \pi - \theta(L_p^0)$, with $\phi_{L^0} + \phi_B > \pi$, such that

$$\|R(\mu + w, B_p)\| \le \frac{c}{|\mu + w|}, \quad |\arg \mu| < \phi_B\,, \qquad \|R(\nu + 1, L_p^0)\| \le \frac{c}{1 + |\nu|}, \quad |\arg \nu| < \phi_{L^0}.$$

an analytic semigroup). Part (b) allows to estimate

$$\|C_w(\mu, \nu)\| \le \frac{c_1}{(1 + |\nu|)|\mu + w|^2} + \|C_w(\mu, \nu)\|\, \frac{c_2}{|\mu + w|}$$

with constants c_1, c_2 independent of w. Taking $w = \max\{w_0, 2c_2\}$, we arrive at

$$\|C_w(\mu,\nu)\| \leq \frac{2c_1}{(1+|\nu|)|\mu+w|^2} \leq \frac{2c_1}{(1+|\nu|)|\mu|^2}$$

for $|\arg\nu| < \phi_{L^0}$ and $|\arg\mu| < \phi_B$. We can now apply [17, Corollary 2] and deduce that $G + B_w = w + 1 - L_p$ is closed on $D(B_p) \cap D(L_p^0) = W_\mu^{2,p}$. □

4. Further results

In this section we employ the same ideas used before to describe the domain of the Ornstein-Uhlenbeck operator on $L^p(\mathbb{R}^N)$. Even though more general situations can be treated with the same methods, we prefer to deal only with this particular case in order to simplify the exposition. To shorten the notation, we write $\|u\|_p$ for the norm of a function $u \in L^p(\mathbb{R}^N)$. We consider the operator

$$Lu(x) = \frac{1}{2}\sum_{i,j=1}^{N} q_{ij} D_{ij} u(x) + \sum_{i,j=1}^{N} a_{ij} x_i D_j u(x) = \frac{1}{2}\operatorname{tr} QD^2 u(x) + \langle Ax, Du(x)\rangle, \quad x \in \mathbb{R}^N,$$

in $L^p(\mathbb{R}^N)$ (with respect to the Lebesgue measure). The matrix Q is still assumed to be positive but we require only that A is real and nonzero.

It is well known that L with a suitable domain $\mathcal{D}_p$ is the generator of the Ornstein-Uhlenbeck semigroup $T(\cdot)$ on $L^p(\mathbb{R}^N)$ defined in (1.1). For $1 < p < \infty$ the domain $\mathcal{D}_p$ can be described by

$$\mathcal{D}_p = \{u \in L^p(\mathbb{R}^N) \cap W^{2,p}_{loc}(\mathbb{R}^N) : Lu \in L^p(\mathbb{R}^N)\} \tag{4.1}$$

and $C_0^\infty(\mathbb{R}^N)$ is a core for $(L, \mathcal{D}_p)$. We refer, e.g., to [15] for a proof of these properties. A more explicit description of $\mathcal{D}_p$ is given in the next theorem.

Theorem 4.1. *Assume that Q is real, symmetric and positive definite, that $A \neq 0$ is real, and that $1 < p < \infty$. Then*

$$\mathcal{D}_p = \{u \in W^{2,p}(\mathbb{R}^N) : \langle Ax, Du\rangle \in L^p(\mathbb{R}^N)\}.$$

There are positive constants c_1, c_2 such that

$$c_1(\|u\|_p + \|Lu\|_p) \leq \|u\|_{W^{2,p}(\mathbb{R}^N)} + \|\langle Ax, Du\rangle\|_p \leq c_2(\|u\|_p + \|Lu\|_p) \tag{4.2}$$

for every $u \in \mathcal{D}_p$.

Proof. We follow closely the proof of Theorem 3.4.

(a) We decompose $L = L^Q + L^A$, where $L^Q u = \frac{1}{2}\sum_{i,j=1}^N q_{ij} D_{ij} u$ and $L^A u = \langle Ax, Du\rangle$. The operator L^Q with domain $D(L^Q) = W^{2,p}(\mathbb{R}^N)$ generates an analytic semigroup in $L^p(\mathbb{R}^N)$ and has bounded imaginary powers with power angle 0, see e.g. [19, Theorem C]. The operator L^A with domain

$$D(L^A) = \{u \in L^p(\mathbb{R}^N) : L^A u \in L^p(\mathbb{R}^N)\}$$

($L^A u$ is understood in the sense of distributions) generates the C_0–group given by $V(t)f(x) = f(e^{tA}x)$. Observe that

$$\|V(t)f\|_p = e^{-\frac{t}{p}\operatorname{tr} A}\|f\|_p,$$

see [15, Proposition 2.2]. The transference principle, see [4, Theorem 5.8] or [5, §4], shows that $w - L^A$ has bounded imaginary powers with power angle $\pi/2$ for every $w > -\operatorname{tr} A/p$.

(b) Let

$$\mathcal{W}_k = \{u \in W^{k,2}(\mathbb{R}^N) : (1+|x|^2)^{k/2} D^\alpha u \in L^2(\mathbb{R}^N) \text{for } |\alpha| \le k\}.$$

If k is sufficiently large (depending on p) and $u \in \mathcal{W}_k$, then $u \in D(L^Q) \cap D(L^A)$ and $L^Q u \in D(L^A)$, $L^A u \in D(L^Q)$. Moreover,

$$[L^Q, L^A]u = \sum_{i,h} \Big(\sum_j q_{ij} a_{hj}\Big) D_{ih} u$$

for $u \in \mathcal{W}_k$ and therefore the operator $[L^Q, L^A]R(1, L^Q)$ is bounded on $L^p(\mathbb{R}^N)$.

(c) In view of the arguments given in part (c) of the proof of Theorem 3.4, it remains to show that $R(\mu, L^Q)$ and $R(\mu, L^A)$ leave $\mathcal{W}_k$ invariant for large μ and every $k \in \mathbb{N}$.

The invariance of $\mathcal{W}_k$ under $R(\mu, L^Q)$ for $\operatorname{Re}\mu > 0$ can be verified by elementary Fourier transform methods. As regards L^A, we first observe that

$$|x|^k |V(t)u(x)| = |e^{-tA} e^{tA} x|^k \, |u(e^{tAx})| \le M e^{\gamma t} \, |e^{tA}x|^k \, |u(e^{tA}x)|$$

for suitable $M > 0$, $\gamma \in \mathbb{R}$, and hence

$$\||x|^k V(t)u\|_2 \le M e^{(\gamma - \operatorname{tr} A/2)t} \||x|^k u\|_2. \tag{4.3}$$

Since $DV(t)u = e^{tA^*} V(t) Du$, from (4.3) one easily obtains by induction that

$$\||x|^k D_\alpha V(t)u\|_2 \le M_k e^{\gamma_k t} \||x|^k D_\alpha u\|_2 \tag{4.4}$$

for $|\alpha| \le k$ and suitable $M_k, \gamma_k \in \mathbb{R}$. The invariance of $\mathcal{W}_k$ under $R(\mu, L^A)$ (for $\operatorname{Re}\mu$ large) follows since $R(\mu, L^A)$ is the Laplace transform of $V(\cdot)$. □

The above theorem says that the domain $\mathcal{D}_p$ is the intersection of the domains of the diffusion term L^Q and of the drift term L^A and implies the L^p-estimate

$$\|u\|_{W^{2,p}(\mathbb{R}^N)} \le c\,(\|u\|_p + \|Lu\|_p), \quad u \in \mathcal{D}_p,$$

which is analogous to the Calderón–Zygmund inequality for uniformly elliptic operators. We remark that Schauder estimates for Ornstein-Uhlenbeck operators have been obtained by G. Da Prato and A. Lunardi in [8].

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