Sufficiency of Lipschitz Conditions for Strict Solutions of Linear Evolution Equations in Nonreflexive Spaces

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Abstract

In this paper we give sufficient conditions of existence for strict solutions of the nonautonomous linear problem $u'(t) = A(t)\,u(t)$ in a Banach space, in the case $D(A(t)) = D =$ constant. With respect to the classic Kato's sufficient conditions we need only $t \mapsto A(t)$ continuous and of bounded variation in $\mathcal{L}(D, X)$, also if X is not reflexive. The same result can be extended to the case of nonconstant domain, with a common subspace $Y \subseteq D(A(t))$.
The main tool is the Stieltjes integral of operators in Banach spaces.

Keywords: evolutionary processes, CD–systems, functions of bounded variation, strict solutions, Stieltjes integral, Volterra–Stieltjes equations.

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1 Introduction

Consider in the Banach space X the nonautonomous linear Cauchy problem:

$$\begin{cases} u'(t) = A(t)\,u(t), & t > 0 \\ u(s) = x, & x \in X. \end{cases} \tag{1}$$

In [3, 4], Kato (see also [5]) gave sufficient conditions to the existence of an evolutionary process $U(t,s)$, $0 \le s \le t \le T$, related to this equation. These conditions are now classical: in the case of constant domain $D(A(t)) = D$, he assumes density and continuous inclusion of D in X, and the Kato–stability. Furthermore, he requires a regularity condition: $A(\cdot) \in C^1([0,T], \mathcal{L}(D,X))$ in the strong sense ([8], Chp. 4 or [6], Thm. 5.4.8). But this is not the possibly weakest condition; already in [5], Kato treated the case of a reflexive Banach space X, where he asks only for the Lipschitz regularity.

My aim is to show that the strict solutions exists under the weaker assumption that $A(\cdot) \in C([0,T], \mathcal{L}(D,X)) \cap \mathrm{BV}([0,T], \mathcal{L}(D,X))$ (in the norm sense) and so for $A(\cdot) \in \mathrm{Lip}([0,T], \mathcal{L}(D,X))$ in the strong sense. This case seems to me more suitable for applications.

The main tool of the proof is the Stieltjes integral for operators in Banach spaces (see Hönig [2], Diekmann, Gyllenberg and Thieme [1] and the related references). Some definitions and properties are reported in the Appendix.

2 CD–systems

Let X be a Banach space and let I the time interval $[0,T]$, $T>0$.

I recall some definitions (see also Tanabe [8], Def. 4.2.1 and Def. 4.3.1 or [9], Def. 7.2):

Definition 1 (Kato–stability) *A family $\{A(t) \mid t \in [0,T]\}$ of generators of C_0–semigroups is* Kato–stable, *with* stability constants $M \geq 1$, $\omega \in \mathbf{R}$, *if $\rho(A(t)) \supseteq (\omega,\infty)$, $t \in [0,T]$ and, for every finite sequence $0 \leq t_1 \leq t_2 \leq \ldots \leq t_k \leq T$, $k = 1,2,\ldots$, $\lambda > \omega$:*

$$\Big\| \prod_{j=1}^{k} (\lambda \mathrm{I} - A(t_j))^{-1} \Big\| \leq \frac{M}{(\lambda-\omega)^k}, \tag{2}$$

where $\prod_{j=1}^{k}(\lambda \mathrm{I} - A(t_j))^{-1} = (\lambda \mathrm{I} - A(t_k))^{-1} \ldots (\lambda \mathrm{I} - A(t_1))^{-1}$.

Kato–stability is used to prove the well–posedness of Pbm. (1) and the corresponding existence of an evolutionary process (see [5, 6, 8, 9]).

Now we give the following definition of CD–systems, different from the classic one in [5] because of weaker regularity requests and since X is not needed reflexive.

Definition 2 (CD-systems) *A* triplet $\{\mathcal{A}; X, D\}$, *where X and $D \subseteq X$ are Banach spaces, and $\mathcal{A} = \{A(t)\}_{t\in I}$ is a family of closed operator on X, will be called a* CD–system *(or Constant Domain system) if the following conditions are met:*

(A1) *$\{A(t)\}_{t\in I}$ is a Kato–stable family of generators of C_0–semigroups with stability constants M, ω.*

(A2) *D is continuously and densely embedded in X. Moreover, $D(A(t)) = D$ for each $t \in I$.*

(A3) *$t \mapsto A(t) \in C([0,T], \mathcal{L}(D,X)) \cap \mathrm{BV}([0,T], \mathcal{L}(D,X))$.*

In particular, Hyp. (A3) is satisfied if $t \mapsto A(t)\,x \in \mathrm{Lip}([0,T],X)$ for every $x \in D$.

Theorem 3 *Let $(\{A(t)\}_{t\in I}; X, D)$ be a CD–system.*

Then there exists a unique evolution process $U(t,s)$, $0 \leq s \leq t \leq T$, in X satisfying

(E1) $\|U(t,s)\| \leq M\, e^{\omega\,(t-s)}$ *for* $0 \leq s \leq t \leq T$,

(E2) $\displaystyle \frac{\partial^+}{\partial t} U(t,s)\, y \Big|_{t=s} = A(s)\, y \qquad \text{for } y \in D,\ 0 \le s \le T,$

(E3) $\displaystyle \frac{\partial}{\partial s} U(t,s)\, y = -U(t,s)\, A(s)\, y \qquad \text{for } y \in D,\ 0 \le s \le t \le T.$

(The derivative in (E2) and in (E3) are in the strong sense in X).

Furthermore, the following properties hold:

(E4) $U(t,s)\, D \subseteq D$ *per* $0 \le s \le t \le T$,

(E5) for $y \in D$, $U(t,s)\, y$ *is continuous in* D *for* $0 \le s \le t \le T$.

Finally, $U(t,s)\, x$ *is the unique strict solution for* Pbm. 1 *when* $x \in D$, *i. e. for each* $x \in D$ *we have* $u(\cdot) = U(\cdot, s)\, x \in C^1([s,T], X) \cap C([s,T], D)$ *and* $u(\cdot)$ *satisfies* Pbm. 1 *for each* $0 \le s \le t \le T$.

Proof: the proof of (E1)–(E3) is the classical one, given in [6], Thm 5.3.1 (see also [8, 9]).

The second part, concerning the regularity of the evolutionary process, follows the scheme of the proof of [6], Thm. 5.4.6, but using the Stieltjes integral (see the enclosed Appendix for the properties and the corresponding integral equations).

We define $W(t,s)$ as the solution of the following integral equation (Thm. 13): for $x \in X$ and $\lambda > \omega$,

$$W(t,s)\, x = U(t,s)\, x + \int_s^t W(t,r)\, dA(r)\ (A(r) - \lambda)^{-1}\, U(r,s)\, x, \tag{3}$$

Then define $Q(t,r) = U(t,r)\, (A(r) - \lambda)^{-1}$. We prove that $Q(t,s)\, x$ satisfies the Stieltjes integral equation

$$Q(t,s)\, x = (A(t) - \lambda)^{-1} U(t,s)\, x + \int_s^t Q(t,r)\, dA(r)\, (A(r) - \lambda)^{-1} U(r,s)\, x, \tag{4}$$

for each $x \in X$. In fact, take $x \in D$ and consider the family of operator $U_n(t,s)$ i. e. the approximation of $U(t,s)$ as finite product of C_0–semigroups, given as in [6], Eq. (5.3.5):

$$U_n(t,s) = \begin{cases} e^{(t-s)\, A(t_j^n)} & \text{for } t_j^n \le s \le t \le t_{j+1}^n, \\ e^{(t-t_k^n)\, A(t_i^n)} \left[\prod_{j=l+1}^{k-1} e^{(T/n)\, A(t_j^n)}\right] e^{(t_{l+1}^n - s)\, A(t_l^n)}, & \end{cases}$$

for $k > l$, $t_k^n \le t \le t_{k+1}^n$, $t_l^n \le s \le t_{l+1}^n$, where $t_k^n = (k/n)\, T$, $k = 0, \ldots, n$, is a subdivision of $[0,T]$ with $t_0^n = 0$, $t_n^n = T$.

For [6], (5.3.10), we have for $x \in D$ that $\frac{\partial}{\partial t}(U_n(t,s)\, x) = A_n(t)\, U_n(t,s)\, x$ for each $0 \le s \le t \le T$, $t \ne t_k^n$, $k = 0, 1, \ldots, n$, and $r \mapsto U_n(r,s)\, x$ is absolutely continuous and differentiable except for a finite number of values of r; therefore we have

$$-Q(t,t)\,U_n(t,s)\,x + Q(t,s)\,x = -\int_s^t d\left(Q(t,r)\,U_n(r,s)\,x\right) =$$

$$= -\int_s^t d\left(U(t,r)\,(A(r)-\lambda)^{-1}\right)\,U_n(r,s)\,x -$$

$$-\int_s^t U(t,r)\,(A(r)-\lambda)^{-1} d\left(U_n(r,s)\,x\right) =$$

$$= \int_s^t Q(t,r)\,\left[A(r)-A_n(r)\right]\,U_n(r,s)\,x +$$

$$+\int_s^t Q(t,r)\,dA(r)(A(r)-\lambda)^{-1}U_n(r,s)\,x. \tag{5}$$

For [6], (5.3.4) and (5.3.15) we have $U(t,s)\,x = \lim_{n\to\infty} U_n(t,s)\,x$ and

$$\lim_{n\to\infty}\sup_{0\le r\le T} \|A_n(r)-A(r)\|_{\mathcal{L}(D,X)} = 0 :$$

then

$$Q(t,s)\,x = (A(t)-\lambda)^{-1}U(t,s)\,x + \int_s^t Q(t,r)\,dA(r)\,(A(r)-\lambda)^{-1}U(r,s)\,x,$$

that is Eq. (4). For the density of D, this equation can be extended to each $x\in X$.

Because of (3), Eq. (4) is satisfied also by $(A(t)-\lambda)^{-1}W(t,s)\,x$; since the solutions of this class of integral equations are unique, we conclude that

$$U(t,s)\,(A(s)-\lambda)^{-1} = Q(t,s) = (A(t)-\lambda)^{-1}W(t,s)$$

and

$$U(t,s) = (A(t)-\lambda)^{-1}\,W(t,s)\,(A(s)-\lambda).$$

Then $U(t,s)$ is well–defined in D and $U(t,s)\,x$ is continuous in D for $0\le s\le t\le T$.

Finally, for $x\in D$, $U(t,s)\,x$ is the strict solution of (1): the proof is standard [6].

In the course of the proof we have obtained the following result that seems of independent interest. It is used by Tanaka [7] with different assumptions.

Corollary 4 *Let $(\{A(t)\}_{0\le t\le T}, X, D)$ be a CD–system and let $U(t,s)$, $0\le s\le t\le T$, be the corresponding evolution process.*

Then, for each $x\in X$ and $\lambda>\omega$,

$$U(t,s)\,(A(s)-\lambda)^{-1}\,x = (A(t)-\lambda)^{-1}U(t,s)\,x +$$

$$+\int_s^t U(t,r)\,(A(r)-\lambda)^{-1}dA(r)\,(A(r)-\lambda)^{-1}U(r,s)\,x. \tag{6}$$

2.1 Extensions: General Case and Inhomogeneous Problem

Thm. 3 is extended to the case of non–constant domain. Let Y be a Banach space, such that $Y \subseteq D(A(t))$ for each $0 \le t \le T$, like in [6], Chpp. 5.3–5.4. Some of the argumentations given in [6] work still under weaker hypotheses. We will use the terminology of [6], Chp. 5.

(H_1) $\{A(t)\}_{t\in[0,T]}$ is a family of closed operators in X.

(H_2) Y is $A(t)$–admissible for $t \in [0,T]$ and the family $\{\}$ of parts $\tilde{A}(t)$ of $A(t)$ in Y is stable in Y.
First of all, one can prove, with analogous proofs to [6], Lemma 5.4.4, that condition $(\mathrm{H}_2)^+$ in [6], Thm. 5.4.3 can be replaced by

$(H_2)^*$ There is a family $\{Q(t)\}_{t\in[0,T]}$ of isomorphisms of Y onto X such that $t \mapsto Q(t) \in C([0,T], \mathcal{L}(Y,X)) \cap \mathrm{BV}([0,T], \mathcal{L}(Y,X))$ and

$$Q(t)\, A(t)\, Q(t)^{-1} = A(t) + B(t)$$

where $B(t)$, $0 \le t \le T$, is a strongly continuous family of bounded operators in X.

Analogously we can weaken the hypotheses of Thm. 5.4.6–7.

Lemma 5 *The conditions (H_1) and $(H_2)^*$ imply the condition (H_2) in* [6]*; in particular Y is $A(t)$–admissible for each $t \in [0,T]$.*

Theorem 6 *Let $A(t)$, $0 \le t \le T$, be the infinitesimal generator of a C_0–semigroup in X. If the family $\{A(t)\}_{t\in[0,T]}$ satisfies the conditions (H_1), $(H_2)^*$ and (H_3) then there exists a unique evolution system $U(t,s)$, $0 \le s \le t \le T$, in X, satisfying (E_1)–(E_5) of* [6], Chp. 5.

Finally, for each $x \in Y$, Pbm. 1 *has a unique strict Y–solution u on $s \le t \le T$, i. e. for $x \in Y$ we have that $u \in C^1([s,T],X) \cap C([s,T],Y)$ and* Pbm. 1 *is satisfied in X for every $0 \le s \le t \le T$ (cmp.* [6], Def. 4.1*).*

Sketch of the proof: proceed like in the proof of Thm. 3, using

$$\begin{aligned} W(t,s)\,x &= U(t,s)\,x + \int_s^t W(t,r)\,B(r)\,U(r,s)\,x\,dr + \\ &\quad + \int_s^t W(t,r)\,dQ(r)\,Q(r)^{-1}U(r,s)\,x \end{aligned} \tag{7}$$

instead of Eq.(3) or [6], Eq. (5.4.14), and $Q(t,r) := U(t,r)\,Q(r)^{-1}$.

Also the variation–of–constant formula hold with these weaker hypotheses. For [6], Thm. 5.5.2 we have: *assume hypotheses of* Thm. 6*. If $f \in C([0,T],Y)$, then for every $x \in Y$ the initial–value problem*

$$\begin{cases} u'(t) = A(t)\,u(t) + f(t), & \text{for } 0 \le s < t \le T, \\ u(s) = x \end{cases} \tag{8}$$

possesses a unique strict Y–solution given by

$$u(t) = U(t,s)\,x + \int_s^t U(t,r)\,f(r)\,dr, \tag{9}$$

i. e. for every $x \in Y$ we have $u \in C^1([s,T],X) \cap C([s,T],Y)$ and u satisfies Pbm. 8 *for each $0 \leq s \leq t \leq T$.*

Appendix

A The Stieltjes Integral for Operators in Banach Spaces

The Stieltjes integral in Banach spaces is treated in [2]; see also [1]. However I give here some definitions to the case $U \in C([0,T],\mathcal{L}(X))$, $A \in \mathrm{BV}([0,T],\mathcal{L}(X,Y))$ and $f \in C([0,T],Y)$. Proofs are very standard and will be not reported.

I extend the definitions to the case where we have three vector–valued functions.

A.1 Definition

Let $[0,T]$ be a time interval, $T > 0$.

Consider a sub–interval $[s,t] \subseteq [0,T]$, $s < t$ and a sequence of partitions of $[s,t]$: take the set $\mathcal{P}_n = \{r_1, r_2, \dots r_n\}$, $n \geq 1$ with $r_{k-1} < r_k$ for $2 \leq k \leq n$. We say that this set *subhordinates*, or, with abuse of notation, *is a partition* of $[s,t]$. Take $r_0 := s$.

We denote the sequence of the partitions by $\mathcal{P} = \{\mathcal{P}_n \mid n \geq 1\}$ and we will take it becoming finer and finer.

Let X, Y be Banach spaces and let us take the functions $U\colon [0,T] \to \mathcal{L}(X)$, $A\colon [0,T] \to \mathcal{L}(Y,X)$, $f\colon [0,T] \to Y$.

Definition 7 *The pair of functions U and f is said* Stieltjes integrable *with respect to A, if there exists the limit*

$$\int_s^t U(r)\,dA(r)\,f(r) := \lim_{n\to\infty} \sum_{k=1}^n U(r_k)\,[A(r_k) - A(r_{k-1})]\,f(r_k), \tag{10}$$

independent of the sequence of $\{\mathcal{P}_n \mid n \geq 1\}$, taken finer and finer. The value of this limit is said Stieltjes integral of the pair U, f with respect to A.

Theorem 8 *Assume $A \in \mathrm{BV}([0,T],\mathcal{L}(X,Y))$.*

Then every pair $U \in C([0,T],\mathcal{L}(X))$, $f \in C([0,T],Y)$ is Stieltjes integrable with respect to f.

A.2 Some Properties

Proposition 9 *The Stieltjes integral for functions on intervals in* $[0,T]$*, where it exists, is a linear for each involved function.*

Theorem 10 *If* $f\colon[0,T]\to Y$*,* $A\colon[0,T]\to\mathcal{L}(Y,X)$ *and* $U\colon[0,T]\to\mathcal{L}(X)$*, the corresponding Stieltjes integral exists, and we have the estimate:*

$$\Big\|\int_s^t U(r)\,dA(r)\,f(r)\Big\|_X \le \\ \le \max_{s\le r\le t}\|U(r)\|_{\mathcal{L}(X)}\ \max_{s\le r\le t}\|f(r)\|_Y\cdot V_s^t[A]_{\mathcal{L}(Y,X)}. \tag{11}$$

Proposition 11 *The following properties hold:*

1. *if* $G\in C^1(X,Y)$ *and* $f\in\mathrm{BV}([0,T],X)$*, then*

$$\int_s^t dG(f(r)) = \int_s^t G'(f(r))\,df(r); \tag{12}$$

2. *If* $A(\cdot)\in\mathrm{BV}([0,T],\mathcal{L}(Y,X))$ *and* $0\in\rho(A(t))$ *for each* $0\le t\le T$*, then* $A(\cdot)^{-1}\in\mathrm{BV}([0,T],\mathcal{L}(X,Y))$ *and*

$$d\left(A(r)^{-1}\right) = -A(r)^{-1}dA(r)\,A(r)^{-1},$$

i. e. for $U\in C([0,T],\mathcal{L}(X))$*,* $f\in C([0,T],X)$ *we have*

$$\int_s^t U(r)\,d\left(A(r)^{-1}\right)\,f(r) = -\int_s^t U(r)\,A(r)^{-1}dA(r)\,A(r)^{-1}f(r). \tag{13}$$

Lemma 12 *Assume* $A(\cdot)\in C(I,\mathcal{L}(Y,X))\cap\mathrm{BV}(I,\mathcal{L}(Y,X))$*; then* $t\mapsto V_s^t[A]\in C(I)$ *and in particular*

$$\lim_{t\to s^+} V_s^t[A] = 0. \tag{14}$$

B Volterra–Stieltjes Integral Equations

Following the lines of [2], Chp. 3, we state:

Theorem 13 *Assume that* $U(t,s)$ *is an evolutionary process in* X*,* $g\in C([0,T],Y)$ *and* $A\in C([0,T],\mathcal{L}(Y,X))\cap\mathrm{BV}([0,T],\mathcal{L}(Y,X))$*. Then, for every* $x\in X$*, the integral equation*

$$W(t,s)\,x = U(t,s)\,x + \int_s^t W(t,r)\,dA(r)\,g(r)\,x\,dr \tag{15}$$

admit a unique continuous solution $\tilde{W}(t,s)\,x$ *on* $[s,T]$*, linear and uniformly bounded in* X *for each* $s\le t\le T$*.*

Proof: it follows from a contraction argument, using Eq. (11) and Lemma 12.

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