THEORY AND APPLICATIONS OF ONE-SIDED COUPLED OPERATOR MATRICES

Lectures by Klaus-J. Engel and Rainer Nagel

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ABSTRACT. These notes are meant as an introduction to and a survey on the theory of one-sided coupled operator matrices as developed recently by K.-J. Engel. We show how the theory can be applied to Volterra integro-differential equations, delay differential equations, and boundary initial value problems. In each case we concentrate on well-posedness and asymptotic behavior of the solutions.

1. INTRODUCTION

Of concern is an abstract Cauchy problem

$$\text{(ACP)} \qquad \begin{cases} \dot{\mathcal{U}}(t) = \mathcal{A}\,\mathcal{U}(t), & t \geq 0, \\ \mathcal{U}(0) = \mathcal{U}_0, \end{cases}$$

on a product Banach space $\mathcal{X} = X \times Y$. It is well-known that (ACP) can be studied by means of the theory of one-parameter semigroups of bounded linear operators. In particular, (ACP) is *well-posed* (which, roughly speaking, means that for all $\mathcal{U}_0 \in D(\mathcal{A})$ there exists a unique solution of (ACP), which depends continuously on the initial data) if and only if $\mathcal{A}$ generates a strongly continuous semigroup on $\mathcal{X}$ (see [EN00, Thm.II.6.7, Def.II.6.8 and Cor.II.6.9]). Thus, it is our goal to check the generator property of $\mathcal{A}$ (e.g. by using the Hille–Yosida Theorem), and then to discuss the asymptotic behavior of the solutions by means of spectral theory as, e.g., in [EN00, Chapters IV and V].

If we are given an operator $\mathcal{A}$ on a product space $\mathcal{X} = X \times Y$, one can wonder whether it is possible to represent it as an operator matrix and subsequently to characterize properties of $\mathcal{A}$ by means of properties of the matrix entries. Having reduced in this way a "two-dimensional" problem on $\mathcal{X} = X \times Y$ to an equivalent "one-dimensional" problem on the factor spaces X and/or Y there is the hope to obtain conditions that in practice are easier to check. This is the underlying idea of the subsequent results.

Such results can be obtained as a consequence of the theory of *one-sided coupled operator matrices* mainly developed by K.-J. Engel (see [En97a], [En97b], [En98a], [En98b], [En99], [En01]). Although this theory applies to many more initial value problems (including second order Cauchy problems and boundary value problems), we begin by introducing two special classes of equations that will be referred to in the following: Volterra integro-differential equations and delay differential equations.

2. VOLTERRA AND DELAY DIFFERENTIAL EQUATIONS

We consider the following *Volterra integro-differential equation*

$$\text{(VIDE)} \qquad \begin{cases} \dot{u}(t) = Au(t) + \displaystyle\int_0^t C(t-s)Au(s)ds + f(t), & t \geq 0, \\ u(0) = u_0, \end{cases}$$

where X is a Banach space, A is an operator on X, $u, f, C(\cdot)x \in F(\mathbb{R}_+, X)$ for all $x \in D(A)$ (where F denotes some function space such as $F = C_0$, L^p, etc.) and $u_0 \in X$. In order to obtain an appropriate (ACP) for this equation, we define a new state space $\mathcal{X} := X \times F(\mathbb{R}_+, X)$ and a new operator

$$\mathcal{A} := \begin{pmatrix} A & \delta_0 \\ C(\cdot) & \frac{d}{ds} \end{pmatrix}, \quad D(\mathcal{A}) := D(A) \times D\left(\frac{d}{ds}\right) \tag{2.1}$$

on $\mathcal{X}$, where δ_0 is the point evaluation in 0 and $\frac{d}{ds}$ denotes the generator of the left translation semigroup on $F(\mathbb{R}_+, X)$ (for a thorough treatment see [EN00, §VI.7.]). ■

As second prototype we consider the *abstract delay differential equation*

$$\text{(ADDE)} \qquad \begin{cases} \dot{u}(t) = Au(t) + \Phi u_t, & t \geq 0, \\ u(s) = h(s), & s \in [-1, 0], \end{cases}$$

on a Banach space X, where A is an operator on X, $u : [-1, +\infty) \to X$,

$$u_t : [-1, 0] \ni s \mapsto u(t + s) \in X$$

is the *history function* associated to u, $h \in L^p([-1, 0], X)$ and $\Phi : W^{1,p}([-1, 0], X) \to X$ a *delay operator*. Again, we reformulate this problem as (ACP) on a product space $\mathcal{X} := X \times L^p([-1, 0], X)$, for the operator

$$\mathcal{A} := \begin{pmatrix} A & \Phi \\ 0 & \frac{d}{ds} \end{pmatrix}, \quad D(\mathcal{A}) := \left\{ \begin{pmatrix} x \\ f \end{pmatrix} \in D(A) \times W^{1,p}([-1, 0], X) \ : \ f(0) = x \right\}. \tag{2.2}$$

For the definition of classical solutions of (ADDE), and for more results about this class of differential equations, we refer to [EN00, §VI.6] or [BP01] ∎

3. GENERAL FRAMEWORK

Our aim is to develop a general matrix theory which allows to treat the examples from the previous section (and many other) in a unified and systematic way.

To that purpose we will always assume that

- X and Y are Banach spaces, and we denote by I_X and I_Y the identity operators on X and Y, respectively,
- $\mathcal{X} := X \times Y$, and we denote by π_X and π_Y the projections from $\mathcal{X}$ onto X and Y, respectively,
- $\tilde{A} : D(\tilde{A}) \subseteq X \to X$ and $D : D(D) \subseteq Y \to Y$ are closed linear operators,
- $B : [D(D)] \to X$ and $L : [D(\tilde{A})] \to Y$ are bounded linear operators.

Here, for a given closed operator C on a Banach space Z, $[D(C)]$ denotes the Banach space $(D(C), \|\cdot\|_C)$ equipped with the graph norm

$$\| z \|_C := \| z \| + \| Cz \|, \quad z \in D(C).$$

Definition 1. *With the above notations, we call the operator*

$$\mathcal{A} := \begin{pmatrix} \tilde{A} & B \\ 0 & D \end{pmatrix} \begin{pmatrix} I_X & 0 \\ L & I_Y \end{pmatrix}$$

with domain

$$D(\mathcal{A}) := \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in D(\tilde{A}) \times Y \ : \ Lx + y \in D(D) \right\}$$

the associated one-sided coupled (short, osc) operator matrix on $\mathcal{X}$.

4. CHARACTERIZATION OF ONE-SIDED COUPLED OPERATOR MATRICES

It is now a rather natural question to ask which operators $\mathcal{A}$ on a product space $\mathcal{X}$ are osc. For invertible operators the answer is provided by the following result (cf. [En98b]).

Proposition 2. *Let X, Y be Banach spaces and $(\mathcal{A}, D(\mathcal{A}))$ be a linear operator on $\mathcal{X} = X \times Y$. Define the operators B and D by*

$$D(B) = D(D) := \left\{ y \in Y : \begin{pmatrix} 0 \\ y \end{pmatrix} \in D(\mathcal{A}) \right\},$$
$$By := \pi_X \left[\mathcal{A} \begin{pmatrix} 0 \\ y \end{pmatrix} \right], \qquad Dy := \pi_Y \left[\mathcal{A} \begin{pmatrix} 0 \\ y \end{pmatrix} \right].$$

If $\mathcal{A}$ is invertible and $\mathcal{A}^{-1} = \begin{pmatrix} U & V \\ W & S \end{pmatrix}$, then the following properties are equivalent.

(a) $\mathcal{A}$ is an osc operator matrix and D is invertible.

(b) U is injective and $\operatorname{rg}(V) \subseteq \operatorname{rg}(U)$.

(c) D is invertible.

Proof. Observe first that $\mathcal{A}\binom{0}{y} = \binom{By}{Dy}$ for $y \in D(B) = D(D)$, hence

$$\begin{pmatrix} 0 \\ y \end{pmatrix} = \mathcal{A}^{-1} \begin{pmatrix} By \\ Dy \end{pmatrix} = \begin{pmatrix} U & V \\ W & S \end{pmatrix} \begin{pmatrix} By \\ Dy \end{pmatrix} = \begin{pmatrix} UBy + VDy \\ WBy + SDy \end{pmatrix}. \tag{4.1}$$

From this it follows that

$$UB + VD = 0_{D(D)} \quad \text{and} \quad WB + SD = I_{D(D)}. \tag{4.2}$$

We now prove that (a)$\Rightarrow$(c)$\Rightarrow$(b)$\Rightarrow$(a), where the first implication is obvious.

(c)$\Rightarrow$(b). Since D is invertible, we obtain from (4.2) that $V = -UBD^{-1}$ and therefore $\operatorname{rg}(V) \subseteq \operatorname{rg}(U)$. To show the injectivity of U take $x \in \ker(U)$. Then $\mathcal{A}^{-1}\binom{x}{0} = \binom{0}{Wx} \in D(\mathcal{A})$. This implies $Wx \in D(D)$ and by

$$\begin{pmatrix} BWx \\ DWx \end{pmatrix} = \mathcal{A} \begin{pmatrix} 0 \\ Wx \end{pmatrix} = \begin{pmatrix} x \\ 0 \end{pmatrix},$$

we obtain $x = BWx$ and $Wx \in \ker(D) = \{0\}$. Hence, $x = 0$ proving that U is injective.

(b)$\Rightarrow$(a). For $\tilde{A} := U^{-1}$ with domain $D(\tilde{A}) := \operatorname{rg}(U)$ and $L := -WU^{-1}$, we define the matrix

$$\tilde{\mathcal{A}} := \begin{pmatrix} \tilde{A} & B \\ 0 & D \end{pmatrix} \begin{pmatrix} I_X & 0 \\ L & I_Y \end{pmatrix},$$
$$D(\tilde{\mathcal{A}}) := \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in D(\tilde{A}) \times Y : Lx + y \in D(D) \right\}.$$

We first prove that $\mathcal{A} = \tilde{\mathcal{A}}$ by showing that $\mathcal{A}^{-1}$ is the inverse of $\tilde{\mathcal{A}}$. In fact, for $\binom{x}{y} \in D(\tilde{\mathcal{A}})$ we have

$$\begin{aligned}
\mathcal{A}^{-1}\tilde{\mathcal{A}}\begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} U & V \\ W & S \end{pmatrix}\begin{pmatrix} \tilde{A} & B \\ 0 & D \end{pmatrix}\begin{pmatrix} x \\ Lx+y \end{pmatrix} \\
&= \begin{pmatrix} U & V \\ W & S \end{pmatrix}\begin{pmatrix} U^{-1}x + B(Lx+y) \\ D(Lx+y) \end{pmatrix} \\
&= \begin{pmatrix} x + UB(Lx+y) + VD(Lx+y) \\ WU^{-1}x + WB(Lx+y) + SD(Lx+y) \end{pmatrix} \\
&= \begin{pmatrix} x + (UB+VD)(Lx+y) \\ WU^{-1}x + (WB+SD)(-WU^{-1}x+y) \end{pmatrix} \overset{(4.2)}{=} \begin{pmatrix} x \\ y \end{pmatrix}
\end{aligned}$$

and hence $\tilde{\mathcal{A}} \subseteq \mathcal{A}$. To show equality it suffices to verify that $D(\mathcal{A}) \subseteq D(\tilde{\mathcal{A}})$, i.e.,

$$\mathcal{A}^{-1}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} Ux+Vy \\ Wx+Sy \end{pmatrix} \in D(\tilde{\mathcal{A}}) \tag{4.3}$$

for all $\binom{x}{y} \in \mathcal{X}$. To this end, we observe that the assumption $\mathrm{rg}(V) \subset \mathrm{rg}(U)$ implies $Ux+Vy \in \mathrm{rg}(U) = D(\tilde{A})$. Moreover, for arbitrary $y \in Y$ the element $-U^{-1}Vy \in X$ is well defined and

$$\mathcal{A}^{-1}\begin{pmatrix} -U^{-1}Vy \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ (-WU^{-1}V+S)y \end{pmatrix} \in D(\mathcal{A}) \tag{4.4}$$

gives $(-WU^{-1}V+S)y = (LV+S)y \in D(D)$. Using these facts we obtain

$$L[Ux+Vy] + [Wx+Sy] = (LV+S)y - WU^{-1}Ux + Wx = (LV+S)y \in D(D)$$

proving (4.3).

At this point it only remains to prove invertibility of operator D and to verify that the operators $\tilde{A}$, B, D and L satisfy our general assumptions. To this end, we first note that $\tilde{A} = U^{-1}$ is invertible, hence closed and that $L = -W\tilde{A} : [D(\tilde{A})] \to Y$ is bounded. Next, by the first equality in (4.2) it follows that $B = -U^{-1}VD$. Here the operator $U^{-1}V : Y \to X$ is bounded since $\mathrm{rg}(V) \subseteq \mathrm{rg}(U)$ by assumption. The second equality in (4.2) then implies $(S - WU^{-1}V)Dy = y$ for all $y \in D(D)$. On the other hand, multiplying (4.4) from the left by $\mathcal{A}$ gives $D(S - WU^{-1}V)y = y$ for all $y \in Y$. This shows that D is invertible, hence closed, and that $B = -U^{-1}VD : [D(D)] \to X$ is bounded as claimed. □

As a consequence we obtain a large class of osc operator matrices.

Corollary 3. *Assume that* $\dim X < \infty$. *If* $\mathcal{A}$ *is the generator of a strongly continuous semigroup on* $\mathcal{X} = X \times Y$, *then* $\mathcal{A} - \lambda$ *is an osc operator matrix for* λ *large enough.*

Proof. By the Hille–Yosida Theorem, $\lambda R(\lambda, \mathcal{A})$ exists for λ large and converges strongly to $I_{\mathcal{X}}$ as $\lambda \to +\infty$. In particular

$$\lambda R(\lambda, \mathcal{A})\begin{pmatrix} x \\ 0 \end{pmatrix} =: \lambda\begin{pmatrix} U_\lambda & V_\lambda \\ W_\lambda & S_\lambda \end{pmatrix}\begin{pmatrix} x \\ 0 \end{pmatrix} = \begin{pmatrix} \lambda U_\lambda x \\ \lambda W_\lambda x \end{pmatrix} \to \begin{pmatrix} x \\ 0 \end{pmatrix}$$

for all $x \in X$. Since X is finite dimensional, this implies $\lambda U_\lambda \to I_X$ with respect to the operator norm. Hence U_λ is invertible for λ large enough and Proposition 2 (b) gives the assertion. $\square$

Examples. Let us consider again the operators associated to the equations from §2.

- (VIDE). For every $\lambda \in \rho\left(\frac{d}{ds}\right)$ one can easily see that

$$\mathcal{A} - \lambda = \begin{pmatrix} A-\lambda & \delta_0 \\ C(\cdot) & \frac{d}{ds}-\lambda \end{pmatrix} = \begin{pmatrix} \tilde{A}_\lambda - \lambda & \delta_0 \\ 0 & D-\lambda \end{pmatrix} \begin{pmatrix} I_X & 0 \\ L_\lambda & I_{F(\mathbb{R}_+,X)} \end{pmatrix}, \tag{4.6}$$

  where $\tilde{A}_\lambda := A + \delta_0 R\left(\lambda, \frac{d}{ds}\right) C(\cdot)$, $L_\lambda := -R(\lambda, \frac{d}{ds})C(\cdot)$, and $D = \frac{d}{ds}$ with domain depending on the choice of the function space F (see [EN00, I.4.16]). Hence, $\mathcal{A} - \lambda$ is an osc operator matrix. ■

- (ADDE). The matrix $\mathcal{A}$ introduced in (ADDE) is an osc operator matrix on $\mathcal{X} := X \times L^p([-1,0],X)$. In fact, we can decompose it as

$$\mathcal{A} = \begin{pmatrix} \tilde{A} & \Phi \\ 0 & D \end{pmatrix} \begin{pmatrix} I_X & 0 \\ L & I_{L^p} \end{pmatrix}, \tag{4.5}$$

  where $\tilde{A} := A - \Phi L$, $D = \frac{d}{ds}$ with domain $D(D) = \{f \in W^{1,p}([-1,0],X) : f(0) = 0\}$, and $L := -\mathbb{1} \otimes I_X$, i.e., Lx is the function of constant value $-x$. By (2.2) we have

$$\begin{pmatrix} x \\ f \end{pmatrix} \in D(\mathcal{A}) \iff x \in D(A), \quad f \in W^{1,p}([-1,0],X) \quad \text{and} \quad f(0) = x.$$

  Since the latter condition is equivalent to $f - \mathbb{1} \otimes x = Lx + f \in D(D)$, the domain of $\mathcal{A}$ according to Definition 1 actually coincides with the domain introduced in §2 for (ADDE). Moreover, for $\binom{x}{f} \in D(\mathcal{A})$ we obtain

$$\begin{aligned} \begin{pmatrix} \tilde{A} & \Phi \\ 0 & D \end{pmatrix} \begin{pmatrix} I_X & 0 \\ L & I_Y \end{pmatrix} \begin{pmatrix} x \\ f \end{pmatrix} &= \begin{pmatrix} A - \Phi L & \Phi \\ 0 & \frac{d}{ds} \end{pmatrix} \begin{pmatrix} x \\ Lx + f \end{pmatrix} \\ &= \begin{pmatrix} Ax - \Phi Lx + \Phi Lx + \Phi f \\ (Lx+f)' \end{pmatrix} = \begin{pmatrix} Ax + \Phi f \\ f' \end{pmatrix} = \mathcal{A}\begin{pmatrix} x \\ f \end{pmatrix} \end{aligned}$$

  proving the representation of $\mathcal{A}$ as in (4.5).
  We remark that, in this case, the operator D defined in Proposition 2 equals $\frac{d}{ds}$ with domain

$$D\left(\frac{d}{ds}\right) = \left\{ f \in L^p([-1,0],X) : \begin{pmatrix} 0 \\ f \end{pmatrix} \in D(\mathcal{A}) \right\} = \left\{ f \in W^{1,p}([-1,0],X) : f(0) = 0 \right\}.$$

Therefore, D is the generator of the nilpotent (left) translation semigroup (see [EN00, I.4.17]) and is invertible, since $\sigma(D)=\emptyset$. Note that we cannot use Proposition 2 in order to show that $\mathcal{A}$ is osc, since in general it is not invertible.

- Let us now give an example of a *non*-osc operator matrix. Take $\mathcal{X}=L^p[0,1]\times L^p[0,1]$ and $\tilde{\mathcal{A}}$ defined by

$$\tilde{\mathcal{A}}:=\begin{pmatrix}\frac{d}{ds} & 0\\ 0 & -\frac{d}{ds}\end{pmatrix},\quad D(\tilde{\mathcal{A}})=\left\{\begin{pmatrix}f\\ g\end{pmatrix}\in W^{1,p}[0,1]\times W^{1,p}[0,1]\,:\,f(0)=g(0),f(1)=g(1)\right\},$$

which can be identified with the generator of the 'counter-clockwise' shift (semi)group on the circle.
Then the operator $\mathcal{A}:=\tilde{\mathcal{A}}-I_{\mathcal{X}}$ is invertible since it generates a uniformly exponentially stable semigroup. Moreover, the operator D defined in Proposition 2 is given by

$$Dg=\pi_Y\left[\mathcal{A}\begin{pmatrix}0\\ g\end{pmatrix}\right]=-\left(\frac{d}{ds}+I_{L^p[0,1]}\right)g\quad\text{with domain}$$
$$D(D)=\left\{g\in L^p[0,1]\,:\,\begin{pmatrix}0\\ g\end{pmatrix}\in D(\mathcal{A})\right\}=\left\{g\in W^{1,p}[0,1]\,:\,g(0)=g(1)=0\right\}.$$

Observe that $\sigma(D)=\mathbb{C}$, hence D is not invertible and, by Proposition 2, $\mathcal{A}$ is not osc. ∎

5. SPECTRAL THEORY FOR ONE-SIDED COUPLED OPERATOR MATRICES

We now look for a connection between the spectrum of an osc matrix $\mathcal{A}$ on $\mathcal{X}=X\times Y$ and the spectra of certain associated operators on the factor spaces X and Y. Let

$$\mathcal{A}:=\begin{pmatrix}\tilde{A} & B\\ 0 & D\end{pmatrix}\begin{pmatrix}I_X & 0\\ L & I_Y\end{pmatrix}$$

be an osc operator matrix and for all $\lambda\in\rho(D)$ define

$$A_\lambda:=\tilde{A}+\lambda BR(\lambda,D)L,\quad D(A_\lambda):=D(\tilde{A}).$$

Theorem 4. *With the above notation and for every $\lambda\in\rho(D)$ we have*

$$\lambda\in\rho(\mathcal{A})\iff\lambda\in\rho(A_\lambda).$$

In this case

$$R(\lambda,\mathcal{A})=\begin{pmatrix}R(\lambda,A_\lambda) & R(\lambda,A_\lambda)BR(\lambda,D)\\ -L_\lambda R(\lambda,A_\lambda) & R(\lambda,D)-L_\lambda R(\lambda,A_\lambda)BR(\lambda,D)\end{pmatrix},\tag{5.1}$$

where $L_\lambda := -DR(\lambda, D)L$.

Proof. It is not difficult to verify that

$$\lambda - \mathcal{A} = \begin{pmatrix} I_X & -BR(\lambda,D) \\ 0 & I_Y \end{pmatrix} \begin{pmatrix} \lambda - A_\lambda & 0 \\ 0 & \lambda - D \end{pmatrix} \begin{pmatrix} I_X & 0 \\ L_\lambda & I_Y \end{pmatrix} =: \mathcal{L}_\lambda \mathcal{Q}_\lambda \mathcal{R}_\lambda.$$

Observe now that $\mathcal{L}_\lambda$ is invertible on $X \times Y$ and $\mathcal{R}_\lambda$ is invertible and bounded on $[D(\tilde{A})] \times Y$, while $\mathcal{Q}_\lambda$ is invertible if and only if $\lambda - A_\lambda$ is invertible. Hence we conclude that $\lambda - \mathcal{A}$ is invertible if and only if $\lambda - A_\lambda$ is invertible.

To obtain the resolvent we compute

$$\begin{aligned} R(\lambda, \mathcal{A}) = \mathcal{R}_\lambda^{-1} \mathcal{Q}_\lambda^{-1} \mathcal{L}_\lambda^{-1} &= \begin{pmatrix} I_X & 0 \\ -L_\lambda & I_Y \end{pmatrix} \begin{pmatrix} R(\lambda, A_\lambda) & 0 \\ 0 & R(\lambda, D) \end{pmatrix} \begin{pmatrix} I_X & BR(\lambda, D) \\ 0 & I_Y \end{pmatrix} \\ &= \begin{pmatrix} R(\lambda, A_\lambda) & R(\lambda, A_\lambda)BR(\lambda, D) \\ -L_\lambda R(\lambda, A_\lambda) & R(\lambda, D) - L_\lambda R(\lambda, A_\lambda)BR(\lambda, D) \end{pmatrix}. \end{aligned}$$ □

Remark. If $\dim X < \infty$, then for $\lambda \in \rho(D)$ we have

$$\lambda \in \sigma(\mathcal{A}) \iff \lambda \in P\sigma(A_\lambda) \iff \det(\lambda - A_\lambda) = 0,$$

i.e., we obtain a *characteristic equation* for the spectral values of $\mathcal{A}$, cf. [Na97], [En99].

Examples. With Theorem 4 in mind we can now examine the spectra of our operators from §2.

- (VIDE). Using the representation (4.6) of $\mathcal{A} - \lambda$, we obtain by Theorem 4 for $\lambda \in \rho\left(\frac{d}{ds}\right)$ the characterization

  $$\lambda \in \sigma(\mathcal{A}) \iff \lambda \in \sigma\left(A + \delta_0 R(\lambda, \frac{d}{ds})C(\cdot)\right).$$

  For $\operatorname{Re}\lambda > \omega\left(\frac{d}{ds}\right)$, using the integral representation of the resolvent of A, this is equivalent to

  $$\lambda \in \sigma\left(A + \int_0^\infty e^{-\lambda s} C(s)\, ds\right).$$ ■

  If X is finite dimensional (typically, $X = \mathbb{C}^n$), by the previous remark we obtain the characteristic equation

  $$\lambda \in \sigma(\mathcal{A}) \iff \det\left(\lambda - A - \int_0^\infty e^{-\lambda s} C(s)\, ds\right) = 0,$$

  for the spectral values of $\mathcal{A}$ satisfying $\operatorname{Re}\lambda > \omega\left(\frac{d}{ds}\right)$.

- (ADDE). The condition $\lambda \in \rho(D) = \mathbb{C}$ is always satisfied and Theorem 4 yields
$$\lambda \in \sigma(\mathcal{A}) \iff \lambda \in \sigma(A_\lambda) = \sigma\left(A + \Phi\left(\mathbb{1} \otimes I_X\right) - \lambda \Phi R\left(\lambda, \frac{d}{ds}\right)\left(\mathbb{1} \otimes I_X\right)\right).$$
In order to simplify this characterization, we show that $\mathbb{1}\otimes I_X - \lambda R(\lambda, \frac{d}{ds})(\mathbb{1}\otimes I_X) = \varepsilon_\lambda \otimes I_X$, where $\varepsilon_\lambda \otimes I_X : X \to W^{1,p}([-1,0], X)$, $((\varepsilon_\lambda \otimes I_X)(x))(s) = e^{\lambda s}x$. In fact, using the explicit formula for the resolvent of the first derivative (see [EN00, II.2.10]) we obtain for all $x \in X$ and $s \in [-1, 0]$ that
$$\begin{aligned}[(\varepsilon_\lambda \otimes I_X)x](s) = e^{\lambda s}x &= x - \lambda \int_s^0 e^{-\lambda(\tau - s)} x \, d\tau \\ &= x - \lambda \int_s^0 e^{\lambda k} dk \, x \\ &= \left[\mathbb{1} \otimes I_X - \lambda R\left(\lambda, \frac{d}{ds}\right)\left(\mathbb{1} \otimes I_X\right)\right] x.\end{aligned}$$
Therefore, we have
$$\lambda \in \sigma(\mathcal{A}) \iff \lambda \in \sigma\left(A + \Phi\left(\varepsilon_\lambda \otimes I_X\right)\right).$$
Moreover, if $\dim X < \infty$, by the above remark it holds that
$$\lambda \in \sigma(\mathcal{A}) \iff \det\left(\lambda - A - \Phi\left(\varepsilon_\lambda \otimes I_X\right)\right) = 0,$$
i.e., the spectral values of $\mathcal{A}$ are again the zeros of a characteristic equation. ■

6. ONE-SIDED COUPLED GENERATORS OF STRONGLY CONTINUOUS SEMIGROUPS

We have already characterized osc operator matrices and also mentioned several spectral properties they inherit from their entries. We will now try to give some conditions which imply that a given osc operator matrix generates a strongly continuous semigroup on the product space $\mathcal{X}$.

A motivation for this task is the fact that, as a rule, *wellposedness of (ACP) for the osc operator matrix $\mathcal{A}$ is equivalent to wellposedness of the original initial value problem* (in a suitable sense). A proof of this claim for the particular cases of Volterra integro-differential equations and delay differential equations can be found in [EN00, §§VI.6–7], but similar results have actually been proven in a general context of abstract boundary/initial value problems in [CENN01] and [Mu01].

Checking the generator property for an osc operator matrix $\mathcal{A}$ is particularly easy if the first factor of $\mathcal{A}$ in Definition 1 is a diagonal matrix, i.e., if $B = 0$. The general case will then be obtained by means of similarity and perturbation arguments.

Theorem 5. *Assume that $\mathcal{A}$ is an osc operator matrix such that $B = 0$, i.e.,*

$$\mathcal{A} = \begin{pmatrix} \tilde{A} & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} I_X & 0 \\ L & I_Y \end{pmatrix},$$

defined on the usual domain introduced in Definition 1. Then $\mathcal{A}$ is the generator of a strongly continuous semigroup $(\mathcal{T}(t))_{t\geq 0}$ on $\mathcal{X}$ if and only if the following conditions hold.

(i) $\tilde{A}$ and D generate strongly continuous semigroup $(T(t))_{t\geq 0}$ and $(S(t))_{t\geq 0}$ on X and Y, respectively, and

(ii) for all $t \geq 0$ the operator

$$\tilde{Q}(t) : D(\tilde{A}^2) \to Y, \qquad \tilde{Q}(t)x := D\int_0^t S(t-s)LT(s)x\, ds, \tag{6.1}$$

extends to a bounded linear operator $Q(t)$ from X to Y, and this extension satisfies

$$\limsup_{t\to 0^+} \| Q(t) \|_{\mathcal{L}(X,Y)} < \infty.$$

In this case, $(\mathcal{T}(t))_{t\geq 0}$ is given by

$$\mathcal{T}(t) = \begin{pmatrix} T(t) & 0 \\ Q(t) & S(t) \end{pmatrix}. \tag{6.2}$$

Remarks.

1. We note that the operators $\tilde{Q}(t)$ are well defined. In fact, the function $\mathbb{R}_+ \ni s \mapsto LT(s)x$ is differentiable for all $x \in D(\tilde{A}^2)$ which implies, as one can see by integrating by parts, that $\int_0^t S(t-s)LT(s)x\, ds \in D(D)$.
2. We also remark that there are examples of generators $\tilde{A}$ and D such that the family of operators $\tilde{Q}(t)$ defined in (6.1) does not satisfy the assumptions in (ii). Take, e.g., for A an unbounded generator of a strongly continuous group on a Banach space X. Then for

$$\mathcal{A} := \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} \begin{pmatrix} I_X & 0 \\ I_X & I_X \end{pmatrix}$$

condition (i) of Theorem 5 is satisfied, while (ii) is not, and hence $\mathcal{A}$ is no generator on $\mathcal{X} = X \times X$.

Proof. We first assume that $\mathcal{A}$ is a generator. Since $B = 0$, the representation of the resolvent in Theorem 4 becomes

$$R(\lambda, \mathcal{A}) = \begin{pmatrix} R(\lambda, \tilde{A}) & 0 \\ DR(\lambda, D)LR(\lambda, \tilde{A}) & R(\lambda, D) \end{pmatrix}.$$

Hence, the powers are

$$R(\lambda,\mathcal{A})^n = \begin{pmatrix} R(\lambda,\tilde{A})^n & 0 \\ * & R(\lambda,D)^n \end{pmatrix} \quad \text{for all } n \in \mathbb{N}.$$

Therefore, both $\| R(\lambda,\tilde{A})^n \|$ and $\| R(\lambda,D)^n \|$ are dominated by $\| R(\lambda,\mathcal{A})^n \|$, which by the general version of the Hille–Yosida Theorem (see [EN00, Thm.II.3.8]) satisfies an estimate of the form

$$\| R(\lambda,\mathcal{A})^n \| \leq \frac{M}{(\mathrm{Re}\lambda-\omega)^n}$$

for all $\lambda \in \mathbb{C}$ such that $\mathrm{Re}\lambda > \omega$ for some ω sufficiently large. Hence, in order to apply the Hille–Yosida Theorem to $\tilde{A}$ and D we only need that both operators are densely defined. This follows for $\tilde{A}$ from the fact that $D(\mathcal{A}) \subseteq D(\tilde{A}) \times Y$ is dense in $\mathcal{X} = X \times Y$, hence $D(\tilde{A})$ must be dense in X. To show the denseness of $D(D)$ in Y, we fix some $f \in Y$ and consider

$$\lambda R(\lambda,\mathcal{A})\begin{pmatrix} 0 \\ f \end{pmatrix} = \begin{pmatrix} 0 \\ \lambda R(\lambda,D)f \end{pmatrix} \to \begin{pmatrix} 0 \\ f \end{pmatrix}$$

as $\lambda \to +\infty$. Since $\lambda R(\lambda,D)f \in D(D)$, this shows that $D(D)$ is dense in Y and thus $\tilde{A}$ and D are generators of strongly continuous semigroups $(T(t))_{t\geq 0}$ and $(S(t))_{t\geq 0}$ on X and Y, respectively.

In order to prove the claim on the operators $Q(t)$ we note that for λ with real part large enough the Laplace transform of the semigroup $\mathcal{T}(\cdot)$ coincides with the resolvent $R(\lambda,\mathcal{A})$ of its generator. In particular, we obtain for the Laplace transform of the lower left entry $\mathcal{T}(\cdot)_{21}$ of $\mathcal{T}(\cdot)$ the equality

$$\mathcal{L}(\mathcal{T}(\cdot)_{21})(\lambda) = R(\lambda,\mathcal{A})_{21} = DR(\lambda,D)LR(\lambda,\tilde{A}).$$

On the other hand, the convolution theorem for the Laplace transform implies that

$$\mathcal{L}(\tilde{Q}(\cdot)x)(\lambda) = DR(\lambda,D)LR(\lambda,\tilde{A})x$$

for all $x \in D(\tilde{A}^2)$ and $\mathrm{Re}\,\lambda$ sufficiently large. This proves that $\mathcal{L}(\tilde{Q}(\cdot)x)(\lambda) = \mathcal{L}(\mathcal{T}(\cdot)_{21}x)(\lambda)$ for all $x \in D(\tilde{A}^2)$ and $\mathrm{Re}\,\lambda$ large. The denseness of the domain $D(\tilde{A}^2)$ in X (see [EN00, Prop.II.1.8]) and the injectivity of the Laplace transformation then imply that $\mathcal{T}(t)_{21} = Q(t)$. The remaining assertion on the boundedness of $\|Q(t)\|$ close to $t=0$ then follows from the boundedness of $(\mathcal{T}(t))_{t\geq 0}$ as $t \to 0^+$.

To show the converse implication, let us first define $\mathcal{T}(t)$ as in (6.2). We are going to show that the operator family $(\mathcal{T}(t))_{t\geq 0}$ is a strongly continuous semigroup and then verify that its generator is $\mathcal{A}$.

In fact, for $t, s \in \mathbb{R}_+$, and $x \in D(\tilde{A}^2)$, $y \in Y$ we have

$$
\begin{aligned}
&\mathcal{T}(t)\mathcal{T}(s)\begin{pmatrix} x \\ y \end{pmatrix} \\
&\quad = \begin{pmatrix} T(t) & 0 \\ Q(t) & S(t) \end{pmatrix}\begin{pmatrix} T(s) & 0 \\ Q(s) & S(s) \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \\
&\quad = \begin{pmatrix} T(t) & 0 \\ Q(t) & S(t) \end{pmatrix}\begin{pmatrix} T(s)x \\ D\int_0^s S(s-r)LT(r)x\,dr + S(s)y \end{pmatrix} \\
&\quad = \begin{pmatrix} T(t)T(s)x \\ D\int_0^t S(t-q)LT(q)T(s)x\,dq + S(t)D\int_0^s S(s-r)LT(r)x\,dr + S(t)S(s)y \end{pmatrix} \\
&\quad \overset{r=q+s}{=} \begin{pmatrix} T(t+s)x \\ D\int_s^{t+s} S(t+s-r)LT(r)x\,dr + D\int_0^s S(t+s-r)LT(r)x\,dr + S(t+s)y \end{pmatrix} \\
&\quad = \begin{pmatrix} T(t+s)x \\ D\int_0^{t+s} S(t+s-r)LT(r)x\,dr + S(t+s)y \end{pmatrix} \\
&\quad = \mathcal{T}(t+s)\begin{pmatrix} x \\ y \end{pmatrix}.
\end{aligned}
$$

From [EN00, Prop. II.1.8] it follows that $D(\tilde{A}^2) \times Y \subseteq \mathcal{X}$ is dense. Since the operators $\mathcal{T}(t)$ and $\mathcal{T}(s)$ are bounded, this proves the semigroup property $\mathcal{T}(t)\mathcal{T}(s) = \mathcal{T}(t+s)$. Moreover,

$$
\mathcal{T}(0) = \begin{pmatrix} T(0) & 0 \\ Q(0) & S(0) \end{pmatrix} = \begin{pmatrix} I_X & 0 \\ 0 & I_Y \end{pmatrix} = I_{\mathcal{X}},
$$

since $Q(0)x = \tilde{Q}(0)x = 0$ for all $x \in D(\tilde{A}^2)$ and hence $Q(0) = 0$ by the denseness of $D(\tilde{A}^2)$ in X.

To show strong continuity of $(\mathcal{T}(t))_{t\geq 0}$, i.e.,

$$
\lim_{t\to 0^+} \mathcal{T}(t)\begin{pmatrix} x \\ y \end{pmatrix} \overset{(6.2)}{=} \begin{pmatrix} \lim\limits_{t\to 0^+} T(t)x \\ \lim\limits_{t\to 0^+}\Big[Q(t)x + S(t)y\Big] \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}
$$

for all $\binom{x}{y} \in \mathcal{X}$, it suffices to verify that operators $Q(t)$ converge to 0 as $t \to 0^+$. To this end we fix some $\lambda \in \rho(D) \cap \rho(\tilde{A})$ and choose $x \in D(\tilde{A}^2)$. Then, integrating by parts and using the fact that $(e^{-\lambda t}T(t))_{t\geq 0}$ and $(e^{-\lambda t}S(t))_{t\geq 0}$ are strongly continuous semigroups with generators $\tilde{A}-\lambda$ and $D-\lambda$, respectively, we obtain

$$\begin{aligned}
Q(t)x &= e^{\lambda t} D \int_0^t e^{-(t-s)\lambda} S(t-s) L e^{-\lambda s} T(s)x \, ds \\
&= e^{\lambda t} D(D-\lambda)^{-1} \Bigg(\Big[-e^{-(t-s)\lambda} S(t-s) L e^{-\lambda s} T(s)x \Big]_{s=0}^{s=t} \\
&\qquad\qquad + \int_0^t e^{-(t-s)\lambda} S(t-s) L e^{-\lambda s} T(s)(\tilde{A}-\lambda)x \, ds \Bigg) \\
&= D(D-\lambda)^{-1} \Big[S(t)Lx - LT(t)x + \int_0^t S(t-s) L (\tilde{A}-\lambda)^{-1} T(s)(\tilde{A}-\lambda)^2 x \, ds \Big] \\
&= D(D-\lambda)^{-1} \Big[S(t)Lx - L(\lambda - A)^{-1} T(t)(\lambda - A)x \\
&\qquad\qquad + \int_0^t S(t-s) L (\tilde{A}-\lambda)^{-1} T(s)(\tilde{A}-\lambda)^2 x \, ds \Big].
\end{aligned}$$

Since $LR(\lambda, \tilde{A})$ and $DR(\lambda, D)$ are bounded operators,

$$\begin{aligned}
\| Q(t)x \| \leq \| D(D-\lambda)^{-1} \| \cdot \Big[&\underbrace{\| S(t)Lx - L(\lambda-A)^{-1}T(t)(\lambda-A)x \|}_{\to 0 \text{ as } t\to 0^+} \\
&+ \underbrace{\int_0^t \| S(t-s)L(\tilde{A}-\lambda)^{-1}T(s) \| \cdot \| (\tilde{A}-\lambda)^2 x \| \, ds}_{\to 0 \text{ as } t\to 0^+} \Big].
\end{aligned} \tag{6.3}$$

This shows that $\lim_{t\to 0^+} Q(t) = 0$ on $D(\tilde{A}^2)$. The assumption on the boundedness of $\| Q(t) \|$ as $t \to 0^+$ and the density of $D(\tilde{A}^2)$ in X then yield $\lim_{t\to 0^+} Q(t) = 0$ on the whole space X.

Finally, we have to show that $\mathcal{A}$ is actually the generator of $(\mathcal{T}(t))_{t\geq 0}$. To this end it suffices to note that, as already mentioned above, the Laplace transform of $Q(\cdot)$ for $\operatorname{Re}\lambda$ sufficiently large coincides with the Laplace transform of the lower left entry of the resolvent $R(\lambda, \mathcal{A})$. □

Proposition 6. *Assume that $\tilde{A}$ is bounded on X. Then an osc operator matrix $\mathcal{A}$ generates a strongly continuous semigroup on $\mathcal{X}$ if and only if the operator $D + LB$, defined on $D(D)$, generates a strongly continuous semigroup on Y.*

Proof. We begin by remarking that under the above assumptions L is bounded and hence the matrix

$$\mathcal{S} := \begin{pmatrix} I_X & 0 \\ L & I_Y \end{pmatrix}$$

is bounded and invertible on $\mathcal{X}$. Therefore, $\mathcal{A}$ is similar to

$$\mathcal{A}^{\mathcal{S}} := \begin{pmatrix} I_X & 0 \\ L & I_Y \end{pmatrix} \begin{pmatrix} \tilde{A} & B \\ 0 & D \end{pmatrix} = \mathcal{S}\mathcal{A}\mathcal{S}^{-1}.$$

However,

$$ {}^s\!\mathcal{A} = \begin{pmatrix} \tilde{A} & B \\ L\tilde{A} & D+LB \end{pmatrix} = \begin{pmatrix} 0 & B \\ 0 & D+LB \end{pmatrix} + \begin{pmatrix} \tilde{A} & 0 \\ L\tilde{A} & 0 \end{pmatrix} =: \tilde{\mathcal{A}} + \mathcal{B}, $$

where $\mathcal{B}$ is bounded and $D(\tilde{\mathcal{A}}) = X \times D(D)$.

Hence, by bounded perturbation and similarity arguments, we can conclude that $\mathcal{A}$ is a generator if and only if the matrix operator $\tilde{\mathcal{A}}$ is a generator. By inverting in Theorem 5 the order of the spaces X and Y (obtaining in this way a generation result for upper instead of lower triangular matrices), one can see that this is the case if and only if $D + LB$ is a generator on Y. □

Example. We now apply the above results to delay differential equations.

Let A be bounded. Then, due to the assumptions on the various operators, also $\tilde{A} = A - \Phi L$ is bounded and Proposition 6 applies. Hence, the osc operator matrix $\mathcal{A}$ associated to (ADDE) generates a strongly continuous semigroup on $\mathcal{X}$ if and only if

$$ D + L\Phi = \frac{d}{ds} - (\mathbb{1} \otimes I_X)\Phi \quad \text{defined on} \quad D\left(\frac{d}{ds}\right) = \{f \in W^{1,p}([-1,0], X) \,:\, f(0) = 0\} $$

generates a strongly continuous semigroup on $L^p([-1,0], X)$.

In particular, if $\dim X < \infty$, by the Desch–Schappacher perturbation theorem one can prove that $\frac{d}{ds} + (\mathbb{1} \otimes I_X)\Phi$ is always a generator, hence (ADDE) is always well-posed. This applies, for instance, if Φ is a linear combination of Dirac measures on the interval $[-1, 0]$. For the details and further results see [En99, Expl.3.16] and [En97b].

We also emphasize that a different approach is needed when $\dim X = \infty$ and Φ is unbounded. We refer the reader to [BP01] for further reading. ■

Exercise. Find conditions such that the osc operator matrix associated to (VIDE) on $\mathcal{X} = X \times C_0(\mathbb{R}_+, X)$ and on $\mathcal{X} = X \times L^p(\mathbb{R}_+, X)$, respectively, generates a strongly continuous semigroup. (Hint: See [En99].)

7. ONE-SIDED COUPLED OPERATOR MATRICES AND POSITIVITY

Definition 7. *A strongly continuous semigroup $(\mathcal{T}(t))_{t\geq 0}$ on a Banach lattice $\mathcal{X}$ is called positive if each operator $\mathcal{T}(t)$ is positive, i.e.,*

$$ \mathcal{X} \ni f \geq 0 \quad \Longrightarrow \quad \mathcal{T}(t)f \geq 0 \quad \text{for each} \quad t \geq 0. $$

A thorough discussion of positive semigroups can be found in [Nag86], some fundamental results also in [EN00, §VI.1.b].

There are many characterizations of positive semigroups via properties of its generator. For example, it is not difficult to show that a strongly continuous semigroup is positive if and only if the resolvent of its generator is positive for sufficiently large λ (cf.[EN00, Thm.VI.1.8]). However, we would like to use our "matrix decomposition" of osc operators to give a characterization in terms of the matrix entries.

Theorem 8. *Let an osc operator matrix*

$$\mathcal{A} = \begin{pmatrix} \tilde{A} & B \\ 0 & D \end{pmatrix} \begin{pmatrix} I_X & 0 \\ L & I_Y \end{pmatrix}, \tag{7.2}$$

with $L \in \mathcal{L}(X,Y)$, generate a semigroup $(\mathcal{T}(t))_{t\geq 0}$ and let $A_\lambda := \tilde{A} + \lambda BR(\lambda, D)L$. Then the following properties are equivalent.

(a) $(\mathcal{T}(t))_{t\geq 0}$ is positive.

(b) There exists $\lambda_0 \in \mathbb{R}$ such that

(i) $R(\lambda, A_\lambda) \geq 0$ for all $\lambda > \lambda_0$,

(ii) $R(\lambda, D) \geq 0$ for all $\lambda > \lambda_0$,

(iii) $By \geq 0$ for all $y \in D(D)_+$ and $L^*D^*y^* \geq 0$ for all $y^* \in D(D^*)_+$.

Remarks.

1. By D^* and L^* in (b.iii) we denote the adjoints of the operators D and L. Observe that the operator DL (which is the lower left entry of the matrix one obtains by formally computing the product in (7.1)) does not make sense in general (or, more precisely, may have domain $\{0\}$ as in the case of the matrix associated to the delay equation (ADDE)), while the operator L^*D^* has domain $D(D^*)$ since L is bounded.
2. One can show that the assumption (b.iii) is equivalent to

   (iii') $By \geq 0$ for all $y \in D(D)_+$ and $-L_\lambda \geq 0$ for all $\lambda > \lambda_0$,

   where $L_\lambda = -DR(\lambda, D)L$, avoiding in this way the use of the adjoint operators D^* and L^*.

Sketch of Proof. The proof is quite involved, hence we only give the main steps and refer the reader to [En97a, Thm.3.2] for the details.

(b)⇒(a). Recalling the explicit formula (5.1) for resolvent of $\mathcal{A}$, it suffices to observe that under assumption (b) (with (iii) replaced by (iii'), cf. the remark above) it holds that $R(\lambda, \mathcal{A}) \geq 0$ for all $\lambda > \lambda_0$. Therefore $(\mathcal{T}(t))_{t\geq 0}$ is positive by Theorem VI.1.8 in [EN00].

(a)$\Rightarrow$(b). Assume $(\mathcal{T}(t))_{t\geq 0}$ to be positive. Then there exists a suitable λ_0 such that $R(\lambda, \mathcal{A}) \geq 0$ for all $\lambda > \lambda_0$. This implies that each entry of the resolvent operator matrix must be positive (to check this, it suffices to take $x \in X$, $y \in Y$, and then consider $R(\lambda, \mathcal{A})\binom{x}{0}$ and $R(\lambda, \mathcal{A})\binom{0}{y}$). Hence, in particular the upper-left entry $R(\lambda, A_\lambda)$ is positive for λ large enough proving (b.i).

(b.iii) follows from the *positive minimum principle* applied to $\mathcal{A}$ and its adjoint $\mathcal{A}^*$, see [En97a, Lemma 2.7] for details.

In order to prove (b.ii) consider

$$\mathcal{A}_v = \mathcal{A} - \begin{pmatrix} vI_X & 0 \\ 0 & 0 \end{pmatrix}, \quad v \geq 0,$$

and observe that with $\mathcal{A}$ also $\mathcal{A}_v$ generates a positive semigroup. Hence, the resolvent $R(\lambda, \mathcal{A}_v)$ is positive for large λ and, by some further calculations, one obtains for the lower right entry of $R(\lambda, \mathcal{A}_v)$

$$0 \leq \lim_{v\to\infty} R(\lambda, \mathcal{A}_v)_{22} = R(\lambda, D),$$

which shows (ii). $\square$

One important reason to consider positive semigroups is the fact that positivity greatly simplifies the study of the asymptotic behavior. Recall that the *spectral bound* of a closed operator A is defined as

$$s(A) := \sup\{\operatorname{Re}\lambda \,:\, \lambda \in \sigma(A)\}.$$

In case A is the generator of a strongly continuous semigroup $(T(t))_{t\geq 0}$, it is always dominated by the *growth bound*

$$\omega_0 := \inf\left\{w \in \mathbb{R} \,:\, \exists M_w \geq 1 : \| T(t) \| \leq M_w e^{wt} \ \ \forall\, t \geq 0\right\}.$$

These two bounds differ in general, but are equal in the special case of positive semigroups on L^p spaces (cf. [EN00, Thm.VI.1.15]). Also, it is usually difficult to determine $s(A)$. However, if A generates a positive semigroup, then $s(A) \in \sigma(A)$ (cf. [EN00, Thm.VI.1.10]) and hence it suffices to calculate the real spectral values of A in order to find $s(A)$. Another nice property says that a positive strongly continuous semigroup is exponentially stable if and only if the spectral bound of its generator is strictly negative (cf. [EN00, Prop.VI.1.14]). Combining the general theory of positive semigroups with the above results, one obtains the following characterization.

Corollary 9. *A positive strongly continuous semigroup $(\mathcal{T}(t))_{t\geq 0}$ generated by an osc operator matrix $\mathcal{A}$ (as in Theorem 8) is exponentially stable if and only if the spectral bounds $s(\tilde{A})$ and $s(D)$ satisfy*

$$s(\tilde{A}) < 0 \quad \textit{and} \quad s(D) < 0.$$

For a proof we refer to [En97a, Thm.4.1].

Examples.

- (ADDE). Let $(\mathcal{T}(t))_{t\geq 0}$ be the semigroup generated by the operator $\mathcal{A}$ defined in (ADDE). Keeping in mind that, for $D = \frac{d}{ds}$ defined on $W^{1,p}([-1,0],X)$ we have $s(D) = -\infty$, and that $R(\lambda, D)$ is positive for all $\lambda \in \mathbb{C}$, Theorem 8 reads as follows in this case.

  $(\mathcal{T}(t))_{t\geq 0}$ is positive if and only if the following two conditions hold.

  (i) There exists $\lambda_0 \in \mathbb{R}$ such that $R(\lambda, A + \Phi(\varepsilon_\lambda \otimes I_X)) \geq 0$ for all $\lambda \geq \lambda_0$,

  (ii) $\Phi f \geq 0$ for all $f \in D(D)_+ = \{f \in W^{1,p}([-1,0],X) \,:\, f \geq 0,\, f(0) = 0\}$.

  It is not difficult to show that both conditions are satisfied if

  $$(A\delta_0 + \Phi) f \geq 0 \quad \text{for all} \quad 0 \leq f \in W^{1,p}([-1,0],X).$$

  From Corollary 9 we obtain the following stability result for (ADDE).

  If $(\mathcal{T}(t))_{t\geq 0}$ is positive, then

  $(\mathcal{T}(t))_{t\geq 0}$ is exponentially stable if and only if $s(A + \Phi(\mathbb{1} \otimes I_X)) < 0$. ∎

- Consider now a special case of (ADDE) where we take $\Phi = \Psi\delta_{-r}$ for some $r \in [0,1]$ and $\Psi \in \mathcal{L}(X)$. Then the delay differential equation becomes

  $$\begin{cases} \dot{u}(t) = Au(t) + \Psi u(t-r), & t \geq 0, \\ u(s) = h(s), & s \in [-1,0]. \end{cases}$$

  Assume that the associated osc operator matrix $\mathcal{A}$ generates a positive semigroup. Then the solution of this equation is exponentially stable if and only if

  $$s\left(A + \Psi\delta_{-r}(\mathbb{1} \otimes I_X)\right) = s(A + \Psi) < 0.$$

  This is exactly the case when the solution of the *undelayed* equation

  $$\begin{cases} \dot{u}(t) = (A + \Psi)\, u(t), & t \geq 0, \\ u(s) = h(s), & s \in [-1,0] \end{cases}$$

  is exponentially stable! Hence,

  the stability of the solutions is independent of the delay r.

  This result deeply relies on the positivity assumptions and is in sharp contrast to the general (nonpositive) situation (see [BP01]). ∎

Exercise. Find conditions such that the osc operator matrix associated to (VIDE) on $\mathcal{X} = X \times C_0(\mathbb{R}_+, X)$ or on $\mathcal{X} = X \times L^p(\mathbb{R}_+, X)$, respectively, generates a positive strongly continuous semigroup and discuss its stability properties. (Hint: See [En01].)

8. APPLICATION TO INITIAL BOUNDARY VALUE PROBLEMS

We conclude these notes by discussing a further application of the theory of osc operator matrices to *abstract initial boundary value problems* (see [KMN01]). We call the problem

$$\text{(ABFS)} \qquad \begin{cases} \dot{x}(t) = Ax(t) + Bu(t), & t \geq 0, \\ \dot{u}(t) = D_m u(t), & t \geq 0, \\ x(t) = \mathcal{P}u(t), & t \geq 0, \\ x(0) = x_0, & \\ u(0) = u_0. & \end{cases}$$

an *abstract boundary feedback system.* Here,

- A is a linear operator on the (Banach) *boundary space* ∂X,
- D_m is linear on the (Banach) *state space* X and is called *maximal operator*,
- $B : X \to \partial X$ is a linear *feedback operator*,
- $\mathcal{P} : D(D_m) \to \partial X$ is a linear *boundary operator.*

We will denote by $\mathcal{X}$ the product space $\partial X \times X$. In order to apply the theory of osc operator matrices, we impose the following set of assumptions:

(G_1) A generates a strongly continuous semigroup $(T(t))_{t\geq 0}$ on ∂X,

(G_2) the restriction $D_0 := D_m|_{\ker \mathcal{P}}$ generates a strongly continuous semigroup $(S(t))_{t\geq 0}$ on X,

(G_3) the boundary operator $\mathcal{P}$ is surjective,

(G_4) the operator $\binom{\mathcal{P}}{D_m} : D(D_m) \subset X \to \mathcal{X}$ is closed.

As before, our aim is to rewrite (ABFS) in terms of a suitable abstract Cauchy problem on the product space $\mathcal{X}$. We refer the reader to [CENN01] for a concrete interpretation of (ABFS), as well as for a thorough treatment of this section's topic.

The following key lemma is essentially due to Greiner ([Gr87, Lemma 1.2], see also [CENN01, Lemma 2.2]).

Lemma 10. *Let $\lambda \in \rho(D_0)$. Then the restriction $\mathcal{P}|_{\ker(\lambda - D_m)}$ is invertible and its inverse*

$$\mathcal{D}_\lambda := (\mathcal{P}|_{\ker(\lambda - D_m)})^{-1} : \partial X \to \ker(\lambda - D_m) \subseteq X,$$

called the associated Dirichlet operator, is bounded.
Moreover, in the case where A_0 is invertible, $\mathcal{D}_0$ is related to $\mathcal{D}_\lambda$ by

$$\mathcal{D}_\lambda = (I - \lambda R(\lambda, A_0))\mathcal{D}_0.$$

We first discuss a simplified (unperturbed) version of (ABFS) by taking $B = 0$, and introduce the operator matrix

$$\mathcal{A}_0 := \begin{pmatrix} A & 0 \\ 0 & D_m \end{pmatrix} \qquad \text{with domain} \qquad D(\mathcal{A}_0) := \left\{ \begin{pmatrix} x \\ u \end{pmatrix} \in D(A) \times D(D_m) \,:\, \mathcal{P}u = x \right\}$$

on the product space $\mathcal{X}$.

Lemma 11. *Let $\lambda \in \rho(D_0)$. Then, $D(\mathcal{A}_0)$ defined as above coincides with*

$$\mathcal{H} := \left\{ \begin{pmatrix} x \\ u \end{pmatrix} \in D(A) \times X \,:\, -\mathcal{D}_\lambda x + u \in D(D_0) \right\}. \tag{8.1}$$

Moreover, the identity

$$\lambda - \mathcal{A}_0 = \begin{pmatrix} \lambda - A & 0 \\ 0 & \lambda - D_0 \end{pmatrix} \begin{pmatrix} I_{\partial X} & 0 \\ -\mathcal{D}_\lambda & I_X \end{pmatrix} =: \mathcal{Q}_\lambda \mathcal{R}_\lambda \tag{8.2}$$

holds.

Proof. To show the first claim, let us take a vector $\binom{x}{u} \in \mathcal{X}$. Then, $\binom{x}{u} \in \mathcal{H}$ if and only if $x \in D(A)$ and $u - \mathcal{D}_\lambda x \in \ker\mathcal{P}$. But $\mathcal{D}_\lambda x \in \ker(\lambda - D_m) \subset D(D_m)$, and hence $u - \mathcal{D}_\lambda x \in D(D_0) = \ker\mathcal{P}$ if and only if $u \in D(D_m)$ and $\mathcal{P}(u - \mathcal{D}_\lambda x) = \mathcal{P}u - x = 0$, i.e., $\mathcal{P}u = x$. This proves (8.1).

It is clear that $\mathcal{H} = D(\mathcal{Q}_\lambda \mathcal{R}_\lambda)$ since $\binom{x}{u} \in D(\mathcal{Q}_\lambda \mathcal{R}_\lambda)$ if and only if

$$\mathcal{R}_\lambda \begin{pmatrix} x \\ u \end{pmatrix} = \begin{pmatrix} x \\ -\mathcal{D}_\lambda x + u \end{pmatrix} \in D(\mathcal{Q}_\lambda) = D(A) \times D(D_0).$$

To show (8.2) we take $\binom{x}{u} \in D(\mathcal{Q}_\lambda \mathcal{R}_\lambda)$, and obtain

$$\mathcal{Q}_\lambda \mathcal{R}_\lambda \begin{pmatrix} x \\ u \end{pmatrix} = \begin{pmatrix} \lambda - A & 0 \\ 0 & \lambda - D_m \end{pmatrix} \begin{pmatrix} x \\ -\mathcal{D}_\lambda x + u \end{pmatrix} = \begin{pmatrix} (\lambda - A)x \\ (\lambda - D_m)u \end{pmatrix} = (\lambda - \mathcal{A}_0) \begin{pmatrix} x \\ u \end{pmatrix}. \qquad \square$$

Note that Lemma 11 simply says that $\lambda - \mathcal{A}_0$ is an osc operator matrix for $\lambda \in \rho(D_0)$. In particular, $\mathcal{A}_0$ is osc whenever D_0 is invertible.

We now study the generator property for $\mathcal{A}_0$.

Theorem 12. *Assume that D_0 is invertible. Then $\mathcal{A}_0$ generates a strongly continuous semigroup $(\mathcal{T}_0(t))_{t \geq 0}$ on $\mathcal{X}$ if and only if for all $t \geq 0$ the operator*

$$\tilde{Q}(t) : D(A^2) \to X, \qquad \tilde{Q}(t)x := -D_0 \int_0^t S(t-s)\mathcal{D}_0 T(s)x \, ds$$

extends to a bounded linear operator $Q(t) : \partial X \to X$, and this extension satisfies

$$\limsup_{t\to 0^+} \| Q(t) \|_{\mathcal{L}(\partial X, X)} < \infty.$$

In this case, the explicit form of $(\mathcal{T}_0(t))_{t\geq 0}$ is given as in (6.2) by

$$\mathcal{T}_0(t) = \begin{pmatrix} T(t) & 0 \\ Q(t) & S(t) \end{pmatrix}.$$

Proof. By Lemma 11, the claim is only a consequence of Theorem 5. □

Corollary 13. *Assume that A and D_0 generate analytic semigroups on ∂X and X, respectively. Then $\mathcal{A}_0$ generates an analytic semigroup on $\mathcal{X}$.*

The proof is based on similarity and perturbation arguments, and can be found in [CENN01, Corollary 2.8].

We now turn to the original version of (ABFS). The standard perturbation theory allows us to consider the particular case where the feedback operator B is bounded. In that case, we look at the operator matrix $\mathcal{A} := \mathcal{A}_0 + \mathcal{B}$, where

$$\mathcal{B} := \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix}$$

is a bounded operator on $\mathcal{X}$.

It is now easy to check the osc property for the matrix $\lambda - \mathcal{A}$.

Lemma 14. *Let $\lambda \in \rho(D_0)$. Then $\lambda - \mathcal{A}$ is an osc operator matrix. More precisely, $\lambda - \mathcal{A}$ can be factorized as*

$$\lambda - \mathcal{A} = \begin{pmatrix} \lambda - A_\lambda & -B \\ 0 & \lambda - D_0 \end{pmatrix} \begin{pmatrix} I_{\partial X} & 0 \\ -\mathcal{D}_\lambda & I_X \end{pmatrix}, \tag{8.3}$$

where $A_\lambda = A + B\mathcal{D}_\lambda$.

Proof. Taking into account the identity (8.2), a formal computation yields

$$\begin{aligned} \lambda - \mathcal{A} &= \mathcal{Q}_\lambda \mathcal{R}_\lambda - \mathcal{B} = (\mathcal{Q}_\lambda - \mathcal{B}\mathcal{R}_\lambda^{-1})\mathcal{R}_\lambda = \left[\begin{pmatrix} \lambda - A & 0 \\ 0 & \lambda - D_0 \end{pmatrix} - \begin{pmatrix} B\mathcal{D}_\lambda & B \\ 0 & 0 \end{pmatrix} \right] \mathcal{R}_\lambda \\ &= \begin{pmatrix} \lambda - A - B\mathcal{D}_\lambda & -B \\ 0 & \lambda - D_0 \end{pmatrix} \begin{pmatrix} I_{\partial X} & 0 \\ -\mathcal{D}_\lambda & I_X \end{pmatrix} = \begin{pmatrix} \lambda - A_\lambda & -B \\ 0 & \lambda - D_0 \end{pmatrix} \begin{pmatrix} I_{\partial X} & 0 \\ -\mathcal{D}_\lambda & I_X \end{pmatrix}. \end{aligned}$$

Since $\mathcal{B}$ is bounded, the domain of $\lambda - \mathcal{A}$ does not change and is again of the form (8.1). □

Lemma 14 claims that $\lambda - \mathcal{A}$ is osc for all $\lambda \in \rho(D_0)$. Hence, we obtain spectral properties of $\mathcal{A}$ analogous to those stated in Theorem 4.

Proposition 15. *For every $\lambda \in \rho(D_0)$, one has the equivalence*

$$\lambda \in \rho(\mathcal{A}) \iff \lambda \in \rho(A + B\mathcal{D}_\lambda).$$

Proof. Of course, $\lambda \in \rho(\mathcal{A})$ if and only if $0 \in \rho(\lambda - \mathcal{A})$. Since, by Lemma 14, $\lambda - \mathcal{A}$ is osc, we now only apply Theorem 4, whose claim in this case reads by (8.3) as

$$0 \in \rho(\lambda - \mathcal{A}) \iff 0 \in \rho(\lambda - A_\lambda).$$

Using the explicit expression of A_λ completes the proof. □

We remark that in the particular case of the unperturbed problem, i.e., for $B = 0$, we obtain a version of (5.1) that reads

$$R(\lambda, \mathcal{A}_0) = \begin{pmatrix} R(\lambda, A) & 0 \\ -\mathcal{D}_\lambda R(\lambda, A) & R(\lambda, D_0) \end{pmatrix}. \tag{8.4}$$

The next question that naturally arises is, how the stability of a system is affected by a feedback operator. We can provide an answer in the special case of positive semigroups (compare with Theorem 8 and Corollary 9).

Proposition 16. *Let ∂X and X be Banach lattices and let $\mathcal{A}$ generate a strongly continuous semigroup $(\mathcal{T}(t))_{t\geq 0}$ on $\mathcal{X} = \partial X \times X$. Assume further that A and D_0 generate positive strongly continuous semigroups on ∂X and X, respectively. Then the following hold.*

(i) If the feedback operator B is positive and the Dirichlet operators $\mathcal{D}_\lambda$ are positive for λ sufficiently large, then $(\mathcal{T}(t))_{t\geq 0}$ is a positive semigroup.

(ii) Let $(\mathcal{T}(t))_{t\geq 0}$ be positive. Then it is exponentially stable if (and only if) the spectral bounds $s(A + B\mathcal{D}_0)$ and $s(D_0)$ are both strictly negative.

Proof. If $\mathcal{D}_\lambda \geq 0$ for λ large, then (8.4) implies that $\mathcal{A}_0$ has positive resolvent, i.e., generates a positive semigroup. Since $\mathcal{B} \geq 0$, the Phillips–Dyson expansion (cf. [EN00, Thm.III.1.10]) for the semigroup $(\mathcal{T}(t))_{t\geq 0}$ generated by $\mathcal{A} = \mathcal{A}_0 + \mathcal{B}$ consists of positive terms only, hence it is positive as claimed.

For (ii), we first note that D_0 is invertible, and hence, by Lemma 14, $\mathcal{A}$ is an osc operator matrix. The claim is then just a reformulation of Corollary 9.

For a proof of the "only if" part we refer to [CENN01, Thm.5.4]. □

The approach discussed in this section is quite general. However, let us also mention one concrete application.

Example. We consider a heat equation on a bounded domain $\Omega \subseteq \mathbb{R}^n$ with dynamic boundary and assume that we can control the internal dynamic by a boundary feedback. More precisely, we consider the system

$$\text{(HEBF)} \qquad \begin{cases} \dot{u}(t,z) = \Delta_{\partial\Omega} u(t,z) - \mu(z)\, u(t,z) + \phi(t,z), & t \geq 0, \ \ z \in \partial\Omega, \\ \dot{u}(t,x) = \Delta_{\Omega} u(t,x), & t \geq 0, \ \ x \in \Omega, \\ u(0,z) = f(z), & z \in \partial\Omega, \\ u(0,x) = g(x), & x \in \Omega. \end{cases}$$

Here, $\phi : \mathbb{R}_+ \times \partial\Omega \to \mathbb{R}$ is defined as

$$\phi(t,z) := \int_{\Omega} k(z,x) u(t,x) dx \qquad \text{for all } t \geq 0 \text{ and } z \in \partial\Omega,$$

with kernel $k \in L^2(\partial\Omega \times \Omega)$.

We assume that μ is an essentially bounded non-negative function on $\partial\Omega$, and the initial data u_0 and u_1 are in $L^2(\partial\Omega)$ and $L^2(\partial\Omega)$, respectively. We also denote by $\Delta_{\partial\Omega}$ the Laplace–Beltrami operator on the boundary $\partial\Omega$, which becomes a compact manifold (without boundaries) under suitable regularity assumptions on the domain Ω.

Next we show that the above evolution equation is governed by a strongly continuous semigroup. To this aim, we reformulate (HEBF) as (ABFS).

Indeed, we consider the state space $X := L^2(\Omega)$ and the boundary space $\partial X := L^2(\partial\Omega)$, and then define the following operators:

- $Av := \Delta_{\partial\Omega} v - \mu(\cdot) v$ with domain $D(A) := \left\{ v \in H^{\frac{1}{2}}(\partial\Omega) \cap H^2_{\mathrm{loc}}(\partial\Omega) \, : \, \Delta_{\partial\Omega} v \in L^2(\partial\Omega) \right\}$,
- $Bv := \int_{\Omega} k(x,\cdot) v(x) dx$ mapping $L^2(\Omega)$ on $L^2(\partial\Omega)$,
- $D_m v := \Delta_{\Omega} v$ with domain $D(D_m) := \left\{ v \in H^{\frac{1}{2}}(\Omega) \cap H^2_{\mathrm{loc}}(\Omega) \, : \, \Delta_{\Omega} v \in L^2(\Omega) \right\}$,
- $\mathcal{P} v := v|_{L^2(\partial\Omega)}$ for $v \in D(D_m)$.

Remark. In the definition of $\mathcal{P}$, and also in the sequel, we consider traces of L^2 functions. For a complete survey of this field, we refer the reader to [LM72, Chap. 2] and emphasize that, in particular, D_m is defined on the maximal domain such that the traces of its elements exist as $L^2(\partial\Omega)$ functions. This is also an explanation for the notion of *maximal operator.*

It is now obvious that (HEBF) can formally be rewritten as an abstract boundary feedback system (ABFS) with initial data $(x_0, u_0) = (f, g)$. In order to apply the above generation and stability results, we still have to check that all assumptions are fulfilled.

We again refer the reader to [CENN01] for details about surjectivity of $\mathcal{P}$ and closedness of $\binom{\mathcal{P}}{D_m}$ and only point out that these proofs require non-trivial results about interpolation spaces (as a reference, see [LM72]) and estimates for the Laplacian which are well-known in the context of partial differential equations, see for instance [GT77, Chap. 8].

Our goal is now to show that Corollary 13 applies. The Laplace–Beltrami operator $(\Delta_{\partial\Omega}, D(A))$ on the manifold $\partial\Omega$ needs no boundary condition to generate an analytic semigroup, as does its bounded perturbation A. While $(D_m, D(D_m))$ is *not* a generator, its restriction D_0 is nothing but the Dirichlet-Laplacian on $L^2(\Omega)$, and hence generates an analytic semigroup. These generation results essentially use self-adjointness of the Laplace(–Beltrami) operator, under suitable assumptions, see, e.g., [Ta96].

By Corollary 13, the osc operator matrix $\mathcal{A}_0$ is the generator of an analytic semigroup on $\mathcal{X}$.

Moreover, B is a bounded operator, and hence $\mathcal{B} = \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix}$ is a bounded perturbation. Therefore, by standard perturbation theorems (see [EN00, §§III.1–2]) also the operator matrix $\mathcal{A} = \mathcal{A}_0 + \mathcal{B}$ is the generator of an analytic semigroup.

Both the Laplace–Beltrami operator on $\partial\Omega$ and the perturbed Laplace operator on Ω generate positive semigroups in an L^2-setting. Since it can be shown, using the Hopf maximum principle, that all Dirichlet operators $\mathcal{D}_\lambda$, $\lambda > 0$, are positive, it follows from Proposition 16(i) that $(\mathcal{T}(t))_{t\geq 0}$ is positive whenever the integral kernel k is a.e. positive.

Finally, under the above positivity assumptions, it is well-known that the Dirichlet Laplacian has strictly negative spectral bound. Therefore, as a consequence of Proposition 16(ii) we obtain that $(\mathcal{T}(t))_{t\geq 0}$ is exponentially stable if and only if

$$s(A + B\mathcal{D}_0) < 0.$$

Roughly speaking, this means that the asymptotics of the semigroup governing our evolution equation (HEBF) is not influenced by the internal dynamics of the problem.

Summing up, we can say that

stability of the problem is independent of the (internal) diffusion. ∎

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