A short proof for the stability theorem for positive semigroups on $L_p(\mu)$.

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Abstract: We give a short proof showing that the growthbound of a positive semigroup on $L_p(\mu)$ equals the spectral bound of its generator. It is based on a new boundedness theorem for positive convolution operators on $L_p(L_q)$. We also give a counterexample, showing that Gearhart's result does not extend from Hilbert spaces to $L_p(\mu)$–spaces.

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1. The Results:

Let T_t be a c_0–semigroup on $L_p(\Omega, \mu), 1 \le p < \infty$, with generator A. T_t is called positive, if $f \ge 0$ implies $T_t f \ge 0$ for all t. The spectral bound $s(A)$ of A is defined by

$$s(A) = \sup\{\mathrm{Re}\ \lambda : \lambda \in \sigma(A)\}$$

and the growth bound of T_t is given by

$$\omega(T_t) = \inf\{\omega :\ \exists\, C < \infty \text{ with } \|T_t\| \le C e^{\omega t} \text{ for all } t \ge 0\}.$$

In [9] the following theorem was proved.

Theorem 1: If T_t is a positive c_0–semigroup on $L_p(\Omega, \mu)$, $1 \le p < \infty$, then $s(A) = \omega(T_t)$.

The case $p = 2$ is due to Gearhart and Greiner–Nagel, the case $p = 1$ is due to Derndinger (see [7], [8], or [3], theorem 9.5 and 9.7), but the general case remained an open problem for about 10 years. The proof in [9] used a new spectral mapping theorem for the evolutionary semigroup $I \otimes T_t$ on $L_q(L_p)$ by Latushkin and Montgomery–Smith [5] and an extrapolation procedure for the Yosida approximation of T_t. In [6] S. Montgomery–Smith simplified the proof by replacing the extrapolation procedure by a direct resolvent estimate.

In this note we give a new and simple proof of theorem 1 that is based on a boundedness result for positive convolutions on mixed norm spaces $L_p(L_q)$, which may be of independent interest (see theorem 2 below). With this convolution result we can reduce theorem 1 to a well known characterization of the spectral bound in terms of weak integrability ([3], theorem 7.4).

Finally, we point out, that recent counterexamples concerning stability of semigroups (see e. g. [1]) can be „transplanted“ onto L_p–spaces. At the end of this note we give an example of a semigroup T_t on $L_p(0, 1), 1 < p < \infty, p \ne 2$, for which $s(A) = s_\infty(A) < \omega(T_t)$. Here

$$s_\infty(A) = \inf\{\omega : R(\lambda, A) \text{ is uniformly bounded for } \lambda, \mathrm{Re}\ \lambda \ge \omega\}.$$

This shows that Gearhart's spectral mapping theorem on Hilbert space (see e. g. [3], theorem 9.6), does not extend to L_p–spaces for $1 < p < \infty$, $p \ne 2$, and answers negatively question (IV) on page 147 of [8].

To state our convolution result we need the following notation. For $1 \le p, q < \infty$ and $f \in L_{1,\mathrm{loc}}(\mathbb{R}_+ \times \Omega)$ put

$$\begin{aligned} \|f\|_{p,q} &= \Big(\int_\Omega \Big(\int_{\mathbb{R}} |f(t,\omega)|^q dt\Big)^{\frac{p}{q}} d\mu(\omega)\Big)^{\frac{1}{p}} \\ L_p(L_q) &= \{f \in L_{1,\mathrm{loc}}(\mathbb{R}_+, \Omega) : \|f\|_{p,q} < \infty\}. \end{aligned}$$

Note that for $p = q$ we have by Fubinis theorem $L_p(L_p) = L_p(\mathbb{R}, L_p(\Omega, \mu))$. For these mixed norm spaces we have the following convolution result:

Theorem 2: For a fixed $1 \le p < \infty$, let $t \in \mathbb{R} \to K(t)$ be a function of positive operators on $L_p(\Omega, \mu)$ such that $t \to K(t)f$ is locally Bochner integrable for $f \in L_p(\Omega, \mu)$. Assume that for all $0 \leq h \in L_p(\Omega, \mu)$ and $0 \leq g \in L_{p'}(\Omega, \mu)$ we have

$$\int_{\mathbb{R}} < g, K(t)h >_{L_p} dt \leq C\|g\|_{L_p'} \cdot \|h\|_{L_p}$$

then the convolution integral

$$\mathcal{K}\, f(t) = \int_{-\infty}^{\infty} K(t-s)(f(s))ds$$

defined for stepfunctions $f : \mathbb{R} \to L_p(\Omega)$ extends to a bounded operator on $L_p(L_q), 1 \leq q \leq \infty$, with $\|\mathcal{K}\, f\|_{p,q} \leq C\|f\|_{p,q}$.

2. The Proofs:

Proof of theorem 1: Since $s(A) \leq \omega(T_t)$ is always true, we only have to show that $s(A) < 0$ implies $\omega(T_t) < 0$, or, by a result of Pazy ([3], Prop. 9.4.), that $s(A) < 0$ implies that for all $f \in L_p(\Omega, \mu)$

$$\int_0^{\infty} \|T_t f\|_{L_p}^p \, dt < \infty. \tag{1}$$

This claim can be reformulated as a convolution result. Indeed, for a fixed $\alpha > \omega(T_t)$

$$\int_0^t T_{t-s}\left(e^{-\alpha s}T_s f\right) ds = \frac{1}{\alpha}\left(1 - e^{-\alpha t}\right) T_t\, x.$$

Put $K(t) = T_t$ for $t \geq 0$ and $K(t) = 0$ for $t < \infty$ and $f(t) = e^{-\alpha t}T_t f$ for $t \geq 0$ and $f(t) = 0$ for $t < 0$. Then for $t \geq 1$ there is a constant D such that

$$\|T_t x\| \leq D \,\| \int_{-\infty}^{\infty} K(t-s)(f(s))ds\|. \tag{2}$$

The function $t \to K(t)$ satisfies the assumption of theorem 2 since by theorem 7.4 of [3] we have for all $0 \leq g \in L_{p'}$ and $0 \leq f \in L_p$ that

$$\int_0^{\infty} < g, T_t f > dt \leq \|R(0, A)\| \, \|g\|_{L_{p'}} \|f\|_{L_p}.$$

Since $f \in L_p(\mathbb{R}, L_p(\Omega))$ we obtain from theorem 2

$$\int \| \int K(t-s)(f(s))ds\|_{L_p}^p d(t) \leq C \int \|f(s)\|_{L_p}^p ds \leq C_1 \|f\|^p. \tag{3}$$

Now (3) together with (2) implies (1) and the proof is complete. Alternatively, one can obtain (3) from the estimate in the proof of theorem 1 in [6]. ∎

Proof of theorem 2: First we check the claim for $q = 1$. Given a stepfunction $f : \mathbb{R} \to L_p(\Omega)$ with $f \geq 0$ and a $0 \leq g \in L_{p'}(\Omega)$ we have for all $N \in \mathbb{N}$

$$< g, \int_{-N}^{N} \mathcal{K}\, f(t)dt >_{L_p} \;=\; \int_{-N}^{N} < g, \mathcal{K}\, f(t) >_{L_p} dt$$

$$
\begin{aligned}
&= \int_{-N}^{N} < g, \int K(s) f(t-s) ds > dt \\
&= \int \int_{-N}^{N} < g, K(s) f(t-s) > dt\, ds \\
&= \int < g, K(s) [\int_{-N}^{N} f(t-s) dt] > ds \\
&\leq \int < g, K(s) h > ds \leq C \|g\|_{L_{p'}} \cdot \|h\|_{L_p}
\end{aligned}
$$

by assumption, where $h = \int f(t)dt$ with $\|h\|_{L_p} = \|f\|_{p,1}$. Since such stepfunctions are dense in $L_p(L_1)$ we can extend $\mathcal{K}$ to $L_p(L_1)$ with

$$\|\mathcal{K} f\|_{p,1} \leq C \|f\|_{p,1}.$$

For $q = \infty$ and a stepfunction $f : \Omega \to L_\infty(\mathbb{R})$, $f(t, \omega) = \sum_k g_k(t) \chi_{A_k}(\omega)$, with $f \geq 0$ the integral

$$\int_{-N}^{N} K(s) f(t-s) ds = \sum_k \int_{-N}^{N} g_k(t-s) K(s) [\chi_{A_k}(\omega)]$$

exists. Put $h(\omega) = \operatorname{ess\,sup}_t f(t, \omega)$ with $\|h\|_{L_p} = \|f\|_{p,\infty}$. For all $g \in L_{p'}$ with $g \geq 0$, $\|g\|_{L_{p'}} = 1$ and $N \in \mathbb{N}$ we have

$$
\begin{aligned}
< g, \int_{-N}^{N} K(s) f(t-s) ds > \leq < g, \int_{-N}^{N} K(s) h ds >_{L_p} \leq & \int < g, K(s) h > ds \\
& \leq C \|g\|_{L_{p'}} \cdot \|h\|_{L_p} = C \|f\|_{p,\infty}.
\end{aligned}
$$

Since stepfunctions with countably many values are dense in $L_p(L_\infty)$ we can extend $\mathcal{K}$ to $L_p(L_\infty)$ by Fatou's Lemma and continuity so that $\|\mathcal{K} f\|_{p,\infty} \leq C \|f\|_{p,\infty}$.

Interpolating in the scale $L_p(L_q)$, $\quad 1 \leq q \leq \infty$, gives the general claim according to [2], theorem 5.1.2. ■

Example: Let $X = L_p(1, \infty) \cap L_2(1, \infty)$ with norm $\|f\| = \|f\|_2 + \|f\|_p$ for $2 < p < \infty$. Consider the semigroup $(S_t f)(x) = f(xe^t), t \geq 0$, with generator $(Bf)(x) = x(\frac{d}{dx} f)(x)$ on a suitable domain and $(R(0, B)f)(x) = \int_x^\infty f(y) \frac{dy}{y}$. One can check that $s(B) = -\frac{1}{2} < -\frac{1}{p} = \omega(S_t)(cf.\ [1])$. Since S_t is positive, we also have $s_\infty(B) = s(B)$ (see [3], Corollary 7.5)

By [4] 2.e.8(ii) and section 2.f. there is an isomorphism J of X onto $L_p[0, 1]$ (essentially given by a stochastic integral with respect to the poisson process). Then the semigroup

$T_t = JS_tJ^{-1}$ on $L_p[0,1]$ with generator $A = JBJ^{-1}$ on $D(A) = J(D(B))$ still satisfies $s_\infty(A) = -\dfrac{1}{2} < -\dfrac{1}{p} = \omega(T_t)$.
If $1 < p < 2$ we take the dual of T_t on $L_{p'}$, to obtain a similar example.

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