On Nonautonomous Higher Order Cauchy Problems

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Abstract

We study the existence and uniqueness of classical solutions to the following nonautonomous higher order Cauchy problem

$$\begin{cases} u^{(n+1)}(t) = A(t)u^{(n)}(t) + B_1(t)u^{(n-1)}(t) + ... + B_n(t)u(t) + f(t), \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 0 \leq s \leq t \leq T, \\ u^{(i)}(0) = x_i \in E, \ i = 0, 1, ..., n \end{cases}$$

by using operator matrices. The results cover most of the known results about existence and uniqueness of higher order Cauchy problem. Examples and applications are also given.

1 Introduction

Consider the differential equation

$$(nACP_n) \begin{cases} u^{(n+1)}(t) = A(t)u^{(n)}(t) + B_1(t)u^{(n-1)}(t) + ... + B_n(t)u(t), \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 0 \leq s \leq t \leq T, \\ u^{(i)}(0) = x_i \in E, \ i = 0, 1, ..., n, \end{cases} \tag{1.1}$$

and its inhomogeneous variant

$$(inACP_n) \begin{cases} u^{(n+1)}(t) = A(t)u^{(n)}(t) + B_1(t)u^{(n-1)}(t) + ... + B_n(t)u(t) + f(t), \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 0 \leq s \leq t \leq T, \\ u^{(i)}(0) = x_i \in E, \ i = 0, 1, ..., n. \end{cases} \tag{1.2}$$

Here, $A(t)$ are closed linear operators in a Banach space E with $D(A(t)) \equiv D$, $D(B(t)) \supseteq D$ for all $0 \leq t \leq T$ and f is a continuous function from $[0, T]$ to E.

Definition 1.1 *A function $u(\cdot)$ is said to be a solution of $(nACP_n)$ if it is (n+1)–times continuously differentiable on $[0, T]$, $A(t)u^{(n)}(t)$ and $B_i(t)u^{(n-i)}(t)$ are defined and continuous in t and (1.2) is satisfied.*

In [8], Neubrander considered the autonomous case

$$\begin{cases} u^{(n+1)}(t) = Au^{(n)}(t) + +B_1u^{(n-1)}(t) + ... + B_nu(t), \quad t \geq 0, \\ u^{(i)}(0) = x_i \in E, \ i = 0, 1, ..., n, \end{cases} \tag{1.3}$$

showing the "*wellposedness*" of (1.3) and applying his result to, e.g., strongly damped wave equations.

The pioneer work on nonautonomous higher order Cauchy problems was done by Obrecht. In [11] and [12] he proved the existence and uniqueness of solutions under rather strong "parabolic" hypotheses. Recently, Oka (see [13]) obtained the existence and uniqueness of solutions for second order equations under certain "hyperbolic" conditions. In [9] we obtained similar results under the "hyperbolic" as well as the "parabolic" case.

In this paper, we extend the result in [8] to nonautonomous equations and the results in [9] and [13] to higher order equations by using the theory of operator matrices in a systematic way. This method is quite simple and works equally well in the hyperbolic and parabolic case, as well as in homogeneous and inhomogeneous case. For the hyperbolic case, our result extends Oka's [13] . Moreover, we show in Corollary 2.10 that if $D(A(t)) = D(B_i(t))$, $i = 1, 2, ..., n$, then the results in [11] and [12] follow from our (parabolic) Theorem 2.8 .

For our purpose, we assume that there exists an invertible operator A from D onto E with $A^{-1} \in L(E)$. We then introduce new variables by defining

$$\begin{aligned} v_0 : &= Au \\ v_1 : &= Au' \\ .. \ &. \ .. \\ v_{n-1} : &= Au^{(n-1)} \\ v_n : &= u^{(n)}. \end{aligned}$$

In this way we obtain from (1.1) the following system

$$\begin{aligned} v_0' = Au' &= v_1 \\ v_1' = Au'' &= v_2 \\ \cdots \ &. \ .. \\ v_{n-1}' = Au^{(n)} &= Av_n \\ v_n' = u^{(n+1)} &= A(t)v_n + B_1(t)A^{-1}v_{n-1} + ... + B_n(t)A^{-1}v_0. \end{aligned}$$

This can be written in matrix form as

$$\begin{cases} \mathcal{V}'(t) = \mathcal{C}(t)\mathcal{V}(t), \quad 0 \le t \le T, \\ \mathcal{V}(0) = \mathcal{V}_0, \end{cases} \tag{1.4}$$

on E^{n+1}, where $\mathcal{V}(t) := (v_0(t), v_1(t), ..., v_n(t))^T$, $\mathcal{V}_0 := (Ax_0, Ax_1, ...,, Ax_{n-1}, x_n)^T$ and

$$\mathcal{C}(t) := \begin{pmatrix} 0 & I & 0 & \cdots & \cdots & 0 \\ 0 & 0 & I & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ \vdots & & & \ddots & I & 0 \\ 0 & \cdots & & \cdots & 0 & A \\ B_n(t)A^{-1} & \cdots & & \cdots & B_1(t)A^{-1} & A(t) \end{pmatrix} \tag{1.5}$$

with $D(\mathcal{C}(t)) := E \times E \times ... \times E \times D$.

In Lemma 2.1 we will show the equivalence between the solution of (1.1) and the solution of (1.4). This lemma allows us to find the solutions for a higher order problem by studying a corresponding first order problem on an appropriate product space.

In Section 2 we will show the existence and uniqueness for equations (1.1) and (1.2) in the hyperbolic case (Theorem 2.5 and 2.7) as well as in the parabolic case (Theorem 2.8). Finally, in Section 3, by Theorem 3.2 and 3.3 we consider nonautonomous higher order Cauchy problems, where $D(A(t))$ and $D(B_i(t))$ are not dense in E.

Throughout this paper the operators $A(t)$ are closed, $D(A(t)) \equiv D$ and $D(B_i(t)) \supseteq D$. Note that the product space $E_1 \times E_2 \times ... \times E_n$ of Banach spaces E_1, E_2,..., E_n, is again a Banach space with the norm $\|(x_1, x_2, ..., x_n)^T\| := \sum_1^n \|x_i\|$.

As a first step, we collect some basic facts concerning the wellposedness of first order nonautonomous Cauchy problems (see, e.g., [14])

$$(nCP) \begin{cases} u'(t) = A(t)u(t), & 0 \le t \le T, \\ u(0) = u_0 \end{cases} \tag{1.6}$$

for an operator family $\{A(t)\}_{0 \le t \le T}$ on a Banach space E.

Definition 1.2 *The problem (nCP) is called* **wellposed** *if there exists a unique, strongly continuous* **evolution family** *$\{U(t,s)\}_{0 \le s \le t \le T}$ of bounded, linear operators on E satisfying*

1. *$U(t,t) = Id$ for all $0 \le t \le T$,*
2. *$U(t,s)U(s,r) = U(t,r)$ for all $0 \le r \le s \le t \le T$,*
3. *$(t,s) \mapsto U(t,s)$ is strongly continuous,*

and for every $s \in [0,T]$ there exists a dense subspace $Y_s \subseteq E$ satisfying $U(t,s)Y_s \subseteq D(A(t))$ for every $0 \le s \le t \le T$ such that

$$\frac{\partial}{\partial t} U(t,s)x = A(t)U(t,s)x$$

and

$$\frac{\partial}{\partial s} U(t,s)x = -U(t,s)A(s)x$$

for all $x \in Y_s$.

If problem (nCP) is wellposed, then, for every initial value $u_0 \in Y_0$, $u(t) := U(t,0)u_0$ is the unique solution. There are several conditions implying wellposedness of nonautonomous Cauchy problems. Here, we recall two results, which usually are referred to Kato ([5]) and Tanabe ([15]), see also Pazy ([14], Theorem ... and ...)

Theorem 1.3 *Assume that the family of operators $\{A(t)\}_{0 \le t \le T}$ satisfies the* **hyperbolic condition***, i.e.,*

(H1) The domain $D := D(A(t))$ is independent of t and dense in E, and is a Banach space for the norm $\|.\|_D$. Moreover, there exists $c_0 > 0$ such that

$$c_0^{-1}\|x\|_D \le \|x\| + \|A(t)x\| \le c_0\|x\|_D$$

for all $t \in [0,T]$ and $x \in D$.

(H2) The family $(A(t))_{t\in[0,T]}$ is **stable**, *i.e., there exist constants $M \geq 1$ and $\omega \in \mathbb{R}$ such that*

$$(\omega,\infty) \subset \rho(A(t)) \quad \text{for all } t \in [0,T]$$

and

$$\left\| \prod_{j=1}^{k} R(\lambda, A(t_j)) \right\| \leq M(\lambda-\omega)^{-k} \quad \text{for all } \lambda > \omega \tag{1.7}$$

and any finite sequence $0 \leq t_1 \leq t_2 \leq \ldots \leq t_k \leq T$.

(H3) The mapping $t \mapsto A(t)y$ is continuously differentiable in E for every $y \in D$.

Then the Cauchy problem (nCP) is wellposed with evolution family $\{U(t,s)\}_{0\leq t\leq T}$ and for every $u_0 \in D$ the map $t \mapsto U(t,0)u_0$ is the unique classical solution of (nCP).

Theorem 1.4 *Assume that the family $\{A(t)\}_{0\leq t\leq T}$ satisfies the* **parabolic condition**, *i.e.,*

(P1) The domain $D := D(A(t))$ is independent of t and dense in E, and is a Banach space for the norm $\|.\|_D$. Moreover, there exists $c_0 > 0$ such that

$$c_0^{-1}\|x\|_D \leq \|x\| + \|A(t)x\| \leq c_0\|x\|_D$$

for all $t \in [0,T]$ and $x \in D$.

(P2) For all $t \in [0,T]$, the resolvent $R(\lambda, A(t))$ of $A(t)$ exists for all λ with $Re\lambda \geq 0$ and there is a constant $M > 0$ such that

$$\|R(\lambda, A(t))\| \leq \frac{M}{|\lambda|+1} \text{ for } Re\lambda \geq 0.$$

(P3) There exist constants $L \geq 0$ and $0 < \alpha \leq 1$ such that

$$\|(A(t)-A(s))A(0)^{-1}\| \leq L|t-s|^{\alpha} \quad \text{for all } t,s \in [0,T]. \tag{1.8}$$

Then the Cauchy problem (nCP) is wellposed with evolution family $\{U(t,s)\}_{0\leq t\leq T}$ and for every $u_0 \in D$ the map $t \mapsto U(t,0)u_0$ is the unique classical solution of (nCP).

For further information on semigroup theory and the theory of operator matrices, we refer to, e.g., [4], [6], [14] and [3], [7].

2 Higher order Cauchy problems

We now turn to the homogeneous higher order Cauchy problem

$$(nACP_n)\begin{cases} u^{(n+1)}(t) = A(t)u^{(n)}(t) + B_1(t)u^{(n-1)}(t) + \ldots + B_n(t)u(t), \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 0 \leq s \leq t \leq T, \\ u^{(i)}(0) = x_i \in E, \;\; i = 0,1,\ldots,n, \end{cases} \tag{2.1}$$

and the related first order problem on E^{n+1}

$$\begin{cases} \mathcal{V}'(t) = \mathcal{C}(t)\mathcal{V}(t), \quad 0 \leq t \leq T, \\ \mathcal{V}(0) = \mathcal{V}_0, \end{cases} \tag{2.2}$$

where $\mathcal{V}(t) := (v_0(t), v_1(t), ..., v_n(t))^T$, $\mathcal{V}_0 := (Ax_0, Ax_1, ..., , Ax_{n-1}, x_n)^T$ and $\mathcal{C}(t)$ is given by (1.5).

The following lemma is the key tool of our approach.

Lemma 2.1 *Assume that there exist a bijective operator A from D onto E. For $x_i \in D$ (i=0,1,..., n) the following statments hold true.*

(1) *If $u(\cdot)$ is a solution of* (2.1), *then $u^{(i)}(t) \in D$, $(i = 0, 1, ..., n-1)$ and $(Au, Au', ..., Au^{(n-1)}, u^{(n)})^T$ is a solution of* (2.2).

(2) *Conversely, assume that $(v_0, v_1, ..., v_n)^T$ is a solution of* (2.2). *Moreover, set $u_0 := v_n$ and define $u_k : [0,T] \mapsto E$ by $u_k(t) := x_{n-k} + \int_0^t u_{k-1}(s)ds$, (k= 1,2,...,n). Then $u := u_n$ is a solution of* (2.1).

Proof (1)$\Rightarrow$(2): Let $u(\cdot)$ be a solution of (2.1). We show first that we have $u^{(k)}(t) \in D$ and

$$\frac{d}{dt}(Au^{(k-1)}(t)) = Au^{(k)}(t) \tag{2.3}$$

for $k = 0, 1, ..., n$. To this end we assume that for some $k \in \{1, 2, ..., n\}$, $u^{(k)}(t) \in D$ and that $t \to Au^{(k)}(t)$ is continuous. Then we obtain

$$\int_0^t Au^{(k)}(s)ds = A\int_0^t u^{(k)}(s)ds = A(u^{(k-1)}(t) - x_{k-1}).$$

Since $x_{k-1} \in D$, it follows that $u^{(k-1)}(t) \in D$ and (2.3) holds. However, we have by assumption $u^{(n)}(t) \in D$ for every $t \in [0,T]$ and $t \mapsto Au^{(n)}(t)$ is continuous. Hence it follows that $u^{(n-1)}(t) \in D$ and $\frac{d}{dt}(Au^{(n-1)}(t)) = Au^{(n)}(t)$. By iterating this procedure we obtain that the above statement holds for all $k = n-1, n-2, ..., 1, 0$.

Now let $(v_0, v_1, ..., v_n)^T := (Au, Au', ..., Au^{(n-1)}, u^{(n)})^T$. Then we have $v_0' = v_1$, $v_1' = v_2$,..., $v_{n-1}' = v_n$, $v_n' = \sum_{k=1}^n B_k(t)A^{-1}(t)v_1$ and $(v_0(0), v_1(0), ..., v_n(0))^T = (Ax_0, Ax_1, ..., Ax_{n-1}, x_n)^T$, which proves that $(v_0, v_1, ..., v_n)^T$ is a solution of (2.2).

(2)$\Rightarrow$(1): Let $(v_0, v_1, ..., v_n)^T$ be a solution of (2.2) and u_i be defined as above. We show first by induction that $v_{n-k} = Au_k$ for $k = 1, 2, ..., n$ $(*)$.

For $n = 1$, since $(v_0, v_1, ..., v_n)^T$ is the solution of (2.2), we have $v_{n-1}' = Av_n$, and thus, $v_{n-1}(t) = Ax_{n-1} + \int_0^t Av_n(s)ds = A(x_{n-1} + \int_0^t v_n(s)ds) = Au_1(t)$.

Next, assume that the above statement is true up to $(k-1)$. Then, from $v_{n-k}' = Av_{n-k+1}$, we conclude

$$v_{n-k}(t) = Ax_{n-k} + \int_0^t Av_{n-k+1}(s)ds = A(x_{n-k} + \int_0^t u_{k-1}(s)ds) = Au_{n-k}(t).$$

Thus the assertion $(*)$ is true for k as well, hence our claim is proved.

Using this fact and the definition of u_k we obtain that $u^{(k)} = u_{n-k}$ for $k = 0, 1, ..., n$ and

$$\begin{aligned} u^{(n+1)}(t) = u_0'(t) &= v_n'(t) \\ &= A(t)v_n(t) + B_1(t)A^{-1}v_{n-1}(t) + ... + B_n(t)A^{-1}v_0(t) \\ &= A(t)v_n(t) + B_1(t)u_1(t) + ... + B_n(t)u_n(t) \\ &= A(t)u^{(n)}(t) + B_1(t)u^{(n-1)}(t) + ... + B_n(t)u(t). \end{aligned}$$

Moreover, the definition of u_k implies $u^{(k)}(0) = u_{n-k}(0) = x_k$ for $k = 0, 1, ..., n$. Therefore we conclude that $u(\cdot)$ is a solution of (2.1) and the lemma is proved. □

The above lemma shows that to study the higher order problem (2.1) it suffices to study the first order problem (2.2). As a further preparation we need the following lemmas.

Lemma 2.2 ([14], 5.2 , Theorem 2.3) *Let $\{A(t)\}_{0\le t\le T}$ be stable and $\{B(t)\}_{0\le t\le T}$ be a family of uniformly bounded operators. Then the family $\{A(t) + B(t)\}_{0\le t\le T}$ with $D(A(t) + B(t)) := D(A(t))$ is stable as well.*

Lemma 2.3 ([5], Prop. 4.4) *Let $\{A(t)\}_{0\le t\le T}$ be a stable family on E and $\{S(t)\}_{0\le t\le T}$ be a family of isomorphisms $S(t) \in L(E)$, which is strongly continuously differentiable. Then the family $\{\tilde{A}(t)\}_{0\le t\le T} := \{S(t)A(t)S^{-1}(t)\}_{0\le t\le T}$ with $D(\tilde{A}(t)) = \{x : S^{-1}(t)x \in D(A(t))\}$ is stable as well.*

Lemma 2.4 *Let $\{A(t)\}_{0\le t\le T}$ be a stable, strongly differentiable family on a Banach space E_2 with $D(A(t)) = D$ and B be a invertible operator from E_2 to a Banach space E_1 with $D(B) \supseteq D$. Then the family $\{\mathcal{A}(t)\}_{0\le t\le T}$, defined on $E_1 \times E_2$ by $\mathcal{A}(t) := \begin{pmatrix} 0 & B \\ 0 & A(t) \end{pmatrix}$, $D(\mathcal{A}(t)) := E \times D$, is stable as well.*

Proof Choose some $\lambda \in \varrho(A(t))$ and write $\mathcal{A}(t)$ as

$$\begin{aligned} \mathcal{A}(t) = \begin{pmatrix} 0 & B \\ 0 & A(t) \end{pmatrix} &= \begin{pmatrix} I & -BR(\lambda, A(t)) \\ 0 & I \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & A(t) - \lambda \end{pmatrix} \begin{pmatrix} I & BR(\lambda, A(t)) \\ 0 & I \end{pmatrix} \\ &\quad + \begin{pmatrix} 0 & 0 \\ 0 & \lambda \end{pmatrix}. \end{aligned}$$

Note that, by assumption, $\{\begin{pmatrix} 0 & 0 \\ 0 & A(t) - \lambda \end{pmatrix}\}_{0\le t\le T}$ with constant domain $E \times D$ is a stable family on $E_1 \times E_2$. Moreover, we know that if $A(\cdot)$ is strongly continuously differentiable with bounded, invertible operators $A^{-1}(\cdot)$, then $A^{-1}(\cdot)$ is strongly continuously differentiable as well. Thus, the strongly continuous differentiability of $BR(\lambda, A(t))$ follows from that of $(A(t) - \lambda)B^{-1}$ and the equality $BR(\lambda, A(t)) = [(A(t) - \lambda)B^{-1}]^{-1}$. Using Lemma 2.2 and Lemma 2.3 we then obtain that $\{\mathcal{A}(t)\}_{0\le t\le T}$ is stable. □

We are now in a position to prove the wellposedness of problem (2.2).

Theorem 2.5 *Suppose that the family $\{A(t)\}_{0\le t\le T}$ fullfils the hyperbolic condition and that the operator family $\{B_i(t)\}_{0\le t\le T}$, $(i = 1, 2, ..., n)$ is D–bounded and strongly continuously differentiable. Then problem (2.2) is wellposed on E^{n+1}.*

Proof We choose $(A(0)+\lambda)$ for some $\lambda \in \varrho(-A(0))$ as the bijective operator A. Then the family $\{\mathcal{C}(t)\}_{0\le t\le T}$ in (1.5), by assumption, is strongly continuously differentiable. Thus, in order to verify the hyperbolicity of $\{\mathcal{C}(t)\}$, it suffices to show its stability . However, by Lemma 2.2 for this it suffices to show that the family $\{\begin{pmatrix} 0 & A \\ 0 & A(t) \end{pmatrix}\}$ is stable. Since this follows from Lemma 2.4, the theorem is proved. □

As an immediate consequence of the above theorem, we obtain an autonomous result from [8].

Corollary 2.6 ([8], Theorem 5) *Let A be a densely defined linear operator and let $B_1, B_2, ..., B_n$ be closed linear operators with $D(A) \subset D(B_k)$ for $1 \le k \le n$. Consider the autonomous problem*

$$\begin{cases} u^{(n+1)}(t) = Au^{(n)}(t) + B_1 u^{(n-1)}(t) + ... + B_n u(t), & t \ge 0, \\ u^{(i)}(0) = u_i \in E \quad 0 \le i \le n. \end{cases} \tag{2.4}$$

Then the following statments are equivalent:

1. *A is the generator of a strongly continuous semigroup.*
2. *A has nonempty resolvent set and for every initial value $(x_0, x_1, ..., x_n) \in (D(A))^{n+1}$ the Cauchy problem (2.4) has a unique solution $u(\cdot)$ in*

$$C^{n+1}(\mathbb{R}_+, E) \cap C^n(\mathbb{R}_+, [D(A)]).$$

3. *A has nonempty resolvent set and for every initial value $(x_0, x_1, ..., x_n) \in (D(A))^{n+1}$ the Cauchy problem (2.4) has a unique solution $u(\cdot)$ in $C^{n+1}(\mathbb{R}_+, E)$ such that the following conditions hold.*
   (a) $u^{(n-k)}(\cdot) : \mathbb{R}_+ \to D(B_k)$ for $0 \le k \le n$ $(B_0 := A)$,
   (b) $B_k u^{(n-k)}(\cdot)$ are continuous for $1 \le k \le n$.

We now turn to the inhomogeneous equation

$$\begin{cases} u^{(n+1)}(t) = A(t)u^n(t) + B_1(t)u^{(n-1)}(t) + ... + B_n(t)u(t) + f(t), \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 0 \le s \le t \le T, \\ u^{(i)}(0) = x_i \in E, \; i = 0, 1, ..., n. \end{cases} \tag{2.5}$$

To this end we recall that we considered in [10] the first order inhomogeneous equation

$$\begin{cases} u'(t) = A(t)u(t) + f(t), & 0 \le t \le T, \\ u(0) = u_0 \end{cases} \tag{2.6}$$

for a hyperbolic family $\{A(t)\}$ and $f \in W^{1,1}(\mathbb{R}_+, E)$. By showing that the family $\{\mathcal{A}(t)\}_{0\le t\le T} := \{\begin{pmatrix} A(t) & \delta_0 \\ 0 & \frac{d}{dx} \end{pmatrix}\}_{0\le t\le T}$ with $D(\mathcal{A}(t)) := D \times W^{1,1}(\mathbb{R}_+, E)$ satisfies the hyperbolic condition, we obtained that the equation

$$\begin{cases} \mathcal{U}'(t) = \mathcal{A}(t)\mathcal{U}(t), & 0 \le t \le T, \\ \mathcal{U}(0) = \binom{u_0}{f} \end{cases} \tag{2.7}$$

has a unique solution for every $u_0 \in D$ and all $f \in W^{1,1}(\mathbb{R}_+, E)$. Moreover, the first component of this solution is the solution of (2.6). By combining this result with Lemma 2.1 we arrrive at the following.

Theorem 2.7 *Suppose that the assumptions of Theorem 2.5 are satisfied. If $x_i \in D$ $(i = 0, 1, ..., n)$ and $f \in W^{1,1}(\mathbb{R}_+, E)$, then equation (2.5) has a unique solution.*

Proof On $E^{n+1} \times L^1(\mathbb{R}_+, E)$ we consider the equation

$$\left\{\begin{array}{l} \begin{pmatrix} v_0 \\ . \\ . \\ v_n \\ \Phi \end{pmatrix}'(t) = \mathcal{D}(t) \begin{pmatrix} v_0 \\ . \\ . \\ v_n \\ \Phi \end{pmatrix}(t), \quad 0 \leq t \leq T, \\ \qquad \begin{pmatrix} v_0 \\ . \\ . \\ v_n \\ \Phi \end{pmatrix}(0) = \begin{pmatrix} Ax_0 \\ . \\ . \\ x_n \\ f \end{pmatrix}, \end{array}\right. \tag{2.8}$$

where

$$\mathcal{D}(t) := \begin{pmatrix} 0 & I & 0 & \cdots & & \cdots & 0 \\ 0 & 0 & I & \ddots & & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & & \\ & & & \ddots & I & \ddots & 0 \\ 0 & \cdots & & \cdots & 0 & A & 0 \\ B_n(t)A^{-1} & \cdots & & \cdots & B_1(t)A^{-1} & A(t) & \delta_0 \\ 0 & \cdots & & & \cdots & 0 & \frac{d}{dx} \end{pmatrix}$$

with domain $D(\mathcal{D}(t)) := E^n \times D \times W^{1,1}(\mathbb{R}_+, E)$.
To show the hyperbolic condition for family $\{\mathcal{D}(t)\}_{0 \leq t \leq T}$ it suffices to show that the family $\left\{\begin{pmatrix} (0) & \begin{pmatrix} A & 0 \end{pmatrix} \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} A(t) & \delta_0 \\ 0 & \frac{d}{dx} \end{pmatrix} \end{pmatrix}\right\}_{0 \leq t \leq T}$ is stable on $E \times (E \times L^1(\mathbb{R}_+, E))$. This, however, follows from Lemma 2.4 and the fact that $\left\{\begin{pmatrix} A(t) & \delta_0 \\ 0 & \frac{d}{dx} \end{pmatrix}\right\}$ fulfills the hyperbolic condition.

Let now $\mathcal{V}(\cdot) := (v_0(\cdot), ..., v_n(\cdot), \phi(\cdot))^T$ be the solution of equation (2.8). Obviously $\phi(t) = T_r(t)f$, where $T_r(t)f(\theta) := f(t+\theta)$. Set $u_0 := v_n$ and define u_k by $u_k(t) := x_{n-k} + \int_0^t u_{k-1}(s)ds$ $(k = 0, 1, ..., n)$. In the same way as in Lemma 2.1 we obtain

$$\begin{aligned} u^{(n+1)}(t) &= A(t)u^{(n)}(t) + B_1(t)u^{(n-1)}(t) + ... + B_n(t)u(t) + \Phi(t)(0) \\ &= A(t)u^{(n)}(t) + B_1(t)u^{(n-1)}(t) + ... + B_n(t)u(t) + f(t). \end{aligned}$$

Since the uniqueness of this solution follows from the uniqueness of the solution of the homogeneous equation, the proof is complete. □

We are now interested in the nonautonomous higher order equation in case the operators $A(t)$ generate analytic semigroups.

Theorem 2.8 *Suppose that the family $\{A(t)\}_{0\le t\le T}$ fulfills the parabolic condition and $t\mapsto B_i(t)$ is Hölder continuous with respect to the norm in $L(D,E)$, i.e., there exist $0<\gamma_i\le 1$ such that*

$$\|(B_i(t)-B_i(s))x\| \le L_i|t-s|^{\gamma_i}\|x\|_D \quad \text{for all } t,s\in[0,T], x\in D, \quad i=1,...,n. \tag{2.9}$$

Then, for every $x_i\in D$ and every Hölder–continuous function f, equation (2.5) has a unique solution.

Before proving the theorem, we need a perturbation lemma for the wellposedness under the parabolic condition.

Lemma 2.9 *Let the operators $A(t)$, $0\le t\le T$, satisfy the* **parabolic condition** *and let $B(t)$, $0\le t\le T$, be bounded operators satisfying*

$$\|B(t)-B(s)\| \le K \mid t-s\mid^{\beta} \text{ for some } 0<\beta\le 1 \text{ and for all } s,t\in[0,T]. \tag{2.10}$$

Then the problem

$$\begin{cases} \dot u(t) = (A(t)+B(t))u(t), & 0\le t\le T,\\ u(s)=u_s\in E \end{cases} \tag{2.11}$$

with $D(A(t)+B(t)) := D(A(t))$ is wellposed.

Proof Inequality (2.10) implies $\|B(t)\|\le C$ for all $0\le t\le T$. From the classical perturbation theorem for analytic semigroups ([14], Corollary 3.2.2) we know that $A(t)+B(t)$ is the generator of an analytic C_0-semigroup for every t.

We observe next that $(nCAP)$ is wellposed for the family $\{A(t)\}$ if and only if it is wellposed for $\{A(t)-\lambda\}$ for some fixed $\lambda\in\mathbb{C}$. In fact, the evolution system for this translated problem is given by $\{U_\lambda(t,s)\}_{0\le s\le t\le T}$ where $U_\lambda(t,s) := e^{-\lambda(t-s)}U(t,s)$. Therefore, by the Hille–Yosida theorem, we can, without loss of generality, assume that $\|R(\lambda,A(t))\|\le \frac{K}{|Re\lambda-\lambda_0|}$ for a constant $\lambda_0 < -CK$. Then, for $\mathrm{Re}\lambda\ge 0$, we have

$$\|B(t)R(\lambda,A(t))\| \le \|B(t)\|\|R(\lambda,A(t))^{-1}\| \le C\frac{K}{Re\lambda-\lambda_0} < \frac{CK}{-\lambda_0} = C_0 < 1.$$

Thus $(I-B(t)R(\lambda,A(t)))$ is invertible and

$$\begin{aligned}
\|R(\lambda,A(t)+B(t))\| &= \|R(\lambda,A(t))[I-B(t)R(\lambda,A(t))]^{-1}\|\\
&\le \|R(\lambda,A(t))\|\|(I-B(t)R(\lambda,A(t))^{-1}\|\\
&\le \frac{M}{\mid\lambda\mid+1}\sum_0^\infty C_0^n = \frac{M'}{\mid\lambda\mid+1}
\end{aligned}$$

for some constant M'. This shows that *(P1)* and *(P2)* of the **parabolic condition** are satisfied. We now check part *(P3)* of this condition by estimating

$$\begin{aligned}
&\|(A(t)+B(t)-A(s)-B(s))(A(0)+B(0))^{-1}\| \le\\
\le\ & \|(A(t)-A(s))(A(0)+B(0))^{-1}\| + \|(B(t)-B(s))(A(0)+B(0))^{-1}\|\\
\le\ & \|(A(t)-A(s))A(0)^{-1}\|\cdot\|A(0)(A(0)+B(0))^{-1}\| + \|(B(t)-B(s))\|\cdot\|(A(0)+B(0))^{-1}\|\\
\le\ & \|A(0)(A(0)+B(0))^{-1}\|L\mid t-s\mid^{\alpha} + \|(A(0)+B(0))^{-1}\|K\mid t-s\mid^{\beta}\\
\le\ & \tilde L\mid t-s\mid^{min(\alpha,\beta)}
\end{aligned}$$

for a suitable constant $\tilde{L}$. So the family $\{A(t)+B(t)\}$ satisfies the parabolic condition and problem (2.11) is wellposed. $\square$

Proof of Theorem 2.8 Suppose $\|(f(t)-f(s))\| \leq L_0|t-s|^\beta$ for all $t,s \in [0,T]$. We denote by $H^\beta([0,T],E)$ the space of all Hölder continuous functions on $[0,T]$, which is a Banach space for the norm

$$\|f\| := sup_{0\leq t\neq s\leq T}\frac{\|f(t)-f(s)\|}{|t-s|^\beta}.$$

On $E^{n+1}\times H^\beta([0,T],E)$ we consider the equation

$$\begin{cases} \begin{pmatrix} v_0 \\ . \\ . \\ v_n \\ \phi \end{pmatrix}'(t) = \mathcal{E}(t)\begin{pmatrix} v_0 \\ . \\ . \\ v_n \\ \phi \end{pmatrix}(t), \quad 0\leq t\leq T, \\ \begin{pmatrix} v_0 \\ . \\ . \\ v_n \\ \phi \end{pmatrix}(0) = \begin{pmatrix} Ax_0 \\ . \\ . \\ x_n \\ f \end{pmatrix}. \end{cases} \tag{2.12}$$

Here we take

$$\mathcal{E}(t) := \begin{pmatrix} 0 & I & 0 & \cdots & & \cdots & 0 \\ 0 & 0 & I & \ddots & & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & & \\ & & & \ddots & I & 0 & 0 \\ 0 & \cdots & & \cdots & 0 & A & 0 \\ B_n(t)A^{-1} & \cdots & & \cdots & B_1(t)A^{-1} & A(t) & \delta_t \\ 0 & \cdots & & & & \cdots & 0 \end{pmatrix}$$

with domain $D(\mathcal{E}(t)) := E^{n+1}\times H^\beta([0,T],E)$. In order to show the parabolicity of $\{\mathcal{E}(t)\}$, by Lemma 2.9, we only have to verify that of $\{\mathcal{B}(t)\}_{0\leq t\leq T} := \{\begin{pmatrix} -I & A \\ 0 & A(t)\end{pmatrix}\}$ with $D(\mathcal{B}(t)) := E\times D$ on $E\times E$. But this follows since $\{A(t)\}$ fulfills the parabolic condition (see also [9], Theorem 2.6).

Let now $\mathcal{V}(\cdot) := (v_0(\cdot),...,v_n(\cdot),\phi(\cdot))^T$ be the solution of equation (2.12). Then obviously $\phi(t)\equiv f$. Set $u_0 = v_n$ and define $u_k : [0T]\mapsto E$ by $u_k(t) := x_{n-k}+\int_0^t u_{k-1}(s)ds$, $(k=1,...n)$. In the same way as in Lemma 2.1 we then obtain

$$\begin{aligned} u^{(n+1)}(t) &= A(t)u^{(n)}(t)+B_1(t)u^{(n-1)}(t)+...+B_n(t)u(t)+\Phi(t)(t) \\ &= A(t)u^{(n)}(t)+B_1(t)u^{(n-1)}(t)+...+B_n(t)u(t)+f(t). \end{aligned}$$

$\square$

Example

In [11] and [12], the following time–dependent abstract higher order equation

$$\begin{cases} u^{(n)} + \sum_{k=0}^{n-1} A_k(t)u^{(k)} = f(t), \\ u^{(h)}(0) = x_h, \quad h = 0, \ldots n-1 \end{cases} \tag{2.13}$$

was considered. It was proved under a series of hypotheses that it has a unique (classical) solution. In case $D(A_k(t)) \equiv D$ is time independent, these hypotheses reduce to

(O1) *The operators $A_k(t)$ are closed with dense domain D for all $t \in [0,T]$ and there exist $\alpha_1, \alpha_2 \in \mathbb{R}_+$ such that*

$$\alpha_1 \|A_k(t)x\| \leq \|A_i(s)x\| \leq \alpha_2 \|A_k(t)x\|$$

for all $s, t \in [0,T], 0 \leq i, k \leq n-1$ and all $x \in D$.

Moreover, set $A_n(t) := I$ and define

$$P(\lambda, t) := \sum_{k=0}^{n} \lambda^k A_k(t) \text{ for } \lambda \in \mathbb{C}, \ t \in [0,T].$$

The expression $P(\lambda, t)$ will be considered as an operator in E with domain $D(P) := D$. We shall say that $\lambda \in \mathbb{C}$ belongs to the resolvent set of the operator pencil $P(\lambda, t)$ (and we write $\lambda \in \varrho(P(\cdot, t))$), if $P(\lambda, t)$ has an everywhere defined bounded inverse, which we denote by $P^{-1}(\lambda, t)$.

(O2) *There exist $M \in \mathbb{R}_+$ and $\theta \in]\frac{\pi}{2}, \pi[$ so that*

(a) $S_\theta = \{z \in \mathbb{C} : |argz| \leq \theta\} \cup \{0\} \subset \varrho(P(\cdot, t))$ for every $t \in [0,T]$.

(b) $\|A_k(t)P^{-1}(\lambda, t)\| \leq M_1(1 + |\lambda|)^{-k}$ for all $\lambda \in S_\theta$, $k \in \{0, 1, \ldots, n\}$ and for every $t \in [0,T]$.

(O3) *There exist constants $M_2 \in \mathbb{R}_+$ and $\alpha \in]0,1]$ such that*

$$\|(A_k(t) - A_k(s))x\| \leq M_2 |t-s|^\alpha \|A_k(s)x\|$$

for every $x \in D, k \in \{0, 1, \ldots, n-1\}$ and $t, s \in [0,T]$.

From (H2)(b), by setting $\lambda = 0$, we have that $A_0(t)$ is invertible with $\|A_0^{-1}(t)\| \leq M_1$.

For $\lambda \neq 0$, we set

$$K(\lambda, t) := \frac{1}{\lambda^{n-1}} P(\lambda, t) A_0^{-1}(t) = (\lambda + A_{n-1}(t))A_0^{-1}(t) + \frac{1}{\lambda^{n-1}} \sum_{k=0}^{n-2} \lambda^k A_k(t) A_0^{-1}(t).$$

Then $K(\lambda, t)$ is a bounded, invertible operator in E with $K^{-1}(\lambda, t) = \lambda^{n-1} A_0(t) P^{-1}(\lambda, t)$. Therefore, if $\lambda \in S_\theta$ and $|\lambda| > M_1$, then

$$\|(\lambda + A(t))A_0^{-1}(t) - K(\lambda, t)\| = \frac{1}{|\lambda^{n-1}|} \|\sum_{k=0}^{n-2} \lambda^k A_k(t) A_0^{-1}(t)\| < \frac{1}{\|K^{-1}(\lambda, t)\|}.$$

It follows that $(\lambda + A(t))A_0^{-1}(t)$ and thus, $\lambda + A(t)$ are invertible.
Also from (H2)(b) we obtain that $\|P(\lambda,t)x\| \geq M_1^{-1}(1+|\lambda|)^n\|x\|$ and from (H2)(b) using (H1) we conclude

$$\|A_k(t)x\| \leq M_3(1+|\lambda|)^{-n+1}\|P(\lambda,t)x\| \text{ for every } x \in D \text{ and } k \in \{0,1,...,n-1\}.$$

Thus, for every $x \in D$ and $\lambda \in S_\theta$ with $|\lambda| > M_1$, we obtain

$$\begin{aligned}
\|(\lambda + A(t))x\| &= \|\frac{1}{\lambda^{n-1}}(P(\lambda,t) - \sum_{k=0}^{n-2}\lambda^k A_k(t))x\| \\
&\geq \frac{1}{|\lambda^{n-1}|}(\|P(\lambda,t)x\| - \|\sum_{k=0}^{n-2}\lambda^k A_k(t)x\|) \\
&\geq \frac{1}{|\lambda|^{n-1}}(\|P(\lambda,t)x\| - M(1+|\lambda|)^{-1}\|P(\lambda,t)x\|) \\
&\geq \frac{1}{|\lambda|^{n-1}}(1 - M(1+|\lambda|)^{-1})M_1^{-1}(1+|\lambda|)^n\|x\| \\
&\geq (M')^{-1}(1+|\lambda|)\|x\|
\end{aligned}$$

for a suitable constant M'. This implies $\|(\lambda+A(t))^{-1}\| \leq M'(1+|\lambda|)^{-1}$ for all $\lambda \in S_\theta \cap \{|z| > M_1\}$.
From these considerations we obtain the following consequence.

Corollary 2.10 *In the case $D(A_k(t)) \equiv D$, $(k \in \{0,1,...n-1\})$, the hypotheses* (H1), (H2), *and* (H3) *in* [12] *imply the assumptions of* Theorem 2.8. *Therefore, for every initial value in D^n and every Hölder–continuous function f, there exists a unique solution of Equation* (2.5).

3 In non–dense case

This section is devoted to nonautonomous higher order Cauchy problems, in which $D(A(t)) = D$ is **not dense** in E. For the first order case, such initial value problems were studied by several autors, e.g. Acquistapace and Terreni [1], Da Prato and Sinestrari [2], Tanaka [16] . The following result was obtained by Tanaka [16].

Theorem 3.1 ([16], Theorem 1.8) *For the initial value problem* (1.6), *we suppose that the family $\{A(t)\}$ satisfies the* **hyperbolic condition** *except the density of D. Then there is an evolution family $\{U(t,s)\}_{0\leq s\leq t\leq T}$ on $\overline{D}$ with following properties.*

(1) $U(t,s)D(s) \subset D(t)$ for all $0 \leq s \leq t \leq T$, where the set $D(r)$ is defined by

$$D(r) := \{x \in D : A(r)x \in \overline{D}\}.$$

(2) The mapping $t \to U(t,s)x$ is continuously differentiable in E on $[s,T]$ and $\frac{\partial}{\partial t}U(t,s)x = A(t)U(t,s)x$ for $x \in D(s)$ and $t \in [s,T]$.

We see immediately that $u(t) = U(t,0)x_0$ is the unique solution of (1.6) for $x_0 \in D$ with $A(0)x_0 \in \overline{D}$. Using this theorem, we obtain a modification of Theorem 2.7, which extends the result in [13] .

Theorem 3.2 *Suppose that the assumptions of* Theorem 2.7 *are satisfied except the density of* D*. Then* (2.5) *has a unique solution for every* $f \in W^{1,1}(R_+, E)$ *and* $x_i \in D$ $(i = 0, 1, ...n)$ *satisfying* $A(0)x_n + \sum_1^n B_i(0)x_{n-i} + f(0) \in \overline{D}$*.*

Proof Consider equation (2.8), where the family $\{\mathcal{D}(t)\}$ satisfies the hyperbolic condition except the density of $D(\mathcal{D}(t)) = E^n \times D \times W^{1,1}(\mathbb{R}_+, E)$. Therefore, by Theorem 3.1 there is a unique classsical solution for (2.8) for every initial value $\mathcal{U}(0) := (Ax_0, Ax_1, ..., Ax_{n-1}, x_n, f)^T \in D(\mathcal{D}(t))$ satisfying $\mathcal{D}(0)(\mathcal{U}(0)) \in \overline{D(\mathcal{D}(t))} = E^n \times \overline{D} \times L^1(\mathbb{R}_+, E)$. This just means that (2.8) has a unique classsical solution for every $x_i \in D$, $f \in W^{1,1}(\mathbb{R}_+, E)$ and $A(0)x_n + \sum_1^n B_i(0)x_{n-i} + f(0) \in \overline{D}$.

By Lemma 2.1, we conclude that (2.5) has a unique solution for every $f \in W^{1,1}(R_+, E)$ and $x_i \in D$ $(i = 0, 1, ...n)$ satisfying $A(0)x_n + \sum_1^n B_i(0)x_{n-i} + f(0) \in \overline{D}$. □

Analogously, we have a result for the parabolic case. It can be proved using Theorem 2.8 and the result in [1] on the nonautonomous parabolic Cauchy problem with nondense domain.

Theorem 3.3 *Suppose that the assumptions of* Theorem 2.8 *are satisfied except the density of* D*. Then* (2.5) *has a unique solution for every Hölder–continuous* f *and* $x_i \in D$ $(i = 0, 1, ...n)$ *satisfying* $A(0)x_n + \sum_1^n B_i(0)x_{n-i} + f(0) \in \overline{D}$*.*

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