Characteristic Equations for the Spectrum of Generators

Rainer Nagel*

Abstract

We propose an abstract framework for the computation of the spectrum $\sigma(A)$ of a linear operator $A : D(A) \subset X \to X$ on a Banach space X through a condition in a smaller Banach space X_1. If this space is finite dimensional this yields a characteristic equation for $\sigma(A)$. The method is tested for delay, integro-differential and population equations and is applicable to control and stability questions.

1. Motivation

The spectrum $\sigma(A)$ of the generator A of a strongly continuous semigroup on a Banach space X characterizes various qualitative and, in particular, asymptotic properties of the semigroup. We only mention that the negativity of the spectral bound of A, i.e.,

$$\mathrm{s}(A) := \sup \{\mathrm{Re}\,\lambda : \lambda \in \sigma(A)\} < 0$$

implies uniform exponential stability for eventually norm continuous semigroups on Banach spaces or for positive semigroups on L^p- spaces (see [Ne], Chapter 3.5). We refer to the monographs [Na1] and [Ne] where these and many more relations between spectrum and asymptotics are discussed systematically.

It thus remains a fundamental task to find or at least to estimate the spectrum $\sigma(A)$ of semigroup generators A. By definition, this is a problem in the given Banach space. However, in many situations it can be solved by finding the zeros of a certain complex function. More precisely, there exists a function $\xi_A : \mathbb{C} \to \mathbb{C}$ such that the spectral values of A are the zeros of a so-called *characteristic equation*, i.e.,

$$\sigma(A) = \{\lambda \in \mathbb{C} : \xi_A(\lambda) = 0\}.$$

Thus the infinite dimensional (linear) problem of computing $\sigma(A)$ is reduced to a one-dimensional (non linear) problem. Here are some first examples for this phenomenon.

1.1 Examples

(i) Take the finite dimensional space $X := \mathbb{C}^n$ and a complex $n \times n$-matrix $A := (a_{ij})_{n\times n}$. Then

$$\sigma(A) = \{\lambda \in \mathbb{C} : p_A(\lambda) = 0\}$$

for the characteristic polynomial $p_A(\lambda) := \det(\lambda - A)$.

(ii) In the semigroup approach to delay equations one considers the Banach space $X := C[-1, 0]$ and the operator $Af := f'$ on the domain

$$D(A) := \left\{f \in C^1[-1, 0] : f'(0) = Lf\right\}$$

for some continuous linear form $L \in X'$. Then it is well known that

$$\sigma(A) = \{\lambda \in \mathbb{C} : L(\epsilon_\lambda) - \lambda = 0\},$$

where $\epsilon_\lambda(s) := e^{\lambda s}$ for $s \in [-1, 0]$.

These and many more examples lead to the question we intend to study in this paper.

1.2 Problem

For which operators $A : D(A) \subseteq X \to X$ is it possible to characterize (parts of) the spectrum $\sigma(A)$ through a "characteristic equation" in a smaller, preferably finite dimensional space?

In Section 2 we propose an abstract framework for this problem and then show in Section 3 its usefulness by a series of examples.

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2. Theory

We use perturbation methods and, in particular, the following well-known observation.

Let $A_0 : D(A_0) \subset X \to X$ be a closed, linear operator with nonempty resolvent set $\rho(A_0)$. For a linear operator $B : D(A_0) \subset X \to X$ satisfying $BR(\lambda, A_0) \in \mathcal{L}(X)$ for some (all) $\lambda \in \rho(A_0)$ define the perturbed operator

$$A := A_0 + B$$

on the domain $D(A) := D(A_0)$. Then the identity

$$\lambda - A_0 - B = [1 - BR(\lambda, A_0)](\lambda - A_0)$$

implies the following characterization of $\sigma(A) \cap \rho(A_0)$.

2.1 Lemma

For $\lambda \in \rho(A_0)$ one has the equivalence

$$\lambda \in \sigma(A) \iff 1 \in \sigma(BR(\lambda, A_0)).$$

Another technique we shall employ in Section 3 consists in taking the "part" of a given operator $A : D(A) \subset X \to X$. For this we suppose that a Banach space Y is continuously sandwiched between the domain $D(A)$ with the graph norm $\|\cdot\|_A$ and the given Banach space X, i.e.,

$$D(A) \hookrightarrow Y \hookrightarrow X.$$

Then the **part** of A in Y is

$$A_Y x := Ax$$

with domain

$$D(A_Y) := \{x \in D(A) : Ax \in Y\}.$$

If A is closed then A_Y is closed as well and one has coincidence between the spectra $\sigma(A)$ and $\sigma(A_Y)$ (see [Ar], Proposition 1.1).

2.2 Lemma

If $\rho(A) \neq \emptyset$ then one has

$$\sigma(A) = \sigma(A_Y).$$

After these preparatory lemmas we formulate the abstract framework which we will use to discuss Problem 1.2.

2.3 Assumptions

Throughout this section we assume the following.

- (A_1) *The Banach space $\mathcal{X} = \mathcal{X}_1 \times \mathcal{X}_2$ is the product of two Banach spaces $\mathcal{X}_1$ and $\mathcal{X}_2$.*
- (A_2) *For the operator $\mathcal{A}_0 : D(\mathcal{A}_0) \subseteq \mathcal{X} \to \mathcal{X}$ and $\lambda \in \rho(\mathcal{A}_0)$ we have an explicit representation of the resolvent*

$$R(\lambda, \mathcal{A}_0) = (R_{ij}(\lambda))_{2\times 2}$$

  in operator matrix form with respect to the product $\mathcal{X}_1 \times \mathcal{X}_2$.
- (A_3) *The perturbation $\mathcal{B} : D(\mathcal{A}_0) \subseteq \mathcal{X} \to \mathcal{X}$ satisfies $\mathcal{B}R(\lambda, \mathcal{A}_0) \in \mathcal{L}(\mathcal{X})$ for all $\lambda \in \rho(\mathcal{A}_0)$.*
- (A_4) *The operator $\mathcal{A} : D(\mathcal{A}_0) \subseteq \mathcal{X} \to \mathcal{X}$ is defined as the sum*

$$\mathcal{A} := \mathcal{A}_0 + \mathcal{B}.$$

It is evident from these assumptions that we will use terminology and results from the theory of operator matrices. However, only some elementary facts from that theory will be sufficient and we refer the interested reader to K.-J. Engel's survey article [En-3] and his book manuscript [En-2] for more sophisticated results. In particular, it is shown there (see also [Na-2]) that – using Schur complements – the spectrum of a 2×2-operator matrix can be computed in one of the factor spaces once the spectrum of the diagonal entries is known. Surprisingly in all concrete applications studied in Section 3 the following special situation occurs.

2.4 Theorem
Let the operators $\mathcal{A}, \mathcal{A}_0$ and $\mathcal{B}$ on the product space $\mathcal{X} = \mathcal{X}_1 \times \mathcal{X}_2$ satisfy assumptions (A_1)–(A_4). If $\mathcal{B}$ is of the form

$$\mathcal{B} = \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix}$$

then for each $\lambda \in \rho(\mathcal{A}_0)$ one has the equivalence

$$\lambda \in \sigma(\mathcal{A}) \text{ in } \mathcal{X} \iff 1 \in \sigma\left(BR_{21}(\lambda)\right) \text{ in } \mathcal{X}_1.$$

Proof. The assumption (A_3) guarantees that we can apply Lemma 2.1. To do this we use the standard matrix rules to compute

$$\begin{aligned} \mathcal{B}R(\lambda, \mathcal{A}_0) &= \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix} \begin{pmatrix} R_{11}(\lambda) & R_{12}(\lambda) \\ R_{21}(\lambda) & R_{22}(\lambda) \end{pmatrix} \\ &= \begin{pmatrix} BR_{21}(\lambda) & BR_{22}(\lambda) \\ 0 & 0 \end{pmatrix}. \end{aligned}$$

Hence, $\lambda \in \rho(\mathcal{A})$ if and only if

$$Id_{\mathcal{X}} - \mathcal{B}R(\lambda, \mathcal{A}_0) = \begin{pmatrix} Id_{\mathcal{X}_1} - BR_{21}(\lambda) & -BR_{22}(\lambda) \\ 0 & Id_{\mathcal{X}_2} \end{pmatrix}$$

is invertible. This is the case if and only if $Id_{\mathcal{X}_2}$ and $Id_{\mathcal{X}_1} - BR_{21}(\lambda)$ are invertible, i.e.,

$$1 \in \rho(BR_{21}(\lambda)).$$

□

The condition

$$1 \in \sigma(BR_{21}(\lambda))$$

might be called the **characteristic equation** of $\mathcal{A}$ with respect to $\mathcal{X}_1$. It gives an answer to Problem 1.2 and permits to compute values of $\sigma(\mathcal{A})$ in the smaller space $\mathcal{X}_1$. If this space is finite dimensional the condition reduces to a scalar equation.

2.5 Corollary
In the situation of Theorem 2.4 assume $\dim \mathcal{X}_1 < \infty$. *If we define*

$$\xi_{\mathcal{A}}(\lambda) := \det(1 - BR_{21}(\lambda))$$

then

$$\sigma(\mathcal{A}) \cap \rho(\mathcal{A}_0) = \{\lambda \in \rho(\mathcal{A}_0) : \xi_{\mathcal{A}}(\lambda) = 0\}.$$

In particular, if $\sigma(\mathcal{A}_0) = \emptyset$ one has

$$\sigma(\mathcal{A}) = \{\lambda \in \mathbb{C} : \xi_{\mathcal{A}}(\lambda) = 0\}.$$

Note that we did not assume dense domain for our operator $\mathcal{A}$. This is necessary since in many applications (see 3.3 and 3.4 below) we start with a (densely defined) semigroup generator, but then extend it to a Hille-Yosida operator with non dense domain in some extrapolation space. It is only this extension which produces, in a quite surprising way, situations for which Assumptions (A_1)–(A_4) are satisfied. However, having characterized the spectrum of the extended operator, we obtain information on the spectrum of the original operator by applying Lemma 2.2. A special case occurring frequently in the examples below is stated explicitly.

2.6 Corollary
In the situation of Theorem 2.4 assume that

$$D(\mathcal{A}) \hookrightarrow \mathcal{Y} := \{0\} \times \mathcal{X}_2 \hookrightarrow \mathcal{X}_1 \times \mathcal{X}_2 = \mathcal{X}$$

and that the operator $\mathcal{A}$ on $\mathcal{X}$ and its part $\mathcal{A}_{\mathcal{Y}}$ in $\mathcal{Y}$ satisfy $\rho(\mathcal{A}) \cap \rho(\mathcal{A}_{\mathcal{Y}}) \neq \emptyset$. Then for $\lambda \in \rho(\mathcal{A}_0)$ one has

$$\lambda \in \sigma(\mathcal{A}_{\mathcal{Y}}) \iff 1 \in \sigma\left(BR_{21}(\lambda)\right).$$

Proof. Lemma 2.2 implies $\sigma(\mathcal{A}) = \sigma(\mathcal{A}_{\mathcal{Y}})$, hence the assertion follows from Theorem 2.4. $\square$

2.7 Comments

(i) It was the special form of the perturbing operator $\mathcal{B}$ as a matrix $\mathcal{B} = \left(\begin{smallmatrix} 0 & B \\ 0 & 0 \end{smallmatrix}\right)$ which produced the characteristic equation in the space $\mathcal{X}_1$. This is not such a restrictive assumption as it might seem. In fact, let us start with a full operator matrix

$$\mathcal{A} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

on $\mathcal{X} = \mathcal{X}_1 \times \mathcal{X}_2$ and decompose it as $\mathcal{A} = \mathcal{A}_0 + \mathcal{B}$ with

$$\mathcal{A}_0 := \begin{pmatrix} A & 0 \\ C & D \end{pmatrix} \qquad \text{and} \qquad \mathcal{B} := \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix}.$$

Then it has been shown in [Na-2], see also [En-2], that for “nice” domains the spectrum, resolvent and, if it exists, the semigroup generated by the triangular matrix $\mathcal{A}_0$ can be calculated explicitly. So the non trivial part of the spectrum of the full operator matrix can be obtained by Theorem 2.4.

(ii) We return to the situation of Corollary 2.6 and observe that the domain of the part $\mathcal{A}_{\mathcal{Y}}$ in $\mathcal{Y}$ is

$$D(\mathcal{A}_{\mathcal{Y}}) = \left\{ \begin{pmatrix} 0 \\ x \end{pmatrix} \in D(\mathcal{A}) : Bx = 0 \right\}.$$

This means that the additive perturbation $\mathcal{B}$ of $\mathcal{A}$ becomes a perturbation of the domain. In the Examples 3.3 and 3.4 we invert this procedure and extend an operator with “perturbed domain” to an additively perturbed operator. To this extended operator we can apply our theory.

(iii) A completely different approach to Problem 1.2 and for eigenvalues only is presented in [Ka-VL].

3. Examples

As said before the abstract framework might seem artificial and the results simple. Therefore we are going to present some typical examples which should demonstrate that the Assumptions (A_1)–(A_4) and Theorem 2.4 allow to obtain old and new characteristic equations in a unified way.

3.1 Delay Differential Equations

In the semigroup approach to (Banach space valued) delay equations (with possibly infinite delay) one looks at the space

$$\mathcal{X} := X \times L^1(\mathbb{R}_-, X)$$

for some Banach space X and at the operator

$$\mathcal{A} := \begin{pmatrix} 0 & B \\ 0 & \frac{d}{ds} \end{pmatrix}$$

with domain

$$D(\mathcal{A}) := \left\{ \begin{pmatrix} x \\ f \end{pmatrix} : x \in X, f \in W^{1,1}(\mathbb{R}_-, X), f(0) = x \right\}.$$

For the off–diagonal entry in $\mathcal{A}$ we suppose $B \in \mathcal{L}\left(W^{1,1}(\mathbb{R}_-, X), X\right)$. The “unperturbed” operator

$$\mathcal{A}_0 := \begin{pmatrix} 0 & 0 \\ 0 & \frac{d}{ds} \end{pmatrix} \qquad \text{with domain } D(\mathcal{A}_0) := D(\mathcal{A})$$

generates a strongly continuous semigroup, has spectrum

$$\sigma(\mathcal{A}_0) = \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \leq 0\}$$

and its resolvent is

$$R(\lambda,\mathcal{A}_0)=\begin{pmatrix}\frac{1}{\lambda} & 0\\ \frac{1}{\lambda}Id\otimes\epsilon_\lambda & R(\lambda)\end{pmatrix}\quad \text{for } \operatorname{Re}\lambda>0.$$

Here, $\epsilon_\lambda(s):=e^{\lambda s}$ and

$$R(\lambda)f(s):=\int_s^0 e^{-\lambda(t-s)}f(t)dt$$

for $s\in\mathbb{R}$ and $f\in L^1(\mathbb{R}_-,X)$. If we take as perturbation the operator

$$\mathcal{B}:=\begin{pmatrix}0 & B\\ 0 & 0\end{pmatrix}$$

then the Assumptions (A_1)–(A_4) are satisfied and Theorem 2.4 takes the following form.

Proposition

In the above situation and for $\operatorname{Re}\lambda>0$ *one has*

$$\lambda\in\sigma(\mathcal{A})\iff\lambda\in\sigma(B\circ Id\otimes\epsilon_\lambda).$$

If B *is of the form* $B(\cdot)\in L^\infty(\mathbb{R}_-,\mathcal{L}(X))$ *this is equivalent to*

$$\lambda\in\sigma\left(\int_{-\infty}^0 e^{\lambda t}B(t)dt\right),$$

and, if in addition $\dim X<\infty$, *to*

$$\det\left(\lambda-\int_{-\infty}^0 e^{\lambda t}B(t)dt\right)=0.$$

3.2 Volterra Integro-Differential Equation

We refer to [Na-Si], Section 4 where Volterra integro-differential equations have been solved using semigroups on product spaces. In order to determine the spectrum of the corresponding generator we look at the Banach space

$$\mathcal{X}:=X\times L^1(\mathbb{R}_+,X)$$

and the operator matrix

$$\mathcal{A}:=\begin{pmatrix}A & \delta_0\\ C & \frac{d}{ds}\end{pmatrix}$$

with domain

$$D(\mathcal{A}):=\left\{\begin{pmatrix}x\\ f\end{pmatrix}: x\in D(A), f\in W^{1,1}(\mathbb{R}_+,X)\right\}.$$

Here, $A:D(A)\subseteq X\to X$ is some Hille-Yosida operator with $\mathrm{s}(A)\le 0$ and the operator $C:D(A)\to L^1(\mathbb{R}_+,X)$ is of the form

$$Cx:=a(\cdot)Ax$$

for some scalar-valued, integrable function $a(\cdot)$ of bounded variation on $\mathbb{R}_+$. Then

$$\mathcal{A}_0:=\begin{pmatrix}A & 0\\ C & \frac{d}{ds}\end{pmatrix}$$

with $D(\mathcal{A}_0)=D(\mathcal{A})$ is a Hille-Yosida operator with spectrum

$$\sigma(\mathcal{A}_0)=\{\lambda\in\mathbb{C}:\operatorname{Re}\lambda\le 0\}$$

and resolvent

$$R(\lambda, \mathcal{A}_0) = \begin{pmatrix} R(\lambda, A) & 0 \\ R(\lambda, \frac{d}{ds})CR(\lambda, A) & R(\lambda, \frac{d}{ds}) \end{pmatrix}$$

for $\operatorname{Re}\lambda > 0$. Taking as perturbation

$$\mathcal{B} := \begin{pmatrix} 0 & \delta_0 \\ 0 & 0 \end{pmatrix}$$

all the Assumptions (A_1)–(A_4) are satisfied. It therefore suffices to compute explicitly the resolvent $R(\lambda, \frac{d}{ds})$ as

$$R\left(\lambda, \tfrac{d}{ds}\right) f(s) = \int_s^\infty e^{-\lambda(t-s)} f(t) dt$$

for $\operatorname{Re}\lambda > 0, s \geq 0$ and $f \in L_1(\mathbb{R}_+, X)$ in order to obtain the following characteristic equation for the spectrum of $\mathcal{A}$.

Proposition

In the above situation and for $\operatorname{Re}\lambda > 0$ *one has*

$$\begin{aligned} \lambda \in \sigma(\mathcal{A}) \iff & 1 \in \sigma\left(\left(\int_0^\infty e^{-\lambda t} a(t) dt\right) AR(\lambda, A)\right) \\ \iff & \int_0^\infty e^{-\lambda t} a(t) dt \neq 0 \quad \textit{and} \\ & \left(\int_0^\infty e^{-\lambda t} a(t) dt\right)^{-1} \in \sigma(AR(\lambda, A)). \end{aligned}$$

We leave it to the reader to rewrite the above condition as a scalar characteristic equation in case $\dim X < \infty$. Similar conditions appear in the study of Volterra integral equations but usually are obtained via Laplace transform methods (see [Pr]). Finally we point out that the spectra of $\mathcal{A}$ and its restriction to $\overline{D(\mathcal{A})} = \overline{D(A)} \times L^1(\mathbb{R}_+, X)$ coincide (use the extrapolation theory from [Na-Si], Section 1 and Lemma 2.2).

3.3 Population Equation with Spatial Diffusion

We now compute the spectrum of an operator introduced by Nickel-Rhandi [Ni-Rh] in order to study the behavior of an age-dependent population with spatial diffusion. We refer to [Ni-Rh], Section 5 for more details and only give the definitions without their biological interpretation.

Take $X := L^1(\Omega)$ for some open subset Ω in $\mathbb{R}^N$. On this space we take the Laplace operator A with, e.g., Dirichlet boundary conditions generating the semigroup $\left(e^{tA}\right)_{t \geq 0}$. On the product space

$$\mathcal{X} := X \times L^1(\mathbb{R}_+, X)$$

we define an operator matrix

$$\mathcal{A}_0 := \begin{pmatrix} 0 & -\delta_0 \\ 0 & G \end{pmatrix}$$

with domain

$$D(\mathcal{A}_0) := \{0\} \times D(G).$$

The lower right entry is an operator $G : D(G) \subseteq L^1(\mathbb{R}_+, X) \to L^1(\mathbb{R}_+, X)$ given by

$$Gf(s) := -f'(s) - \mu(s) f(s) + k(s) A f(s), \quad 0 \leq s,$$

on a core of $D(G)$ and we refer to [Ni-Rh] for a precise description.

For the functions μ and k we suppose $0 \leq \mu \in L^\infty_{loc}(\mathbb{R}_+)$ and $0 < \alpha \leq k \in C_b(\mathbb{R}_+)$. Then $\mathcal{A}_0$ is a Hille-Yosida operator and for $\operatorname{Re}\lambda > 0$ one has

$$R(\lambda, \mathcal{A}_0) = \begin{pmatrix} 0 & 0 \\ \tilde{\epsilon}_{-\lambda} & R(\lambda, G_0) \end{pmatrix}$$

with

$$\tilde{\epsilon}_{-\lambda}(s) := e^{-\lambda s} e^{-\int_0^s \mu(\tau)d\tau} e^{(\int_0^s k(\tau)d\tau)A} \in \mathcal{L}(X)$$

and

$$R(\lambda, G_0)f(s) := \int_0^s e^{-\lambda(s-t)} e^{-\int_t^s \mu(\tau)d\tau} e^{(\int_t^s k(\tau))A} f(t)dt$$

for $f \in L^1(\mathbb{R}_+, X)$ and $s \geq 0$. These statements can be verified by direct calculations and are explained by the observation that

$$(\lambda - G)\tilde{\epsilon}_{-\lambda} = 0,$$

while $R(\lambda, G_0)$ is the resolvent of a "weighted" translation semigroup with zero boundary condition (see [Ni-Rh], (5.5)).

Choose now $0 \leq \beta \in L^\infty(\mathbb{R}_+)$ and define $B : L^1(\mathbb{R}_+, X) \to X$ by

$$Bf := \int_0^\infty \beta(t)f(t)dt.$$

Then the operators $\mathcal{A}_0$ and $\mathcal{B} := \left(\begin{smallmatrix} 0 & B \\ 0 & 0 \end{smallmatrix}\right)$ satisfy the Assumptions (A_1)–(A_4) and we obtain the following information on the spectrum of the operator $\mathcal{A} := \mathcal{A}_0 + \mathcal{B}$.

Proposition

In the above situation and for $\operatorname{Re}\lambda > 0$ *one has*

$$\lambda \in \sigma(\mathcal{A}) \iff 1 \in \sigma\left(\int_0^\infty e^{-\lambda t}\beta(t)\mathcal{U}(t)dt\right),$$

where

$$\mathcal{U}(t) := e^{-\int_0^t \mu(\tau)d\tau} e^{\left(\int_0^t k(\tau)d\tau\right)A} \in \mathcal{L}(X).$$

Assuming the spectrum of the Laplace operator A to be known as $\sigma(A) = P\sigma(A) = \{-n^2 : n \in \mathbb{Z}\}$ we obtain a very explicit characterization of $\sigma(\mathcal{A})$.

Corollary

For $\operatorname{Re}\lambda > 0$ *one has*

$$\lambda \in \sigma(\mathcal{A}) \iff 1 = \int_0^\infty e^{-\lambda t}\beta(t)e^{-\int_0^t \mu(\tau)d\tau - n^2\int_0^t k(\tau)d\tau}dt$$

for some $n \in \mathbb{N}$.

3.4 A Cell Equation

In this example we compute the spectrum of an operator which, a priori, is not defined on a product space and therefore seems far from satisfying Assumptions (A_1)–(A_4).

Partial differential equations describing the growth of a cell population in which division takes place have been studied by many authors using semigroup methods (e.g., see [Me-Di], [Gr-Na], [Na-4]).

In this approach one considers the operator

$$Gf(s) := -f'(s) - (\mu(s) + b(s))f(s) + Cf(s)$$

with domain

$$D(G) := \left\{f \in W^{1,1}\left(\tfrac{\alpha}{2}, 1\right) : f\left(\tfrac{\alpha}{2}\right) = 0\right\}$$

generating a strongly continuous semigroup on the space

$$X := L^1\left(\tfrac{\alpha}{2}, 1\right).$$

Here, $\alpha \in (0,1)$ is a constant, $0 \leq \mu, b \in L^\infty\left(\frac{\alpha}{2},1\right)$ and

$$Cf(s) := \begin{cases} 4b(2s)f(2s) & \text{for } \frac{\alpha}{2} \leq s \leq \frac{1}{2} \\ 0 & \text{for } \frac{1}{2} < s \leq 1. \end{cases}$$

By solving differential equations one finds that the spectrum of G is given by the following characteristic equation

$$\sigma(G) = \left\{ \lambda \in \mathbb{C} : 1 = \int_{\frac{\alpha}{2}}^{\frac{1}{2}} 4b(2t) e^{-\int_t^{2t} (\mu(\tau)+b(\tau)+\lambda)d\tau} dt \right\}.$$

We show that this also is a special case of Assumptions (A$_1$)–(A$_4$) and Theorem 2.4.

As product space take

$$\mathcal{X} := \left(\mathbb{C} \times L^1\left(\tfrac{1}{2},1\right)\right) \times L^1\left(\tfrac{\alpha}{2},\tfrac{1}{2}\right) = \mathcal{X}_1 \times \mathcal{X}_2.$$

On $L^1\left(\frac{1}{2},1\right)$ consider the operator

$$D_m f := -f' - (\mu + b) f$$

with maximal domain

$$D(D_m) := W^{1,1}\left(\tfrac{1}{2},1\right).$$

On the space $L^1\left(\frac{\alpha}{2},\frac{1}{2}\right)$, the operator

$$D_0 g := -g' - (\mu + b) g$$

with domain

$$D(D_0) := \left\{ g \in W^{1,1}\left(\tfrac{\alpha}{2},\tfrac{1}{2}\right) : g(\tfrac{\alpha}{2}) = 0 \right\}$$

generates the nilpotent semigroup given by

$$e^{tD_0} := \begin{cases} e^{-\int_{s-t}^{s} (\mu(\tau)+b(\tau))d\tau} g(s-t) & \text{for } s-t \leq \frac{1}{2}, \\ 0 & \text{for } s-t > \frac{1}{2}. \end{cases}$$

Finally, consider C with

$$Cf(s) := 4b(2s)f(2s), \qquad \tfrac{\alpha}{2} \leq s \leq \tfrac{1}{2},$$

as a bounded operator from $L^1\left(\frac{1}{2},1\right)$ into $L^1\left(\frac{\alpha}{2},\frac{1}{2}\right)$ and the Dirac measure $\delta_{\frac{1}{2}}$ as an operator from $W^{1,1}\left(\frac{1}{2},1\right)$ into $\mathbb{C}$. With these operators we define a 3×3-operator matrix

$$\mathcal{A}_0 := \begin{pmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{pmatrix} := \left(\begin{array}{cc|c} 0 & -\delta_{\frac{1}{2}} & 0 \\ 0 & D_m & 0 \\ \hline 0 & C & D_0 \end{array}\right)$$

with domain

$$D(\mathcal{A}_0) := \left(\{0\} \times W^{1,1}\left(\tfrac{1}{2},1\right)\right) \times D(D_0)$$

which will be considered as a 2×2-block matrix in the indicated way.

The resolvent of a triangular operator matrix $\mathcal{A}_0 := \begin{pmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{pmatrix}$ exists for $\lambda \in \rho(A_{11}) \cap \rho(A_{22})$ and is given by the matrix

$$R(\lambda, \mathcal{A}_0) = \begin{pmatrix} R(\lambda, A_{11}) & 0 \\ R(\lambda, A_{22}) A_{21} R(\lambda, A_{11}) & R(\lambda, A_{22}) \end{pmatrix},$$

at least when the lower left entry is bounded (see [Na-2]). This enables us to compute the resolvent of $\mathcal{A}_0$ for each $\lambda \in \mathbb{C}$ as the operator matrix

$$R(\lambda, \mathcal{A}_0) = \left(\begin{array}{cc|c} 0 & 0 & 0 \\ \tilde{\epsilon}_{-\lambda} & R(\lambda) & 0 \\ \hline R(\lambda, D_0) C \tilde{\epsilon}_{-\lambda} & R(\lambda, D_0) C R(\lambda) & R(\lambda, D_0) \end{array}\right).$$

In this 2×2-block matrix, the upper left entry is analogous to the resolvent in Example 3.3. More precisely, one has to take

$$\tilde{\epsilon}_{-\lambda}(s) := e^{-\lambda(s - \frac{1}{2})} e^{-\int_{\frac{1}{2}}^{s} (\mu(\tau) + b(\tau)) d\tau} dt$$

and

$$R(\lambda) f(s) := \int_{\frac{1}{2}}^{s} e^{-\lambda(s-t)} e^{-\int_t^s (\mu(\tau) + b(\tau)) d\tau} f(t) dt \quad \text{for } \tfrac{1}{2} \leq s \leq 1.$$

The lower right entry is the resolvent of the nilpotent, weighted translation semigroup generated by D_0 and can be computed as

$$R(\lambda, D_0) g(s) := \int_{\frac{\alpha}{2}}^{s} e^{-\lambda(s-t)} e^{-\int_t^s (\mu(\tau) + b(\tau)) d\tau} g(t) dt$$

for $\frac{\alpha}{2} \leq s \leq \frac{1}{2}$. From this we obtain

$$C \tilde{\epsilon}_{-\lambda}(s) = 4b(2s) e^{-\lambda(2s - \frac{1}{2})} e^{-\int_{\frac{1}{2}}^{2s} (\mu(\tau) + b(\tau)) d\tau}$$

and therefore

$$\begin{aligned} R(\lambda, D_0) C \tilde{\epsilon}_{-\lambda}(s) &= \int_{\frac{\alpha}{2}}^{s} e^{-\lambda(s-t)} e^{-\int_t^s (\mu(\tau) + b(\tau)) d\tau} 4b(2t) e^{-\lambda(2t - \frac{1}{2}) - \int_{\frac{1}{2}}^{2s} (\mu(\tau) + b(\tau)) d\tau} dt \\ &= \int_{\frac{\alpha}{2}}^{s} 4b(2t) e^{-\lambda(s + t - \frac{1}{2})} e^{-\int_t^s (\mu(\tau) + b(\tau)) d\tau - \int_{\frac{1}{2}}^{2t} (\mu(\tau) + b(\tau)) d\tau} dt \end{aligned}$$

for $\frac{\alpha}{2} \leq s \leq \frac{1}{2}$.
In the last step we take as perturbing operator the matrix

$$\mathcal{B} := \left(\begin{array}{cc|c} 0 & 0 & \delta_{\frac{1}{2}} \\ 0 & 0 & 0 \\ \hline 0 & 0 & 0 \end{array}\right)$$

and obtain an operator

$$\mathcal{A} := \mathcal{A}_0 + \mathcal{B} \quad \text{with domain} \quad D(\mathcal{A}) = D(\mathcal{A}_0)$$

satisfying the Assumptions (A_1)–(A_4).

Before computing its spectrum we should note that

(1) the part $\mathcal{A}_|$ of $\mathcal{A}$ in

$$\mathcal{Y} := \{0\} \times L^1\left(\tfrac{1}{2}, 1\right) \times L^1\left(\tfrac{\alpha}{2}, \tfrac{1}{2}\right)$$

is exactly our initial operator G on $L^1\left(\frac{\alpha}{2}, 1\right)$ and

(2) the spectra $\sigma(\mathcal{A}_|) = \sigma(\mathcal{A})$ and $\sigma(\mathcal{A})$ coincide by Lemma 2.2.

Therefore it remains to compute $\mathcal{B}R(\lambda, \mathcal{A}_0)$ which is an operator matrix of the form

$$\begin{pmatrix} R_{11}(\lambda) & R_{12}(\lambda) & R_{13}(\lambda) \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

with $R_{11}(\lambda) = \delta_{\frac{1}{2}} \circ R(\lambda, D_0)C\tilde{\epsilon}_{-\lambda}$ being a complex number. From the above expressions we therefore obtain the desired result.

Proposition

The spectrum of the cell growth operator is

$$\sigma(G) = \sigma(\mathcal{A}) = \left\{ \lambda \in \mathbb{C} : 1 = \int_{\frac{\alpha}{2}}^{\frac{1}{2}} 4b(2t)e^{-\int_t^{2t}(\mu(\tau)+b(\tau)+\lambda)d\tau}\, dt \right\}.$$

Clearly, the same method also works for the similar but more complicate equations on tumor growth treated in [Gr] or [Gy-We]. However, instead of adding more examples we emphasize the following observation.

The method proposed does not only yield a characteristic equation but it isolates the space and the operator being responsible for the (non trivial part of the) spectrum. This insight allows many applications to perturbation and stability problems and we conclude by giving an example in terms of control theory.

Corollary

Let $\mathcal{A}$ be the above operator on the space

$$\mathcal{X} := \mathbb{C} \times L^1\left(\tfrac{1}{2}, 1\right) \times L^1\left(\tfrac{\alpha}{2}, \tfrac{1}{2}\right).$$

Then there exists a one-dimensional control space $\mathcal{U} = \mathbb{C}$ and a control operator $K \in \mathcal{L}(\mathcal{U}, \mathcal{X})$ such that the control system $(\mathcal{A}, K)$ is stabilizable, i.e., there exists a feed back operator $L \in \mathcal{L}(D(\mathcal{A}), \mathcal{U})$ such that the spectral bound of $\mathcal{A} + KL$ satisfies

$$\mathrm{s}(\mathcal{A} + KL) < 0.$$

Proof. If we take $K : \mathcal{U} \to \mathcal{X}$ with $Ku := \begin{pmatrix} u \\ 0 \\ 0 \end{pmatrix} \in \mathcal{X}$ and $L : \mathcal{X} \to \mathcal{U}$ with $L \begin{pmatrix} x \\ f \\ g \end{pmatrix} := -g\left(\frac{1}{2}\right)$ we obtain that

$$\mathcal{A} + KL = \mathcal{A}_0.$$

Since we have already seen that $\sigma(\mathcal{A}_0) = \emptyset$ we have proved the assertion. $\square$

Both the operators $\mathcal{A}_0$ and $\mathcal{A}$ are Hille-Yosida operators, hence become generators of semigroups if restricted to the closure of its domain, i.e., to $\{0\} \times L^1\left(\frac{1}{2}, 1\right) \times L^1\left(\frac{\alpha}{2}, \frac{1}{2}\right)$.

Looking at the domain of these restrictions one sees that the above corollary is a statement on "boundary stabilization".

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Rainer Nagel
Mathematisches Institut
Universität Tübingen
Auf der Morgenstelle 10
D - 72076 Tübingen
GERMANY

e-mail: rana@michelangelo.mathematik.uni-tuebingen.de