Stability theorems for semi–groups via Multiplier theorems

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1 Introduction

A classical theorem of Liapunov states that if all eigenvalues λ of a matrix A have Re $\lambda < 0$, then all solutions of the Cauchy problem

$$y'(t) = Ay(t), \qquad y(0) = y_0$$

are exponentiell stable, i. e. there are $\varepsilon > 0$ and $M < \infty$ such that $y(t) = e^{tA}y_0$ satisfies

$$\|y(t)\| \leq Me^{-\varepsilon t}. \tag{1.1}$$

In recent years, an analoguous stability theory was developped for c_0–semigroups on infinite dimensional Banach spaces. If A is the generator of a c_0–semigroup $(T_t)_{t\geq 0}$ on a Banach space X, we denote by

$$\begin{aligned} s(A) &= \sup\{\text{Re } \lambda : \lambda \in \sigma(A)\} \qquad \text{and} \\ s_0(A) &= \inf\{\omega : \sup_{\text{Re}\lambda \geq \omega} \|R(\lambda, A)\| < \infty\} \end{aligned} \tag{1.2}$$

the spectral bound of A and the boundedness abscissa of $R(\lambda, A)$ respectively. For every $x \in X$,

$$w(x) = \inf\{\omega : \sup_{t\geq 0} \|e^{-\omega t}T_t x\| < \infty\} \tag{1.3}$$

is the growth bound of the mild solution $y(t) = T_t x$ of

$$y'(t) = Ay(t), \qquad y(0) = x. \tag{1.4}$$

It was shown in [17] and also in [8], [9], that

$$w(x) \leq s_0(A) \quad \text{for all} \quad x \in D(A), \tag{1.5}$$

i. e. if $s_\infty(A) < 0$ and $y(t)$ is a classical solution of (4) (because of $x \in D(A)$), then $y(t)$ is exponentially stable. Since for a matrix A on $X = \mathbb{C}^n$ we have

$$s_\infty(A) = s(A) = \sup\{\text{Re } \lambda : \lambda \text{ eigenvalue of } A\}$$

and $D(A) = X$, (5) seems to be the proper generalization of Liapunov's theorem to the infinite dimensional setting, as far as classical solutions of (4) are concerned.
For mild solutions $y(t) = T_t x$ of (4) with $x \in X$ one has

$$\omega(x) \leq s(A) = s_\infty(A) \quad \text{for all} \quad x \in X \tag{1.6}$$

only in special circumstances. It is well known ([10]) that (6) holds if (T_t) is an analytic semigroup, but more recently it was shown in [15] (see also [9], [16]) that (6) also holds for positive semigroups on L_p–spaces.
If X has additional geometric properties, e. g. if X is uniformly convex or a Banach space of Fourier–type $1 < p \leq 2$, then (cf. [17]) one obtains a result in between (1.5) and (1.6): $w(x) \leq s_0(A)$ for all x in the domain of a fractional power $(-A)^\alpha$, where $0 < \alpha < 1$ depends on the geometry of X.
In this note we give alternative proofs for these results based on Multiplier theorems for operator–valued functions $M(\lambda) \in B(X)$, $\lambda \in \mathbb{R}$, on vector–valued Besov spaces $B^s_{u,v}(\mathbb{R}^n, X)$. These theorems from [4] and discovered independently by Amann [1] assure that

$$\widehat{Kf}(\lambda) = M(\lambda)(\hat{f}(\lambda)), \quad \lambda \in \mathbb{R}^n \tag{1.7}$$

defines a bounded operator K on $B^s_{u,v}(X)$, if M satisfies various conditions of Mihlin type, e. g.

$$\|(1 + |\lambda|^\alpha) D^\alpha M(\lambda)\| \leq C \quad \text{for} \;\; \lambda \in \mathbb{R}^n, \; |\alpha| \leq n + 1. \tag{1.8}$$

Here $\hat{f}$ denotes the Fourier–transform

$$\hat{f}(t) = \int\limits_{\mathbb{R}^n} e^{-it\cdot s} f(s) ds. \tag{1.9}$$

If X has Fourier–type p, $p \leq 2$, i. e. the vector–valued Hausdorff–Young inequality

$$\|\hat{f}\|_{L_{p'}(\mathbb{R}^n, X)} \leq C \|f\|_{L_p(\mathbb{R}^n, X)} \tag{1.10}$$

holds, it is enough to assume that

$$M \in B^{1/p}_{p,1}(\mathbb{R}^n) \tag{1.11}$$

These multiplier theorems are stated preciscely in Section 2 where we also give a short sketch of their proof.

In Section 3 we explain how stability theorems can be reduced to multiplier theorems and we give a new proof of (1.5) and its improvements for Banach spaces of Fourier–type p. An important consequence of this proof and examples in [17] will be, that condition (9) is optimal for Banach spaces of Fourier–type p. We also include a simple proof for the convexity of the growth function $\alpha \to \omega((-A)^{-\alpha}x)$ from [17] (see 3.5. for details), which gives a partial answer to a question of van Neerven concerning individual stability estimates.

Finally, in Section 4 we sketch an alternative proof to the short proof of (6) for positive semigroups on L_p that will appear in [16]. A final example gives a negative answer to a question of van Neerven concerning the growth bound of non–positive semigroups on $L_p(\Omega)$.

2 Operator–valued Fourier Multipliers on Vector–valued Besov spaces

It was observed by G. Pisier that if an operator–valued function $t \in \mathbb{R}^n \to M(t) \in B(X)$ satisfying the Mihlin condition (1.8) defines a bounded operator K on $L_p(\mathbb{R}^n, X)$, $1 < p < \infty$, by

$$\widehat{Kf}(\lambda) = M(\lambda)(\hat{f}(\lambda))$$

then X is necessarily a Hilbert space. To obtain multiplier results for arbitrary Banach spaces, we are therefore lead to replace the Bochner spaces $L_p(\mathbb{R}^n, X)$ by the Besov spaces $B^s_{u,v}(\mathbb{R}^n, X)$. Let us recall their definition. Put

$$I_0 = \{x \in \mathbb{R}^n : |x| \leq 2\}, \quad I_j = \{x \in \mathbb{R}^n : 2^{j-1} \leq |x| \leq 2^{j+1}\},\ j \in \mathbb{N}.$$

We fix a sequence $\phi_j \in C^\infty(\mathbb{R}^n),\ j = 0, 1, 2, \ldots$, such that

$$\mathrm{supp}\phi_j \subset I_j, \qquad \sum_{j=0}^{\infty} \phi_j(s) = 1 \text{ on } \mathbb{R}^n, \qquad \sup_j \int_{\mathbb{R}^n} |\hat{\phi}_j(t)|dt < \infty. \tag{2.1}$$

Then $B^s_{u,v}(\mathbb{R}^n, X)$, $s \in \mathbb{R}, 1 \leq u, v \leq \infty$, consists of all X–valued distributions f for which the Besov norm

$$\|f\|_{B^s_{u,v}} = \left(\sum_{j=0}^{\infty} (2^{js} \|\hat{\phi}_j * f\|_u)^v \right)^{1/v} \tag{2.2}$$

is finite. (Different choices of sequence ϕ_j lead to equivalent norms; for this and other properties of these Besov spaces see e. g. [12], [13].) A well–known equivalent norm on $B^s_{u,v}(\mathbb{R},X)$ for $0<s<1$ is

$$|||f|||_{B^s_{u,v}} = \|f\|_{L_u} + \left(\int\int \left\| \frac{f(t)-f(r)}{|t-r|^s} \right\|^v_{L_u} |t-r|^{-1} dt\, dr \right)^{1/v}.$$

$B^s_{u,v}(\mathbb{R}_+,X)$ is the space of distribution in $B^s_{u,v}(\mathbb{R},X)$ with support in $\mathbb{R}_+$. Our results also depend on the geometry of the underlying Banach space X. We say that X has <u>Fourier–type p</u>, $1\le p\le 2$, if the Fourier–transform (1.9) extends to a bounded operator

$$f\in L_p(\mathbb{R},X)\to \hat{f}\in L_{p'}(\mathbb{R},X), \qquad \frac{1}{p}+\frac{1}{p'}=1, \tag{2.3}$$

(cf.[11]). Obviously, every Banach space has Fourier–type 1. On the other hand, a Banach space has Fourier–type 2 if and only if X is a Hilbert space. A space $L_q(\Omega,\mu)$ has Fourier–type $p=\min(q,\frac{q}{q-1})$ and Bourgain has shown in [3], that every B–convex (in particular every uniformly convex) Banach space has type p for some $p>1$. Also the dual space, every closed subspace and quotient space of a Banach space X has the same Fourier–type as X.
Finally, we fix two functions $\psi_0,\psi_1\in C^\infty(\mathbb{R}^n)$ with

$$\begin{aligned} &\text{supp } \psi_0\subset\{x\in\mathbb{R}^n \;:\; |x|\le 4\}, \qquad \psi_0(x)=1 \text{ for } |x|\le 2\\ &\text{supp } \psi_1\subset\{x\in\mathbb{R}^n \;:\; \frac{1}{4}\le |x|\le 4\}, \qquad \psi_0(x)=1 \text{ for } \frac{1}{2}\le |x|\le 2 \end{aligned} \tag{2.4}$$

In [4] we prove the following Multiplier theorem:

2.1. Theorem: Let X and Y have Fourier–type p, $1\le p\le 2$. Assume that the function $t\in\mathbb{R}^n\to M(t)\in B(X,Y)$ satisfies:

i)
$$\begin{aligned} \|\psi_1(\cdot)M(2^k\cdot)x\|_{B^{n/p}_{p,1}(Y)} &\le C\|x\| \quad \text{for all } x\in X, k\in\mathbb{N}\\ \|\psi_0(\cdot)M(\cdot)x\|_{B^{n/p}_{p,1}(Y)} &\le C\|x\| \quad \text{for all } x\in X. \end{aligned}$$

ii)
$$\begin{aligned} \|\psi_1(\cdot)M^*(2^k\cdot)y^*\|_{B^{n/p}_{p,1}(X^*)} &\le C\|y^*\| \quad \text{for all } y^*\in Y^*, k\in\mathbb{N}\\ \|\psi_0(\cdot)M^*(\cdot)y^*\|_{B^{n/p}_{p,1}(X^*)} &\le C\|y^*\| \quad \text{for all } y^*\in Y^* \end{aligned}$$

Then $\widehat{Kf}(\lambda)=M(\lambda)[\hat{f}(\lambda)]$, $\lambda\in\mathbb{R}^n$, defines a bounded operator

$$K: B^s_{u,v}(\mathbb{R}^n,X)\to B^s_{u,v}(\mathbb{R}^n,Y).$$

for $s \in \mathbb{R}$, $1 \leq u, v \leq \infty$.

2.2. Remark: a) It is checked in [4] that i) and ii) are in particular fulfilled if M satisfies the Mihlin condition

$$\|(1+|\lambda|)^{|\alpha|} D^\alpha M(\lambda)\| \leq C \quad \text{for} \quad \lambda \in \mathbb{R}^n,\ |\alpha| \leq n+1. \tag{2.5}$$

This case was proved independently in [1] by Amann. If X has Fourier–type p, it is enough to have (2.5) for $|\alpha| \leq [\frac{n}{p}]+1$. In particular, if X is uniformly convex it is enough to consider derivatives of order $|\alpha| \leq n$.
Alternatively, it is enough to assume

$$\|D^\alpha M(\lambda)\| \leq C(1+|\lambda|)^{-n} \tag{2.6}$$

for all $|\alpha| \leq n+1$ (or $|\alpha| \leq \frac{n}{p}+1$ if X and Y have Fourier–type p).
b) Cf. [4] one can replace the assumptions i) and ii) by

i') $\|M(\cdot)x\|_{W_p^r(I_j,Y)} \leq C\|x\|_X \quad$ for all $x \in X, j = 0,1,2,\ldots$

ii') $\|M^*(\cdot)y^*\|_{W_p^r(I_j,X^*)} \leq C\|y^*\|_{Y^*} \quad$ for all $y^* \in Y^*, j = 0,1,2\ldots$

for $r \in \mathbb{N}, r > n/p$.

2.3. The proof of theorem 2.1.: We need

1) an estimate for the Fourier–transform (theorem 3.1. in [4]), namely

$$\|\hat{g}\|_{L_1(\mathbb{R}^n,X)} \leq C\|g\|_{B_{p,1}^{n/p}}(\mathbb{R}^n, X)$$

2) the following extention of Youngs inequality (Lemma 3.2. in [4]):
If $k : \mathbb{R}^N \to B(X,Y)$ satisfies

$$\int_{\mathbb{R}^n} \|k(s)x\|_Y ds \leq C\|x\|_X, \quad \int_{\mathbb{R}^n} \|k(s)^*y^*\|_{X^*} ds \leq C\|y^*\|_{Y^*}$$

then $\quad Tf(t) = \int_{\mathbb{R}^n} k(t-s)f(s)ds \quad$ defines a bounded operator

$$T : L_p(\mathbb{R}^n, X) \to L_p(\mathbb{R}^n, Y).$$

Now observe that K restricted to the "blocks" $\varphi_j * f \in L_p(\mathbb{R}^n, X)$ of $B^s_{u,v}(\mathbb{R}^n, X)$ is essentially the convolution with $k_j = [\psi(2^{-j}\cdot)M(\cdot)]^\vee$. Applying 1) to $g = \psi(2^{-j}M(\cdot))$ we obtain that k_j satisfies the assumptions of 2) and, by the definition of the norm (2.2), we obtain the theorem by estimating the "blocks" separately (and uniformly)

$$\|\varphi_j * Kf\| \leq \|k_j * (\varphi_j * f)\| \leq C\|\varphi_j * f\|$$

(see [4] for details). ■

3 Stability of solutions with initial values in $D((-A)^\alpha)$

Let $(T_t)_{t\geq 0}$ be a c_0–semigroup on X with $\|T_t\| \leq Me^{\omega t}$ and $s_0(A) < 0$. Let X_α be the domain of the fractional power $(a - A)^\alpha$, where $\alpha \geq 0$ and $a > \omega$ (see e. g. [14]) with the graph norm. In particular, we have for all $w > \omega$ and $x \in X_\alpha$ that

$$\int_0^\infty \|e^{-wt}T_t x\|^p_{X_\alpha} dt < \infty$$

The basic idea in applying multiplier theorems to stability results is now the following: Observe that

$$\int_0^t T_{t-s}(e^{-ws}T_s x) ds = \frac{1}{\alpha}(1 - e^{-wt})T_t x \tag{3.1}$$

If we could assure, that the operator

$$Kf(t) = \int_0^t T_{t-s}(f(s)) ds = (T_t) \star f \tag{3.2}$$

maps $L_p(\mathbb{R}_+, X_\alpha)$ into $L_p(\mathbb{R}_+, X)$ by applying a multiplier result to

$$M(\lambda) = "\int_0^\infty e^{-i\lambda t} T_t dt" = R(i\lambda, A), \qquad \lambda \in \mathbb{R} \tag{3.3}$$

(where the "integral" should be understood in the distributional sense), then $t \to \frac{1}{\alpha}(1 - e^{-wt})T_t x$ would belong to $L_p(\mathbb{R}_+, X)$. Hence

$$\int_0^\infty \|T_t x\|^p_X dt < \infty$$

and a result of Pazy ([10] Sect. 4.4) would tell us that

$$\|T_t x\|_X \to 0 \quad \text{for } t \to \infty.$$

The proofs below will follow this outline but they will be somewhat complicated by the fact that our Multiplier Theorems do not garanty that K is bounded in Bochner spaces but only in Besov spaces.
With the notation of the introduction we prove first

3.1. Theorem: Let (T_t) be a c_0–semigroup on X. Then

$$w(x) \le s_0(A) \quad \text{for all } x \in D(A).$$

Proof:
Let $s_0(A) < \infty$ and $\|T_t\| \le Me^{\omega t}$. Then for $x \in Y = D(A)$

$$\|R(i\lambda, A)x\| \le \frac{C}{|\lambda|}\|x\|_Y$$

by Lemma 3.3. Also

$$\|\frac{d}{dx}R(i\lambda, A)x\|_X = \|R(i\lambda, A)^2 x\|_X \le \|R(i\lambda, A)\|_{B(X)}\|R(i\lambda, A)x\| \le \frac{C^2}{|\lambda|}\|x\|_Y.$$

Hence

$$\|R(i\lambda, A)\|_{Y\to X} \le \frac{C}{|\lambda|}, \qquad \left\|\frac{d}{d\lambda}R(i\lambda, A)\right\|_{Y\to X} \le \frac{C^2}{|\lambda|} \tag{3.4}$$

and in the same way we can show that $\left\|\frac{d^2}{d\lambda^2}R(i\lambda, A)\right\|_{Y\to X} \le \frac{C^3}{|\lambda|}$.
By remark 2.2.a) $M(\lambda) = R(i\lambda, A) \in B(Y, X)$ satisfies the assumptions of Theorem 2.1. For $f \in C_c(\mathbb{R}_+)$ and $w > \omega$ we have that

$$\begin{aligned}\int_0^\infty e^{-i\lambda t}\left[\int_0^t T_s f(t-s)ds\right]dt &= \int_0^\infty e^{-i(\lambda-w)t}\left[\int_0^t e^{-ws}T_s e^{-w(t-s)}f(t-s)ds\right]dt \\ &= \tilde{R}(\lambda-w, A)[\tilde{f}(\lambda-w)] = R(\lambda, A)[\hat{f}(\lambda)] = M(\lambda)[\hat{f}(\lambda)]\end{aligned}$$

where

$$\begin{aligned}\tilde{R}(\lambda)x &= \int e^{-\lambda t}[e^{-wt}T_t x]dt = R(\lambda+w, A) \\ \tilde{f}(\lambda) &= \int e^{-\lambda t}[e^{-wt}f(t)]dt = \hat{f}(\lambda+w).\end{aligned}$$

Hence Theorem 1.2 implies that for all $\varepsilon > 0$

$$K : B^{\varepsilon}_{u,v}(\mathbb{R}_+, Y) \to B^{\varepsilon}_{u,v}(\mathbb{R}_+, X) \tag{3.5}$$

If $x \in X_{1+2\varepsilon}$, then by Lemma 3.4 we have for $w > \omega$ that

$$(t \to f(t) = e^{-wt} T_t x) \in B^{\varepsilon}_{1,u}(\mathbb{R}_+, X).$$

By (3.1) and (3.5) it follows that

$$(t \to \frac{1}{\alpha}(1 - e^{-wt}) T_t x) \in B^{\varepsilon}_{1,u}(\mathbb{R}_+, X) \subset L_1(\mathbb{R}_+, X).$$

Hence $\int\limits_0^\infty \|T_t x\|_X \, dt < \infty$ and by Pazy's result we have $\|T_t x\| \to 0$ for $t \to \infty$. This shows that

$$\omega_\alpha(A) := \sup\{\omega(x) : x \in X_\alpha\} \le 0$$

for all $\alpha > 1$. Since the function $\alpha \in (0, \infty) \to \omega_\alpha(A) \in \mathbb{R}$ is continuous by Lemma 3.5 it follows that $\omega_1(A) \le 0$ and the proof is complete. ∎

If we make additional assumptions on the geometry of X Theorem 3.1. can be improved as follows:

3.2. Theorem Let X be a Banach space of Fourier–type p with $1 < p \le 2$ and let $(T_t)_{t \ge 0}$ be a c_0–semigroup on X. Then

$$w(x) \le s_0(A) \quad \text{for all } x \in X_{\frac{1}{p} - \frac{1}{p'}}.$$

Remark: To illustrate this result, consider the scale of spaces $X = L_q(\Omega, \mu)$. For the Hilbert space case $q = 2$ we get the best possible result: $\omega(x) \le s_0(A)$ for all $x \in X$. (This is indeed Gearhart's result.) For $q = 1$ we have the worst result: $\omega(x) \le s_0(A)$ is only garantied for $x \in D(A)$. For $1 < q < 2$ the result improves as we get closer to Hilbert–space and for Banach spaces of Fourier type p it was shown in [17] that $x \in X_{\frac{1}{p} - \frac{1}{p'}}$ is indeed best possible (see also Example 4.4 below).

Proof: Let $s_0(A) < 0$ and $\|T_t\| \le M e^{\omega t}$. Put $\frac{1}{r} = \frac{1}{p} - \frac{1}{p'} = \frac{2}{p} - 1$. We will use the following variant of (3.1):

$$\int\limits_0^t s T_s (e^{-w(t-s)} T_{t-s} x) \, ds = g(t) T_t x \tag{3.6}$$

where $g(t) = t/w - \frac{1}{w^2}(1 - e^{-wt})$ is larger than 1 for t large enough. Similarly as the proof of Theorem 3.1. one shows for $f \in C_c(\mathbb{R}, X)$

$$\int_0^\infty e^{-i\lambda t} \left[\int_0^t sT_s f(t-s) ds \right] = R(i\lambda, A)^2 \hat{f}(\lambda). \tag{3.7}$$

Hence

$$Kf(t) = \int_0^t sT_s f(t-s) ds = [M(\lambda)\hat{f}(\lambda)]^\vee, \quad M(\lambda) = R(i\lambda, A)^2$$

In order to apply Theorem 2.1 we want to show that

i) $\|M(\cdot)x\|_{W_p^1(J_k, X)} \le C\|x\|_{X_{\frac{1}{r}}}$

ii) $\|M^*(\cdot)y^*\|_{W_p^1(J_k, X^*_{\frac{1}{r}})} \le C\|y^*\|_{X^*}$

for all $k \in \mathbb{N}_0$.
To check i) note that for $x \in D(A)$ and $\alpha = \frac{1}{r}$ we get from Lemma 3.3 b) and $\dfrac{1}{p} = \dfrac{1}{r} + \dfrac{1}{p'}$

$$\begin{aligned} \left(\int_{2^{k-1}}^{2^{k+2}} \|R(i\lambda, A)^2 x\|^p d\lambda \right)^{\frac{1}{p}} &\le \left(\int_{2^{k-1}}^{2^{k+2}} |\lambda|^{-1} d\lambda \right)^{\frac{1}{r}} \left(\int \left[\|R(i\lambda, A)^2 x\| |\lambda|^{\frac{1}{r}} \right]^{p'} d\lambda \right)^{\frac{1}{p'}} \\ &\le C\|x\|_{X_{\frac{1}{r}}} \end{aligned}$$

Since $M(\lambda)' = -2iR(i\lambda, A)^3 x$ and $\|R(i\lambda, A)^3 x\| \le C\|R(i\lambda, A)^2 x\|$ the same estimate holds for $\dfrac{d}{d\lambda} M(\lambda)$ and we get i). ii) follows in the same way from Lemma 3.3.c.
Now Theorem 2.1 and remark 2.2.b garanties that

$$K : B^\varepsilon_{u,v}(\mathbb{R}_+, X_{\frac{1}{r}}) \to B^\varepsilon_{u,v}(\mathbb{R}_+, X) \tag{3.8}$$

for all $\varepsilon > 0$. If $x \in X_{\frac{1}{r}+2\varepsilon}$, then by Lemma 3.4 we have for $w > \omega$

$$(t \to f(t) = e^{-wt} T_t x) \in B^\varepsilon_{1,u}(\mathbb{R}_+, X).$$

Now (3.6) and (3.8) give

$$(t \to g(t) T_t x) \in B^\varepsilon_{1,u}(\mathbb{R}_+, X) \subset L_1(\mathbb{R}_+, X).$$

By Pazy's result we obtain $\|T_t x\| \to 0$ for $t \to \infty$ and we have shown

$$\omega_\alpha(A) = \sup\{\omega(x) : x \in X_\alpha\} \le 0$$

for $\alpha > \frac{1}{r}$. By Lemma 3.5 $\alpha \to \omega_\alpha(A)$ is continuous and we get $\omega_{\frac{1}{r}}(A) \le 0$ ∎

3.3 Lemma: Let A be the generator of a c_0–semigroup T_t on X with $s_0(A) < 0$.

a) If $x \in X_\alpha,\ 0 \le \alpha \le 1$, then for $\lambda \in \mathbb{R}$

$$\|R(i\lambda, A)x\|_X \le \frac{C}{|\lambda|^\alpha}\|x\|_{X_\alpha}.$$

b) If X has Fourier–type p and $x \in X_\alpha$, then

$$\Big(\int [\|R(i\lambda, A)^2 x\|_X |\lambda|^\alpha]^{p'} d\lambda\Big)^{\frac{1}{p'}} \le C\|x\|_{X_\alpha}.$$

c) If X has Fourier–type p and $x^* \in X^*$, then

$$\Big(\int (\|R(i\lambda, A^*)^2 x^*\|_{X^*_\alpha} |\lambda|^\alpha)^{p'} d\lambda\Big)^{\frac{1}{p'}} \le C\|x^*\|_{X^*}.$$

Proof: a) Since $s_0(A) < \infty$, we have $C = \sup_{\lambda \in \mathbb{R}} \|R(i\lambda, A)\| < \infty$ and

$$\|R(i\lambda, A)x\| \le \frac{1}{|\lambda|}\|R(i\lambda, A)(Ax) + x\| \le \frac{C+1}{|\lambda|}\|x\|_{D(A)}. \tag{3.9}$$

This gives the claim for $\alpha = 0$ and $\alpha = 1$. For $0 < \alpha < 1$ we assume first that the growth bound $w(A)$ of (T_t) is smaller than 0, so that we can apply the inequality

$$\|(-A)^{1-\alpha} y\| \le C\|y\|^\alpha \|Ay\|^{1-\alpha}, \quad y \in D(A). \tag{3.10}$$

With $y = R(i\lambda, A)(-A)^{\alpha-1}x$ we obtain from (1)

$$\begin{aligned}
\|R(i\lambda, A)x\| = \|(-A)^{1-\alpha}y\| &\le C\|R(i\lambda, A)(-A)^{\alpha-1}x\|^\alpha \|AR(i\lambda, A)(-A)^{\alpha-1}x\|^{1-\alpha} \\
&\le \Big[\frac{C+1}{|\lambda|}\|A(-A)^{\alpha-1}x\|\Big]^\alpha \Big[C\|A(-A)^{\alpha-1}x\|\Big]^{1-\alpha} \\
&\le \frac{C}{|\lambda|^\alpha}\|(-A)^\alpha x\|.
\end{aligned}$$

For general (T_t) choose $w > w(A)$ and apply this estimate to $e^{-wt}T_t$

$$\|R(i\lambda + w, A)x\| \le \frac{C}{|\lambda|^\alpha}\|x\|_{X_\alpha}. \tag{3.11}$$

The resolvent equation can be rewritten as

$$R(i\lambda, A) = [I + wR(i\lambda, A)]R(w + i\lambda, A) \quad \text{so, that}$$

$$\|R(i\lambda, A)x\| \leq (1 + wC)\|R(w + i\lambda, A)x\|. \tag{3.12}$$

Now the claim follows from (3.11) and (3.12).
b) Since X_α is isomorph to the graph of $(-A)^\alpha$ as a subspace of $X \times X$, X_α has the same Fourier–type as X. By a)

$$\|R(\lambda, A)^2 x\| \leq \frac{C}{|\lambda|^\alpha}\|R(i\lambda, A)x\|_{X_\alpha}. \tag{3.13}$$

T_t is also a c_0–semigroup on X_α and for w larger than the growth bound of T_t on X_α, we obtain from

$$R(w + i\lambda, A)x = \int e^{-i\lambda t}[e^{-wt}T_t x]dt$$

and the Fourier–type of X_α that

$$\left(\int \|R(w + i\lambda, A)x\|_X^{p'} d\lambda\right)^{\frac{1}{p'}} \leq C\left(\int \|e^{-wt}T_t x\|_{X_\alpha}^p dt\right)^{\frac{1}{p}} \leq D\|x\|_{X_\alpha}. \tag{3.14}$$

Combining (3.13), (3.12) and (3.14) gives the claim.
c) By part a) we have

$$\begin{aligned} \|R(\lambda, A^*)^2 x\|_{X_\alpha^*} &\leq \|R(\lambda, A)^*\|_{X^* \to X_\alpha^*}\|R(\lambda, A^*)x\|_{X^*} \\ &\leq \frac{C}{|\lambda|^\alpha}\|R(\lambda, A^*)x^*\|_{X^*} \end{aligned} \tag{3.15}$$

Since X^* has the same Fourier–type as X we have

$$\left(\int \|R(w + i\lambda, A^*)x^*\|_{X_\alpha^*}^{p'} d\lambda\right)^{\frac{1}{p'}} \leq C\left(\int \|e^{-wt}T_t^* x^*\|_{X^*}^p dt\right)^{\frac{1}{p}} \leq D\|x^*\|_{X^*}. \tag{3.16}$$

Now combine (3.15), (3.12) for A^* and (3.16) to get c). ∎

3.4. Lemma: Let $\varepsilon > 0, \alpha > 0$ and $w > w(A)$. If $x \in X_\beta$ with $\beta > \alpha + \varepsilon$ than the function $t \to e^{-wt}T_t\, x,\ t \geq 0$, belongs to $B_{1,\infty}^\varepsilon(\mathbb{R}_+, X_\alpha)$.

Proof: Choose γ with $\alpha < \gamma(1-\varepsilon) < \beta - \varepsilon$. The map

$$U : X \to L_1(\mathbb{R}_+, X),\ Ux(t) = e^{-wt}T_t x$$

also defines bounded operators

$$\begin{aligned} X_\gamma &\xrightarrow{U} L_1(\mathbb{R}_+, X_\gamma) \subset B^0_{1,\infty}(\mathbb{R}_+, X_\gamma) \\ X_1 &\xrightarrow{U} W^1_1(\mathbb{R}_+, X) \subset B^1_{1,\infty}(\mathbb{R}_+, X). \end{aligned}$$

By interpolation (see [14], [13]) we get

$$(X_\gamma, X_1)_\varepsilon \xrightarrow{U} B^\varepsilon_{1,\infty}(\mathbb{R}_+, (X_\gamma, X)_\varepsilon).$$

Since $\beta > \gamma(1-\varepsilon) + \varepsilon$ and $\alpha < \gamma(1-\varepsilon)$ we have $X_\beta \subset (X_\gamma, X_1)_\varepsilon$ and $(X_\gamma, X)_\varepsilon \subset X_\alpha$, respectively. Now the claim follows. ■

For the convenience of the reader we include the following Lemma from [18], which also gives a simpler proof of Corollary 6 of [17] (or [9] Section 4.1.). The original proof was based on interpolation theory.

3.5. Lemma: Let $a > \omega(T_t)$. Then the functions

$$\begin{aligned} \alpha \in (0,\infty) &\to w_\alpha(x) = w((a-A)^{-\alpha}x), \quad x \in X \text{ fixed} \\ \alpha \in (0,\infty) &\to w_\alpha(A) := \sup\{w((a-A)^{-\alpha}x) : x \in U_x\} \end{aligned}$$

are convex and therefore continuous.

Proof: Put $B^\alpha = (a-A)^{-\alpha}$. Let $0 < \alpha < \beta$ and $\gamma = (1-\theta)\alpha + \theta\beta$ for some $0 < \theta < 1$. For $u > w_\alpha(x)$, $v > w_\beta(x)$, choose a constant M such that $\|T_t(B^\alpha x)\| \le Me^{ut}$, $\|T_t(B^\beta x)\| \le Me^{vt}$. Now we apply the convexity estimate (see [5], theorem 8.1)

$$\|B^{-c}y\| \le \|B^{-a}y\|^{1-\theta}\|B^{-b}y\|^\theta, \quad a,b > 0,\ c = (1-\theta)a + \theta b$$

for $y \in D(B^{-1})$ to $y = BT_t x$ and $a = 1-\alpha, b = 1-\beta, c = 1-\gamma$. Since T_t and B^α commute we find

$$\begin{aligned} \|T_t B^\gamma x\| &\le C\|B^\alpha T_t x\|^{1-\theta}\|B^\beta T_t x\|^\theta \\ &\le MCe^{u(1-\theta)t}e^{v\theta t}. \end{aligned}$$

Hence $w_\gamma(x) \le (1-\theta)w_\alpha(x) + \theta w_\beta(x)$. The function $w_\alpha(A)$ is just the supremum of a set of convex functions. ■

3.6. Remark: Lemma 3.5 also provides a partial answer to question on page of [9]. Indeed, the result of [18] or Corollary 4.3. of [9] and the continuity of $\alpha \to \omega(R(a,A)^\alpha x)$ imply the following result:
Let X be a Banach space of Fourier–type $1 \le p \le 2$. If $x \in X_{\frac{1}{p}}$ and the local resolvent $\lambda \to R(\lambda, A)x$ admits a bounded holomorphic extention $F(\lambda)$ to the open right half–plane, then $\omega(x) \le 0$.

3.7. Remark: The spaces $B^{\frac{n}{p}}_{p,1},(\mathbb{R}^n, X)$ form a scale since by [13], Remark 6,

$$B^{\frac{n}{p}}_{p,1}(\mathbb{R}^n, X) \subset B^{\frac{n}{q}}_{q,1}(\mathbb{R}^n, X) \quad \text{if} \quad p < q.$$

The assumption of Theorem 2.1 is sharp in the sense, that the Theorem does not hold anymore for all Banach spaces X and Y of Fourier–type p, if we replace in its assumption the space $B^{n/p}_{p,1}(Y)$ by the larger space $B^{n/q}_{q,1}(Y)$ for some $q > p$.
Otherwise one could prove that for every c_0–semigroup T_t with generator A on a Banach space X of Fourier–type p, one has

$$\omega(x) \le s_0(A) \quad \text{for all} \quad x \in X_\alpha, \ \alpha \ge \frac{1}{q} - \frac{1}{p'}. \tag{3.17}$$

(Indeed, the proof of Theorem 3.2 shows that for $\alpha \ge \frac{1}{q} - \frac{1}{p'}$, $M(\lambda) = R(\lambda i, A)^2$ satisfies i) and ii) for the space $W^1_q(X_\alpha)$ in the place of $W^1_p(X_\alpha)$, and one could apply the multiplier Theorem to get (3.17)). But it was shown in [17] that (3.17) fails for $\alpha < \frac{1}{p} - \frac{1}{p'}$.

4 Stability of positive semigroups in $L_p(\mu)$.

Combing ideas from [7] and [16] we give a simple proof of the following theorem from [15]:

4.1. Theorem: If T_t is a positive semigroup on $L_p(\Omega, \mu), 1 \le p < \infty$, with generator A, then $s(A)$ equals the growth bound of (T_t).

Assume $s(A) < 0$. According to the discussion in the beginning of Section 3, we only have to show that there is a constant $C < \infty$ such that for

$$K_N f(t) = \int_0^N T_s(f(t-s))ds, \quad f \in L_p(\mathbb{R}_+, L_p(\Omega,\mu))$$

we have

$$\|K_N f\|_{L_p(L_p)} \leq C\|f\|_{L_p(L_p)} \quad \text{for all } N.$$

For this we need the following result from [7]:

<u>**4.2. Lemma:**</u> Let X and Y be Banach function spaces and $P : X \to Y$ a positive linear operator... If $F : [0,N] \to X$ is a Bochner–measurable function then

$$(\int |PF(t)|^p dt)^{\frac{1}{p}} \leq P(\int |F(t)|^p dt)^{\frac{1}{p}}.$$

We will apply this lemma to $X = L_p([0,N], L_p(\mu))$, $Y = L_p(\mu)$ and

$$Pf = \int_0^N T_s(f(s))ds, \qquad \text{for} \qquad f \in L_p([0,N], L_p(\mu))$$

Define $F : [0,N] \to X$ by $F(t)(s) = f(t-s)\cdot\chi_{\mathbb{R}_+}(t-s)$. Then by Fubini and 4.2

$$\begin{aligned}
\|K_N f\|_{L_p(\mathbb{R}_+, L_p(\mu))} &= \|(\int |\int_0^N T_s f(t-s)ds|^p dt)^{\frac{1}{p}}\|_{L_p(\mu)} \\
&= \|(\int |PF(t)|^p dt)^{\frac{1}{p}}\|_Y \leq \|P(\int |F(t)|^p dt)^{\frac{1}{p}}\|_Y \\
&= \|\int_0^N T_s(\int |f(t-s)|^p dt)^{\frac{1}{p}} ds\| = \|\int_0^N T_s g ds\| \\
&\leq \|R(0,A)g\| \leq \|R(0,A)\| \cdot \|g\|
\end{aligned}$$

with $g = (\int |f(t)|^p dt)^{\frac{1}{p}}$ and $\|g\|_{L_p(\mu)} = \|f\|_{L_p(\mathbb{R}_+, L_p(\mu))}$.

<u>**4.3 Remark:**</u> For a positive semigroup we have that $s_0(A) = s(A)$. It is therefore asked in ([9], question (iv) on page 147) if $s_0(A) = \omega(T_t)$ for all semigroups on $L_p(\mu)$–spaces. This is indeed true for $p = 2$ according to a result of Gearhart (see e. g. [9]). But for $p \neq 2$ the answer is negative, as shown by the following example from [16].

4.4 Example: Let $X = L_p(1,\infty) \cap L_2(1,\infty)$ with norm $\|f\| = \|f\|_2 + \|f\|_p$ for $2 < p < \infty$. Consider the semigroup $(S_t f)(x) = f(xe^t), t \geq 0$, with generator $(Bf)(x) = x(\frac{d}{dx} f)(x)$ on a suitable domain and $(R(0,B)f)(x) = \int\limits_x^\infty f(y) \frac{dy}{y}$.
One can check that $s(B) = -\frac{1}{2} < -\frac{1}{p} = \omega(S_t)$ (cf. [2]). Since S_t is positive, we also have $s_0(B) = s(B)$.
By [6] 2.e.8(ii) and section 2.f. there is an isomorphism J of X onto $L_p[0,1]$ (essentially given by a stochastic integral with respect to the poisson process). Then the semigroup $T_t = JS_tJ^{-1}$ on $L_p[0,1]$ with generator $A = JBJ^{-1}$ on $D(A) = J(D(B))$ still satisfies $s_\infty(A) = -\frac{1}{2} < -\frac{1}{p} = \omega(T_t)$.
If $1 < p < 2$ we take the dual of T_t on $L_{p'}$, to obtain a similar example.

References

[1] H. Amann, *Operator–valued Fourier Multipliers, vector–valued Besov spaces, and applications.* Submitted.

[2] W. Arendt, *Spectrum and growth of positive semigroups*, in: G. Ferreyra, G. R. Goldstein and F. Neubrander: Evolution equations, Lecture Notes in Pure and Applied Math. 168, Marcel Dekker, 1994.

[3] J. Bourgain, *Vector–valued Hausdorff–Young inequalities and applications.* Springer Lect. Notes in Math. 1317, Springer Verlag (1988), 239–249.

[4] M. Jung and L. Weis, *Fourier Multiplier theorems for Operator–valued functions.* In preparation.

[5] H. Komatsu, *Fractional powers of operators*, Pac. J. of Math. 19, (1966), 285–346.

[6] J. Lindenstrauss and L. Tzafriri, *Classical Banach Spaces II*, Springer 1979.

[7] S. Montgomery–Smith, *Stability and dichotomy of positive operators on* L_p. To appear in Proc. Amer. Math. Soc.

[8] J. van Neerven, *Individual stability of* c_0 *–semigroups and uniform boundedness of the local resolvent.* To appear in Semigroup Forum.

[9] J. van Neerven, *The Asymptotic Behaviour of Semigroups of Linear operators.* Operator theory, Advances and Applications, Vol. 88 Birkhäuser.

[10] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations.* Springer Verlag (1983).

[11] J. Peetre, *Sur la transformation de Fourier des fonctions à valeurs vectorielles.* Rend. Sem. Mat. Univ. Padova 42, 1969, 15–26.

[12] A. Pelczynski and M. Wojciechowski, *Molecular decompositions and embedding theorems for vector–valued Sobolev spaces with gradient norm.* Studia Mathematica 107, 61–100, 1993.

[13] H. G. Schmeisser, *Vector–valued Sobolev and Besov spaces.* In B.–W. Schulze and H. Triebel, ed. Seminar Analysis of the Karl–Weierstraß–Institute 1985/86, pages 4–44, Leipzig 1987, Teubner Verlag.

[14] H. Triebel, *Theory of function spaces*, Birkhäuser Verlag, Basel–Boston–Stuttgart, 1983.

[15] L. Weis, *The stability of positive semigroups on L_p–spaces.* Proc. AMS 123 (1995), 3089–3094.

[16] L. Weis, *A short proof for the stability theorem for positive semigroups on $L_p(\mu)$.* To appear in Proc. AMS.

[17] L. Weis and V.Wrobel, *Asymptotic behaviour of c_0–semigroups in Banach spaces.* Proc. Amer. Math. Soc. 124 (1996), 3663–3671.

[18] L. Weis and V. Wrobel, *Optimal individual stability estimates for C_0–semigroups in Banach spaces*, preprint.

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