THE SPECTRAL MAPPING THEOREM FOR EVOLUTION SEMIGROUPS ON L^p ASSOCIATED WITH STRONGLY CONTINUOUS COCYCLES

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ABSTRACT. In this note we show the spectral mapping theorem for the evolution semigroup on $L^p(\Theta, \mu; X)$, $1 \le p < \infty$, associated with a strongly continuous cocycle on a Banach space over a continuous flow on a locally compact metric space.

1. INTRODUCTION

In this note we prove the spectral mapping theorem

$$\sigma(T(t)) \setminus \{0\} = e^{t\sigma(G)}, \quad t \ge 0, \tag{1}$$

for the evolution semigroup $\mathcal{T} = (T(t))_{t\ge 0}$, $T(t) = e^{tG}$, defined on the space $E = L^p(\Theta, \mu; X)$, $1 \le p < \infty$, by

$$(T(t)f)(\theta) = \left(\tfrac{\mu\circ\varphi^{-t}}{d\mu}(\theta)\right)^{\frac{1}{p}} \Phi(\varphi^{-t}\theta, t) f(\varphi^{-t}\theta), \quad \theta \in \Theta,\ f \in E. \tag{2}$$

Here and throughout the paper we impose the following *assumptions*: Θ is a locally compact metric space, μ is a σ-finite regular Borel measure on Θ being positive on open sets, and X is a Banach space. Further, $\varphi : \Theta \times \mathbb{R} \to \Theta : (\theta, t) \mapsto \varphi^t\theta$ is assumed to be a continuous flow and $\Phi : \Theta \times \mathbb{R}_+ \to \mathcal{L}(X)$ is a strongly continuous, exponentially bounded (semi)cocycle over φ. This means that $\Phi(\theta, t+s) = \Phi(\varphi^s\theta, t)\Phi(\theta, s)$ and $\Phi(\theta, 0) = I$ for $\theta \in \Theta$ and $t, s \ge 0$, the function $(\theta, t) \mapsto \Phi(\theta, t) \in \mathcal{L}(X)$ is strongly continuous, $\mathcal{L}(X)$ is the set of bounded on X linear operators, and $\|\Phi(\theta, t)\| \le Me^{wt}$ for $t \ge 0$, $\theta \in \Theta$, and constants $M \ge 1$ and $w \in \mathbb{R}$ We assume that the measure μ is *quasi-invariant* with respect to the flow φ^t, i.e., the Radon-Nikodim derivative $d\mu \circ \varphi^t/d\mu$ belongs to $L^\infty(\Theta)$ and is uniformly bounded for $t \in \mathbb{R}$ In addition, we suppose that the set of aperiodic points of φ^t (i.e., $\varphi^t\theta \ne \theta$ for all $t \ne 0$) has full measure in Θ. This implies that the aperiodic points are dense in Θ. In Lemma 1 we will see that $\mathcal{T}$ is a strongly continuous semigroup on $E = L^p(\Theta, \mu; X)$. We denote the generator of $\mathcal{T}$ by $(G, D(G))$ and set $T = T(1)$.

Let us recall that cocycles arise as solution operators for equations like

$$u'(t) = A(\varphi^t\theta)u(t), \quad \theta \in \Theta,\ t \ge 0, \tag{3}$$

where $A(\theta)$ is a linear operator. Such equations occur, for instance, if one linearizes a non-linear equation over an invariant set Θ or if one considers the "hull" of a given non-autonomous equation, see, e.g., [20] and the references therein. Moreover, in

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the special case $\Theta = \mathbb{R}$, $\varphi^t\theta = \theta + t$, and $\theta = 0$ equation (3) yields the evolution equation $u'(t) = A(t)u(t)$, $t \geq 0$.

Our result generalizes spectral mapping theorems for evolution semigroups obtained by a number of authors and is proved by a different method. Such theorems were first shown in [5, 6, 9] on the spaces $L^2(\Theta, \mu; X)$ and $C(\Theta, X)$ for a finite dimensional space X and a smooth compact manifold Θ. The main tool was the so-called Mañe Lemma, see [6] and (4) below, which does not hold in infinite dimensions. Also, μ was assumed to be φ^t-invariant, which leads to additional restrictions on the dynamics of φ^t and the spectrum of $T(t)$, cf. [5, Prop. 1.4]. In [14] the case of a uniformly continuous cocycle, $p = 2$, and a Hilbert space X was considered. The main instrument to prove (1) was the Gearhart-Prüß spectral mapping theorem (see, e.g., [16, 22]) that does not hold for Banach spaces.

For Banach spaces and strongly continuous cocycles the spectral mapping theorem for evolution semigroups was proved in [12, Thm. 3.3] (see also [19]) on the space $C_0(\Theta; X)$ of continuous functions vanishing at infinity. One of the motivations for proving this result in [12] was to develop a technique to obtain the spectral mapping theorem for the group generated by the kinematic dynamo operator on the space of divergence-free vector fields, see [7]. Though this goal was achieved, the proofs in [7, 12] work only for the sup-norm.

L^p-proofs for strongly continuous cocycles on Banach spaces were obtained in [11, 17, 18] (see also [22] and, for $X = \mathbb{C}^n$, [2, 3]) in the special case of the flow of translations $\varphi^t\theta = \theta + t$ on $\Theta = \mathbb{R}$ and $\Phi(\theta, t) = U(\theta + t, \theta)$, where $\{U(t, s)\}_{t \geq s}$ is an *evolution family* (roughly speaking, the propagator for a nonautonomous abstract Cauchy problem).

In the above mentioned papers the spectral mapping theorems were essential steps to characterize exponential dichotomy of cocycles (or evolution families) in terms of spectral properties of T or G. These results can be used, e.g., to show robustness of dichotomy, see [12, 13, 21].

The main tool in our approach in the present paper is to "localize" approximate eigenfunctions of T in order to construct approximate eigenfunctions of G. This gives a replacement for the Mañe Lemma in infinite dimensions. In a sense, this idea goes back as far as to [15]. To achieve this localization we use *weighted shift operators* $\pi_\theta(T)$ acting on the sequence space $\ell^p = \ell^p(\mathbb{Z}, X)$ by the rule

$$\pi_\theta(T)v = (\Phi(\varphi^{k-1}\theta, 1)x_{k-1})_{k\in\mathbb{Z}}, \quad v = (x_k)_{k\in\mathbb{Z}} \in \ell^p.$$

The appearence of the operators $\pi_\theta(T)$ is related to a deep algebraic structure in the background; in fact, π_θ define a representation of a certain *weighted translation algebra* of operators of the type (2) in ℓ^p, see [1], [8], [11], [12], [13], [14], and the references therein.

2. Proofs

Descriptively, our strategy of the proof of (1) is as follows. By standard arguments (see, e.g., [12, Thm. 3.3]), involving rescaling, the spectral inclusion theorem, and the spectral mapping theorem for the residual spectra, see [16, A-III], it suffices to show that $1 \in A\sigma(T)$ implies $i\mathbb{R} \subseteq A\sigma(G)$ for the approximate point spectrum defined by $A\sigma(B) = \{\lambda \in \mathbb{C} \colon (\lambda - B)f_n \to 0 \text{ for } \|f_n\| = 1\}$ for a closed operator B. So we assume $1 \in A\sigma(T)$ and give an explicit construction for an "almost-eigenfunction"

g of the generator G. This function is supported in a long and thin tube, i.e., a "flow-box" formed by trajectories of φ^t.

As a preparation we prove in Proposition 4 that uniform injectivity of $I-T$ on $L^p(\Theta,\mu;X)$ is equivalent to that of $I-\pi_\theta(T)$ on ℓ^p uniformly for all $\theta\in\Theta$. This result is of interest by itself, but we need it as a replacement of the Mañe Lemma, mentioned in the Introduction.

The Mañe Lemma holds for $\dim X<\infty$ and says that $1\in A\sigma(T)$ on the space of continuous functions if and only if there exists a point $\theta_0\in\Theta$ and a vector $x_0\in X$, $\|x_0\|=1$, such that

$$\sup\{\|\Phi(\theta_0,t)x_0\|:\ t\in\mathbb{R}\}<\infty, \tag{4}$$

see, e.g., [6, p.158]. In finite dimensions, see [6, 7], the "almost-eigenfunction" g is constructed by propagating x_0 along the orbit through θ_0 and "spreading" it in a tube around the orbit.

Instead, in infinite dimensions and L^p, we construct g below by choosing a point θ_0 where $I-\pi_{\theta_0}(T)$ "almost" looses its uniform injectivity, and spreading the corresponding almost-eigenvector $v=(x_k)_{k\in\mathbb{Z}}\in\ell^p$ for $\pi_{\theta_0}(T)$ in a tube around the orbit through θ_0.

To start, we remark that for a given flow φ^t a quasi-invariant regular Borel measure μ with $\operatorname{supp}\mu=\Theta$ always exists, see e.g. [1, §9]. For brevity we denote the Radon-Nikodim derivative by $J^t(\theta)=\frac{d\mu\circ\varphi^t}{d\mu}(\theta)$. Notice that

$$J^{t+s}(\theta)=J^t(\varphi^s\theta)J^s(\theta)\quad\text{and}\quad \big(J^t(\theta)\big)^{-1}=J^{-t}(\varphi^t\theta)>0 \tag{5}$$

for $t,s\in\mathbb{R}$ and a.e. $\theta\in\Theta$. We first show that $\mathcal{T}$ is in fact a $\mathcal{C}_0$–semigroup, cf. [14].

Lemma 1. *Under the assumptions listed in the Introduction, (2) defines a strongly continuous semigroup $\mathcal{T}=(T(t))_{t\geq 0}$ on E.*

Proof. Clearly, $\mathcal{T}$ is a semigroup in $\mathcal{L}(E)$. Define isomteries $T_0(t)$ on E by $T_0(t)f(\theta)=J^{-t}(\theta)^{\frac{1}{p}}f(\varphi^{-t}\theta)$. Since

$$\|T(t)f-f\|\leq Me^{wt}\,\|T_0(t)f-f\|+\|(\Phi(\varphi^{-t}\cdot,t)-I)f\|,$$

$f\in E$, it suffices to show strong continuity of $T_0(t)$. We claim that

$$\text{for } t_n\to 0\ \exists \text{ a subsequence } t_k \text{ so that}\quad J^{t_k}(\theta)\to 1\ \text{ for a.e. } \theta. \tag{6}$$

Otherwise, there is $t_k\to 0$ and a compact set W of positive measure so that $|J^{t_k}(\theta)-1|\geq\delta>0$ for $\theta\in W$. We may assume that $J^{t_k}(\theta)\geq 1$. Hence, $\mu(\varphi^{t_k}W)-\mu(W)\geq\delta\mu(W)$. However, $\mu(\varphi^{t_k}W)\to\mu(W)$ by the dominated convergence theorem. Therefore (6) must hold.

Now, if $T_0(t)$ were not strongly continuous then we would find a continuous function f with compact support and a sequence $t_n\to 0$ so that $\|T_0(t_n)f-f\|\geq\delta>0$. Using (6) we easily get a contradiction. $\square$

Next, we give two technical lemmas concerning flows.

Lemma 2. *Let $\theta_0\in\Theta$ be aperiodic and $n\in\mathbb{N}$ There exists an open set U and a set Σ such that $\theta_0\in\Sigma\subseteq U$, $\overline{U}$ is compact, and for all $\theta\in U$ there is a unique number $t(\theta)\in(-n,n)$ with $\sigma:=\varphi^{t(\theta)}\theta\in\Sigma$. Moreover, the map $\theta\mapsto t(\theta)$ is continuous.*

The proof of this lemma can be found in [4, Thm. IV.2.11]. The set U is called *tube* of length n with *section* Σ at θ_0. The next result is a simple consequence of the Halmos-Rokhlin lemma.

Lemma 3. *Consider $0 \leq g_j \in L^1(\Theta, \mu; \mathbb{R})$, $j = 1, \cdots, m$. Then for every $\epsilon > 0$ and $n \in \mathbb{N}$ there is a measurable set $F \subseteq \Theta$ such that $\varphi^k F \cap \varphi^l F = \emptyset$ for $k \neq l$, $k, l \in \{-n, \cdots, n\}$, and*

$$\int_{\Theta \setminus U} g_j(\theta)\, d\mu(\theta) \leq \epsilon, \qquad j = 1, \cdots, m,$$

where $U = \bigcup_{|k| \leq n} \varphi^k F$.

Proof. Fix $\epsilon > 0$ and $n \in \mathbb{N}$. There is a set $\Theta_0 \subseteq \Theta$ of finite measure so that $\int_{\Theta \setminus \Theta_0} g_j(\theta)\, d\mu(\theta) \leq \epsilon/2$ for $j = 1, \cdots, m$. Choose $0 < \alpha \in L^1(\Theta, \mu; \mathbb{R}) \cap L^\infty(\Theta, \mu; \mathbb{R})$ with $\alpha = 1$ on Θ_0. Set $\nu := \alpha\mu$. Notice that $g_j \in L^1(\Theta, \nu; \mathbb{R})$. We now apply the Halmos-Rokhlin lemma, see e.g. [10, Thm. 1.11], to the aperiodic map φ^1 on the finite measure space (Θ, ν). So, for each $\delta > 0$, we derive the existence of a measurable subset F_δ of Θ satisfying $\varphi^k F_\delta \cap \varphi^l F_\delta = \emptyset$ for $k \neq l \in \{-n, \cdots, n\}$ and $\nu(\Theta \setminus U_\delta) \leq \delta$, where $U_\delta = \bigcup_{|k| \leq n} \varphi^k F_\delta$. Hence,

$$\begin{aligned}
\int_{\Theta \setminus U_\delta} g_j(\theta)\, d\mu(\theta) &\leq \int_{\Theta_0 \cap (\Theta \setminus U_\delta)} \alpha(\theta) g_j(\theta)\, d\mu(\theta) + \int_{\Theta \setminus \Theta_0} g_j(\theta)\, d\mu(\theta) \\
&\leq \int_{\Theta \setminus U_\delta} g_j(\theta)\, d\nu(\theta) + \frac{\epsilon}{2} \leq \epsilon
\end{aligned}$$

for $j = 1, \cdots, m$ and $\delta > 0$ small enough. Setting $F := F_\delta$ yields the assertion. □

Proposition 4. *Under the above assumptions the following assertions are equivalent.*

(a) $\|(I - T)f\|_E \geq c\, \|f\|_E$ *for all* $f \in E$ *and a constant* $c > 0$.
(b) $\|(I - \pi_\theta(T))v\|_{\ell^p} \geq c\, \|v\|_{\ell^p}$ *for all* $\theta \in \Theta$, $v \in \ell^p$, *and a constant* $c > 0$.

Proof. **(a)⇒(b).** Consider first an aperiodic point $\theta_0 \in \Theta$ and a sequence $v = (x_k) \in \ell^p$ with $x_k = 0$ for $|k| > n$. Fix $\epsilon > 0$ and choose an open set $U \ni \theta_0$ of finite measure such that $\varphi^k U \cap \varphi^l U = \emptyset$ for $k \neq l \in \{-n, \ldots, n+1\}$ and

$$\|\Phi(\varphi^k \theta, 1)x_k - \Phi(\varphi^k \theta_0, 1)x_k\| \leq \epsilon \quad \text{for } |k| \leq n,\ \theta \in U. \tag{7}$$

We define $f(\theta) := \left(J^{-k}(\theta)\right)^{\frac{1}{p}} x_k$ if $\theta \in \varphi^k U$, $|k| \leq n$, and $f(\theta) := 0$ otherwise. Clearly, $f \in E$ and

$$\|f\|_E^p = \sum_{k=-n}^{n} \|x_k\|^p \int_{\varphi^k U} J^{-k}(\theta)\, d\mu(\theta) = \mu(U)\|v\|_{\ell^p}^p. \tag{8}$$

On the other hand, using (5) and (7) we compute

$$
\begin{aligned}
\|(I-T)f\|_E^p &= \int_{\bigcup_{k=-n}^{n+1} \varphi^k U} \left\| f(\theta) - \left(J^{-1}(\theta)\right)^{\frac{1}{p}} \Phi(\varphi^{-1}\theta, 1) f(\varphi^{-1}\theta) \right\|^p d\mu(\theta) \\
&= \int_U \sum_{k=-n}^{n+1} \Big\| \left(J^k(\theta)\right)^{-\frac{1}{p}} x_k - \left(J^1(\varphi^{k-1}\theta)\, J^{k-1}(\theta)\right)^{-\frac{1}{p}} \cdot \\
&\qquad\qquad\qquad \Phi(\varphi^{k-1}\theta, 1) x_{k-1} \Big\|^p J^k(\theta)\, d\mu(\theta) \\
&= \int_U \sum_{k=-n}^{n+1} \|x_k - \Phi(\varphi^{k-1}\theta, 1) x_{k-1}\|^p \, d\mu(\theta) \\
&\le \int_U \sum_{k=-n}^{n+1} \big(\|x_k - \Phi(\varphi^{k-1}\theta_0, 1) x_{k-1}\| \\
&\qquad + \|\Phi(\varphi^{k-1}\theta_0, 1) x_{k-1} - \Phi(\varphi^{k-1}\theta, 1) x_{k-1}\| \big)^p \, d\mu(\theta) \\
&\le \int_U \sum_{k=-n}^{n+1} \left(\|x_k - \Phi(\varphi^{k-1}\theta_0, 1) x_{k-1}\| + \epsilon \right)^p d\mu(\theta) \\
&= \mu(U) \sum_{k=-n}^{n+1} \left(\|x_k - \Phi(\varphi^{k-1}\theta_0, 1) x_{k-1}\| + \epsilon \right)^p .
\end{aligned}
$$

Then (8) and (a) imply

$$
\begin{aligned}
c^p \, \|v\|_{\ell^p}^p &= \frac{c^p}{\mu(U)} \|f\|_E^p \le \frac{1}{\mu(U)} \|(I-T)f\|_E^p \\
&\le \sum_{k=-n}^{n+1} \left(\|x_k - \Phi(\varphi^{k-1}\theta_0, 1) x_{k-1}\| + \epsilon \right)^p .
\end{aligned}
$$

Letting $\epsilon \to 0$, we derive $\|(1-\pi_\theta)v\|_{\ell^p} \ge c\|v\|_{\ell^p}$. Thus we have proved (b) for finitely supported $v \in \ell^p$ and aperiodic θ, which are dense in Θ by assumption. The assertion now follows by straightforward approximation arguments.

(b)⇒(a). Fix $f \in E$ and $\epsilon > 0$. By Lemma 3 there are $n \in \mathbb{N}$ and a measurable set $F \subseteq \Theta$ such that the sets $\varphi^k F$, $|k| \le n$, are disjoint and

$$
\|f\|_E^p \le \epsilon + \int_{\bigcup_{|k|\le n} \varphi^k F} \|f(\theta)\|^p \, d\mu(\theta), \tag{9}
$$

$$
\epsilon \ge \int_{\varphi^{n+1} F} \|Tf(\theta)\|^p \, d\mu(\theta) . \tag{10}
$$

For almost every $\theta \in F$ we define $v(\theta) = (x_k(\theta)) \in \ell^p$ by

$$
x_k(\theta) := \begin{cases} \left(J^k(\theta)\right)^{\frac{1}{p}} f(\varphi^k \theta), & \text{if } |k| \le n, \\ 0, & \text{otherwise.} \end{cases}
$$

Then (5), (b), and estimates (9)–(10) yield

$$
\begin{aligned}
&\|f - Tf\|_E^p \\
&\quad \geq \int_{\bigcup_{|k|\leq n} \varphi^k F} \left\| f(\theta) - \left(J^{-1}(\theta)\right)^{\frac{1}{p}} \Phi(\varphi^{-1}\theta, 1) f(\varphi^{-1}\theta)\right\|^p \, d\mu(\theta) \\
&\quad = \int_F \sum_{k=-n}^{n} \left\| f(\varphi^k\theta) - \left(J^{-1}(\varphi^k\theta)\right)^{\frac{1}{p}} \Phi(\varphi^{k-1}\theta, 1) f(\varphi^{k-1}\theta)\right\|^p J^k(\theta)\, d\mu(\theta) \\
&\quad = \int_F \sum_{k=-n}^{n} \|x_k(\theta) - \Phi(\varphi^{k-1}\theta, 1)x_{k-1}(\theta)\|^p d\mu(\theta) \pm \int_F \|\Phi(\varphi^n\theta, 1)x_n(\theta)\|^p \, d\mu(\theta) \\
&\quad = \int_F \|(I - \pi_\theta(T))v(\theta)\|_{\ell^p}^p \, d\mu(\theta) - \int_{\varphi^{n+1}F} \|Tf(\theta)\|^p \, d\mu(\theta) \\
&\quad \geq -\epsilon + c^p \int_F \|v(\theta)\|_{\ell^p}^p \, d\mu(\theta) \\
&\quad = -\epsilon + c^p \int_{\bigcup_{|k|\leq n} \varphi^k F} \|f(\theta)\|^p d\mu(\theta) \\
&\quad \geq -(c^p + 1)\epsilon + c^p \|f\|_E^p.
\end{aligned}
$$

Since $\epsilon > 0$ is arbitrary, assertion (b) is proved. $\square$

We can now show our main result. Let $\mathbb{T} = \{\lambda \in \mathbb{C} \colon |\lambda| = 1\}$.

Theorem 5. *Suppose the assumptions listed in the Introduction. Then the spectral mapping theorem $e^{t\sigma(G)} = \sigma(T(t)) \setminus \{0\}$ holds. Also, $\sigma(T(t)) = \mathbb{T} \cdot \sigma(T(t))$ for $t > 0$ and $\sigma(G) = \sigma(G) + i\mathbb{R}$*

Proof. **Step 1.** Assume $1 \in A\sigma(T)$. Proposition 4 implies that for all $n \in \mathbb{N}$ there are vectors $v^n \in \ell^p$ and points $\theta^n \in \Theta$ such that

$$
\|(I - \pi_{\theta^n}(T))v^n\|_{\ell^p} < \frac{1}{n} \|v^n\|_{\ell^p}. \tag{11}
$$

Since the aperiodic points are dense in Θ we can assume that θ^n are aperiodic. Fix $n \geq 2$, $v = (x_k) := v^n$ and set $\theta_k := \varphi^k \theta^n$, $k \in \mathbb{Z}$. Because of $\pi_{\theta_0}(T)v \in \ell^p$ there is $N \in \mathbb{N}$ such that $N \geq n$ and

$$
\frac{1}{2} \sum_{k=-\infty}^{\infty} \|\Phi(\theta_k, 1)x_k\|^p \leq \sum_{k=-N}^{N-1} \|\Phi(\theta_k, 1)x_k\|^p. \tag{12}
$$

Due to Lemma 2 we can choose a tube U of length $2N$ with a section Σ at θ_0 such that $\mu(U) < \infty$,

$$
\|\Phi(\varphi^{k-1}\sigma, 1)x_{k-1} - x_k\| \leq 2\, \|\Phi(\theta_{k-1}, 1)x_{k-1} - x_k\| \qquad \text{and} \tag{13}
$$

$$
\frac{1}{2}\, \|\Phi(\theta_{k-1}, 1)x_{k-1}\| \leq \|\Phi(\varphi^{k-1}\sigma, 1)x_{k-1}\| \leq 2\, \|\Phi(\theta_{k-1}, 1)x_{k-1}\| \tag{14}
$$

for all $\sigma \in \Sigma$ and $|k| \leq 2N$. Finally, (11) yields

$$
\sum_{k=-\infty}^{\infty} \|\Phi(\theta_{k-1}, 1)x_{k-1} - x_k\|^p < \frac{2^p}{n^p} \sum_{k=-\infty}^{\infty} \|\Phi(\theta_{k-1}, 1)x_{k-1}\|^p. \tag{15}
$$

We now construct an approximate eigenfunction g of G. Set $U_0 = \{\theta = \varphi^t\sigma : \sigma \in \Sigma, 0 < t \le 1\}$ and $U_1 = \{\varphi^t\sigma : \sigma \in \Sigma, \frac{3}{4} < t \le 1\}$. Choose a smooth function $\alpha : [0,1] \to [0,1]$ such that $\alpha = 0$ on $[0, \frac{1}{4}]$, $\alpha = 1$ on $[\frac{3}{4}, 1]$, and $\|\alpha'\|_\infty \le 3$. Choose a smooth function $\beta : \mathbb{R} \to [0,1]$ such that $\beta = 1$ on $[-N, N]$, $\operatorname{supp}\beta \subseteq (-2N, 2N)$, and $\|\beta'\|_\infty \le \frac{2}{N}$. For $\theta \in U$ we set $\theta = \varphi^{k+\tau}\sigma$, where $\sigma \in \Sigma$ and $t = k + \tau \in (-2N, 2N)$ with $k = -2N, \dots, 2N-1$ and $0 \le \tau < 1$. For each $\xi \in \mathbb{R}$ we define a function $g \in E$ by

$$g(\theta) = \beta(t)e^{-i\xi t}\left(J^{-t}(\theta)\right)^{\frac{1}{p}} \Phi(\varphi^k\sigma, \tau)[(1-\alpha(\tau))\Phi(\varphi^{k-1}\sigma, 1)x_{k-1} + \alpha(\tau)x_k]$$

for $\theta \in U$ and $g(\theta) = 0$ for $\theta \notin U$.

Step 2. We show $g \in D(G)$ and compute Gg. Consider $\theta \in \Theta$ and $h > 0$ with $\varphi^{-h}\theta = \varphi^{k+\tau-h}\sigma \in U$. First, assume $\tau > 0$. For $h < \tau$ and $t = k + \tau$ follows

$$\begin{aligned}
(T(h)g)(\theta) &= \left(J^{-h}(\theta)\right)^{\frac{1}{p}} \Phi(\varphi^{-h}\theta, h) g(\varphi^{-h}\theta) \\
&= \beta(t-h)e^{i\xi(h-t)}\left(J^{-t+h}(\varphi^{-h}\theta)J^{-h}(\theta)\right)^{\frac{1}{p}} \Phi(\varphi^{k+\tau-h}\sigma, h)\Phi(\varphi^k\sigma, \tau - h) \\
&\qquad \cdot [(1-\alpha(\tau-h))\Phi(\varphi^{k-1}\sigma, 1)x_{k-1} + \alpha(\tau-h)x_k] \\
&= \beta(t-h)e^{i\xi h}e^{-i\xi t}\left(J^{-t}(\theta)\right)^{\frac{1}{p}} \Phi(\varphi^k\sigma, \tau) \\
&\qquad \cdot [(1-\alpha(\tau-h))\Phi(\varphi^{k-1}\sigma, 1)x_{k-1} + \alpha(\tau-h)x_k].
\end{aligned}$$

If $\tau = 0$, we obtain for small $h > 0$

$$\begin{aligned}
T(h)g(\theta) &= \beta(t-h)e^{i\xi(h-t)}\left(J^{-t}(\theta)\right)^{\frac{1}{p}} \Phi(\varphi^{k-h}\sigma, h)\Phi(\varphi^{k-1}\sigma, 1-h)x_{k-1} \\
&= \beta(t-h)e^{i\xi h}e^{-i\xi t}\left(J^{-t}(\theta)\right)^{\frac{1}{p}} \Phi(\varphi^{k-1}\sigma, 1)x_{k-1}.
\end{aligned}$$

This implies $T(h)g(\theta) = g(\theta) = 0$ for h small enough and $\theta \notin U$,

$$\begin{aligned}
&\lim_{h \searrow 0} h^{-1}(T(h)g(\theta) - g(\theta)) \\
&= \lim_{h \searrow 0} h^{-1}(\beta(t-h)e^{i\xi h} - \beta(t))e^{-i\xi t}\left(J^{-t}(\theta)\right)^{\frac{1}{p}} \Phi(\varphi^{k-1}\sigma, 1)x_{k-1} \\
&= i\xi g(\theta) - \beta'(t)e^{-i\xi t}\left(J^{-t}(\theta)\right)^{\frac{1}{p}} \Phi(\varphi^{k-1}\sigma, 1)x_{k-1}
\end{aligned}$$

for $\theta \in U$ with $\tau = 0$, and

$$\begin{aligned}
&\lim_{h \searrow 0} h^{-1}(T(h)g(\theta) - g(\theta)) \\
&= e^{-i\xi t}\left(J^{-t}(\theta)\right)^{\frac{1}{p}} \Phi(\varphi^k\sigma, \tau) \lim_{h \searrow 0} \Big\{\beta(t-h)\Big[\big(h^{-1}(e^{i\xi h} - 1) - h^{-1}(e^{i\xi h} \\
&\qquad \cdot\alpha(\tau-h) - \alpha(\tau))\big)\Phi(\varphi^{k-1}\sigma, 1)x_{k-1} + h^{-1}(e^{i\xi h}\alpha(\tau-h) - \alpha(\tau))x_k\Big] \\
&\qquad + h^{-1}(\beta(t-h) - \beta(t))\big[(1-\alpha(\tau))\Phi(\varphi^{k-1}\sigma, 1)x_{k-1} + \alpha(\tau)x_k\big]\Big\} \\
&= i\xi g(\theta) + e^{-i\xi t}\left(J^{-t}(\theta)\right)^{\frac{1}{p}} \Phi(\varphi^k\sigma, \tau)\big[\beta(t)\alpha'(\tau)(\Phi(\varphi^{k-1}\sigma, 1)x_{k-1} - x_k) \\
&\qquad - \beta'(t)((1-\alpha(\tau))\Phi(\varphi^{k-1}\sigma, 1)x_{k-1} + \alpha(\tau)x_k)\big]
\end{aligned}$$

for $\theta \in U$ with $\tau \neq 0$. Since the above limits do not depend on θ we derive $g \in D(G)$ and

$$\begin{aligned} Gg(\theta) - i\xi g(\theta) &= e^{-i\xi t}\left(J^{-t}(\theta)\right)^{\frac{1}{p}} \Phi(\varphi^k\sigma, \tau)\left[(\beta(t)\alpha'(\tau) + \beta'(t)\alpha(\tau))\right. \\ &\quad \left. \cdot (\Phi(\varphi^{k-1}\sigma, 1)x_{k-1} - x_k) - \beta'(t)\Phi(\varphi^{k-1}\sigma, 1)x_{k-1})\right] \end{aligned}$$

for $\theta \in U$ and $Gg(\theta) - i\xi g(\theta) = 0$ for $\theta \notin U$.

Step 3. It remains to show $\|Gg - i\xi g\|_E < \frac{\tilde{C}}{n}\|g\|_E$ for a constant $\tilde{C}$ independent of n and g. Let $M := \sup\{\|\Phi(\theta, \tau)\| : 0 \leq \tau \leq 1, \theta \in \Theta\}$ and $C := \sup\{J^t(\theta) : t \in \mathbb{R}, \theta \in \Theta\}$. By our assumptions $M, C < \infty$. Moreover, $\mu(U_1) \geq \frac{1}{4C}\,\mu(U_0)$. Then (13) and (14) yield

$$\begin{aligned} \|Gg(\theta) - i\xi g(\theta)\|^p &\leq CM^p\Big(8\,\|\Phi(\theta_{k-1}, 1)x_{k-1} - x_k\| + \frac{4}{N}\|\Phi(\theta_{k-1}, 1)x_{k-1}\|\Big)^p \\ &\leq C_1\Big(\|\Phi(\theta_{k-1}, 1)x_{k-1} - x_k\| + \frac{1}{n}\|\Phi(\theta_{k-1})x_{k-1}\|\Big)^p \end{aligned}$$

for $\theta = \varphi^{k+\tau}\sigma \in U$, where C_1 is a constant not depending on k, n, and g. Consequently,

$$\begin{aligned} \|Gg - i\xi g\|_E &= \left(\int_{\bigcup_{k=-2N}^{2N-1}\varphi^k U_0} \|Gg(\theta) - i\xi g(\theta)\|^p\, d\mu(\theta)\right)^{\frac{1}{p}} \\ &\leq C_1^{\frac{1}{p}}\left[\sum_{k=-\infty}^{\infty} \mu(\varphi^k U_0)\,\Big(\|\Phi(\theta_{k-1}, 1)x_{k-1} - x_k\|\right. \\ &\qquad \left. + \frac{1}{n}\,\|\Phi(\theta_{k-1}, 1)x_{k-1}\|\Big)^p\right]^{\frac{1}{p}} \\ &< (CC_1\mu(U_0))^{\frac{1}{p}}\,\frac{3}{n}\left(\sum_{k=-\infty}^{\infty}\|\Phi(\theta_k, 1)x_k\|^p\right)^{\frac{1}{p}}. \end{aligned} \tag{16}$$

In the last step we have used (15). On the other hand we have

$$g(\theta) = e^{-i\xi t}\left(J^{-t}(\theta)\right)^{\frac{1}{p}}\Phi(\varphi^k\sigma, \tau)x_k$$

for $\theta \in \varphi^k U_1$, $k = -N, \ldots, N-1$. Moreover,

$$\|\Phi(\theta, 1)x\| \leq \|\Phi(\varphi^\tau\theta, 1-\tau)\|\,\|\Phi(\theta, \tau)x\| \leq M\|\Phi(\theta, \tau)x\|$$

holds for all $\theta \in \Theta$, $0 \leq \tau \leq 1$, and $x \in X$. Thus (5) and (14) imply

$$\begin{aligned} \|g(\theta)\|^p &= \left\|\left(J^{-t}(\theta)\right)^{\frac{1}{p}}\Phi(\varphi^k\sigma, \tau)x_k\right\|^p \geq \frac{1}{CM^p}\|\Phi(\varphi^k\sigma, 1)x_k\|^p \\ &\geq \frac{1}{C(2M)^p}\|\Phi(\theta_k, 1)x_k\|^p \end{aligned}$$

for $\theta \in \varphi^k U_1$, $k = -N, \dots, N-1$. Using (12) we conclude

$$\begin{aligned} \|g\|_E^p &\geq \int_{\bigcup_{k=-N}^{N-1} \varphi^k U_1} \|g(\theta)\|^p \, d\mu(\theta) \\ &\geq \frac{1}{C(2M)^p} \sum_{k=-N}^{N-1} \mu(\varphi^k U_1) \|\Phi(\theta_k, 1) x_k\|^p \\ &\geq \frac{\mu(U_0)}{8C^3 \, (2M)^p} \sum_{k=-\infty}^{\infty} \|\Phi(\theta_k, 1) x_k\|^p. \end{aligned} \tag{17}$$

Combining (16) and (17) establishes the theorem. □

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