A VECTOR-VALUED OPERATIONAL CALCULUS AND ABSTRACT CAUCHY PROBLEMS

A Dissertation

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Prelude

Oliver Heaviside,
Electromagnetic Theory, Vol II, §239,
London 1895.

"We have now to consider a number of problems which can be solved at once without going to the elaborate theory of Fourier series and integrals. In doing this, we shall have, primarily, to work by instinct, not by rigorous rules. We have to find out first *how* things go in the mathematics as well as in the physics. When we have learnt the go of it we may be able to see our way to an understanding of the meaning of the processes, and bring them into alignment with other processes. And I must here write a caution. I may have to point out sometimes that my method leads to solutions much more simply than Fourier's method. I may, therefore, appear to be disparaging and endeavouring to supersede his work. But it is nothing of that sort. In a complete treatise on diffusion Fourier's and other methods would come side by side – not as antagonists, but as mutual friends helping one another. The limitations of space forbid this, and I must necessarily keep Fourier series and integrals rather in the background. But this is not to be misunderstood in the sense suggested. No one admires Fourier more than I do. It is the only entertaining mathematical work I ever saw. Its lucidity has always been admired. But it was more than lucid. It was luminous. Its light showed a crowd of followers the way to a heap of new physical problems.

The reader who may think that mathematics is all found out, and can be put in a cut-and-dried form like Euclid, in propositions and corollaries, is very much mistaken; and if he expects a similar systematic exposition here he will be disappointed. The virtues of the academical system of rigorous mathematical training are well known. But it has its faults. A very serious one (perhaps a necessary one) is that it checks instead of stimulating any originality the students may possess, by keeping him in regular grooves. Outsiders may find that there are other grooves just as good, and perhaps a great deal better, for their purposes. Now, as my grooves are not the conventional ones, there is no need for any formal treatment. Such would be quite improper for our purpose, and would not be favourable to rapid acquisition and comprehension. For it is in mathematics just as in the real world; you must observe and experiment to find out the go of it. All experimentation is deductive work in a sense, only it is done by trial and error, followed by new deductions and changes of direction to suit circumstances. Only afterwards, when the go of it is known, is any formal expression possible. Nothing could be more fatal to progress than to make fixed rules and conventions at the beginning, and then go by mere deduction. You would be fettered by your own conventions, and be in the same fix as the House of Commons with respect to the dispatch of business, stopped by its own rules.

But the reader may object, 'Surely the author has got to know the go of it already, and can therefore eliminate the preliminary irregularity and make it logical, not experimental?' So he has in great measure, but he knows better. It is not the proper way under the circumstances, being an unnatural way. It is ever so much easier to the reader to find the go of it first, and it is the natural way.

The reader may then be able a little later to see the inner meaning of it himself, with a little assistance. To this extend, however, the historical method can be departed from to the reader's profit. There is no occasion whatever (nor would there be space) to describe the failures which make up the bulk of experimental work. He can be led into successful grooves at once. Of course, I do not write for rigourists (although their attention would be delightful) but for a wider circle of readers who have fewer prejudices, although their mathematical knowledge may be that of the rigourists as a straw to a haystack. It is possible to carry waggon-loads of mathematics under your hat, and yet know nothing whatever about the operational solution of physical differential equations."

Anonymous Fellow of the Royal Society
to Sir Edmund T. Whittaker.
See, J.L.B. Cooper [Co].

"There was a sort of tradition that a Fellow of the Royal Society could print almost anything he liked in the *Proceedings* untroubled by referees: but when Heaviside had published two papers on his symbolic methods, we felt the line had to be drawn somewhere, so we put a stop to it."

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Abstract

Initial and boundary value problems for linear differential and integro-differential equations are at the heart of mathematical analysis. About 100 years ago, Oliver Heaviside promoted a set of formal, algebraic rules which allow a complete analysis of a large class of such problems. Although Heaviside's operational calculus was entirely heuristic in nature, it almost always led to correct results. This encouraged many mathematicians to search for a solid mathematical foundation for Heaviside's method, resulting in two competing mathematical theories:

(a) Laplace transform theory for functions, distributions and other generalized functions,

(b) J. Mikusinski's field of convolution quotients of continuous functions.

In this dissertation we will investigate a unifying approach to Heaviside's operational calculus which allows us to extend the method to vector-valued functions. The main components are

(a) a new approach to generalized functions, considering them not primarily as functionals on a space of test functions or as convolution quotients in Mikusinski's quotient field, but as limits of continuous functions in appropriate norms, and

(b) an asymptotic extension of the classical Laplace transform allowing the transform of functions and generalized functions of arbitrary growth at infinity.

The mathematics are based on a careful analysis of the convolution transform $f \to k \star f$. This is done via a new inversion formula for the Laplace transform, which enables us to extend Titchmarsh's injectivity theorem and Foias' dense range theorem for the convolution transform to Banach space valued functions. The abstract results are applied to abstract Cauchy problems. We indicate the manner in which the operational methods can be employed to obtain existence and uniqueness results for initial value problems for differential equations in Banach spaces.

I. Introduction

Oliver Heaviside's 1893 classic "Electromagnetic Theory" [He1-3] proposed formal rules governing manipulations of the differential operator, such that linear differential equations are transformed into algebraic ones; however, his work is not free from debate to say the least. Many mathematicians questioned the platform of his operational calculus. K. Yosida ([Yo], preface) notes, "... the explanation of this operator p (the operator of differentiation) as given by him was difficult to understand and to use, and the range of the validity of his calculus remains unclear still now, although it was widely noticed that his calculus gives correct results in general."

Though O. Heaviside may deny K. Yosida's assessment of the rigor of understanding and explaining the calculus, he was unconcerned with remarks on the strength of the foundation of his work (see the prelude). O. Heaviside clearly illustrates his utilitarianism in respect to the calculus in many remarks throughout his work.

In the 1930's, G. Doetsch, D. V. Widder, and many other mathematicians began to strive for the mathematical foundation of Heaviside's operational calculus by virtue of the Laplace transform $\int_0^\infty e^{-\lambda t} f(t)\, dt$. However, as K. Yosida ([Yo], preface) remarks, "the use of such integrals naturally confronts restrictions concerning the growth behaviour of the numerical function f."

Among the goals of this dissertation is the construction of a solid mathematical foundation of Heaviside's operational calculus. The basis of this foundation lies strictly upon Laplace transform methods and fully extends the operational calculus to Banach space valued functions. J. C. Vignaux's [Vi] asymptotic version of the Laplace integral $\int e^{-\lambda t} f(t)\, dt$, which does not require any growth conditions on the locally integrable function f, provides the mathematical mechanism of this work.

For numerical functions, a mathematically sound, algebraic foundation of Heaviside's operational method was given in 1949, when J. Mikusinski introduced the theory of convolution quotients as a basis for Heaviside's operational calculus. Due to Titchmarsh's Theorem ([Ti], Theorem VII), the continuous functions on $[0, \infty)$, with addition and the convolution

$$k \star f : t \mapsto \int_0^t k(t-s) f(s)\, ds$$

as product, form a ring with no zero divisors, and thus can be extended to a field consisting of the convolution quotients $\frac{f}{g}$.(1) As anticipated by O. Heaviside, this theory has been successfully applied to Volterra integral equations,

(1) Despite Mikusinski's contributions, Heaviside's original operational calculus has all but vanished from modern mathematics. The standard treatment of operational calculus is now almost exclusively based on the Laplace transform. During the last two decades, one of the few proponents of Heaviside's original operational method was Professor Kôsaku Yosida. H. Komatsu remarks in [Ko]: " It has always been controversial whether or not Operational Calculus is a mathematics. As far as we know every textbook on Operational Calculus starts with a discussion on this issue. When the Mathematical Society of Japan revised its Encyclopedic Dictionary of Mathematics about ten years ago, the chief editor Professor K. Itô asked opinions of foreign scholars about the then second edition. In his reply a french mathematician wrote 'Operational Calculus has no value of being mentioned; it is a bad succedaneum of distributions and is very far from being useful.' Incidentally Professor Yosida was the author of that item. He liked Operational Calculus very much on the contrary. [...] His fondness for Operational Calculus comes probably from his belief that a good mathematics must be not only beautiful but also useful."

integro-differential equations, some nonlinear integral equations, difference equations, differential-difference equations, delay equations, partial differential equations, and naturally to linear ordinary differential equations, in particular to Laplace's equation

$$(a_2 t + b_2)y''(t) + (a_1(t) + b_1)y'(t) + (a_0(t) + b_0)y(t) = g(t). \qquad (LE)$$

The above shown equation contains as special cases the Bessel, Airy, Laguerre, Hermite, Euler, and the confluent hypergeometric differential equation. See, for example, the books *"Operational Calculus"* by J. Mikusinski ([Mi2]), *"Operational Calculus: A Theory of Hyperfunctions"* by K. Yosida ([Yo]), or L. Berg's *"Einführung in die Operatorenrechnung"* [Ber].

Of course, Mikusinski's method cannot be extended to functions with values in an arbitrary Banach space since the convolution between two vector-valued functions is in general no longer defined. However, by an extension of Titchmarsh's theorem to vector-valued functions (see Theorem II.3.12), the space $C([0, \infty); X)$ forms a torsion free[(2)] module over the convolution ring of scalar functions. We can extend this module to a vector space,[(3)] considering the set of ordered pairs

$$\mathcal{M} := \{\frac{k \star f}{m} : k, m \in C[0, \infty), m \neq 0, f \in C([0, \infty); X)\},$$

and set $\frac{k_1 \star f_1}{m_1} = \frac{k_2 \star f_2}{m_2}$ if and only if $m_2 \star k_1 \star f_1 = m_1 \star k_2 \star f_2$. If one defines

(2) Torsion free means that $k \star f = 0$ implies that f or k is 0.

(3) For an introduction to module theory, see for example W. A. Adkins and S. H. Weintraub [Ad-We].

addition of two elements via

$$\frac{k_1 \star f_1}{m_1} + \frac{k_2 \star f_2}{m_2} = \frac{1}{m_1 \star m_2} \left(m_2 \star k_1 \star f_1 + m_1 \star k_2 \star f_2 \right),$$

then $\mathcal{M}$ becomes a vector space over Mikusinski's quotient field, a vector space of generalized vector-valued functions.

The abstract nature of the convolution quotients presents a problem in the application of the purely algebraic approach. C. Foias provided an extremely useful analytic result which helps us to understand the nature of such quotients. He showed in 1961 ([Fo]) that Mikusinski's convolution quotients can be approximated by continuous functions; i.e., for k, m in the Frechét space $C[0, \infty)$ with $k(0) = 0$ and $0 \in supp(m)$ (i.e., m is not identically zero on $[0, \epsilon)$ for all $\epsilon > 0$), there exist $h_n \in C[0, \infty)$ such that $h_n \star m \to k$. Thus, the continuous functions h_n "approximate" the convolution quotient $\frac{k}{m}$. Unfortunately, Foias' proof of the density of the image of the convolution transform in $C_0[0, \infty)$ was done by contradiction, as a result it does not lead to a concrete approximating sequence of a given quotient. The same holds true for the proofs of Foias' density theorem by J. Mikusinski [Mi3], W. Kierat and K. Skornik [Ki-Sk], or K. Skornik [Sk]. We will give a constructive proof of Foias' density theorem based on a new inversion formula for the Laplace transform, which holds for the Banach space valued case as well as for the convolution of certain operator families $K(t) \in \mathcal{L}(X)^{(4)}$ with vector-valued functions.

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(4) We denote by $\mathcal{L}(X, Y)$ the Banach space of all bounded linear operators between two Banach spaces X and Y. If $X = Y$, then we write $\mathcal{L}(X)$ in lieu of $\mathcal{L}(X, X)$. Throughout this dissertation, X will always stand for a Banach space.

Our approach to a mathematical foundation of Heaviside's operational calculus is not algebraic but entirely analytic in nature. Motivated by Foias' theorem we view generalized functions primarily as limits of continuous functions in appropriate topologies, rather than defining them as linear functionals on a space of test functions or as convolution quotients. For example, we define Banach spaces of vector-valued generalized functions on the interval $[0,1]$ to be the completion of $C([0,1];X)$ equipped with a norm $|||\cdot|||$ which is weaker than the supremums norm $\|\cdot\|_\infty$; i.e., $|||f||| \leq \|f\|_\infty$ for all $f \in C([0,1];X)$. Generalized function spaces are of particular interest in applications to differential equations, where the weaker norm is defined by a convolution operator; i.e.,

$$|||f||| := \|K \star f\|_\infty = \sup_{t\in[0,1]} \|\int_0^t K(t-s)f(s)\,ds\|,$$

where K is a strongly continuous operator family in $\mathcal{L}(X,Y)$. In order for $|||\cdot|||$ to be a norm, the convolution operator $f \to K \star f$ has to be injective on $C([0,1];X)$. We give a proof of Titchmarsh's injectivity theorem of the convolution transform that holds for Banach space valued functions $f \in C([0,1];X)$, as well as for certain operator families $K(t) \in \mathcal{L}(X,Y)$; in particular, it holds for numerical continuous functions $k(t)$ with $0 \in supp(k)$ and strongly continuous semigroups $K(t)$.

Besides generalized functions (defined as limits of continuous functions), a second major aspect of our approach to Heaviside's operational calculus is an extension of the classical Laplace transform theory, developed by G. Doetsch [Do1-3] and D. V. Widder [Wi1-2], to asymptotic Laplace transforms. Based on Poincare's method of asymptotic power series (see, for example, R. Remmert [Re], p.294), asymptotic Laplace transforms extend the classical Laplace transform to include functions of arbitrary growth at infinity, while maintaining all essential operational properties of the classical Laplace transform. The Argentinean mathematicians J. C. Vignaux and M. Cotlar ([Vi],[Vi-Co]) first considered asymptotic Laplace transforms in 1939 and 1944. Further contributions were made by W. A. Ditkin (1958) [Di], L. Berg (1962) [Ber], Y. I. Lyubich (1966) [Ly], and M. Deakin (1993) [De]. We follow and extend the approach taken by G. Lumer and F. Neubrander in [Lu-Ne]. Asymptotic Laplace transforms are certain equivalence classes of analytic or meromorphic functions, and as such, they can be multiplied and divided in the obvious manner. The scalar valued asymptotic Laplace transforms form a field, and the vector-valued ones form a vector space over that field. Combining generalized functions and asymptotic Laplace transforms, one can formulate an "operational calculus" for vector-valued functions. Since convolution is transformed by the asymptotic Laplace transform into multiplication, Mikusinski's quotient field has a one-to-one correspondence to the multiplicative field of asymptotic Laplace transforms. The same holds for the vector space of vector-valued functions over the convolution field of scalar functions and the vector space of vector-valued asymptotic Laplace transforms.

We will demonstrate the use of an operational calculus based on asymptotic Laplace transforms by applying it to Laplace's differential equation (LE) as well as to abstract linear initial value problems

$$u'(t) = Au(t), \quad u(0) = x, \tag{ACP}$$

where A is a linear operator with domain and range in some Banach space X, where the graph is not necessarily closed in $X \times X$. In particular, we introduce and study the notion of a generalized limit solution of the so-called "Abstract Cauchy Problem" (ACP). The notion of a limit solution resembles the "bonnes solutions" introduced by Ph. Bénilan (see [Ben] or [Ben-Cr-Pa]). They turn out to be helpful in explaining the structure of integrated, k-generalized, C-regularized, distributional, ultradistributional and hyperfunction solutions of (ACP).

Furthermore, we discuss existence, uniqueness and regularity properties of the solutions of (ACP) in terms of the characteristic equation

$$(\lambda I - A)y(\lambda) = x + r(\lambda), \tag{CE}$$

where r is a remainder term of exponential decay. The existence, uniqueness and regularity of a solution $u = u(t)$ of (ACP) depends on the existence of local asymptotic resolvents $y(\lambda)$ solving (CE), their regions of analyticity and the growth therein. This gives us a fine gauge to study (ACP).

This dissertation grew out of the following recent papers and research work by the author, G. Lumer, and F. Neubrander.

- B. Bäumer and F. Neubrander: Laplace transform methods for evolution equations. *Conference del Seminario di Matematica dell' Universitá di Bari* **259** (1994), 27–60.
- B. Bäumer and F. Neubrander: Existence and uniqueness of solutions of ordinary linear differential equations in Banach spaces. Submitted.
- G. Lumer and F. Neubrander: Asymptotic Laplace transforms. Preprint.

It is a great pleasure to acknowledge the influence of Prof. G. Lumer's recent research work on the convolution transform, extending Titchmarsh's injectivity theorem and Foias' dense range theorem to vector-valued functions in several variables. Although I know from his results only by hearsay, the knowledge about the existence of such results was a great help and stimulation to find proofs of my own. The proofs of Titchmarsh's and Foias' theorem given here were developed independently, with methods disjoint from those used by Professor G. Lumer. We believe that the Laplace transform methods developed (see Section II.3) for the proofs are of independent interest and can be considered as one of the main results of this work.

II. Generalized Functions

II.1 Basic Concepts and Examples

> "It has been a standard tactic of the analyst, since the dawn of analysis, that, when forced to deal with a 'bad' function, he should try to approximate it with 'nice' ones, study the latter and prove that some of the properties in which he happens to be interested, if valid for the approximating nice functions, would carry over to their limit. Of course, the concept of a 'bad' function has evolved in time, with the resulting effect that the set of functions considered 'good' has steadily increased (but so has the set of functions, or , more generally, of 'function-like' objects, considered 'bad'). We might imagine that Taylor and Mac Laurin felt ill at ease when confronted with analytic functions, and that is why they strove to approximate them by polynomials, whereas for our purposes here, from the local point of view, analytic functions will be regarded as the nicest type of functions (right after polynomials, which retain their supremacy); later on, nondifferentiable continuous functions and the functions which are only measurable would be regarded as bad (they still are), and approximation techniques were devised to deal with them (e.g., approximation by step functions). As we shall see in Part II, functions can become so bad as to stop being functions: they become Dirac's 'function' and measures, and in distribution theory we shall be dealing with derivatives of arbitrary order of measures. In any one of these situations, it will help to have at our disposal approximation techniques, so as to approximate those objects by very smooth functions."
>
> F. Treves [Tr], p.150.

Usually generalized functions are defined as functionals on a space of test functions. Starting from this definition, one tries then to shed light on their local structure, for example, by characterizing them (locally) as (pseudo)-derivatives of continuous functions. In other words, the local structure theorems for generalized functions give regularising functions k such that $k \star \phi$ is a continuous function f;[(1)] or, at

[(1)] For example, for any distribution $\phi \in \mathcal{D}'(\Omega)$, $\Omega \subset \mathbb{R}$ open, and any relatively compact subset $W \subset \Omega$ with $\overline{W} \subset \Omega$, there exists $n \in \mathbb{N}$ such that $\phi \star \frac{t^n}{n!} \in C(W)$. See, for example, [Bar], p.70.

least formally, that $\phi = \frac{1}{\hat{k}(D)} f$, where $\hat{k}(\lambda) := \int_0^\infty e^{-\lambda t} k(t)\, dt$ and D denotes the first derivative operator.

In this section we will explore an alternative approach to *generalized functions*, taking the local structure theorems as definitions. The following guidelines (see also W. Rudin's comments in [Ru], p.149) will give us some directions for the definition and construction of particular classes of generalized functions.

(a) Every continuous[(2)] function should be a generalized function.

(b) Every generalized function should have derivatives which are again generalized functions. For differentiable functions, the new notion of derivative should coincide with the old one.

(c) The usual formal rules of calculus should hold.

(d) There should be a supply of convergence theorems that is adequate for handling the usual limit processes.

Our approach to generalized functions is related to a classical approach, suggested by S. Bochner [Bo] and J. Mikusinski and R. Sikorski ([Mi-Si]), regarding generalized functions as an equivalence class of approximating functions. As mentioned above, this approach is different to the predominant approach via functionals originated by L. Schwartz [Schw] and extended by I. M. Gelfand and G. E. Shilov [Ge-Sh], or the approach taken by J. Mikusinski [Mi], G. Temple [Te], and M. J. Lighthill [Li]. Comparing the three approaches, G. Temple ([Te],

(2) This requirement is rather arbitrary. It would be perfectly justifiable to require (as in the above quote of F. Treves) that C^∞-functions, analytic functions or just polynomials are seeds for generalized functions. However, in many of our considerations it will not matter if we start with polynomials or continuous functions since the resulting generalized function spaces will coincide.

p.180) remarks: " [...] if any reasonable meaning can be attached to the concept of the derivatives $D^p f$ of a continuous function f, then there are at least three different constructive definitions of that concept, namely,

- (Schwartz) $D^p f$ is the continuous linear functional T over the space $(\mathcal{D})$ of test functions ϕ, specified by $T(\phi) = (-1)^p (f D^p \phi)$;
- (Mikusinski) $D^p f$ is the class of equivalent regular approximations g_n such that $g_n(x)$ is indefinitely differentiable and $(g_n \phi) \to (-1)^p (f D^p \phi)$;
- (Bochner) $D^p f$ is the class of equivalent regular approximations g_n such that $g_n(x)$ is indefinitely differentiable, $g_n = D^p f_n$ and $f_n \to f$ uniformly as $n \to \infty$.

[...] Each of these three representations has its advantages and disadvantages. The third definition (*Bochner*) is the simplest since it does not require the use of the test functions ϕ, but it suffers from an aesthetic disadvantage when the number of independent variables is greater than one [...]."

G. Temple clarified and simplified J. Mikusinski's approach. This was documented by M. J. Lighthill [Li], who remarks in the introduction: "Now, Laurant Schwartz in his *Théorie des Distributions* has evolved a rigourous theory of these, while Professor Temple has given a version of the theory (generalized functions) which appears to be more readily intelligible to students."

J. Mikusinski and R. Sikorski also investigated the method suggested by Bochner in their book [Mi-Si]. Their approach, however, as well as the one taken by G. Temple and M. J. Lighthill, had one further drawback: working with defining sequences is not as aesthetic as working with a linear functional, a well defined single mathematical object. For example, E. J. Beltrami in [Bel] was commenting on Mikusinski's approach by claiming that "the approach of Mikusinski is to work with the concretely defined approximating class $\{\phi_i\}$ rather than with the more ideal object Φ. It is somewhat like considering irrational numbers only by virtue of their approximation by rationals (which, in fact, is what is done in numerical analysis)."

J. Mikusinski and R. Sikorski noted themselves in their preface of [Mi-Si]: "[...] Notre but est de présenter la théorie des distributions d'une manière simple, accessible également aux physiciens et aux ingénieurs. Afin d'atteindre ce but, nous avons abandonné les méthodes d'analyse fonctionelle et nous avons utilisé le fait que les distributions se laissent approximer par des functions."

The novelty in our approach is that we bring back the "abandoned functional analytic" aspect by regarding sequences that converge in the above sense of Bochner as elements of a completion of a function space under a new topology. The "unaesthetic" defining sequences vanish in the background of this approach and we can readily consider properties of generalized functions without worrying about the nature of the equivalence class of defining sequences.

In order to define a generalized function space, we adhere to condition (a). Thus, if X is a Banach space and $\Omega \subset \mathbb{R}^n$, we say that any completion[(3)] of the space of continuous functions $f : \Omega \to X$ equipped with a weaker topology

(3) The completion of a topological vector space can be obtained by taking the quotient space of all Cauchy sequences modulo sequences converging to zero. See, for example, H. H. Schaefer [Scha]

than the one of uniform convergence on compact subsets of Ω is called a space of *generalized functions* from Ω into X.

To obtain not just a complete topological space, but – more conveniently – a Banach spaces of generalized functions the following construction is useful. Let X be normed vector space, let Y be a Banach space and $\nu : X \to Y$ be a sublinear operator; i.e., ν is continuous, $\nu(x) = 0$ if and only if $x = 0$, $\|\nu(\lambda x)\| = \|\lambda\nu(x)\|$, and $\|\nu(x+z)\| \leq \|\nu(x)\| + \|\nu(z)\|$ for all $x, y, z \in X$ and $\lambda \in \mathbb{C}$. A weaker norm on X is then given by

$$|||x||| := \|\nu(x)\|.$$

We call the completion of X equipped with the new norm the ν-extension of X, denoted by $\overline{X}^{\nu}$.[4]

Example 1.1 (L^p-spaces). Consider $X := C[0,1]$, the space of continuous functions on $[0,1]$ equipped with the supremums norm. For $1 \leq p < \infty$ define

$$\nu_p f : t \mapsto \left(\int_0^t |f(s)|^p \, ds \right)^{1/p}.$$

Since ν_p is a sublinear operator from X into X,

$$\|f\|_p := \|\nu_p f\|_\infty = \sup_{t\in[0,1]} \left|\int_0^t |f(s)|^p \, ds\right|^{1/p} = \left(\int_0^1 |f(s)|^p \, ds \right)^{1/p}$$

defines a norm on $C[0,1]$ and $\overline{C[0,1]}^{\nu_p}$ is isometrically isomorphic to $L^p[0,1]$.

It is worthwhile to note that one does not need the definition of the Lebesgue integral nor any notion of measurability to define antiderivatives or definite integrals on intervals of $\overline{C[0,1]}^{\nu_p}$-functions. Let f be a continuous function and If be

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(4) Since we will not consider closed sets of incomplete spaces, this abuse of notation should not cause any confusion.

given by $t \mapsto \int_0^t f(s)\,ds$, where the integral is taken in the Riemann sense. Then

$$\|If\|_\infty \leq \|f\|_1 \leq \|f\|_p$$

for all $f \in C[0,1]$. Thus I extends to a continuous linear operator from $\overline{C[0,1]}^{\nu_p}$ into $X = C[0,1]$ (endowed with the supremums norm); i.e., if $f \in \overline{C[0,1]}^{\nu_p}$ and $f_n \in C[0,1]$ with $f_n \to f$ with respect to the p-norm, then the continuous functions

$$g_n : t \mapsto \int_0^t f_n(s)\,ds$$

converge uniformly to the antiderivative $If := g \in C[0,1]$ of f. We can now define the definite integral of $f \in \overline{C[0,1]}^{\nu_p}$ on an interval (a,b) by

$$\int_a^b f(s)\,ds := g(b) - g(a) = \lim g_n(b) - g_n(a) = \lim \int_a^b f_n(s)\,ds,$$

where the latter integral is taken in the Riemann sense. $\diamondsuit$

Clearly, following the above construction, any bounded, injective, linear operator T from $C([0,1];X)$ into a Banach space Y will yield a space of generalized functions by setting

$$\|f\|_T := \|Tf\|$$

and completing $C([0,1];X)$ with respect to the norm $\|\cdot\|_T$. In the context of semigroup theory, G. DaPrato and G. Grisvard [DaP-Gr], R. Nagel [Na1-2], and T. Walther [Wa], have used a similar construction to obtain extrapolation spaces which include, for example, the C^{-n} spaces below as a special case. For further references in this direction, see A. Verrusio [Ve].

For us the most important examples for such operators T are the antiderivative operator $T : f \to \int_0^{(\cdot)} f(s)\,ds$ and, more generally, the convolution operators

$$T_k : f \mapsto k \star f := \int_0^{(\cdot)} k(\cdot - s) f(s)\,ds.$$

In the following example we will give an outline of the main ideas behind our construction of generalized function spaces by studying the antiderivative operator T. For the general convolution case, we refer to the Sections II.2, II.3, and II.4.

Example 1.2 (C_0^{-n}-spaces). We consider next Banach spaces consisting of the n^{th}-derivatives of continuous functions. For simplicity we start with $n = 1$. Let $C[0,1]$ be equipped with the supremums norm $\|\cdot\|_\infty$. Consider the antiderivative operator $Tf : t \mapsto \int_0^t f(s)\,ds$. Then T is in $\mathcal{L}(X)$ and is one-to-one. Define $\|f\|_T := \|Tf\|_\infty$ and consider the completion of $C[0,1]$ under the new norm, in the following denoted as $\overline{C[0,1]}^T$. Then

$$\begin{array}{ccc}
\overline{C[0,1]}^T & \xrightarrow[\text{isom.isom.}]{\tilde{T}} & \overline{Im(T)} = C_0[0,1] \\
\hookuparrow & & \cup \\
C[0,1] & \xrightarrow[\text{injective}]{T} & Im(T) =: C_0^1[0,1].
\end{array}$$

We show first that the operator T can be extended to an isometric isomorphism $\tilde{T}$ between $\overline{C[0,1]}^T$ and $C_0[0,1]$. In order to prove this fact, we start by showing that T can be extended to a bounded linear operator $\tilde{T}$ on $\overline{C[0,1]}^T$. Let $f \in \overline{C[0,1]}^T$. Thus, by definition of a completion, there exists a sequence $(f_n)_{n\in\mathbb{N}} \subset C[0,1]$ such that[5]

$$f =: [f_n] = (f_n)_{n\in\mathbb{N}} + O,$$

where $O = \{(h_n) \subset C[0,1] : Th_n \to 0\}$ is the set of zero-sequences, and $(f_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in the T-norm; i.e., Tf_n is Cauchy in $C[0,1]$. Hence $Tf_n \to v$ for some $v \in C[0,1]$. Let $(g_n)_{n\in\mathbb{N}}$ be another sequence with $f = T - \lim g_n$. With the same argument, $w := \lim Tg_n$ exists and

$$\|v - w\| = \lim \|Tf_n - Tg_n\| = \lim \|f_n - g_n\|_T = \|f - f\|_T = 0.$$

Thus, $\tilde{T}f := \lim Tf_n$ is well defined. Clearly, $\tilde{T}$ is linear and $\|\tilde{T}f\| = \lim \|Tf_n\| = \lim \|f_n\|_T = \|f\|_T$ for all $f \in \overline{C[0,1]}^T$. Thus $\tilde{T}$ is one-to-one and maps isometrically into $C_0[0,1]$.

To show that $\tilde{T}$ is onto, observe that every function $g \in C_0[0,1] = \overline{Im(T)}$ can be approximated by functions $g_n \in C_0^1[0,1] = Im(T)$ and thus $g = \lim g_n = \lim Tg_n'$. The sequence $(g_n')_{n\in\mathbb{N}}$ is Cauchy with respect to the norm $\|\cdot\|_T$ and thus there exists an element $g' \in \overline{C[0,1]}^T$ with

$$\tilde{T}g' = \lim Tg_n' = \lim g_n = g.$$

Hence, the antiderivative operator $\tilde{T}$ is an isometric isomorphism between $\overline{C[0,1]}^T$ and $C_0[0,1]$.

---

(5) The notation $[f_n]$ was introduced by J. Mikusinski and R. Sikorski [Mi-Si] and will be used if we want to stress the approximative nature of the generalized function. Using the natural embedding, we can also say that $f = T - \lim f_n$.

We call g' the *generalized derivative* of g and denote by

$$C_0^{-1}[0,1] := \overline{C[0,1]}^T$$

the space of generalized derivatives of continuous functions in $C_0[0,1]$. Note that the generalized derivative is so far only defined for continuous functions f with $f(0) = 0$.

Since the antiderivative operator is an isomorphism, we also have that every generalized function $f \in C_0^{-1}$ is integrable on intervals; i.e., if $f = [f_n]$ with $f_n \in C[0,1]$, then $\tilde{T}f := g \in C_0[0,1]$, $Tf_n := g_n \in C_0[0,1]$, and

$$\int_a^b f(s)\,ds := g(b) - g(a) = \lim_{n\to\infty} g_n(b) - g_n(a) = \lim_{n\to\infty} \int_a^b f_n(s)\,ds,$$

where the latter integral can be taken in the Riemann sense. It is easy to see that this construction also works for functions with values in a Banach space X. Thus, we can formulate the following version of the Fundamental Theorem of Calculus.

Theorem 1.3 (Fundamental Theorem of Calculus). *Let X be a Banach space and $g \in C_0([0,1];X)$. Then g is differentiable, $g' \in C_0^{-1}([0,1];X)$ is integrable and*

$$\int_a^b g'(s)\,ds = g(b) - g(a)$$

for all $0 \le a \le b \le 1$. Moreover, any $f \in C_0^{-1}([0,1];X)$ is integrable, $g(t) := \int_0^t f(s)\,ds \in C_0([0,1];X)$ and $g' = f$.

Next we want to shed some light on the nature of the generalized derivatives. Referring to our guidelines $(a) - (d)$ we ask the question, whether this notion of a generalized derivative coincides with the classic notion of a derivative. Clearly, the generalized derivative of a differentiable function $f \in C_0^1[0,1]$ is its derivative modulo 0-sequences; i.e., sequences $(f_n)_{n\in\mathbb{N}} \subset C_0[0,1]$ with the property that

$$\sup_{t\in[0,1]} \left|\int_0^t f_n(s)\,ds\right| \to 0.$$

A nontrivial example for such a zero-sequence is given by the functions $f_n : t \mapsto e^{int}$ since $Tf_n(t) = \frac{1}{in}\left(e^{int}-1\right) \to 0$.

Secondly, note that

$$L^p[0,1] \subset C_0^{-1}[0,1]$$

for all $1 \le p \le \infty$, since

$$\|f\|_{-1} = \|Tf\| \le \|f\|_{L^1} \le \|f\|_{L^p} \le \|f\|_\infty.$$

Moreover, consider the above sequence of functions $f_n : t \mapsto e^{int}$. Then f_n does not converge in L^1 since

$$\int_0^1 |f_n(t) - f_{n+1}(t)|\,dt = \int_0^1 |e^{int} - e^{i(n+1)t}|\,dt = \int_0^1 |1-e^{it}|\,dt > 0$$

for all $n \in \mathbb{N}$. Therefore, the space $C_0^{-1}[0,1]$ is strictly larger than $L^1[0,1]$ and thus strictly larger than $L^p[0,1]$ for all $1 \le p \le \infty$. In the following we identify functions in $L^p[0,1]$ with the corresponding generalized functions in $C_0^{-1}[0,1]$.

Further it is important to notice that the space $P[0,1]$, the space of all polynomials on $[0,1]$ (or just the liner span of monomials $t \mapsto t^{\alpha_n}$ satisfying the Müntz condition $\sum_{n=1}^\infty \frac{1}{\alpha_n} = \infty$) is dense[(6)] in $\overline{C[0,1]}^T$. The reason is that $P[0,1]$ is

[(6)] Following E. Hille and R. S. Phillips, [Hi-Ph], we say that a set M is dense in a closed set X_0 if $X_0 = \overline{M \cap X_0}$.

dense in $C_0[0,1]$ – in the supremums norm and thus in the L^1-norm and in the $C_0^{-1}[0,1]$-norm. The space $C_0[0,1]$ itself is dense in $C[0,1]$ in the L^1-norm and thus in the $C_0^{-1}[0,1]$-norm. Hence

$$\overline{P[0,1]}^T = \overline{C_0[0,1]}^T = \overline{C[0,1]}^T = \overline{L^p[0,1]}^T = C_0^{-1}[0,1].$$

We employ these embeddings to show that the notion of a generalized derivative coincides also with derivatives of an absolutely continuous function. Let f be a any absolutely continuous function with $f(0)=0$. Since $C[0,1]$ is dense in $L^1[0,1]$ there exists a sequence $(g_n)_{n\in\mathbb{N}} \subset C[0,1]$ such that $g_n \to f'$ in L^1-norm, and thus also in $\|\cdot\|_T$. Thus $f' = [g_n]$ can be identified with an element in $C_0^{-1}[0,1]$. Furthermore, since $\int_0^t g_n(s)\,ds \to f(t)$ uniformly in t, it is the generalized derivative of f.

Another important class of examples are generalized derivatives of Banach space valued functions. Define $f:[0,1]\to L^\infty[0,1] =: X$ via $t\mapsto \chi_{[0,t]}$. Then f is not almost separably valued and hence $f\notin L^1([0,1];X)$. However, f is Riemann-integrable and $\int_0^t f(s)\,ds = (t-\cdot)\chi_{[0,t]} = \lim_{n\to\infty}\int_0^t f_n(s)\,ds$, where

$$f_n(t) := s \mapsto \begin{cases} 1 & \text{if } 0\le s\le t-\frac{1}{n}, \\ n(t-s) & \text{if } t-1/n \le s\le t, \\ 0 & \text{else,} \end{cases}$$

and the limit is uniform in t. Thus $f: t\mapsto \chi_{[0,t]}$ can be identified with an element in $C_0^{-1}([0,1];X)$.

As another example, for $t\in[0,1]$, define $T(t): L^\infty([0,\infty) \to L^\infty[0,\infty)$ via

$$T(t)h : s \mapsto h(t+s).$$

Then the operator family $(T(t))_{t\in[0,1]}$ is not strongly Bochner integrable since $T(\cdot)h$ is not separately valued for all discontinuous h. But similar to the previous example, $T(\cdot)h$ can be identified with an element in $C_0^{-1}([0,1];X)$, and thus $(T(t))_{t\in[0,1]}$ is strongly "integrable."

Next, we want to comment on item (d) of our guidelines. Clearly, through the approximative nature of the generalized functions we have already an abundance of convergence theorems.[(7)] On top of them, we will show that the embeddings are compact. Let (f_n) be a bounded sequence of continuous functions. Then the sequence (Tf_n) is equicontinuous and thus, by the theorem of Arzela-Ascoli, there exists a subsequence (f_{n_k}), such that Tf_{n_k} converges; i.e., f_{n_k} converges in $C_0^{-1}[0,1]$. Thus every bounded sequence of continuous functions has a convergent subsequence in $C_0^{-1}[0,1]$, which means that the embedding of $C[0,1]$ into $C_0^{-1}[0,1]$ is compact. In Section II.2 we will show that any embedding into a generalized function space, that was obtained via a compact operator with a dense image, is compact.

Next, we want to investigate, whether the usual formal rules of calculus hold (see part (c) of our guidelines). In order to talk about the derivative of a product or a composition we have to define products and compositions of generalized functions first.

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(7) For example, if $C[0,1] \ni f_n \to f$ in $C_0^{-1}[0,1]$, then f_n might not converge pointwise as the example $f_n(t) := e^{int}$ shows, but f_n converges towards f in the mean; i.e., for all $0 \le a < b \le 1$,

$$\frac{1}{b-a}\int_a^b f_n(s)\,ds \to \frac{1}{b-a}\int_a^b f(s)\,ds.$$

Clearly, we would like to define a product on $C_0^{-1}[0,1]$ that coincides with the product of continuous functions. The naive approach would be to multiply the approximating sequences and hope that they converge in the $\|\cdot\|_T$, but that does not work even for the embedded continuous functions in $C_0^{-1}[0,1]$. For example, let $f_n : t \mapsto \sin(2^n \pi t)$. Then $f_n \to 0$ in $C_0^{-1}[0,1]$. But with the above definition we would have $0 \cdot 0 = [f_n] \cdot [f_n] = [f_n^2] \neq 0$, since

$$\|[f_n^2]\|_T \geq \int_0^1 \sin^2(2^n \pi s)\, ds = \frac{1}{2^n \pi} \int_0^{2^n \pi} \sin^2 s\, ds = \frac{2^n}{2^n \pi} \int_0^{\pi} \sin^2 s\, ds \not\to 0.$$

Another hint that defining a meaningful product on all of $C_0^{-1}[0,1]$ might be difficult or impossible, is the fact that even on $L^1[0,1]$ one cannot define a product that stays in $L^1[0,1]$.

The next attempt, mimicking the situation in L^1, is that we try to define the product between a continuous function and a generalized function; i.e., for $f := [f_n] \in C_0^{-1}[0,1]$ and $g \in C[0,1]$ we would like to define

$$fg := [f_n]g := [f_n g] = T - \lim f_n g. \tag{1}$$

Unfortunately, as we will see, the limit does not always exist. In particular, there exists a zero sequence $(f_n)_{n\in\mathbb{N}}$ and a continuous function g such that $0 \cdot g = [f_n] \cdot g$ does not converge, which implies that the product defined as above is not well defined. Let $h : t \mapsto= \sqrt{t} \sin \frac{1}{t}$. Then $h'(t) = \frac{1}{2\sqrt{t}} \sin \frac{1}{t} - \frac{\sqrt{t}}{t^2} \cos \frac{1}{t}$. Let ξ_n, ϵ_n be such that $\frac{1}{(n+1)\pi} < \xi_n < \frac{1}{n\pi}$, $0 < \epsilon_n < \xi_n/2$ and $h'(\xi_n) = h'(\epsilon_n) = 0$. Define

$$f_n : t \mapsto \begin{cases} 0 & \text{for } 0 \leq t \leq \epsilon_n, \\ h'(t) & \text{for } \epsilon_n \leq t \leq \xi_n \\ 0 & \text{for } t \geq \xi_n. \end{cases}$$

Then $f_n \to 0$ in $\|\cdot\|_T$-norm, since

$$\|f_n\|_T \leq \sup_{t\in[\epsilon_n,\xi_n]} |\int_{\epsilon_n}^t h'(s)\,ds| = \sup_{t\in[\epsilon_n,\xi_n]} |\sqrt{t}\sin\frac{1}{t} - \sqrt{\epsilon_n}\sin\frac{1}{\epsilon_n}| \to 0.$$

Let $g : t \mapsto \sqrt{t}\cos\frac{1}{t}$. However, $f_n g$ does not converge since

$$\begin{aligned}\sup_{t\in[0,1]} |\int_0^t f_n(s)g(s)\,ds| \geq & |\int_{\epsilon_n}^{\xi_n} f_n(s)g(s)\,ds| = |\int_{\epsilon_n}^{\xi_n} \frac{1}{2}\sin\frac{1}{s}\cos\frac{1}{s} - \frac{1}{s}\cos^2\frac{1}{s}\,ds| \\ \geq & |\int_{\epsilon_n}^{\xi_n} \frac{1}{s}\cos^2\frac{1}{s}\,ds| - |\int_{\epsilon_n}^{\xi_n} \frac{1}{2}\sin\frac{1}{s}\cos\frac{1}{s}\,ds| \\ = & \int_{1/\xi_n}^{1/\epsilon_n} s\cos^2 s\,ds - |\int_{\epsilon_n}^{\xi_n} \frac{1}{2}\sin\frac{1}{s}\cos\frac{1}{s}\,ds| \to \infty.\end{aligned}$$

However, one can multiply with C^1-functions, taking (1) as definition. Suppose $g \in C^1[0,1]$, $f_n \in C[0,1]$ and $f_n \to f$ in $C_0^{-1}[0,1]$. Let $F_n := Tf_n$ and $F := \tilde{T}f \in C_0[0,1]$. Then, in $C_0[0,1]$,

$$\begin{aligned}T(f_n g) = \int_0^{(\cdot)} f_n(s)g(s)\,ds = F_n g - \int_0^{(\cdot)} F_n(s)g'(s)\,ds \\ \to Fg - \int_0^{(\cdot)} F(s)g'(s)\,ds = Fg - T(Fg').\end{aligned}$$

Thus $f_n g \to fg := \tilde{T}^{-1}(Fg) - Fg'$ in $C_0^{-1}[0,1]$. Since $\tilde{T}^{-1}$ corresponds to differentiation and $f = \tilde{T}^{-1}F$ we have proved the following proposition.

Proposition 1.4. *a) Let $f = [f_n] \in C_0^{-1}[0,1]$ and $g \in C^1[0,1]$. Then $fg := T - \lim f_n g$ exists in $C_0^{-1}[0,1]$.*

b) Let $F \in C_0[0,1]$ and $g \in C^1[0,1]$. Then $(Fg)' = F'g + fg'$.

In a similar fashion, we can define the composition of a C_0^{-1}-function f and a C^2-function g with $g'(t) \neq 0$ for all $t \in [0,1]$. Define

$$f(g) := T - \lim f_n(g).$$

In order to show that the limit exists, we observe that

$$F(g) = \lim F_n(g) = \lim T\left(F_n(g)\right)' = \lim T(f_n(g)g')$$

in $C_0[0,1]$. Thus $\tilde{T}^{-1}\left(F(g)\right) = T - \lim f_n(g)g' \in C_0^{-1}[0,1]$. Since $g'(t) \neq 0$, we have that $\frac{1}{g'} \in C^1[0,1]$, and hence, by the fact that we can multiply with differentiable functions, $\tilde{T}^{-1}\left(F(g)\right)\frac{1}{g'} \in C_0^{-1}[0,1]$ and $T^{-1}\left(F(g)\right)\frac{1}{g'} = T - \lim f_n(g) =: f(g)$. This shows that the following proposition holds.

Proposition 1.5. *a) Let $f = [f_n] \in C_0^{-1}[0,1]$ and $g \in C^2[0,1]$ with $g'(t) \neq 0$ for all $t \in [0,1]$. Then $f(g) := T - \lim f_n(g)$ exists in $C_0^{-1}[0,1]$.*

b) Let $F \in C_0[0,1]$ and $g \in C^2[0,1]$ with $g'(t) \neq 0$ for all $t \in [0,1]$. Then $(F(g))' = F'(g)g'$.

Last, we consider item (b) of our guidelines for constructing generalized function spaces. To acquire the differentiability of every generalized function, we will construct a tower of generalized function spaces by considering powers[8] of T. The derivative of a generalized function will then be a generalized function in a generalized function space of higher order. For the antiderivative operator T, the operators T^2, T^n are also bounded and one-to-one on $C[0,1]$. With the same argument as above we can define $C^{-n}[0,1] := \overline{C[0,1]}^{T^n}$. If we identify $C_0[0,1]$ with its embedding in $C_0^{-1}[0,1]$, the fact that $\tilde{T}$ is an isometry between $C_0^{-1}[0,1]$ and $C_0[0,1])$ implies in particular that $\tilde{T} \in \mathcal{L}(C_0^{-1}[0,1])$ and that $\tilde{T}$ is one-to-one. We will show in Section II.2 that $\overline{C_0^{-1}[0,1]}^{\tilde{T}} = \overline{C[0,1]}^{T^2} = C^{-2}[0,1]$. In particular, we

[8] We give here the discrete version. Fractional powers T^α of T for $\alpha > -1$, which are convolutions with $\frac{t^\alpha}{\Gamma(\alpha+1)}$ are discussed in Section II.4.

can look at $C^{-2}[0,1]$ and, inductively at $C^{-n}[0,1]$, in the following way.

$$
\begin{array}{ccc}
\vdots & & \vdots \\
C^{-n}[0,1] & \xrightarrow[\text{isom. isomorph}]{\tilde{\tilde{\tilde{T}}}} & C^{-n-1}[0,1] \\
\vdots & & \vdots \\
C^{-2}[0,1] & \xrightarrow[\text{isom. isomorph}]{\tilde{\tilde{T}}} & C_0^{-1}[0,1] \\
\rotatebox{90}{$\hookrightarrow$} & & \cup \\
C_0^{-1}[0,1] & \xrightarrow[\text{isom. isomorph}]{\tilde{T}} & C_0[0,1] \\
\rotatebox{90}{$\hookrightarrow$} & & \cup \\
C_0[0,1] & \xrightarrow[\text{isom. isomorph}]{T} & C^1_{0,0}[0,1] \\
\vdots & & \vdots
\end{array}
$$

Here $C^1_{0,0}[0,1] := \{f \in C^1[0,1] : f(0) = f'(0) = 0\}$ with $\|f\|_1 = \|f'\|_\infty$.

Clearly, we can construct similar towers for L^p spaces, completing them with the new norm

$$\|f\|_{-n,p} := \|T^n f\|_p.$$

Since $C_0[0,1] \hookrightarrow L^p[0,1] \hookrightarrow C_0^{-1}[0,1]$, we obtain the embeddings

$$C_0^{-n}[0,1] \hookrightarrow L^{p,-n}[0,1] \hookrightarrow C_0^{-(n+1)}[0,1].$$

It is worthwhile to note that the spaces $C_0^{-n}[0,1]$ have a partial ordering if we set $f \preceq_{-n} g$ if $T^n f \leq T^n g$. They are in fact Banach lattices, and, even more, AM-spaces (see H. H. Schaefer [Scha2] for an introduction to Banach lattices). The positive cones are getting larger if we increase n. For example, $t \mapsto \sin 2\pi t \succ_{-1} 0$ in $C^{-1}[0,1]$, whereas $t \mapsto \cos 2\pi t \not\succ_1 0$, but $t \mapsto \cos 2\pi t \succ_{-2} 0$ in $C^{-2}[0,1]$.

We end the section by giving examples of distributions which are contained in the spaces $C_0^{-n}[0,1]$. First we consider Dirac's δ-function. Since the constant 1-function is in $\overline{C[0,1]}^T = C_0^{-1}[0,1] \subset C^{-2}[0,1]$, and $\tilde{\tilde{T}}$ is an isometric isomorphism between $C^{-2}[0,1]$ and $C_0^{-1}[0,1]$, the generalized derivative $\tilde{\tilde{T}}^{-1}$ of the constant 1-function cannot be zero. We can identify the constant 1-function as the generalized derivative of the function $F(x) = x$. Thus, the generalized derivative of the constant 1-function is the same as the second generalized derivative of F.

Let $(f_n)_{n\in\mathbb{N}}$ be a sequence in $C[0,1]$ such that $T^2 f_n \to F$.(9) Then $(f_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $C^{-2}[0,1]$, and $f_n \to F'' \in C^{-2}[0,1]$.

The generalized function $f := F''$ has the same properties as the functional $\delta_0 - \delta_1$ on $C[0,1]$. To see this, let $g \in C^\infty[0,1]$ with $g'(1) = 0$. Then

$$\begin{aligned}\int_0^1 f(s)g(s)\,ds &:= \lim \int_0^1 f_n(s)g(s)\,ds = \lim \int_0^1 T^2(f_n)(s)g''(s)\,ds \\ &= \int_0^1 s g''(s)\,ds = g(0) - g(1).\end{aligned}$$

Since the functions $g \in C^\infty[0,1]$ with $g'(1) = 0$ are dense in $C[0,1]$ and since the functional L_f with $L_f(g) := \int_0^1 f(s)g(s)\,ds = g(0) - g(1)$ is continuous with respect to the supremums norm on C^∞, we can extend L_f to $C[0,1]$ and $L_f(h) = h(0) - h(1)$ for all $h \in C[0,1]$.

This result is not too surprising. If we identify functions on $[0,1]$ with functions on $(-\infty,\infty)$ by extending the functions with 0, we may identify the constant

(9) For example, let $f_n(x) := \begin{cases} 2n - 2n^2 x & \text{for } 0 \le x \le 1/n \\ 0 & \text{else} \end{cases}$.

1-function with the Heaviside function $H(0) - H(1)$, whose distributional derivative is $\delta_0 - \delta_1$.

In order obtain δ_0 without δ_1 we choose the Frechét space $C[0,1)$. There, the C^∞-functions g with $\lim_{t\to 1} g(t) = \lim_{t\to 1} g'(t) = 0$ are dense, and thus $L_f(h) = h(0)$. Thus, the generalized derivative of the constant 1-function coincides with the Dirac functional on the space $C[0,1)$ and with $\delta_0 - \delta_1$ as functional on $C[0,1]$.

As an other example, we locate the function $t \mapsto \frac{1}{t}$, or, in the language of distributions, the finite part of $\frac{1}{t}$ in the tower of spaces $C_0^{-n}[0,1]$. Since the second derivative of $f : t \mapsto t\ln t - t$ is $\frac{1}{t}$ for $t > 0$, and since $f \in C_0[0,1]$, we check whether the generalized second derivative of f has the distributional properties of the finite part of $\frac{1}{t}H_0$; i.e., whether $\int_0^1 f''(t)g(t)\,dt = \int_0^1 \frac{g(t)-g(0)}{t}\,dt$ for all $g \in C^2[0,1]$ with $supp(g) \in [0,1)$ (See, for example, A. H. Zemanian [Ze], p.18 for a discussion of finite parts).

Let $f'' = [h_n]$ be the second generalized derivative of f, i.e., $h_n \in C[0,1]$ and $T^2 h_n \to f$. Let $g \in C^2[0,1]$ with $supp(g) \in [0,1)$. Then

$$\int_0^1 h_n(t)g(t)\,dt = \int_0^1 T^2(h_n(t))g''(t)\,dt \to \int_0^1 f(t)g''(t)\,dt.$$

However, for all $0 < \epsilon < 1$,

$$\begin{aligned}
\int_0^1 f(t)g''(t)\,dt &= \int_0^\epsilon f(t)g''(t)\,dt + \int_\epsilon^1 f(t)g''(t)\,dt \\
&= \int_0^\epsilon f(t)g''(t)\,dt - f(\epsilon)g'(\epsilon) - \int_\epsilon^1 \ln t g'(t)\,dt \\
&= \int_0^\epsilon f(t)g''(t)\,dt - f(\epsilon)g'(\epsilon) + \ln\epsilon(g(\epsilon) - g(0)) + \int_\epsilon^1 \frac{g(t)-g(0)}{t}\,dt \\
&= \int_0^1 \frac{g(t)-g(0)}{t}\,dt,
\end{aligned}$$

since ϵ was chosen arbitrarily and the first three terms converge to 0 as $\epsilon \to 0$. Thus, the generalized derivative of $t \mapsto \ln t \in L^1[0,1] \subset C_0^{-1}[0,1]$, which is the same as the second generalized derivative of $t \mapsto t\ln t - t \in C_0[0,1]$, corresponds to the finite part distribution $\frac{1}{t}H_0$.

Instead of just looking at the antiderivative operator, we will consider in the following sections the convolution operator T_k with $T_k f := \int_0^{(\cdot)} k(\cdot - s) f(s)\, ds$. E. C. Titchmarsh showed in 1925 ([Ti], Theorem VII) that T_k is one-to-one if $0 \in supp(k)$. C. Foias showed in 1961([Fo]), that under the same condition (i.e., $0 \in supp(k)$), the image of T_k is dense in $L^1[0,1]$. K. Skórnik in [Sk] proved the density of the image in $C_0[0,1]$. Thus the same construction of generalized function spaces works if one takes the convolution operator T_k instead of the antiderivative operator T. Since this is a central point in this dissertation, we will give proofs of vector-valued versions of Titchmarsh's Injectivity Theorem and Foias' Dense Range Theorem in Section II.3. The "towers" of generalized function spaces, defined via T_k, are investigated in Section II.4. They lead to a "continuous" diagram of spaces which allow, if combined with asymptotic Laplace transforms, an operational calculus as powerful as the one developed by J. Mikusinski in [Mi2] or by K. Yosida in [Yo].

II.2 Linear Extensions of Banach Spaces

In this section we collect some basic properties of linear extensions of Banach spaces. A linear extension of a Banach space X is a completion of X with respect to a new topology which has been obtained via a bounded, linear operator $T : X \to Y$

(Y Banach space) that is one-to-one.(1) In other words, the new norm has been obtained by measuring the image of the linear operator with respect to the old norm; i.e.,

$$\|x\|_T := \|Tx\|.$$

The linear extensions, denoted by $\overline{X}^T$, have the following crucial property which explains why they play an important role in many applications.

Theorem 2.1.(2) *Let X, Y be Banach spaces and let $T \in \mathcal{L}(X, Y)$ be one-to-one. On X define a new norm via $\|x\|_T := \|Tx\|$ and let $\overline{X}^T$ denote the completion of the normed vector space $X^T := (X, \|\cdot\|_T)$. Then*

$$\begin{tikzcd} \overline{X}^T \arrow[r, "\tilde{T}", "isom.\ isomorph"'] & \overline{Im(T)} \\ X \arrow[u, hook] \arrow[r, "T"] & Im(T); \arrow[u, phantom, "\cup"] \end{tikzcd}$$

i.e., X is continuously embedded in $\overline{X}^T$, and the operator T extends to an isometric isomorphism $\tilde{T}$ between $\overline{X}^T$ and $\overline{Im(T)}$. In particular, $(X, \|\cdot\|_T)$ is already a Banach space if and only if $Im(T)$ is closed in Y. Moreover, the operator T is compact if and only if every bounded sequence in X has a convergent subsequence in $\overline{X}^T$.

(1) For the sake of clarity and brevity we confine ourselves to Banach spaces. The whole section also holds for spaces equipped with a separating set of semi-norms $\|\cdot\|_\alpha$. All the theorems can be reformulated replacing "Banach space" with "Frechét space" and $\|\cdot\|$ with $\|\cdot\|_\alpha$. Moreover, the operator T may depend on α.

(2) See also R. Nagel [Na] and T. Walther [Wa].

Proof. Clearly, X can be identified with a subset of $\overline{X}^T$ by identifying $x \in X$ with the constant sequence x modulo the set of zero-sequences with respect to $\|\cdot\|_T$. Since

$$\|x\|_T = \|Tx\| \leq \|T\| \, \|x\|$$

for all $x \in X$, the identity is a continuous map from the Banach space $(X, \|\cdot\|)$ into $\overline{X}^T$.

Let $z \in \overline{X}^T$. Then there exist $x_n \in X$ such that $x_n \to z$ in $\overline{X}^T$. This implies that the sequence $(x_n)_{n \in \mathbb{N}}$ is Cauchy in $\overline{X}^T$ or, equivalently, that the sequence $(Tx_n)_{n \in \mathbb{N}}$ is Cauchy in Y. This shows that $y := \lim_{n \to \infty} Tx_n$ exists. Consider another sequence $(v_n)_{n \in \mathbb{N}} \in X$ such that $v_n \to z$ in $\overline{X}^T$. Then, as above, $w := \lim_{n \to \infty} Tv_n$ exists in Y. Now,

$$\|y - w\| = \lim \|Tx_n - Tv_n\| = \lim \|x_n - v_n\|_T = \|z - z\|_T = 0.$$

Thus,

$$\tilde{T}z := \lim_{n \to \infty} Tx_n$$

is a well-defined extension of T mapping $\overline{X}^T$ into $\overline{Im(T)}$. Clearly, $\tilde{T}$ is linear and

$$\|\tilde{T}z\| = \lim \|Tx_n\| = \lim \|x_n\|_T = \|z\|_T$$

for all $z \in \overline{X}^T$. This shows that $\tilde{T} \in \mathcal{L}(\overline{X}^T, \overline{Im(T)})$ maps isometrically and is one-to-one. To show that $\tilde{T}$ is onto, let $y \in \overline{Im(T)}$ and $y_n \in Im(T)$ with $y_n \to y$ in Y. Let $x_n \in X$ with $Tx_n = y_n$. Since $\|y_n\| = \|Tx_n\| = \|x_n\|_T$ it follows that the sequence $(x_n)_{n \in \mathbb{N}}$ is a Cauchy sequence in $\overline{X}^T$. Let $z \in \overline{X}^T$ such that $x_n \to z$

in $\overline{X}^T$. Then

$$\tilde{T}z = \lim Tx_n = \lim y_n = y.$$

Thus $\tilde{T}$ is an isometric isomorphism between $\overline{X}^T$ and $\overline{Im(T)}$.

Suppose $X^T = (X, \|\cdot\|_T)$ is a Banach space. We show that $Im(T)$ is closed. Let $y \in \overline{Im(T)}$. Then there exist $x_n \in X$ such that $\|Tx_n - y\| \to 0$. Since the sequence $(Tx_n)_{n\in\mathbb{N}}$ is Cauchy in Y it follows that the sequence $(x_n)_{n\in\mathbb{N}}$ is Cauchy in X^T. Since X^T is complete, there exists $z \in X$ such that $\|x_n - z\|_T \to 0$, or, equivalently, that $\|Tx_n - Tz\| \to 0$. Thus, $Tz = y$ and therefore $Im(T)$ is closed.

Suppose that $Im(T)$ is closed. Let $(x_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in X^T. Thus $(Tx_n)_{n\in\mathbb{N}}$ is Cauchy and, due to the closedness of $Im(T)$, converges to Tx for some $x \in X$. Hence $x_n \to x$ in X^T and thus every Cauchy sequence in X^T converges. Thus X^T is a Banach space.

The last statement follows from the fact that a map T is compact if and only if for every bounded sequence $(x_n)_{n\in\mathbb{N}}$ in X, the sequence $(Tx_n)_{n\in\mathbb{N}}$ has a convergent subsequence $(Tx_{n_i})_{i\in\mathbb{N}}$; i.e., $(x_{n_i})_{i\in\mathbb{N}}$ is convergent in $\overline{X}^T$. $\diamondsuit$

As explained in the previous section, a typical situation we have in mind is when $X = Y = C[0,1]$, and T is given by the antiderivative operator $T : f \mapsto \int_0^{(\cdot)} f(s)\,ds$. Then T is injective and $\overline{ImT} = C_0[0,1]$. Thus, the derivative operator T^{-1} extends to a bounded linear operator from $C_0[0,1]$ onto $\overline{C[0,1]}^T$. Since we want to do analysis on $\overline{C[0,1]}^T$ we would like to extend typical operations S of analysis (such as integration or convolution) from $C[0,1]$ to $\overline{C[0,1]}^T$. This can be achieved via the mechanism described in the following theorem.

Theorem 2.2. *Let X be a Banach space, and let $T \in \mathcal{L}(X)$ be one-to-one. Let $S \in \mathcal{L}(X)$ with $ST = TS$. Then the following statements hold.*

(a) The operator S has an extension $\tilde{S} \in \mathcal{L}(\overline{X}^T)$ with $\|\tilde{S}\| \leq \|S\|$ and $\tilde{S}\tilde{T} = \tilde{T}\tilde{S}$. Moreover, S is one-to-one if and only if $\tilde{S}$ is one-to-one.

(b) Let S be one-to-one. Then $\overline{X}^{ST}$ is isometrically isomorphic to $\overline{(\overline{X}^T)}^{\tilde{S}}$ $(= \overline{(\overline{X}^S)}^{\tilde{T}})$.

(c) Suppose there exists $V \in \mathcal{L}(X)$ with $VT = TV$, V is one-to-one and $\overline{Im(V)} = \overline{Im(T)}$. Then $\overline{X}^T \subset \overline{X}^V$ if and only if there exists $S \in \mathcal{L}(X^T)$ with S being one-to-one and $ST = TS$ such that $\overline{\overline{X}^T}^S$ is isometrically isomorphic to $\overline{X}^V$. Moreover, S can be chosen to be $\tilde{T}^{-1}V$.

(d) If S is compact then $\tilde{S}$ is compact, and the converse holds if the image of T is dense in X.

Proof. (a) Let $z \in \overline{X}^T$. Then there exist $x_n \in X$ such that $x_n \to z$ in $\overline{X}^T$. Since $\|Sx_n - Sx_m\|_T = \|TSx_n - TSx_m\| = \|STx_n - STx_m\| \leq \|S\|\,\|Tx_n - Tx_m\| = \|S\|\,\|x_n - x_m\|_T$, it follows that there exists $y \in \overline{X}^T$ such that $Sx_n \to y$ in $\overline{X}^T$. Consider another sequence $(v_n)_{n\in\mathbb{N}} \subset X$ such that $v_n \to z$ in $\overline{X}^T$. Then, as above, there exists $w \in \overline{X}^T$ such that $Sv_n \to w$ in $\overline{X}^T$. Now,

$$\begin{aligned}\|y - w\|_T &= \lim \|Sx_n - Sv_n\|_T = \lim \|TSx_n - TSv_n\| \\ &\leq \|S\| \lim \|x_n - v_n\|_T = \|z - z\|_T = 0.\end{aligned}$$

Thus, $\tilde{S}z := T - \lim_{n\to\infty} Sx_n$ is a well-defined extension of S mapping $\overline{X}^T$ into $\overline{X}^T$. Clearly, $\tilde{S}$ is linear and

$$\|\tilde{S}z\|_T = \lim \|Sx_n\|_T \leq \|S\| \lim \|x_n\|_T = \|S\|\,\|z\|_T$$

for all $z \in \overline{X}^T$. This shows that $\tilde{S} \in \mathcal{L}(\overline{X}^T)$ and $\|\tilde{S}\| \leq \|S\|$. The commutativity of $\tilde{S}$ and $\tilde{T}$ follows from

$$\begin{aligned}
\tilde{S}\tilde{T}z &= S\tilde{T}z \text{ (since } \tilde{T}z \in \overline{Im(T)} \subset X) \\
&= S(\lim Tx_n) \text{ (where the limit is taken in } X \text{ and } x_n \to z \text{ in } \overline{X}^T) \\
&= \lim STx_n = \lim TSx_n = \lim \tilde{T}Sx_n \\
&= T - \lim \tilde{T}Sx_n \text{ (if } w = \lim w_n, \text{ then } w = T - \lim w_n) \\
&= \tilde{T}(T - \lim Sx_n) = \tilde{T}\tilde{S}z.
\end{aligned}$$

If $\tilde{S}$ is one-to-one, then trivially, S is one-to-one. Suppose that S is one-to-one. If $0 = \tilde{S}z$, then $Sx_n \to 0$ in $\overline{X}^T$, where $x_n \to z$ in $\overline{X}^T$. Thus $TSx_n = STx_n \to 0$ in X. Since $\lim Tx_n$ exists in X, it follows from the boundedness of S that $S(\lim Tx_n) = \lim STx_n = 0$. Thus $Tx_n \to 0$ in X. This shows that $\|x_n\|_T \to 0$ or, equivalently, $z = 0$. Hence $\tilde{S}$ is one-to-one.

(b) For $x \in \overline{X}^{ST}$ with $x = ST - \lim x_n$ for some $x_n \in X$ define

$$\Phi(x) := \tilde{S} - \lim_{n\to\infty} x_n \text{ in } \overline{X}^T.$$

First we show that $\Phi(x)$ always exists. This follows from

$$\|\tilde{S}x_n - \tilde{S}x_m\|_{\overline{X}^T} = \|TSx_n - TSx_m\|_X = \|STx_n - STx_m\|_X \to 0.$$

Next we show that $\Phi(x)$ is well defined. Suppose $x = ST - \lim x_n = ST - \lim y_n$. Then

$$\lim \|\tilde{S}x_n - \tilde{S}y_n\|_{\overline{X}^T} = \lim \|STx_n - STy_n\|_X = 0.$$

Thus $\tilde{S} - \lim x_n = \tilde{S} - \lim y_n$ in $\overline{X}^T$.

Clearly, Φ is linear. Now we show that Φ is one-to-one. Suppose $\Phi(x) = 0$. Then $\Phi(x) = \tilde{S} - \lim x_n = 0$ in $\overline{X}^T$. implies that

$$\|STx_n\|_X = \|TSx_n\|_X = \|\tilde{S}x_n\|_{\overline{X}^T} \to 0.$$

Thus $x = ST - \lim x_n = 0$.

The fact that Φ is isometric follows from

$$\|x\|_{\overline{X}^{ST}} = \lim \|STx_n\|_X = \lim \|TSx_n\|_X = \lim \|\tilde{S}x_n\|_{\overline{X}^T} = \|\Phi(x)\|_{\overline{(\overline{X}^T)}^{\tilde{S}}}.$$

It remains to be shown that Φ is onto. Let $z \in \overline{\overline{X}^T}^{\tilde{S}}$. Then $z = \tilde{S} - \lim y_n$ for some $y_n \in \overline{X}^T$. Thus $\|\tilde{S}y_n - \tilde{S}y_m\|_{\overline{X}^T} \to 0$ and therefore $\|\tilde{T}\tilde{S}y_n - \tilde{T}\tilde{S}y_m\| \to 0$. For each y_n choose a sequence $(x_{n,k})_{k\in\mathbb{N}} \subset X$, such that $y_n = T - \lim_{k\to\infty} x_{n,k}$ and $\|\tilde{T}y_n - Tx_{n,k}\| \leq \frac{1}{k}$ for all n, k. Thus $\|\tilde{T}y_n - Tx_{n,n}\| \leq \frac{1}{n}$ and thus $\|\tilde{T}y_n - Tx_{n,n}\| \to 0$. Combining this with the fact that $\|\tilde{T}\tilde{S}y_n - \tilde{T}\tilde{S}y_m\| \to 0$ yields

$$\begin{aligned}
&\|STx_{n,n} - STx_{m,m}\| \\
&\qquad \leq \|STx_{n,n} - S\tilde{T}y_n\| + \|S\tilde{T}y_n - S\tilde{T}y_m\| + \|S\tilde{T}y_m - STx_{m,m}\| \\
&\qquad \leq \|S\|\,\|Tx_{n,n} - \tilde{T}y_n\| + \|\tilde{T}\tilde{S}y_n - \tilde{T}\tilde{S}y_m\| + \|S\|\,\|Tx_{m,m} - \tilde{T}y_m\| \to 0.
\end{aligned}$$

Let $x = ST - \lim x_{n,n}$. Then $\Phi(x) = \tilde{S} - \lim x_{n,n}$ in $\overline{X}^T$. Since

$$\|\tilde{S}x_{n,n} - \tilde{S}y_n\|_{\overline{X}^T} = \|\tilde{T}\tilde{S}x_{n,n} - \tilde{T}\tilde{S}y_n\| \leq \|\tilde{S}\|\,\|Tx_{n,n} - \tilde{T}y_n\| \to 0$$

we obtain that $z = \Phi(x)$. Hence Φ is onto.

(c) Suppose $\overline{X}^T \subset \overline{X}^V$. Let $S := \tilde{T}^{-1}V$. Then clearly $S \in \mathcal{L}(\overline{X}^T)$, S is one-to-one and $\tilde{T}S = S\tilde{T} = V$. Since $\|x\|_{\overline{\overline{X}^T}^S} = \|\tilde{T}Sx\| = \|Vx\| = \|x\|_V$ for all

$x \in \overline{X}^T$ we obtain that

$$\overline{\overline{X}^T}^S = \overline{(\overline{X}^T, \|.\|_{\tilde{T}S})} = \overline{(\overline{X}^T, \|.\|_V)} = \overline{X}^V.$$

Suppose $\overline{X}^V = \overline{\overline{X}^T}^S$. Then obviously $\overline{X}^T \subset \overline{X}^V$.

(d) By definition, $\tilde{S}$ is compact if and only if for every $\overline{X}^T$-bounded sequence $(z_n)_{n\in\mathbb{N}}$ in $\overline{X}^T$ there exists a subsequence $(z_{n_i})_{i\in\mathbb{N}}$ such that $\tilde{S}z_{n_i}$ converges in $\overline{X}^T$ or, equivalently, $\tilde{T}\tilde{S}z_{n_i}$ converges in X.

Assume S is compact. Let $(z_n)_{n\in\mathbb{N}}$ be a $\overline{X}^T$-bounded sequence; i.e., $\|\tilde{T}z_n\|$ is bounded. Therefore $\tilde{T}\tilde{S}z_n = \tilde{S}\tilde{T}z_n = S\tilde{T}z_n$. Since S is compact and $\tilde{T}z_n$ is bounded, there exists a subsequence $(z_{n_i})_{i\in\mathbb{N}}$ such that $S\tilde{T}z_{n_i} = \tilde{T}\tilde{S}z_{n_i}$ converges. Thus $\tilde{S}$ is compact.

Suppose $\tilde{S}$ is compact and the image of T is dense in X. Let $(x_n)_{n\in\mathbb{N}}$ be a bounded sequence in X. Then there exists a sequence $(y_n)_{n\in\mathbb{N}} \subset X$ such that $\|Ty_n - x_n\| \leq \frac{1}{n}$. Hence y_n is $\overline{X}^T$-bounded and therefore $\tilde{S}y_n = Sy_n$ has a $\overline{X}^T$ convergent subsequence, i.e., there exists a subsequence $(y_{n_i})_{i\in\mathbb{N}}$, such that $\tilde{T}Sy_{n_i} = STy_{n_i}$ converges. Thus $(Sx_{n_i})_{i\in\mathbb{N}}$ is convergent, and therefore S is compact. $\diamondsuit$

In case that $S = T$, the last theorem leads to the following observation.

Corollary 2.3. *Let X be a Banach space. Let $T \in \mathcal{L}(X)$ be one-to-one, and let $X_0 := \overline{Im(T)}$. The isomorphism $\tilde{T} : \overline{X}^T \to X_0$ also defines a bounded, linear operator from $\overline{X}^T$ into $\overline{X}^T$. As such, it is one-to-one and $\|\tilde{T}\|_{\mathcal{L}(\overline{X}^T)} \leq \|T\|_{\mathcal{L}(X)}$.*

Proof. The statement follows immediately from the fact that $X_0 \hookrightarrow \overline{X}^T$. $\diamondsuit$

Using $\tilde{T}$ of the above corollary, we can extend $\overline{X}^{T}$ to $\overline{\overline{X}^{T}}^{\tilde{T}}$. Iterating this procedure leads to the following tower, where, by Theorem 2.2 (b), $\overline{\overline{X}^{T}}^{\tilde{T}} = \overline{X}^{T^2}$.

Corollary 2.4 (Tower of linear extensions of X). *Let X be a Banach space, and let $T \in \mathcal{L}(X)$ be one-to-one. Then*

$$
\begin{array}{ccc}
\vdots & & \vdots \\
\hookuparrow & & \cup \\
\overline{X}^{T^2} = \overline{\overline{X}^{T}}^{\tilde{T}} & \xrightarrow[\textit{isometric isomorphism}]{\tilde{\tilde{T}}} & \overline{Im(\tilde{T})}^{\tilde{T}} \hat{=} Im(\tilde{\tilde{T}}) \\
\hookuparrow & & \cup \\
\overline{X}^{T} & \xrightarrow[\textit{isometric isomorphism}]{\tilde{T}} & \overline{Im(T)} \hat{=} Im(\tilde{T}) \\
\hookuparrow & & \cup \\
X & \xrightarrow[\textit{isometric isomorphism}]{T} & (Im(T), \|\cdot\|_{T^{-1}}).
\end{array}
$$

Suppose that the image of T is dense in X; i.e., $\overline{Im(T)} = X$. Since $\overline{Im(\tilde{T})}^{\tilde{T}} = \overline{(\overline{Im(T)})}^{\tilde{T}} = \overline{X}^{T}$, the image of $\tilde{T}$ is dense in $\overline{X}^{T}$. Denoting $X_n := \overline{X}^{T^n}$ for $n \geq 0$ and $X_{-n} := Im(T^n)$ equipped with $\|f\|_{-n} := \|T^{-n}f\|$, we obtain the following Sobolev-tower (See also [DaP-Gr], [Na1-2], [Wa]). We identify T with its natural extension or restriction on X_n.[(3)]

[(3)] In the case that T is the antiderivative operator discussed in Example 1.2, we would take $X = C_0[0,1]$. Since $\overline{C_0[0,1]}^{T} = \overline{C[0,1]}^{T}$, the following corollary coincides with the tower-diagram following Proposition 1.5.

Corollary 2.5 (Abstract Sobolev tower). *Let X be a Banach space and let $T \in \mathcal{L}(X)$ be one-to-one with $\overline{Im(T)} = X$. Then*

$$
\begin{tikzcd}
\vdots & \vdots \\
X_n \arrow[u, hook] \arrow[r, "T", "\textit{isometric isomorphism}"'] & X_{n-1} \arrow[u, phantom, "\cup"] \\
\vdots & \vdots \\
X_1 \arrow[u, hook] \arrow[r, "T", "\textit{isometric isomorphism}"'] & X_0 \arrow[u, phantom, "\cup"] \\
X_0 \arrow[u, hook] \arrow[r, "T", "\textit{isometric isomorphism}"'] & X_{-1} \arrow[u, phantom, "\cup"] \\
X_{-1} \arrow[u, hook] \arrow[r, "T", "\textit{isometric isomorphism}"'] & X_{-2}, \arrow[u, phantom, "\cup"] \\
\vdots & \vdots
\end{tikzcd}
$$

In particular, T^j is an isometric isomorphism between X_i and X_{i+j} for all $i, j \in \mathbb{Z}$. If T is compact, so are the embeddings.

In Section II.4 we will apply the results of this section to convolution operators $T_k : f \to k \star f$ acting on generalized function spaces $X_n = C_0^{-n}([0,a]; X)$. Before doing so, we will study in the following section some properties of the convolution operator on spaces $C([0,a]; X)$.

II.3 The Convolution on C([0,a];X)

The convolution transform $T_K : f \mapsto K \star f$, where $(K \star f)(t) = \int_0^t K(t-s) f(s)\, ds$, holds a central place in modern analysis.[(1)] In 1925 E. C. Titchmarsh ([Ti], Theorem VII) showed that the convolution T_k is one-to-one on $L^1[0,a]$ as long as $0 \in supp(k)$ and $k \in L^1[0,a]$. Clearly this implies the injectivity of T_k as an operator on $C[0,1]$. C. Foias showed in 1961 ([Fo]) that with the same condition, the image of the convolution transform is dense in $L^1[0,a]$; K. Skornik [Sk] extended the result to $C_0[0,1]$. Thus, the convolution operator T_k satisfies the conditions of the previous section; i.e., T_k is one-to-one and the range of T_k is dense in $C_0[0,1]$. Therefore, we have the following diagram.

$$
\begin{tikzcd}
\overline{C_0[0,1]}^{T_k} \arrow[r, "\tilde{T}_k", "\text{isom. isomorph}"'] & C_0[0,1] = \overline{Im(T_k)} \\
C[0,1] \arrow[u, hook] \arrow[r, "T_k"] & Im(T_k), \arrow[u, phantom, "\cup"]
\end{tikzcd}
$$

where the extension $\tilde{T}_k$ of the convolution operator is an isometric isomorphism between the generalized function space $\overline{C_0[0,1]}^{T_k}$ and $C_0[0,1]$.

---

(1) W. Kecs noted in the preface of his book "The Convolution Product" ([Ke]) that " The extension of the convolution product in the distribution space created a natural framework for the extension and enrichment of its properties, and it is due to this fact that the convolution product has become a powerful mathematical tool in symbolic calculus, distribution approximation, Fourier series, and the solution of boundary-value problems. The high effectiveness of the convolution product is especially reflected in its properties with respect to the Fourier and Laplace transforms [...]

We will extend Titchmarsh's injectivity theorem and Foias' dense range theorem to convolution operators $T_K : C([0,a];X) \to C([0,a];Y)$, defined by

$$T_K f : t \mapsto \int_0^t K(t-s)f(s)\,ds,$$

where $K(t) \in \mathcal{L}(X,Y)$ for all $t \in [0,\infty)$, $t \mapsto K(t)$ is strongly continuous on $[0,\infty)$, $0 \in supp(K(\cdot)x)$ for all $x \in X$,[(2)] and satisfies a technical condition. These conditions are automatically satisfied for scalar-valued k and uniformly continuous semigroups $(K(t))_{t\geq 0}$ (see Example 3.11). The technical condition can then be weakened considerably, and we will show that Titchmarsh's injectivity theorem and Foias' dense range theorem also hold for strongly continuous semigroups, compositions of strongly continuous operator families that have the injectivity and dense range properties, as well as the composition of such strongly continuous operator families with bounded injective linear operators (see Theorem 3.15).

Contrary to the proofs of Foias' theorem by C. Foias [Fo], K. Skornik [Sk] and J. Mikusinski [Mi2], our proof is constructive and yields an approximating sequence of continuous functions.

The proofs given here are based on a generalization of the Phragmén-Doetsch inversion formula for the Laplace transform which states that if

$$\hat{f}(\lambda) = \int_0^\infty e^{-\lambda t} f(t)\,dt = \lim_n \to \infty \int_0^n e^- \lambda t f(t)\,dt$$

for some exponentially bounded $f \in L^1_{loc}([0,\infty);X)$, then

$$\int_0^t f(s)\,ds = \lim_{k\to\infty} \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n!} e^{tkn} \hat{f}(kn), \tag{1}$$

(2) If $0 \notin suppK(\cdot)x$; i.e., $K(t)x = 0$ for $0 \leq t \leq \epsilon$ for some $\epsilon > 0$, then for all $f \in C[0,a]$ with $f(t) = 0$ for $0 \leq t \leq a-\epsilon$, $K \star f_x = 0$, where $f_x(t) := f(t)x$

where the limit is uniform for all $t \geq 0$ (see [Ba-Ne1], Thm.1.7). The proof of the following extension of this inversion formula was inspired by J. Mikusinski's proof of a uniqueness theorem for the Laplace transform (see [Mi2], Chapter VII). We obtain Mikusinski's uniqueness result as a corollary to the inversion procedure to be described below (see Theorem 3.6). The extraordinary aspect of this new inversion formula for the Laplace transform is that it does not require taking infinite sums like in the Phragmén-Doetsch inversion (1).

We say that a sequence (β_n) satisfies the Müntz condition (M) if there exists $\delta > 0$, such that for all $n \in \mathbb{N}$,

$$\beta_n > 0, \quad \beta_{n+1} - \beta_n > \delta > 0, \text{ and } \sum_{n=1}^{\infty} \frac{1}{\beta_n} = \infty. \tag{M}$$

Theorem 3.1 (Phragmén-Mikusinski inversion). *Let $f \in L^1([0,T];X)$ for some $T > 0$, and $q(\lambda) := \int_0^T e^{-\lambda t} f(t)\,dt$.*(3) *Furthermore, let $(\beta_n)_{n\in\mathbb{N}}$ be a sequence satisfying the Müntz condition (M) and $N_k \in \mathbb{N}$ be such that $\sum_{j=1}^{N_k} \frac{1}{\beta_{kj}} \geq T$. Define*

$$\alpha_{k,n} := \frac{1}{e} \prod_{\substack{j=1 \\ j\neq n}}^{N_k} \frac{\beta_{kj}}{\beta_{kj} - \beta_{kn}} e^{-\frac{\beta_{kn}}{\beta_{kj}}}.$$

Then $|\alpha_{k,n}| \leq e^{2\beta_{kn}/k\delta}$ and

$$\int_0^t f(s)\,ds = \lim_{k\to\infty} \sum_{n=1}^{N_k} \alpha_{k,n} e^{\beta_{kn} t} q(\beta_{kn}),$$

where the limit is uniform for $t \in [0,T]$.

---

(3) We will extend the result later on to the case that $T = \infty$; i.e., we will consider in Corollary 3.3 exponentially bounded $f \in L^1_{loc}([0,\infty);X)$ and $q(\lambda) = \int_0^\infty e^{-\lambda t} f(t)\,dt$.

Proof. Let N_k be such that $c_k := \sum_{j=1}^{N_k} \frac{1}{\beta_{kj}} \geq T$. Such N_k exists since the fact that $\frac{k}{\beta_{kj}} > \sum_{i=1}^{k} \frac{1}{\beta_{kj+i}}$ implies that

$$k \sum_{j=1}^{\infty} \frac{1}{\beta_{kj}} \geq \sum_{j=1}^{\infty} \sum_{i=1}^{k} \frac{1}{\beta_{kj+i}} = \sum_{j=1}^{\infty} \frac{1}{\beta_j} - \sum_{j=1}^{k} \frac{1}{\beta_j} = \infty.$$

Thus, $\sum_{j=1}^{\infty} \frac{1}{\beta_{kj}} = \infty$.

The proof of this theorem is built on the fact that the sequence of functions $\phi_k : \mathbb{R} \to \mathbb{R}^+$ with

$$\phi_k(t) := \begin{cases} 1 \star \beta_k e^{-\beta_k(\cdot)} \star \beta_{2k} e^{-\beta_{2k}(\cdot)} \star \ldots \star \beta_{N_k k} e^{-\beta_{N_k k}(\cdot)}(t + c_k) & \text{for } t \geq -c_k \\ 0 & \text{else} \end{cases}$$

converges pointwise towards the Heaviside function $H(t) = \chi_{(0,\infty)}$ (for all $t \neq 0$). We will show first that $\phi_k(t) = 1 - \sum_{n=1}^{N_k} \alpha_{k,n} e^{-\beta_{kn} t}$ for $t \geq -c_k$. Consider $\psi_k := 1 \star \beta_k e^{-\beta_k(\cdot)} \star \ldots \star \beta_{N_k k} e^{-\beta_{N_k k}(\cdot)}$. Then

$$\begin{aligned} \int_0^{\infty} e^{-\lambda t} \psi_k(t)\, dt =& \frac{1}{\lambda} \frac{\beta_k}{\lambda + \beta_k} \cdots \frac{\beta_{N_k k}}{\lambda + \beta_{N_k k}} \\ =& \gamma_{k,0} \frac{1}{\lambda} + \gamma_{k,1} \frac{1}{\lambda + \beta_k} + \ldots + \gamma_{k,N_k} \frac{1}{\lambda + \beta_{N_k k}} \end{aligned}$$

for some coefficients $\gamma_{k,n}$. Multiplying by λ and putting $\lambda = 0$, we obtain that $\gamma_{k,0} = 1$. Similarly, if we multiply with $\lambda + \beta_{nk}$ and put $\lambda = -\beta_{nk}$, we obtain that

$$\gamma_{k,n} = - \prod_{\substack{j=1 \\ j \neq n}}^{N_k} \frac{\beta_{jk}}{\beta_{jk} - \beta_{nk}}.$$

Since the inverse Laplace transform of $\frac{1}{\lambda + \beta_{nk}}$ is $e^{-\beta_{nk} t}$, we obtain that $\psi_k(t) = 1 + \sum_{n=1}^{N_k} \gamma_{k,n} e^{-\beta_{nk} t}$ for $t \geq 0$. Therefore

$$\phi_k(t) = \psi_k(t + c_k) = 1 + \sum_{n=1}^{N_k} \gamma_{k,n} e^{-\beta_{nk}(t + c_k)}$$

for all $t \geq -c_k$. Let

$$\alpha_{k,n} := -\gamma_{k,n} e^{-\beta_{nk} c_k} = e^{-\beta_{nk} \sum_{j=1}^{N_k} \frac{1}{\beta_{jk}}} \prod_{\substack{j=1 \\ j \neq n}}^{N_k} \frac{\beta_{jk}}{\beta_{jk} - \beta_{nk}}$$

$$= \frac{1}{e} \prod_{\substack{j=1 \\ j \neq n}}^{N_k} \frac{\beta_{jk}}{\beta_{jk} - \beta_{nk}} e^{-\frac{\beta_{nk}}{\beta_{jk}}}.$$

Then $\phi_k(t) = 1 - \sum_{n=1}^{N_k} \alpha_{k,n} e^{-\beta_{nk} t}$ for all $t \geq -c_k$.

Next we show that $|\alpha_{k,n}| \leq e^{\frac{1+\ln 2}{k\delta} \beta_{kn}}$. We have that

$$\ln |\alpha_{k,n}| = -\beta_{nk} \sum_{j=1}^{N_k} \frac{1}{\beta_{jk}} + \sum_{j=1}^{n-1} \ln \frac{\beta_{jk}}{\beta_{nk} - \beta_{jk}} + \sum_{j=n+1}^{N_k} \ln \frac{\beta_{jk}}{\beta_{jk} - \beta_{nk}} =: S_1 + S_2 + S_3.$$

We first look at S_2. Since $\beta_{jk} < \beta_{nk} - k\delta(n-j)$ for $j < n$, and since the function $t \mapsto \frac{t}{\beta_{nk} - t}$ is increasing on $(0, \beta_{nk})$, we know that $\frac{\beta_{jk}}{\beta_{nk} - \beta_{jk}} < \frac{\beta_{nk} - k\delta(n-j)}{\beta_{nk} - (\beta_{nk} - k\delta(n-j))} = \frac{\beta_{nk} - k\delta(n-j)}{k\delta(n-j)}$ and thus

$$S_2 < \sum_{j=1}^{n-1} \ln \frac{\beta_{nk} - k\delta(n-j)}{k\delta(n-j)} = \sum_{j=1}^{n-1} \ln \frac{\beta_{nk} - k\delta j}{k\delta j}.$$

The fact that the function $t \mapsto \frac{\beta_{nk} - k\delta t}{k\delta t}$ is decreasing for $t > 0$ yields

$$S_2 < \int_0^{n-1} \ln \frac{\beta_{nk} - k\delta t}{k\delta t} dt = \frac{\beta_{nk}}{k\delta} \int_0^{k\delta(n-1)/\beta_{kn}} \ln \left(\frac{1}{t} - 1 \right) dt.$$

Now, $\ln(1/t - 1) > 0$ if $t \in (0, 1/2)$ and $\ln(1/t - 1) < 0$, if $t \in (1/2, 1)$. Thus

$$S_2 < \frac{\beta_{nk}}{k\delta} \int_0^{1/2} \ln \left(\frac{1}{t} - 1 \right) dt = -\frac{\beta_{nk}}{k\delta} \int_0^{1/2} t \frac{t}{1-t} \cdot \frac{(-t) - (1-t)}{t^2} dt$$
$$= \frac{\beta_{nk}}{k\delta} \int_0^{1/2} \frac{1}{1-t} dt = \frac{\beta_{nk} \ln 2}{k\delta}.$$

In a similar fashion we find an estimate for $S_1 + S_3$. The function $t \mapsto -\frac{\beta_{nk}}{t} + \ln \frac{t}{t - \beta_{nk}}$ is positive and decreasing on (β_{nk}, ∞), because its derivative

$$\frac{\beta_{nk}}{t^2} + \frac{t - \beta_{nk}}{t} \cdot \frac{t - \beta_{nk} - t}{(t - \beta_{nk})^2} = \frac{\beta_{nk}}{t^2} - \frac{\beta_{nk}}{t(t - \beta_{nk})}$$

is negative and the limit as $t \to \infty$ is equal to 0. Since $\beta_{nk} + k\delta(j-n) < \beta_{jk}$ we obtain that

$$\begin{aligned}
S_1 + S_3 =& -\sum_{j=1}^{N_k} \frac{\beta_{nk}}{\beta_{jk}} + \sum_{j=n+1}^{N_k} \ln \frac{\beta_{jk}}{\beta_{jk} - \beta_{nk}} < \sum_{j=n+1}^{\infty} -\frac{\beta_{nk}}{\beta_{jk}} + \ln \frac{\beta_{jk}}{\beta_{jk} - \beta_{nk}} \\
<& \sum_{j=n+1}^{\infty} \left(-\frac{\beta_{nk}}{\beta_{nk} + k\delta(j-n)} + \ln \frac{\beta_{nk} + k\delta(j-n)}{k\delta(j-n)} \right) \\
=& \sum_{j=1}^{\infty} \left(-\frac{\beta_{nk}}{\beta_{nk} + k\delta j} + \ln \frac{\beta_{nk} + k\delta j}{k\delta j} \right) \\
<& \int_0^{\infty} \left(-\frac{\beta_{nk}}{\beta_{nk} + k\delta t} + \ln \frac{\beta_{nk} + k\delta t}{k\delta t} \right) dt \\
=& \frac{\beta_{nk}}{k\delta} \int_0^{\infty} \left(-\frac{1}{1+t} + \ln \frac{1+t}{t} \right) dt \\
=& \frac{\beta_{nk}}{k\delta} \left(-\frac{t}{1+t} + t \ln \frac{1+t}{t} \right) \Bigg|_{t=0}^{t=\infty} \\
&- \frac{\beta_{nk}}{k\delta} \int_0^{\infty} t \left(\frac{1}{(1+t)^2} + \frac{t}{1+t} \cdot \frac{t-(1+t)}{t^2} \right) dt \\
=& \frac{\beta_{nk}}{k\delta}(-1+1) + \frac{\beta_{nk}}{k\delta} \int_0^{\infty} \frac{1}{(1+t)^2} dt = \frac{\beta_{nk}}{k\delta}.
\end{aligned}$$

Hence $\ln |\alpha_{k,n}| < \frac{\beta_{nk}(1+\ln 2)}{k\delta}$ and thus

$$|\alpha_{k,n}| < e^{\frac{\beta_{nk}(1+\ln 2)}{k\delta}} < e^{\frac{2\beta_{kn}}{k\delta}}.$$

Next, we show that $\phi_k(t) \to 1$ for all $t > 0$. This can be seen via the following inequality. Let $t > 0$, and let k be such that $\frac{2}{k\delta} < t/2$. Then

$$\begin{aligned}
|\phi_k(t) - 1| \leq& \sum_{n=1}^{N_k} |\alpha_{k,n}| e^{-\beta_{kn} t} \leq \sum_{n=1}^{\infty} e^{\frac{2\beta_{kn}}{k\delta}} e^{-\beta_{kn} t} \\
\leq& \sum_{n=1}^{\infty} e^{-\beta_{kn} t/2} \leq \sum_{n=1}^{\infty} e^{-(kn-1)\delta t/2} \\
=& \frac{e^{-(k-1)\delta t/2}}{1 - e^{-k\delta t/2}},
\end{aligned}$$

since $(1 - e^{-k\delta t/2}) \sum_{n=1}^{\infty} e^{-(kn-1)\delta t/2} = e^{-(k-1)\delta t/2}$. Thus $\phi_k(t) \to 1$ as $k \to \infty$, uniformly for $t > \epsilon > 0$.

Notice from the definition of ϕ_k, that ϕ_k as convolution of positive functions is positive. Since

$$\phi_k'(t) := \begin{cases} \beta_k e^{-\beta_k(\cdot)} \star \beta_{2k} e^{-\beta_{2k}(\cdot)} \star \ldots \star \beta_{N_k k} e^{-\beta_{N_k k}(\cdot)}(t + c_k) & \text{for } t \geq -c_k \\ 0 & \text{else} \end{cases}$$

is also positive, we know that ϕ_k is monotonically increasing. Thus

$$\int_0^\infty e^{-t}\phi_k(t)\,dt \to 1.$$

Next, we show that $\int_{-\infty}^\infty e^{-t}\phi_k(t)\,dt \to 1$, which implies – again by the positivity and monotonicity of ϕ_k – that $\phi_k(t) \to 0$ for all $t < 0$ and thus uniformly for all $t < -\epsilon < 0$.

Let k be such that $\beta_{kn} \geq (kn-1)\delta > 1$. Since

$$\frac{\beta_{kn}+1}{\beta_{kn}} = 1 + \frac{1}{\beta_{kn}} < e^{1/\beta_{kn}} < \frac{1}{1 - 1/\beta_{kn}} = \frac{\beta_{kn}}{\beta_{kn}-1},$$

we know that

$$1 < \frac{\beta_{kn}}{\beta_{kn}+1} e^{1/\beta_{kn}} < \frac{\beta_{kn}^2}{\beta_{kn}^2 - 1} < 1 + \frac{1}{(kn-1)^2\delta^2 - 1} < e^{1/((kn-1)^2\delta^2-1)}.$$

Since by the definition of ϕ_k,

$$\begin{aligned} \int_{-\infty}^\infty e^{-t}\phi_k(t)\,dt = \int_{-c_k}^\infty e^{-t}\phi_k(t)\,dt &= e^{c_k}\int_0^\infty e^{-t}\phi_k(t - c_k)\,dt \\ = e^{\sum_{n=1}^{N_k} 1/\beta_{kn}} \prod_{n=1}^{N_k} \frac{\beta_{kn}}{1+\beta_{kn}} &= \prod_{n=1}^{N_k} \frac{\beta_{kn}}{1+\beta_{kn}} e^{1/\beta_{kn}}, \end{aligned}$$

we can conclude that

$$\begin{aligned} 1 \leq \int_{-\infty}^\infty e^{-t}\phi_k(t)\,dt = \prod_{n=1}^{N_k} \frac{\beta_{kn}}{1+\beta_{kn}} e^{1/\beta_{kn}} &\leq \prod_{n=1}^{\infty} e^{1/((kn-1)^2\delta^2-1)} \\ = e^{\sum_{n=1}^\infty \frac{1}{(kn-1)^2\delta^2-1}} &\to e^0 = 1. \end{aligned}$$

Thus $\phi_k(t) \to 0$ for $t < 0$, uniformly for $t < -\epsilon < 0$.

Finally, let $\epsilon > 0$ and k_0 be such that $\phi_k(-\epsilon) + 1 - \phi_k(\epsilon) < \epsilon$ for all $k > k_0$. Then, for $t \in [0, T]$,

$$
\begin{aligned}
&\|\int_0^t f(s)\,ds - \sum_{n=1}^{N_k} \alpha_{k,n} e^{\beta_{kn} t} r(\beta_{kn})\| \\
&= \|\int_0^t f(s)\,ds - \int_0^T \sum_{n=1}^{N_k} \alpha_{k,n} e^{\beta_{kn} t} e^{-\lambda \beta_{kn}} f(\lambda)\,d\lambda\| \\
&= \|\int_0^t f(s)\,ds - \int_0^T (1 - \phi_k(\lambda - t)) f(\lambda)\,d\lambda\| \\
&= \|\int_0^t f(s) - (1 - \phi_k(s-t)) f(s)\,ds - \int_t^T (1 - \phi_k(s-t)) f(s)\,ds\| \\
&\le \|\int_0^{t-\epsilon} \phi_k(s-t) f(s)\,ds\| + \int_{t-\epsilon}^{t+\epsilon} \|f(s)\|\,ds + \|\int_{t+\epsilon}^T (1 - \phi_k(s-t)) f(s)\,ds\| \\
&\le \|f\|_1 \phi_k(-\epsilon) + 2\epsilon \|f\|_1 + \|f\|_1 (1 - \phi_k(\epsilon)) \\
&\le 3\epsilon \|f\|_1 .
\end{aligned}
$$

Thus $\sum_{n=1}^{N_k} \alpha_{k,n} e^{\beta_{kn} t} r(\beta_{kn})$ converges uniformly to $\int_0^t f(s)\,ds$. $\diamondsuit$

We show next that the Phragmén-Mikusinski inversion does not register perturbations of exponential decay T; i.e., if for $t \in [0, T)$,

$$\int_0^t f(s)\,ds = \lim_{k \to \infty} \sum_{n=1}^{N_k} \alpha_{k,n} e^{\beta_{nk} t} q(\beta_{nk})$$

for some function q, then

$$\int_0^t f(s)\,ds = \lim_{k \to \infty} \sum_{n=1}^{N_k} \alpha_{k,n} e^{\beta_{nk} t} \tilde{q}(\beta_{nk})$$

for all perturbed functions $\tilde{q} = q + r$, where r is some perturbation of exponential decay T.

Corollary 3.2. *Let $r : (\omega, \infty) \to X$ be a function, and let $(\beta_n)_{n \in \mathbb{N}} \subset \mathbb{R}^+$ be a sequence of positive numbers satisfying the Müntz condition (M). Suppose r is* *of exponential decay $T > 0$; i.e.,*

$$\limsup_{n\to\infty} \frac{1}{\beta_n} \ln \|r(\beta_n)\| \leq -T.$$

Then

$$\lim_{k\to\infty} \sum_{n=1}^{N_k} \alpha_{k,n} e^{\beta_{nk} t} r(\beta_{nk}) = 0$$

for all $0 \leq t < T$ where $\alpha_{k,n}$ and N_k are as in the previous theorem, and the limit is uniform for all $t \in [0, S]$ and $0 < S < T$.

Proof. Let $t \in [0, T)$. Then $-T < -\frac{2T+t}{3}$. Thus there exists k_0 such that $\|r(\beta_n)\| \leq e^{-\frac{2T+t}{3}\beta_n}$ for all $n \geq k_0$ and such that $2/k_0\delta < (T-t)/3$.

By the previous theorem we know that $|\alpha_{k,n}| < e^{\frac{2}{k\delta}\beta_{kn}} < e^{\frac{T-t}{3}\beta_{nk}}$. Thus

$$\begin{aligned}
\|\sum_{n=1}^{N_k} \alpha_{k,n} e^{\beta_{nk} t} r(\beta_{nk})\| &\leq \sum_{n=1}^{\infty} e^{\frac{T+2t}{3}\beta_{nk}} e^{-\frac{2T+t}{3}\beta_{nk}} \\
&= \sum_{n=1}^{\infty} e^{-\frac{T-t}{3}\beta_{nk}} \leq \sum_{n=1}^{\infty} e^{-\frac{T-t}{3}(\beta_{nk}-\beta_1)} \\
&\leq \sum_{n=1}^{\infty} e^{-\frac{T-t}{3}(nk-1)\delta} = \frac{e^{-\frac{T-t}{3}(k-1)\delta}}{1-e^{-\frac{T-t}{3}k\delta}} \to 0
\end{aligned}$$

as $k \to \infty$, uniformly for all $t \in [0, S]$ for all $0 < S < T$. $\diamondsuit$

In the following two corollaries we will reformulate Theorem 3.1 in terms of Laplace transforms. The resulting "Phragmén-Mikusinski" inversion formulas for the Laplace transform seem to be new.

Corollary 3.3. *Let $f \in L^1_{loc}([0,\infty); X)$ be exponentially bounded and $\hat{f}(\lambda) := \int_0^\infty e^{-\lambda t} f(t)\, dt$ for sufficiently large $Re(\lambda)$. Let $(\beta_n)_{n\in\mathbb{N}}$ be a sequence satisfying the Müntz condition (M). Let $0 < T$; let $N_k \in \mathbb{N}$ be such that $\sum_{n=1}^{N_k} \frac{1}{\beta_{nk}} \geq T$,* *and let $\alpha_{k,n}$ be as in Theorem 3.1. Then, for all $t \in [0,T)$,*

$$\int_0^t f(s)\,ds = \lim_{k\to\infty} \sum_{n=1}^{N_k} \alpha_{k,n} e^{t\beta_{kn}} \hat{f}(\beta_{kn}),$$

where the limit is uniform for $t \in [0,S]$ and all $0 < S < T$.[(4)]

Proof. Let $q(\lambda) := \int_0^T e^{-\lambda t} f(t)\,dt$, and $r(\lambda) := \int_T^\infty e^{-\lambda t} f(t)\,dt$. Then $\hat{f} = q + r$ and

$$\sum_{n=1}^{N_k} \alpha_{k,n} e^{t\beta_{kn}} \hat{f}(\beta_{kn}) = \sum_{n=1}^{N_k} \alpha_{k,n} e^{t\beta_{kn}} q(\beta_{kn}) + \sum_{n=1}^{N_k} \alpha_{k,n} e^{t\beta_{kn}} r(\beta_{kn}).$$

By Theorem 3.1, the first term converges uniformly on $[0,T]$ to $\int_0^t f(s)\,ds$. Since $\|f(t)\| \le Me^{\omega t}$ for almost all $t \ge 0$ and some positive constants M and ω, it follows that

$$\|r(\lambda)\| \le M \int_T^\infty e^{-\lambda t} e^{\omega t}\,dt \le \frac{M}{\lambda-\omega} e^{\omega T} e^{-\lambda T}$$

for all $\lambda > \omega$. Thus r is of exponential decay T and therefore, by Corollary 3.2, the second term converges uniformly to 0 for all $t \in [0,S]$ and $0 < S < T$. $\diamondsuit$

Corollary 3.4. *Let $f \in L^1_{loc}([0,\infty);X)$ be exponentially bounded and $\hat{f}(\lambda) = \int_0^\infty e^{-\lambda t} f(t)\,dt$ for sufficiently large $Re(\lambda)$. Let $(\beta_n)_{n\in\mathbb{N}}$ be a sequence satisfying the Müntz condition (M). Let N_k be such that $\sum_{n=1}^{N_k} \frac{1}{\beta_{nk}} \to \infty$, as $k \to \infty$, and let $\alpha_{k,n}$ be as in Theorem 3.1. Then for all $t \ge 0$,*

$$\int_0^t f(s)\,ds = \lim_{k\to\infty} \sum_{n=1}^{N_k} \alpha_{k,n} e^{t\beta_{kn}} \hat{f}(\beta_{kn}),$$

where the limit is uniform on compact sets.

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(4) It follows from Corollary 3.2 that one can replace $\hat{f}$ by any function q, as long as the difference $\hat{f} - q$ is of exponential decay T.

Proof. Since $\sum_{n=1}^{\infty} \frac{1}{\beta_{kn}} = \infty$ for all $k \in \mathbb{N}$ (see the proof of Theorem 3.1), there exists a sequence $(N_k)_{k\in\mathbb{N}} \subset \mathbb{N}$ such that $\sum_{n=1}^{N_k} \frac{1}{\beta_{nk}} \to \infty$. Thus, for all $T > 0$ there exists k_0 such that $\sum_{n=1}^{N_k} \frac{1}{\beta_{nk}} > T$ for all $k > k_0$. Hence, by the previous corollary,

$$\int_0^t f(s)\,ds = \lim_{k\to\infty} \sum_{n=1}^{N_k} \alpha_{k,n} e^{t\beta_{kn}} \hat{f}(\beta_{kn}),$$

where the limit is uniform in $[0, S]$ for all $0 < S < T$ and all $T > 0$. ♢

If one takes $\beta_n = n$, then the coefficients $\alpha_{k,n}$ defined in Theorem 3.1 reduce to

$$\begin{aligned}
\alpha_{k,n} =& \frac{1}{e} \prod_{\substack{j=1\\ j\neq n}}^{N_k} \frac{\beta_{kj}}{\beta_{kj} - \beta_{kn}} e^{-\frac{\beta_{kn}}{\beta_{kj}}} = e^{-\frac{n}{n}} \prod_{\substack{j=1\\ j\neq n}}^{N_k} \frac{j}{j-n} e^{\frac{-n}{j}} \\
=& \frac{N_k!}{n} \prod_{j=1}^{n-1} \frac{1}{j-n} \prod_{j=n+1}^{N_k} \frac{1}{j-n} e^{-n\sum_{j=1}^{N_k} \frac{1}{j}} = (-1)^{n-1} \binom{N_k}{n} e^{-n\sum_{j=1}^{N_k} \frac{1}{j}}.
\end{aligned}$$

Thus the previous two corollaries yield the following variant of the Phragmén-Doetsch inversion formula.

Corollary 3.5. *Let $f \in L^1_{loc}([0,\infty); X)$ be exponentially bounded and $\hat{f}(\lambda) = \int_0^\infty e^{-\lambda t} f(t)\,dt$. Let $N_k \in \mathbb{N}$ be such that $\frac{1}{k}\sum_{n=1}^{N_k} \frac{1}{n} \geq T$ for all $k \in \mathbb{N}$. Then for all $t \in [0, T]$,*

$$\int_0^t f(s)\,ds = \lim_{k\to\infty} \sum_{n=1}^{N_k} (-1)^{n+1} \binom{N_k}{n} e^{-n\sum_{j=1}^{N_k}\frac{1}{j}} e^{tkn} q(kn),$$

where the limit is uniform on $[0, S]$ for all $0 < S < T$. If the sequence $(N_k)_{n\in\mathbb{N}}$ is such that $\frac{1}{k}\sum_{n=1}^{N_k} \frac{1}{n} \to \infty$, then the limit exists for all $t \geq 0$ and is uniform on compact subsets of $\mathbb{R}$.

Another consequence of the Phragmén-Mikusinski inversion formula is the following statement characterizing the maximal interval $[0, T]$ on which an exponentially bounded L^1_{loc}-function f vanishes in terms of the growth of its Laplace transform $\lambda \mapsto \hat{f}(\lambda)$ at infinity. This Abelian type theorem will be crucial in the proof of Titchmarsh's theorem.

Theorem 3.6. *Let $0 \leq T$ and $\hat{f}(\lambda) := \int_0^\infty e^{-\lambda t} f(t)\,dt$ for some exponentially bounded*(5) *$f \in L^1_{loc}([0,\infty); X)$. Then the following are equivalent.*(6)

(i) Every sequence $(\beta_n)_{n\in\mathbb{N}}$ satisfying the Müntz condition (M), satisfies

$$\limsup_{n\to\infty} \frac{1}{\beta_n} \ln \|\hat{f}(\beta_n)\| = -T.$$

(ii) For every sequence $(\beta_n)_{n\in\mathbb{N}}$ satisfying the Müntz condition (M), there exists a subsequence $(\beta_{n_k})_{k\in\mathbb{N}}$ satisfying the Müntz condition and

$$\lim_{k\to\infty} \frac{1}{\beta_{n_k}} \ln \|\hat{f}(\beta_{n_k})\| = -T.$$

(iii) There exists a sequence $(\beta_n)_{n\in\mathbb{N}}$ satisfying the Müntz condition (M) and

$$\limsup_{n\to\infty} \frac{1}{\beta_n} \ln \|\hat{f}(\beta_n)\| = -T.$$

(iv) $f(t) = 0$ almost everywhere on $[0, T]$ and $T \in supp(f)$.

(v) $\limsup_{\lambda\to\infty} \frac{1}{\lambda} \ln \|\hat{f}(\lambda)\| = -T$.

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(5) In case that f is not exponentially bounded, the theorem still holds if we replace f with the truncated function $\tilde{f}(t) := f(t) \cdot \chi_{[0,S]}$ for some $S > T$.

(6) G. Doetsch ([Do 1], Satz 14.3.1) proved that $\limsup_{\lambda\to\infty} \frac{1}{\lambda} \ln \|\hat{f}(\lambda)\| \leq -T$ is equivalent to the statement $(iv)'$: $f = 0$ on $[0, T]$ a.e. In fact, it follows from the proof below that statement $(iv)'$ is equivalent to the statements $(i) - (iii)$ if $\limsup \frac{1}{\beta_n} \ln \|\hat{f}(\beta_n)\| = -T$ is replaced by $\limsup \frac{1}{\beta_n} \ln \|\hat{f}(\beta_n)\| \leq -T$.

Proof. We show that first that $(i) => (ii) => (iii) => (iv) => (i)$ and then that $(iv) <=> (v)$. Suppose (i) holds. Let $(\beta_n)_{n\in\mathbb{N}}$ be any such sequence. Let $(\beta_n^\epsilon)_{n\in\mathbb{N}}$ be the subsequence that is obtained by dropping the elements of $(\beta_n)_{n\in\mathbb{N}}$ for which $\|r(\beta_n)\| \leq e^{(-T-\epsilon)\beta_n}$. The dropped subsequence $\left(\tilde{\beta}_n^\epsilon\right)$ satisfies

$$\limsup_{n\to\infty} \frac{1}{\tilde{\beta}_n^\epsilon} \ln \|\hat{f}(\tilde{\beta}_n^\epsilon)\| \leq -T - \epsilon.$$

Since (i) is assumed to hold, the dropped sequence $\left(\tilde{\beta}_n^\epsilon\right)$ can not satisfy the Müntz condition. Thus the sum of the reciprocals of the dropped elements is finite, and the sum of the reciprocals of the remaining terms β_n^ϵ is infinite; therefore, $(\beta_n^\epsilon)_{n\in\mathbb{N}}$ still satisfies the Müntz condition. Now we use a diagonal argument. Let $j = 1$ and take the first k_1 elements of $\left(\beta_n^1\right)_{n\in\mathbb{N}}$ such that $\sum_{i=1}^{k_1} \frac{1}{\beta_i^1} > 1$. Continue with elements of the sequence $\left(\beta_n^{1/2}\right)$, picking consecutive elements until $\sum_{j=1}^{k_1} \frac{1}{\beta_j^1} + \sum_{j=k_1}^{k_2} \frac{1}{\beta_j^{1/2}} \geq 2$. Continuing this process we clearly end up with a subsequence having the properties stated in (ii).

Clearly (ii) implies (iii). Suppose (iii) holds. In the case that $T > 0$, combining Corollary 3.3 with Corollary 3.2 we obtain that

$$\int_0^t f(s)\,ds = 0$$

for all $t \in [0,T)$. Hence $f = 0$ a.e. on $[0,T]$. Thus (iii) implies that $f = 0$ a.e. on $[0,T]$ for all $T \geq 0$. Now let $T \geq 0$ and suppose that $f = 0$ almost everywhere on $[0, T+\epsilon]$. Then, for $\lambda > 0$, we have that

$$\|\int_0^\infty e^{-\lambda t} f(t)\,dt\| = \|\int_{T+\epsilon}^\infty e^{-\lambda t} f(t)\,dt\| \leq \frac{M}{\lambda-\omega} e^{\omega(T+\epsilon)} e^{-\lambda(T+\epsilon)}, \qquad (2)$$

where M, ω are such that $\|f(t)\| \leq Me^{\omega t}$ for almost all $t \geq 0$. Thus

$$\limsup_{\lambda\to\infty} \frac{1}{\lambda} \ln \|\hat{f}(\lambda)\| \leq -(T+\epsilon)$$

contradicting (iii). Thus (iv) holds.

Suppose (iv) holds. Then by (2), for $\epsilon = 0$, $\limsup_{\lambda\to\infty} \frac{1}{\lambda} \ln \|\hat{f}(\lambda)\| \leq -T$. Thus, for any sequence (β_n) satisfying the Müntz condition we have that $\limsup_{n\to\infty} \frac{1}{\beta_n} \ln \|\hat{f}(\beta_n)\| \leq -T$. Suppose $\limsup_{n\to\infty} \frac{1}{\beta_n} \ln \|\hat{f}(\beta_n)\| \leq -T-\epsilon$. Then Corollary 3.2 and Corollary 3.3 imply that $\int_0^t f(s)\,ds = 0$ for $t \in [0, T+\epsilon)$, contradicting $T \in supp(f)$. Thus (i) holds.

The equivalence of (iv) and (v) is proved by virtue of the inequality (2). Suppose (iv) holds. Then, by (2), for $\epsilon = 0$, $\limsup_{\lambda\to\infty} \frac{1}{\lambda} \ln \|\hat{f}(\lambda)\| \leq -T$. Suppose $\limsup_{\lambda\to\infty} \frac{1}{\lambda} \ln \|\hat{f}(\lambda)\| < -T$. Then (i) can not hold, and since (iv) implies (i) this contradicts (iv). Thus $\limsup_{\lambda\to\infty} \frac{1}{\lambda} \ln \|\hat{f}(\lambda)\| = -T$.

Suppose (v) holds. Then, by Corollaries 3.2 and 3.3, f vanishes on $[0,T]$. Suppose f vanishes on $[0, T+\epsilon]$. Then (2) implies that $\limsup_{\lambda\to\infty} \frac{1}{\lambda} \ln \|\hat{f}(\lambda)\| \leq -T-\epsilon$, contradicting (v). Thus (iv) holds. $\diamondsuit$

For an exponentially bounded, strongly continuous family of linear operators $(K(t))_{t\geq 0} \subset \mathcal{L}(X,Y)$[7] we obtain similar results. We define the Laplace transform of such an operator family to be the family of linear operators $\left(\hat{K}(\lambda)\right)_{\lambda>\omega}$, where

$$\hat{K}(\lambda)x := \int_0^\infty e^{-\lambda t} K(t)x\,dt.$$

We often write $\int_0^\infty e^{-\lambda t} K(t)\,dt$ instead of $\hat{K}(\lambda)$.

[7] In case the operator family is only defined on a finite interval $[0,T]$ we identify the operator family with its extension $\tilde{K}(t) := \begin{cases} K(t) & \text{for } 0 \leq t \leq T \\ K(T) & \text{else} \end{cases}$ onto $[0,\infty)$.

Theorem 3.7. *Let $(K(t))_{t\geq 0}$ be a strongly continuous, exponentially bounded family of linear operators and let $T \geq 0$. Then the following are equivalent.*

(i) Every sequence $(\beta_n)_{n\in\mathbb{N}}$ satisfying the Müntz condition (M), satisfies

$$\limsup_{n\to\infty} \frac{1}{\beta_n} \ln \|\hat{K}(\beta_n)\| = -T.$$

(ii) Every sequence $(\beta_n)_{n\in\mathbb{N}}$ satisfying the Müntz condition (M) has a subsequence $\left(\beta_{n_j}\right)_{j\in\mathbb{N}}$ satisfying the Müntz condition and

$$\lim_{j\to\infty} \frac{1}{\beta_{n_j}} \ln \|\hat{K}(\beta_{n_j})\| = -T.$$

(iii) There exists a sequence $(\beta_n)_{n\in\mathbb{N}}$ satisfying the Müntz condition (M) and

$$\limsup_{n\to\infty} \frac{1}{\beta_n} \ln \|\hat{K}(\beta_n)\| = -T.$$

(iv) $K(t) = 0$ almost everywhere on $[0, T]$ and $T \in supp(K)$.

(v) $\limsup_{\lambda\to\infty} \frac{1}{\lambda} \ln \|\hat{K}(\lambda)\| = -T$.

Proof. With the same proof as in Theorem 3.6 we show that (i) implies (ii) and clearly, (ii) implies (iii). Suppose (iii) holds; i.e.,

$$\limsup_{n\to\infty} \frac{1}{\beta_n} \ln \|\hat{K}(\beta_n)\| = -T.$$

Then $\limsup_{\beta_n\to\infty} \frac{1}{\beta_n} \ln \|\hat{K}(\beta_n)x\| \leq -T$ for all $x \in X$. Thus, by the previous corollary, $K(t)x = 0$ for all $0 \leq t \leq T$ and all $x \in X$. Thus $K(t) = 0$ on $[0, T]$. Suppose $K(t) = 0$ on $[0, T + \epsilon]$. Then, for $\lambda > 0$, we have that

$$\left\|\int_0^\infty e^{-\lambda t} K(t)\, dt\right\| = \left\|\int_{T+\epsilon}^\infty e^{-\lambda t} K(t)\, dt\right\| \leq \frac{M}{\lambda - \omega} e^{\omega(T+\epsilon)} e^{-\lambda(T+\epsilon)}, \qquad (3)$$

where M, ω are such that $\|K(t)\| \leq Me^{\omega t}$ for almost all $t \geq 0$. Thus

$$\limsup_{\lambda\to\infty} \frac{1}{\lambda} \ln \|\hat{K}(\lambda)\| \leq -(T + \epsilon)$$

contradicting (iii). Thus (iv) holds.

Suppose (iv) holds. Then, by (3), for $\epsilon = 0$, $\limsup_{n\to\infty} \frac{1}{\beta_n} \ln \|\hat{K}(\beta_n)\| \leq -T$ for all sequences (β_n) satisfying the Müntz condition. Suppose

$$\limsup_{n\to\infty} \frac{1}{\beta_n} \ln \|\hat{K}(\beta_n)\| \leq -T - \epsilon$$

and thus $\limsup_{n\to\infty} \frac{1}{\beta_n} \ln \|\hat{K}(\beta_n)x\| \leq -T - \epsilon$ for all $x \in X$. Then the above theorem implies that $\int_0^t K(s)x\, ds = 0$ for $t \in [0, T+\epsilon)$ and all $x \in X$, contradicting $T \in supp(K)$. Thus (i), and with the same argument, (v) holds.

Suppose (v) holds. Then $\limsup_{n\to\infty} \frac{1}{\beta_n} \ln \|\hat{K}(\beta_n)x\| \leq -T$ for all $x \in X$. Then, by Theorem 3.6, K vanishes on $[0, T]$. Suppose K vanishes on $[0, T+\epsilon]$. Then (3) implies that $\limsup_{\lambda\to\infty} \frac{1}{\lambda} \ln \|\hat{K}(\lambda)\| \leq -T - \epsilon$, contradicting (v). Thus (iv) holds. $\diamondsuit$

The Theorems 3.6 and 3.7 provide us with a powerful tool to prove Titchmarsh's theorem. But first we want to establish the some facts about convoluting with a strongly continuous operator family.

Lemma 3.8. *Let $(K(t))_{t\in[0,\infty)} \subset \mathcal{L}(X, Y)$ be a strongly continuous family and $f \in C([0,\infty); X)$. Then $s \mapsto K(t-s)f(s)$ is continuous on $[0, t]$ for all $t > 0$ and the convolution operator $T_K f : t \mapsto \int_0^t K(t-s)f(s)\, ds$ is a bounded linear operator from $C([0, T]; X)$ to $C([0, T]; Y)$ for all $T > 0$. Furthermore, if K and f are exponentially bounded, then for all sufficiently large λ,*

$$\int_0^\infty e^{-\lambda t}(K \star f)(t)\, dt = \int_0^\infty e^{-\lambda t} K(t) \int_0^\infty e^{-\lambda s} f(s)\, ds\, dt = \hat{K}(\lambda)\hat{f}(\lambda).$$

Proof. Let $M_t := \sup_{s\in[0,t]} \|K(s)\|$.[8] The function $s \mapsto K(t-s)f(s)$ is continuous in $[0,t]$ since

$$\begin{aligned}
&\|K(t-(s+h))f(s+h) - K(t-s)f(s)\| \\
&\qquad \leq \|K(t-(s+h))f(s+h) - K(t-(s+h))f(s)\| \\
&\qquad\quad + \|K(t-(s+h))f(s) - K(t-s)f(s)\| \\
&\qquad \leq M_t\|f(s+h) - f(s)\| + \|K(t-(s+h))f(s) - K(t-s)f(s)\| \to 0
\end{aligned}$$

as $h \to 0$. The convolution operator $T_K : C([0,T];X) \to C([0,T];Y)$ is bounded since

$$\|T_K f\| = \sup_{t\in[0,T]} \|\int_0^t K(t-s)f(s)\,ds\| \leq \sup_{t\in[0,T]} M_T \int_0^t \|f(s)\|\,ds \leq M_T T\|f\|.$$

The second statement of the theorem holds since

$$\begin{aligned}
\widehat{K\star f}(\lambda) :=& \int_0^\infty e^{-\lambda t}\int_0^t K(t-s)f(s)\,ds\,dt = \int_0^\infty \int_s^\infty e^{-\lambda t}K(t-s)f(s)\,dt\,ds \\
=& \int_0^\infty \int_0^\infty e^{-\lambda(t+s)}K(t)f(s)\,dt\,ds = \int_0^\infty \int_0^\infty e^{-\lambda t}K(t)e^{-\lambda s}f(s)\,ds\,dt \\
=& \hat{K}(\lambda)\hat{f}(\lambda).
\end{aligned}$$

$\diamondsuit$

As remarked earlier, we can not expect the convolution operator to be injective for a strongly continuous operator families K, assuming that $0 \in supp(K)$. A necessary condition is that $0 \in suppK(\cdot)x$ for all $x \in X$. Otherwise, if $K(t)x = 0$ on $[0,\epsilon]$, the convolution $K \star f_x = 0$ on $[0,T]$ for every scalar valued function f with $f(t) = 0$ on $[0, T-\epsilon]$ and $f_x(t) := f(t)x$. We do not know at this point,

[8] Since K is strongly continuous, there exists for all $x \in X$ and all $t > 0$ a constant M_x such that $\|K(s)x\| \leq M_x$ for all $s \in [0,t]$. By the Principle of Uniform Boundedness, there exists a constant M_t such that $\|K(s)\| \leq M_t$ for all $\in [0,t]$.

whether this condition is also sufficient. We give a technical condition, which combined with the requirement that $0 \in supp(K)$, is a sufficient condition. Once we proved Titchmarsh injectivity theorem and Foias' density theorem for operator families satisfying this technical condition, we give ways to extend Titchmarsh's and Foias' Theorems to a wider class of operator families including, for example, strongly continuous semigroups. For the following definition we recall that

$$K_T(t) := \begin{cases} K(t) & \text{for } 0 \leq t \leq T \leq \infty \\ 0 & \text{else.} \end{cases}$$

Definition 3.9. Let $(K(t))_{t\geq 0} \subset \mathcal{L}(X, Y)$ be a strongly continuous operator family with $K(T_0) \neq 0$ for some $T_0 > 0$. Let $T > T_0$. We say that $(K(t))_{t\geq 0}$ satisfies condition (A_T) if there exists a Müntz sequence $(\beta_n)_{n\in\mathbb{N}}$, such that for all $\epsilon > 0$ there exists a constant N_ϵ with

$$\|\hat{K}_T(\beta_n)\| \, \|x\| \leq e^{\beta_n \epsilon} \|\hat{K}_T(\beta_n)x\| \tag{A_T}$$

for all $n > N_\epsilon$ and all $x \in X$. We say that $(K(t))_{t\geq 0}$ satisfies condition (A) if there exists a $T > T_0$ such that $(K(t))_{t\geq 0}$ satisfies condition (A_T).

Proposition 3.10. *Let $(K(t))_{t\geq 0} \subset \mathcal{L}(X, Y)$ be a strongly continuous operator family satisfying condition (A_{T_0}) for some $T_0 > 0$. Then $(K(t))_{t\geq 0}$ satisfies condition (A_T) for all $T > T_0$. If, in addition, $(K(t))_{t\geq 0}$ is exponentially bounded, then there exists a sequence $(\beta_n)_{n\in\mathbb{N}}$ satisfying the Müntz condition (M), such that for all $\epsilon > 0$ there exists a constant N_ϵ with*

$$\|\hat{K}(\beta_n)\| \, \|x\| \leq e^{\beta_n \epsilon} \|\hat{K}(\beta_n)x\|$$

for all $n > N_\epsilon$ and all $x \in X$.

Proof. Let $T > T_0$, and suppose $(K(t))_{t\geq 0}$ satisfies condition (A_{T_0}). Thus there exists $t \in [0, T_0)$ such that $K(t) \neq 0$. Let (β_n) be a Müntz sequence such that for all $\epsilon > 0$ there exists a constant $N_\epsilon > 0$, and such that

$$\|\hat{K_{T_0}}(\beta_n)\| \, \|x\| \leq e^{\beta_n \epsilon} \|\hat{K_{T_0}}(\beta_n) x\| \tag{4}$$

for all $n > N_\epsilon$ and all $x \in X$. Without loss of generality, it suffices to prove the assertion for all $0 < \epsilon < \frac{T_0 - t}{4}$.

We show first that there exists a subsequence $\left(\beta_{n_j}\right)_{j\in\mathbb{N}}$ such that for all $0 < \epsilon < \frac{T_0-t}{4}$ there exists a constant J_ϵ such that

$$1 - \frac{e^{-\beta_{n_j}(T_0 - 2\epsilon)}}{\|\hat{K_T}(\beta_{n_j})\|} \geq e^{-\beta_{n_j}\epsilon/2}$$

for all $j > J_\epsilon$. By Theorem 3.7 there exists a subsequence of (β_n) such that

$$\lim_{j\to\infty} \frac{1}{\beta_{n_j}} \ln \|\hat{K_T}(\beta_{n_j})\| \geq -t.$$

Let $0 < \epsilon < \frac{T_0-t}{4}$. Hence, there exists $\tilde{J}_\epsilon$ such that

$$\|\hat{K_T}(\beta_{n_j})\| \geq e^{-(t+\epsilon)\beta_{n_j}} \geq e^{-(T_0 - 3\epsilon)\beta_{n_j}}$$

for all $j > \tilde{J}_\epsilon$. Thus

$$1 - \frac{e^{-\beta_{n_j}(T_0-2\epsilon)}}{\|\hat{K_T}(\beta_{n_j})\|} \geq 1 - e^{-\beta_{n_j}\epsilon} \geq e^{-\beta_{n_j}\epsilon/2}$$

for $\beta_{n_j}\epsilon/2 > 1$. Let J_ϵ be such that $\beta_{n_j}\epsilon/2 > 1$ for all $j > J_\epsilon$, and such that $J_\epsilon \geq \tilde{J}_\epsilon$. Then $\left(\beta_{n_j}\right)_{n\in\mathbb{N}}$ satisfies the above inequality for all $j > J_\epsilon$.

We now look for estimates of (4) on both sides. We know that

$$
\begin{aligned}
\|\hat{K_{T_0}}(\beta_n)x\| =& \| \int_0^{T_0} e^{-\beta_n t} K(t)x\,dt \| \\
\leq& \| \int_0^{T} e^{-\beta_n t} K(t)x\,dt \| + \| \int_{T_0}^{T} e^{-\beta_n t} K(t)\,dt \|\,\|x\| \\
\leq& \|\hat{K_T}(\beta_n)x\| + C_1 |e^{-\beta_n T_0} - e^{-\beta_n T}| \|x\|
\end{aligned}
$$

for all $n > N_\epsilon$ and some constant $C_1 > 0$. On the other hand,

$$
\begin{aligned}
\|\hat{K_{T_0}}(\beta_n)\| =& \| \int_0^{T} e^{-\beta_n t} K(t)\,dt - \int_{T_0}^{T} e^{-\beta_n t} K(t)\,dt \| \\
\geq& \|\hat{K_T}(\beta_n)\| - \| \int_{T_0}^{T} e^{-\beta_n t} K(t)\,dt \| \\
\geq& \|\hat{K_T}(\beta_n)\| - C_2 |e^{-\beta_n T_0} - e^{-\beta_n T}|
\end{aligned}
$$

for all $n > N_\epsilon$ and some constant $C_2 > 0$. Thus

$$
\begin{aligned}
(\|\hat{K_T}(\beta_n)\| - C_2 |e^{-\beta_n T_0} - e^{-\beta_n T}| - C_1 e^{\beta_n \epsilon} |e^{-\beta_n T_0} - e^{-\beta_n T}|) \|x\| \\
\leq e^{\beta_n \epsilon} \|\hat{K_T}(\beta_n)x\|
\end{aligned}
\tag{5}
$$

for all $x \in X$ and for all $n > N_\epsilon$.

Let $N_0 > N_\epsilon$ be such that

$$
C_2 |e^{-\beta_n T_0} - e^{-\beta_n T}| + C_1 e^{\beta_n \epsilon} |e^{-\beta_n T_0} - e^{-\beta_n T}| \leq e^{-\beta_n (T_0 - 2\epsilon)}
$$

for all $n > N_0$. Thus (5) reduces to

$$
\|\hat{K_T}(\beta_n)\| \left(1 - \frac{e^{-\beta_n (T_0 - 2\epsilon)}}{\|\hat{K_T}(\beta_n)\|}\right) \|x\| \leq e^{\beta_n \epsilon} \|\hat{K_T}(\beta_n)x\|
$$

for all $n \geq N_0$. Let $N_1 > \max\{N_0, J_\epsilon\}$. Then

$$
e^{-\beta_{n_j} \epsilon/2} \|\hat{K_T}(\beta_{n_j})\| \|x\| \leq e^{\beta_{n_j} \epsilon} \|\hat{K_T}(\beta_{n_j})x\|,
$$

and thus

$$
\|\hat{K}_T(\beta_{n_j})\| \|x\| \leq e^{\beta_{n_j} 3\epsilon/2} \|\hat{K}_T(\beta_{n_j})x\|
$$

for all $x \in X$ and all $j > N_1$.

Let K is an exponentially bounded operator family. Replacing T by ∞ and $\hat{K}_T$ by $\hat{K}$ in the above argument, we obtain the desired properties for K. $\diamondsuit$

Condition (A) is obviously satisfied by continuous scalar valued functions $k \neq 0$, if we identify $k(t)$ with the linear operator $K(t) : x \mapsto k(t)x$. Another important class of strongly continuous operator families that satisfy condition (A) is the class of uniformly continuous semigroups $K(t) = e^{tB}$ with bounded generator B.[(9)]

Example 3.11. *Let $(K(t))_{t\geq 0}$ be a uniformly continuous semigroup. Then $(K(t))_{t\geq 0}$ satisfies condition (A).*

Proof. For a uniformly continuous semigroup $(K(t))_{t\geq 0}$ we know that

$$\int_0^\infty e^{-\lambda t} K(t)\, dt = R(\lambda, B) := (\lambda I - B)^{-1}$$

for some bounded linear operator B and all $\lambda > \omega$ for some $\omega > 0$. Since

$$\|\lambda x - Bx\| \leq \lambda\|x\| + \|B\|\|x\|$$

we obtain, for $x = R(\lambda, B)y$, that

$$\|y\| \leq (\lambda + \|B\|)\|R(\lambda, B)y\|,$$

and thus

$$\|\hat{K}(\lambda)\|\|y\| \leq (\lambda + \|B\|)\|\hat{K}(\lambda)\|\|R(\lambda, B)y\|.$$

Since $\|R(\lambda, B)y\| = \|\hat{K}(\lambda)y\|$ and $\|\hat{K}(\lambda)\| \leq \frac{M}{\lambda-\omega}$ for some constants $M, \omega > 0$, we obtain that

$$\|\hat{K}(\lambda)\|\|y\| \leq e^{\lambda\epsilon}\|\hat{K}(\lambda)y\|$$

for λ large enough. Hence $(K(t))$ satisfies condition (A). $\diamondsuit$

---

(9) See, for example, A. Pazy [Pa] or J. Goldstein [Go] for an introduction to semigroup theory.

Theorem 3.12 (Titchmarsh). *Let $(K(t))_{t\geq 0} \subset \mathcal{L}(X,Y)$ be a strongly continuous operator family satisfying condition (A). Let $T \geq 0$, let $f \in C([0,T];X)$. Then the following are equivalent:*

(i) $K \star f = 0$ *on* $[0,T]$.

(ii) *There exist constants* $0 \leq t_1, t_2 \leq T$ *with* $t_1 + t_2 \geq T$ *such that* $K = 0$ *on* $[0,t_1]$ *and* $f = 0$ *on* $[0,t_2]$.

Proof. By Proposition 3.10 it suffices to show that the theorem holds for exponentially bounded, strongly continuous operator families, since any operator family on $[0,T]$ satisfying condition (A) can be extended to an exponentially bounded operator family satisfying condition (A). Suppose $K \star f = 0$ on $[0,T]$. Then, identifying f with its zero extension on $[0,\infty)$,

$$\limsup_{\lambda\to\infty} \frac{1}{\lambda} \ln \|\widehat{K \star f}(\lambda)\| = \limsup_{\lambda\to\infty} \frac{1}{\lambda} \ln \|\hat{K}(\lambda)\hat{f}(\lambda)\| \leq -T.$$

By Proposition 3.10, for all $\epsilon > 0$ there exists a Müntz sequence $(\beta_n)_{n\in\mathbb{N}}$ and N_ϵ such that

$$e^{-\beta_n(T-2\epsilon)} \geq e^{\beta_n \epsilon}\|\hat{K}(\beta_n)\hat{f}(\beta_n)\| \geq \|\hat{K}(\beta_n)\|\,\|\hat{f}(\beta_n)\|$$

for all $n > N_\epsilon$. The first inequality follows from the estimate above and the second from condition (A). Let

$$t_1 := \limsup_{\lambda\to\infty} \frac{1}{\lambda} \ln \|\hat{K}(\lambda)\| \text{ and } t_2 := \limsup_{\lambda\to\infty} \frac{1}{\lambda} \ln \|\hat{f}(\lambda)\|.$$

Then, by Theorems 3.6 and 3.7, $K = 0$ on $[0,t_1]$ and $f = 0$ on $[0,t_2]$. By Theorem 3.7 there exists a subsequence of $(\beta_n)_{n\in\mathbb{N}}$ satisfying the Müntz condition (M)

such that $t_1 = \lim_{k\to\infty} \frac{1}{\beta_{n_k}} \ln \|\hat{K}(\beta_{n_k})\|$; by Theorem 3.6 there exists a further subsequence, $\left(\beta_{n_{k_j}}\right)$ such that $t_2 = \lim_{j\to\infty} \frac{1}{\beta_{n_{k_j}}} \ln \|\hat{f}(\beta_{n_{k_j}})\|$. Thus

$$-(t_1 + t_2) = \lim_{j\to\infty} \frac{1}{\beta_{n_{k_j}}} \ln \|\hat{K}(\beta_{n_{k_j}})\| \|\hat{f}(\beta_{n_{k_j}})\| \le -T + 2\epsilon$$

for all $\epsilon > 0$. Hence $t_1 + t_2 \ge T$.

Suppose (ii) holds. By Theorems 3.6 and 3.7, we know that

$$\limsup_{\lambda\to\infty} \frac{1}{\lambda} \ln \|\hat{K}(\lambda)\| \le -t_1 \text{ and } \limsup_{\lambda\to\infty} \frac{1}{\lambda} \ln \|\hat{f}(\lambda)\| \le -t_2.$$

Thus,

$$\begin{aligned} \limsup_{\lambda\to\infty} \frac{1}{\lambda} \ln \|\widehat{K \star f}(\lambda)\| &\le \limsup_{\lambda\to\infty} \frac{1}{\lambda} \ln \|\hat{K}(\lambda)\| \|\hat{f}(\lambda)\| \\ &\le -(t_1 + t_2) \le -T. \end{aligned}$$

Therefore, by Theorem 3.6, $K \star f \equiv 0$ on $[0, T]$. $\diamondsuit$

This theorem yields the following corollary.

Corollary 3.13 (Injectivity of the convolution transform). *Let $(K(t))_{t\ge 0}$ be a strongly continuous family of linear operators with $0 \in supp(K)$[10] that satisfies condition (A), and let $f \in C([0, T]; X)$ for some $T > 0$. Then $K \star f = 0$ on $[0, T]$ implies that $f = 0$ on $[0, T]$.*

Proof. Suppose $K \star f = 0$ on $[0, T]$. Then, by the previous theorem, $K = 0$ on $[0, t_1]$ and $f = 0$ on $[0, T - t_1]$ for some $t_1 \ge 0$. Since $0 \in supp(K)$ we obtain that $t_1 = 0$ and thus $f = 0$ on $[0, T]$. $\diamondsuit$

[10] Recall that $0 \in supp(K)$ if there exist sequences $t_n \ge 0$ and $x_n \in X$ such that $t_n \to 0$ and $K(t_n)x_n \ne 0$.

After showing the injectivity of the convolution transform, we consider the density of its image. This is known in the scalar-valued case as Foias' dense range theorem (see [Fo]). The proof of Foias is based on the Hahn-Banach theorem and the Riesz representation theorem of the dual of $L^1[0,T]$. We give a different proof which is constructive and yields an approximating sequence of continuous functions in the image of the convolution transform. Moreover, we extend Foias' dense range theorem to the Banach space valued setting.

Of course, we can not expect a dense image if we do not have a condition on the range of $K(t)$. Suppose $(K(t))_{t\geq 0}$ with $0 \in supp(K)$ is exponentially bounded and satisfies condition (A); i.e., there exists a sequence $(\beta_n)_{n\in\mathbb{N}}$ satisfying the Müntz condition (M) and $\|\hat{K}(\beta_n)\|\|x\| \leq e^{\beta_n \epsilon}\|\hat{K}(\beta_n)x\|$ for all $\epsilon > 0$, $x \in X$, and $n > N_\epsilon$. Let $\epsilon > 0$. Since $0 \in supp(K)$, by Theorem 3.7 there exists a subsequence of $(\beta_n)_{n\in\mathbb{N}}$ satisfying the Müntz condition (M) with $\|\hat{K}(\beta_{n_k})\| \geq e^{-\beta_{n_k}\epsilon}$. Thus, by condition (A) there exists a constant N_ϵ such that

$$\|x\|e^{-\beta_{n_k}\epsilon} \leq e^{\beta_{n_k}\epsilon}\|\hat{K}(\beta_{n_k})x\|$$

for all $x \in X$ and all $k > \tilde{N}_\epsilon$. Thus $\hat{K}(\beta_{n_k})$ is one-to-one and for $y \in Im(\hat{K}(\beta_{nk}))$,

$$\|\hat{K}(\beta_{n_k})^{-1}y\| \leq e^{2\beta_{n_k}\epsilon}\|y\|. \tag{6}$$

Hence, $\hat{K}(\beta_{n_k})^{-1}$ is a bounded linear operator and $Im(\hat{K}(\beta_{n_k})) = \mathcal{D}(\hat{K}(\beta_{n_k})^{-1})$ is a closed subspace of Y. We denote this subspace as $Y_{\beta_{n_k}}$ and the intersection of the ranges with $\tilde{Y} := \bigcap_{k\geq\tilde{N}_\epsilon} Y_{\beta_{n_k}}$, with (β_{n_k}) as a *defining* Müntz sequence.

Theorem 3.14 (Foias' Dense Range Theorem). *Let $(K(t))_{t\geq 0} \subset \mathcal{L}(X,Y)$ be a strongly continuous operator family with $0 \in supp(K)$ satisfying condition (A). Let $\tilde{K}$ be an exponentially bounded strongly continuous operator family that coincides with K on $[0,T]$. Let β_n be a Müntz sequence defining $\tilde{Y}$. Then for all $f \in C_0([0,T];\tilde{Y})$ there exist $g_n \in C([0,T];X)$ such that $K \star g_n \to f$.*[11]

Proof. It suffices to show that the image is dense for $\tilde{K}$, since by Proposition 3.10, $\tilde{K}$ also satisfies condition (A) and, by definition, coincides with K on $[0,T]$. Let $f \in C_0([0,T];\tilde{Y})$. Let $\epsilon > 0$ and Let

$$f_\epsilon(t) := \begin{cases} 0 & \text{for } 0 \leq t \leq \epsilon \\ \frac{1}{\epsilon}\left(\int_{t-\epsilon}^{t} f(s)\,ds - \int_0^\epsilon f(s)\,ds\right) & \text{else.} \end{cases}$$

Then

$$\|f - f_\epsilon\| \leq \sup_{t\in[0,\epsilon]} \|f(t)\| + \sup_{t\in[\epsilon,T]} \frac{1}{\epsilon}\int_{t-\epsilon}^{t} \|f(t) - f(s)\|\,ds.$$

Since f is uniformly continuous we have that $f_\epsilon \to f$. Thus, it suffices to show that we can approximate each f_ϵ. Let

$$h(t) := \begin{cases} \frac{f(t+\epsilon)-f(t)}{\epsilon} & \text{for } 0 \leq t \leq T-\epsilon \\ \frac{f(T)-f(T-\epsilon)}{\epsilon} & \text{else.} \end{cases}$$

Then $h \in C([0,T];\tilde{Y})$ and we identify h with its zero extension on $C([0,\infty);\tilde{Y})$. Then, in particular, $\hat{h}(\beta_n) \in \tilde{Y}$ for all $n \in \mathbb{N}$.

---

(11) If $f \in C_0([0,\infty);\tilde{Y})$, then the functions $\tilde{g}_n \in C([0,\infty);X)$ that satisfy

$$\sup_{t\in[0,n]} \|(K \star \tilde{g}_n)(t) - f(t)\| \leq \frac{1}{n}$$

approximate f in the Frechét topology of $C_0([0,\infty);\tilde{Y})$.

Since $(\beta_n)_{n\in\mathbb{N}}$ is a defining Müntz sequence there exists a constant N_ϵ such that $\|\hat{\tilde{K}}(\beta_n)^{-1}y\| \leq e^{\beta_n 2\epsilon/3}\|y\|$ for all $n > N_\epsilon$ and $y \in \tilde{Y}$.

Let

$$g_k(t) := \sum_{n=1}^{N_k} \alpha_{k,n} e^{(t-\epsilon)\beta_{kn}} \hat{\tilde{K}}(\beta_{kn})^{-1}\hat{h}(\beta_{kn}),$$

where $\alpha_{k,n}$ and N_k are as in Theorem 3.1. Then

$$\begin{aligned}
K \star g_k(t) &= \sum_{n=1}^{N_k} \alpha_{k,n} \int_0^t K(s) e^{(t-s-\epsilon)\beta_{kn}} \hat{\tilde{K}}(\beta_{kn})^{-1}\hat{h}(\beta_{kn})\,ds \\
&= \sum_{n=1}^{N_k} \alpha_{k,n} e^{(t-\epsilon)\beta_{kn}} \int_0^t e^{-s\beta_{kn}} \tilde{K}(s) \hat{\tilde{K}}(\beta_{kn})^{-1}\hat{h}(\beta_{kn})\,ds \\
&= \sum_{n=1}^{N_k} \alpha_{k,n} e^{(t-\epsilon)\beta_{kn}} \int_0^\infty e^{-s\beta_{kn}} \tilde{K}(s) \hat{\tilde{K}}(\beta_{kn})^{-1}\hat{h}(\beta_{kn})\,ds \\
&\quad - \sum_{n=1}^{N_k} \alpha_{k,n} e^{(t-\epsilon)\beta_{kn}} \int_t^\infty e^{-s\beta_{kn}} \tilde{K}(s) \hat{\tilde{K}}(\beta_{kn})^{-1}\hat{h}(\beta_{kn})\,ds \\
&= \sum_{n=1}^{N_k} \alpha_{k,n} e^{(t-\epsilon)\beta_{kn}} \hat{h}(\beta_{kn}) \\
&\quad - \sum_{n=1}^{N_k} \alpha_{k,n} e^{(t-\epsilon)\beta_{kn}} \int_t^\infty e^{-s\beta_{kn}} \tilde{K}(s) \hat{\tilde{K}}(\beta_{kn})^{-1}\hat{h}(\beta_{kn})\,ds.
\end{aligned}$$

Since

$$e^{-\epsilon\beta_{kn}}\hat{h}(\beta_{kn}) = \int_0^\infty e^{-\beta_{kn}(t+\epsilon)} h(t)\,dt = \int_\epsilon^\infty e^{-\beta_{kn}t} h(t-\epsilon)\,dt,$$

by Theorem 3.1, the first term converges uniformly for $T \in [0,T]$ to

$$\begin{cases} 0 & \text{for } 0 \leq t \leq \epsilon \\ \int_\epsilon^t h(s-\epsilon)\,ds & \text{else.} \end{cases}$$

But

$$\begin{aligned}
\int_\epsilon^t h(s-\epsilon)\,ds &= \frac{1}{\epsilon}\int_\epsilon^t f(s) - f(s-\epsilon)\,ds = \frac{1}{\epsilon}\left(\int_\epsilon^t f(s) - \int_0^{t-\epsilon} f(s)\,ds\right) \\
&= \frac{1}{\epsilon}\int_{t-\epsilon}^t f(s)\,ds - \int_0^\epsilon f(s)\,ds = f_\epsilon(t).
\end{aligned}$$

Thus the first term converges to f_ϵ, uniformly on $[0,T]$.

We conclude the proof by showing that the second term converges to zero. Choose k such that $2/k\delta < \epsilon/4$. Recall that $\|\beta_{kn}\| \geq (kn-1)\delta$. Then

$$\begin{aligned}
&\|\sum_{n=1}^{N_k} \alpha_{k,n} e^{(t-\epsilon)\beta_{kn}} \int_t^\infty e^{-s\beta_{kn}} \tilde{K}(s)\hat{\tilde{K}}(\beta_{kn})^{-1}\hat{h}(\beta_{kn})\,ds\| \\
&\leq \sum_{n=1}^{\infty} e^{2\beta_{kn}/k\delta} e^{(t-\epsilon)\beta_{kn}} \int_t^\infty e^{-s\beta_{kn}} \|\tilde{K}(s)\hat{\tilde{K}}(\beta_{kn})^{-1}\hat{h}(\beta_{kn})\|\,ds \\
&\leq \sum_{n=1}^{\infty} e^{2\beta_{kn}/k\delta+(t-\epsilon)\beta_{kn}} \int_t^\infty e^{-s\beta_{kn}} \|\tilde{K}(s)\| e^{\beta_{kn}2\epsilon/3}\|\hat{h}(\beta_{kn})\|\,ds \\
&\leq \frac{Me^{\omega t}\|\hat{h}(\beta_{kn})\|}{\beta_{kn}-\omega} \sum_{n=1}^{\infty} e^{2\beta_{kn}/k\delta+(t-\epsilon)\beta_{kn}} e^{-\beta_{kn}t} e^{\beta_{kn}2\epsilon/3} \\
&\leq C\sum_{n=1}^{\infty} e^{\beta_{kn}(2/k\delta-\epsilon/3)} \leq C\sum_{n=1}^{\infty} e^{-(kn-1)\delta\epsilon/12} \\
&= C\frac{e^{-(k-1)\delta\epsilon/12}}{1-e^{-k\delta\epsilon/12}} \to 0.
\end{aligned}$$

$\diamondsuit$

As seen in inequality (6), the condition (A) implies that $\hat{\tilde{K}}(\beta_n)^{-1} \in \mathcal{L}(\tilde{Y}, X)$ for some Müntz sequence $(\beta_n)_{n\in\mathbb{N}}$. This shows that a strongly continuous semigroup satisfies condition (A) if and only if K is uniformly continuous. We will now develop tools to extend the class of strongly continuous operator families that yield an injective convolution product or a convolution product with a dense range.

Suppose $(K(t))$ with $0 \in supp(K)$ satisfies condition (A) and thus, by Corollary 3.13, yields an injective convolution operator T_K. Let C be any bounded, injective linear operator. Then the operator family $(CK(t))$ does not, in general, satisfy condition (A). However, T_{CK} is also injective, since $CK \star f = 0$ implies that $K \star f = 0$. Furthermore, if T_{K_1} and T_{K_2} are injective, then $T_{K_1 \star K_2}$ is injective.

Theorem 3.15. *Let $(K_1(t))_{t\in[0,\infty)} \subset \mathcal{L}(X,Y)$ and $(K_2(t))_{t\in[0,\infty)} \subset \mathcal{L}(Y,Z)$ be strongly continuous operator families and let $C \in \mathcal{L}(Y,Z)$ be a bounded linear operator with a dense range. Suppose the convolution transforms $T_{K_1} : C([0,T];X) \to C([0,T];Y)$ and $T_{K_2} : C([0,T];Y) \to C([0,T];Z)$ have dense ranges. Then the convolution operators $T_{CK}, T_{K_2\star K_1} : C([0,T];X) \to C([0,T];Z)$ have dense ranges in $C([0,T];Z)$.*

Proof. First we show that for all $f \in C([0,T];Z)$ and $\epsilon > 0$ there exists $g \in C([0,T];Y)$ such that $\sup_{t\in[0,T]} \|f(t) - Cg(t)\|_Z < \epsilon$. Let $\{t_j\}$ be a partition of $[0,T]$ such that $\sup_{t\in[t_j,t_{j+1}]} \|f(t) - f(t_j)\|_Z < \epsilon$ for all j. Since the range of C is dense, we can find $y_j \in Y$ such that $\|Cy_j - f(t_j)\| < \epsilon$. For $t \in [t_j, t_{j+1}]$ define

$$g(t) := (1 - \frac{t-t_j}{t_{j+1}-t_j})y_j + \frac{t-t_j}{t_{j+1}-t_j})y_{j+1}.$$

Then

$$\begin{aligned}
\sup_{t\in[0,T]} \|Cg(t) - f(t)\| &= \sup_j \sup_{t\in[t_j,t_{j+1}]} \|Cg(t) - f(t)\| \\
&\leq \sup_j \sup_{t\in[t_j,t_{j+1}]} \|Cg(t) - Cy_j\| + \|Cy_j - f(t_j)\| + \|f(t_j) - f(t)\| \\
&\leq \frac{t-t_j}{t_{j+1}-t_j}\|Cy_{j+1} - Cy_j\| + 2\epsilon \\
&\leq 2\epsilon + \sup_j \sup_{t\in[t_j,t_{j+1}]} \|Cy_{j+1} - f(t_{j+1})\| \\
&\quad + \|f(t_{j+1}) - f(t_j)\| + \|f(t_j) - Cy_j\| \leq 5\epsilon.
\end{aligned}$$

Let $h \in C([0,T];X)$ be such that $\|T_{K_1}h - g\| \leq \epsilon$. Then

$$\begin{aligned}
\|T_{CK_1}h - f\| &= \sup_{t\in[0,T]} \|\int_0^t CK_1(t-s)h(s)\,ds - f(t)\| \\
&\leq \sup_{t\in[0,T]} \|C\int_0^t K_1(t-s)h(s)\,ds - Cg(t)\| + \|Cg(t) - f(t)\| \\
&\leq \|C\|\epsilon + 5\epsilon = (\|C\| + 5)\epsilon.
\end{aligned}$$

Thus the image of T_{CK_1} is dense.

In order to show that the range of $T_{K_2 \star K_1}$ is dense, let $f \in C([0,T];Z)$. Then there exists $g \in C([0,T];Y)$ such that $\|K_2 \star g\| \leq \epsilon$. Pick $h \in C([0,T];Z)$ such that $\|K_1 \star h - g\| \leq \epsilon$. Then

$$\begin{aligned}
\|T_{K_2 \star K_1} h - f\| &= \sup_{t\in[0,T]} \| \int_0^t K_2(t-s) K_1 \star h(s)\, ds - f(t) \\
&\leq \sup_{t\in[0,T]} \| \int_0^t K_2(t-s) K_1 \star h(s)\, ds \int_0^t K_2(t-s) g(s)\, ds \| \\
&\quad + \| \int_0^t K_2(t-s) g(s)\, ds - f(t) \| \\
&\leq \int_0^t \|K_2(t-s)\|\, \|K_1 \star h(s) - g(s)\|\, ds + \epsilon \leq TM\epsilon + \epsilon,
\end{aligned}$$

where $M = \sup_{t\in[0,T]} \|K_2(t)\|$. Thus the range of $T_{K_2 \star K_1}$ is dense in $C([0,T];Z)$ $\diamondsuit$

As an application of the previous theorem we show that for strongly continuous semigroups the image of the convolution transform is dense.

Proposition 3.16. *Let $(K(t))_{t\geq 0}$ be a strongly continuous semigroup on a Banach space X and $T > 0$. Then the convolution operator $T_K : C([0,T];X) \to C([0,T];X)$ defined by $T_K f := \int_0^t K(t-s) f(s)\, ds$ is one-to-one and has dense range.*[12]

Proof. Let A be the generator of $(K(t))_{t\geq 0}$ and let λ_0 be in the resolvent set of A. We show first that

$$(F(t))_{t\geq 0} := \left((\lambda_0 - A) \int_0^t K(s)\, ds \right)_{t\geq 0}$$

(12) Since for all $g \in C([0,\infty);X)$, and for all $T > 0, \epsilon > 0$ there exist $f_T \in C([0,\infty);X)$ such that $\sup_{t\in[0,T]} \| \int_0^t K(t-s) f_T(s)\, ds - g(t) \| < \epsilon$, the image is also dense in the Frechét space $C([0,\infty);X)$.

satisfies condition (A). By definition, $F(t) = \lambda_0(1 \star K(t)) - K(t) + Id$. Thus $F(t)$ is strongly continuous. Furthermore, $\hat{F}(\lambda) = \int_0^\infty e^{-\lambda t} F(t)\, dt = \frac{\lambda_0 - A}{\lambda} R(\lambda, A)$ and hence $\hat{F}(\lambda)^{-1} = \lambda(\lambda - A)R(\lambda_0, A)$. Therefore

$$\begin{aligned}
\|x\| =& \|\hat{F}(\lambda)(\lambda(\lambda - A)R(\lambda_0, A)x)\| = \|(\lambda(\lambda - A)R(\lambda_0, A)\hat{F}(\lambda)x\| \\
\leq& \lambda\|(\lambda - \lambda_0)R(\lambda_0, A) + Id\|\|\hat{F}(\lambda)x\| \\
\leq& \lambda(\lambda\|R(\lambda_0, A)\| + \lambda_0\|R(\lambda_0, A)\| + 1)\|\hat{F}(\lambda)x\| \leq \lambda^2 C\|\hat{F}(\lambda)x\|
\end{aligned}$$

for some constant $C > 0$ and all $\lambda > \omega$ for some $\omega > 0$. Since

$$\|\hat{F}(\lambda)\| \leq \lambda_0(\frac{1}{\lambda}\|R(\lambda, A)\| + \|R(\lambda, A)\| + 1) \leq \frac{\tilde{M}}{\lambda}$$

we obtain $\|x\|\|\hat{F}(\lambda)\| \leq \lambda M\tilde{M}\|\hat{F}(\lambda)x\|$. Hence, the operator family $(F(t))_{t\in[0,T]}$ satisfies condition (A) and it is easy to see that $0 \in supp(F)$. Thus T_F is one-to-one (by Titchmarsh's theorem), and the image of T_F is dense in $C([0,T]; X)$ for all $T > 0$ (by Foias' theorem). By the previous theorem, the operator $R(\lambda_0, A)F = 1 \star K$ also has these properties. But if $(1 \star K) \star f_n$ converges to f, so does $K \star (1 \star f_n)$, and if $(1 \star K) \star f = 0$, then $f = 0$. Thus the convolution product with a strongly continuous semigroup has a dense range and is one-to-one.

Clearly, these statements can also be proved directly using semigroup methods without referring to Titchmarsh's and Foias' theorems. We will give the direct proofs next.

To see that $T_K f$ is one-to-one, observe that $\widehat{T_K f} = R(\lambda, A)\hat{f}(\lambda)$, where we identify f with its zero continuation onto $[0, \infty)$. Thus, if $T_K f = 0$, then $\hat{f} = 0$ for all sufficiently large λ, and thus $f = 0$.

For the semigroup proof of the density of the range of $T_K f$ in $C_0([0,T]; X)$ we show first that $C_0^1([0,T]; [\mathcal{D}(A)])$ is dense in $C_0([0,T]; X)$, where $[\mathcal{D}(A)]$ denotes the Banach space consisting of the domain $\mathcal{D}(A)$ of the generator A endowed with the graph norm $\|x\|_A := \|x\| + \|Ax\|$. It follows from the Hille-Yosida theorem that the operators $\lambda R(\lambda, A)$ are uniformly bounded $(\lambda > \lambda_0)$ and that $\lambda R(\lambda, A)x = R(\lambda, A)Ax + x \to x$ for all $x \in \mathcal{D}(A)$ as $\lambda \to \infty$. Thus by the density of the domain $\mathcal{D}(A)$, it follows from the Banach-Steinhaus theorem that $\lambda R(\lambda, A)x \to x$ for all $x \in X$, where the convergence is uniform for x in compact subsets of X. Let $g \in C_0[0, T]; X)$. Then

$$g_n : t \mapsto \frac{1}{h} \int_t^{t+h} g(s)\, ds - \frac{1}{h} \int_0^h g(s)\, ds \in C_0^1([0, T]; X)$$

and $g_n \to g$ as $h \to \infty$ uniformly on $[0, T]$. Let

$$g_{h,\lambda} : t \mapsto \lambda R(\lambda, A) g_h(t) \in C_0^1([0, T]; [\mathcal{D}(A)]).$$

Since $C_h := \{g_h(t) : t \in [0, T]\}$ is a compact subset of X, it follows that $g_{h,\lambda} \to g_h$ as $\lambda \to \infty$. Now for $f_{h,\lambda} \in C([0, T]; X)$, we have that $T_K f_{h,\lambda} = g_{h,\lambda}$ if and only if $\hat{T}_K \hat{f}_{h,\lambda}$ or $\hat{f}_{h,\lambda}(\mu) = (\mu - A)\hat{g}_{h,\lambda}(\mu) = \mu \hat{g}_{h,\lambda}(\mu) - \widehat{Ag_{h,\lambda}}(\mu)$. Thus, $f_{h,\lambda} : t \mapsto g'_{h,\lambda}(t) - Ag_{h,\lambda}(t) \in C([0, T]; X)$ satisfies $T_K f_{h,\lambda} \to g$. Thus, the range of T_K is dense in $C_0([0, T]; X)$. $\Diamond$

Consider the inhomogeneous abstract Cauchy problem

$$u'(t) = Au(t) + f(t), \quad u(0) = 0, \qquad (ICP)$$

where A is the generator of a strongly continuous semigroup $K(t)$ on a Banach space X, and where $f \in C([0, \infty); X)$ is a forcing term. Then the unique mild solution of (ICP) is given by

$$u(t) := \int_0^t K(t-s)f(s)\,ds = (T_K f)(t)$$

(see, for example A. Pazy [Pa], or J. Goldstein [Go]). By the previous proposition, the range of $T_K f$ is dense in $C_0([0,T];X)$. Thus for any given target function $z \in C_0([0,T];X)$ and any $\epsilon > 0$, there exists a forcing term $f_\epsilon \in C([0,T];X)$ such that the solution $u_\epsilon = T_K f_\epsilon$ of the evolutionary system described by (ICP) satisfies

$$\|T_K f_\epsilon - z\| = \sup_{t\in[0,T]} \|u_\epsilon(t) - z(t)\| < \epsilon.$$

Therefore any evolutionary system governed by a strongly continuous semigroup can be steered by appropriate forcing terms $f_\epsilon \in C([0,T];X)$ to any preassigned target orbit $z \in C_0([0,T];X)$.

Furthermore, since $\|T_K f_\epsilon - z\| = \|u_\epsilon - z\| < \epsilon$, it follows that there exists a generalized forcing term $f_0 \in \overline{C([0,T];X)}^{T_K}$ such that $f_\epsilon \to f_0 \in \overline{C([0,T];X)}^{T_K}$ and

$$z(t) = (\tilde{T}_K f_0)(t) = \lim_{\epsilon\to 0}(T_K f_\epsilon)(t) = \lim_{\epsilon\to 0} u_\epsilon,$$

uniformly in $t \in [0,T]$, where u_ϵ is the solution of

$$u_\epsilon'(t) = Au_\epsilon(t) + f_\epsilon, \quad u_\epsilon(0) = 0.$$

In this sense, any target $z \in C_0([0,T];X)$ is a "limit solution" or generalized solution of the inhomogeneous problem

$$u'(t) = Au(t) + f_0(t), \quad u(0) = 0,$$

for a unique forcing term $f_0 \in \overline{C([0,T];X)}^{T_K}$.

Having Titchmarsh's and Foias' theorems to our disposal, we do expect similar results for cosine families, solutions of Volterra integral equations and other abstract integro-differential equations that can be treated with the Laplace transform. However, the scope of that investigation is going beyond the range of topics treated in this dissertation. We will return to the concept of "limit solution" in Chapter IV. There we will show how this concept arises naturally in the study of abstract Cauchy problems

$$u'(t) = Au(t); \quad u(0) = x,$$

where the operator A generates an integrated or k-generalized semigroup on a given Banach space X.

II.4 The Vector Space of Generalized Functions

In this section we will show that the generalized functions deduced from the convolution product with a scalar valued function form, in the scalar-valued case a field, and in the vector-valued case a vector space over that field. Extending the domain of the generalized function spaces and considering all k-generalized vector-valued functions on $[\alpha, \infty)$ for some $\alpha \in \mathbb{R}$, we obtain a class of generalized functions that includes the abstract quotient field developed by J. Mikusinski.

But first, we want to shed some light on the structure and interplay between different generalized function spaces that were obtained by the convolution product. In Section II.1 we introduced the notion of a generalized derivative of continuous functions. This led to the following tower of generalized function spaces, where T denotes the antiderivative operator $Tf : t \mapsto \int_0^t f(s)\,ds$.

$$\begin{tikzcd}
\vdots & \vdots \\
C^{-n}[0,1] \arrow[r, "T", "\text{isom. isomorph}"'] & C^{-n-1}[0,1] \\
\vdots & \vdots \\
C^{-2}[0,1] \arrow[r, "T", "\text{isom. isomorph}"'] & C^{-1}[0,1] \\
 & \cup \\
C^{-1}[0,1] \arrow[uu, hook] \arrow[r, "T", "\text{isom. isomorph}"'] & C_0[0,1] \\
 & \cup \\
C_0[0,1] \arrow[uu, hook] \arrow[r, "T", "\text{isom. isomorph}"'] & C_0^1[0,1]
\end{tikzcd}$$

Since integrating is the same as convoluting with the constant one function, we will consider now towers of generalized function spaces generated by convolution operators T_k. In the last section we showed that the convolution operator T_k is one to one and has a dense range for $k \in C[0,T]$ with $0 \in supp(k)$. In the case that $k \in L^1[0,1]$ with $0 \in supp(k)$, the resulting convolution operator $T_k : C([0,1];X) \to C_0([0,1];X)$ is also injective and has a dense range in $C_0[0,1]$, since $1 \star k$ is continuous and $0 \in supp(1 \star k)$. Thus $T_{1\star k}$ is an injective operator with a dense range. But since $k \star f = 0$ implies that $1 \star k \star f = 0$, and hence $f = 0$, we obtain that T_k is injective and since $1 \star k \star f_n \to g$ implies that $k \star (1 \star f_n) \to g$ we have that the range of T_k is dense. Thus the results of Section II.2 apply to $k \in L^1[0,1]$ as well. In particular, Theorem 2.1 yields the following diagram for any $k \in L^1[0,1]$ with $0 \in supp(k)$.

$$
\begin{tikzcd}
\overline{C[0,1]}^{T_k} \arrow[r, "\tilde{T}_k", "\text{isom. isomorph}"'] & C_0[0,1] \\
C[0,1] \arrow[u, hook] \arrow[r, "T_k"] & Im(T_k). \arrow[u, phantom, "\cup"]
\end{tikzcd}
$$

To simplify notation, from now on we will identify the operator $\tilde{T}_k$ with T_k.

As special cases, consider $k_\alpha : t \mapsto \frac{t^{\alpha-1}}{\Gamma(\alpha)}$ for $\alpha > 0$. Then $k_\alpha \in L^1[0,1]$ and we denote with T_α the convolution operator $f \mapsto k_\alpha \star f$. Then $T_\alpha T_\beta = T_{\alpha+\beta}$. Note that in following diagram we consider T_α as an operator from $C_0[0,1]$ into itself. By Theorem 2.2, $\overline{C_0[0,1]}^{T_\beta}$ can be obtained by extending $\overline{C_0[0,1]}^{T_\alpha}$ with $T_{\beta-\alpha}$. Let $0 < \alpha \leq \beta < n$. Then

$$
\begin{tikzcd}
\vdots & \vdots \\
C^{-n}[0,1] \arrow[r, "T_n", "\text{isom. isomorph}"'] & C_0[0,1] \\
\overline{C_0[0,1]}^{T_\beta} \arrow[u, hook] \arrow[r, "T_\beta", "\text{isom. isomorph}"'] & C_0[0,1] \arrow[u, phantom, "||"] \\
\overline{C_0[0,1]}^{T_\alpha} \arrow[u, hook] \arrow[r, "T^\alpha", "\text{isom. isomorph}"'] & C_0[0,1] \arrow[u, phantom, "||"] \\
C_0[0,1] \arrow[u, hook] \arrow[r, "T^\alpha"] & ImT^\alpha \arrow[u, phantom, "\cup"]
\end{tikzcd}
$$

Let $\phi : t \mapsto e^{-\frac{1}{t}}$. Then $\phi^{(n)} := \left(\frac{d}{dt}\right)^n \phi \in C_0[0,1]$, $\phi^{(n)}(0) = 0$, and hence $T_n\phi^{(n)} = \frac{t^{n-1}}{n-1!} \star \phi^{(n)} = \phi$ for all $n \in \mathbb{N}$. Hence, by Theorem 2.2 (c), all the $C_0^{-n}[0,1]$-spaces are contained in the generalized function space $\overline{C[0,1]}^{T_\phi}$. Thus

$$C_0[0,1] \hookrightarrow \overline{C[0,1]}^{T_\alpha} \hookrightarrow \overline{C[0,1]}^{T_\beta} \hookrightarrow C^{-n}[0,1] \hookrightarrow \overline{C[0,1]}^{T_\phi}$$

for all $0 < \alpha < \beta \leq n$. Since distributions are, at least locally, distributional derivatives of some order n, all distributions with support in $(0,1)$ are contained in some $C_0^{-n}[0,1]$ space. The generalized function $T_\phi^{-1}(t \mapsto t)$ cannot be a distribution, since that would imply that $\frac{t^n}{n!} \star T_\phi^{-1}(t \mapsto t)$ is a continuous function f for some $n \in \mathbb{N}$, and therefore that $\frac{t^{n+2}}{n+2!} = e^{-\frac{1}{t}} \star f$. Hence

$$1 = \left(\frac{d}{dt}\right)^{n+2} (e^{-\frac{1}{t}} \star f) = \left(\left(\frac{d}{dt}\right)^{n+2} e^{-\frac{1}{t}}\right) \star f \in C_0[0,1].$$

This is a contradiction. Therefore the space $\overline{C[0,1]}^{T_\phi}$ contains more objects than just distributions.

If we look at arbitrary generalized function spaces deduced from two convolution operators T_{k_1} and T_{k_2}, with $k_1, k_2 \in L^1[0,1]$, Theorem 2.2 yields the following structure.

$$\begin{tikzcd}
\overline{C[0,1]}^{T_{k_1\star k_2}} & \overline{C[0,1]}^{T_{k_2}} \arrow[l, hook'] \\
\overline{C[0,1]}^{T_{k_1}} \arrow[u, hook] & C_0[0,1] \arrow[l, hook'] \arrow[u, hook]
\end{tikzcd}
\qquad
\begin{tikzcd}
\overline{C[0,1]}^{T_{k_1\star k_2}} & & \overline{C[0,1]}^{T_{k_2}} \arrow[ll, "\tilde{T}_{k_1}"', "\text{isom. isomorph}"] \\
\overline{C[0,1]}^{T_{k_1}} \arrow[u, "\tilde{T}_{k_2}", "\text{isom.isom.}"'] & & C_0[0,1]. \arrow[ll, "T_{k_1}"', "\text{isom. isomorph}"] \arrow[u, "\text{isom.isom.}", "T_{k_2}"']
\end{tikzcd}$$

For elements $f \in \overline{C[0,1]}^{T_{k_1}}$ we have that $T_{k_2}T_{k_1}f = T_{k_2\star k_1}f$, whether f is considered as a generalized function in $\overline{C[0,1]}^{T_{k_1}}$ or as its embedding in $\overline{C[0,1]}^{T_{k_1\star k_2}}$.

Thus we identify an element $f \in \overline{C[0,1]}^{T_{k_1}}$ with an element $\tilde{f} \in \overline{C[0,1]}^{T_{k_2}}$, if their embeddings in $\overline{C[0,1]}^{T_{k_1 \star k_2}}$ are the same; i.e., if

$$k_2 \star k_1 \star f = k_1 \star k_2 \star \tilde{f}.$$

This identification enables us to define the convolution between two generalized functions.

Definition 4.1. Let $k_1, k_2 \in L^1[0,1]$ with $0 \in supp(k_1) \cap supp(k_2)$. Let $f \in \overline{C[0,1]}^{k_1}$ and $g \in \overline{C[0,1]}^{k_2}$. Then we define the convolution between the two generalized functions via

$$f \star g := T^{-1}_{k_1 \star k_2} \left(k_1 \star f \star k_2 \star g \right).$$

Clearly, $f \star g \in \overline{C[0,1]}^{k_1 \star k_2}$ is again a generalized function. Next, we show that this convolution is well defined: we show that for $f \in \overline{C[0,1]}^{k_1} \cap \overline{C[0,1]}^{k_2}$ the convolution $f \star g$ is independent of the representative for f; i.e., it does not matter whether f is considered as an element of $\overline{C[0,1]}^{k_1}$ or as an element of $\overline{C[0,1]}^{k_2}$. We know that

$$f = T^{-1}_{k_1 \star k_2} \left(k_1 \star k_2 \star f \right) \in \overline{C[0,1]}^{k_1 \star k_2}.$$

Thus, for $g \in \overline{C[0,1]}^{k}$ for some $k \in L^1[0,1]$ with $0 \in supp(k)$ we obtain that

$$f \star g = T^{-1}_{k_1 \star k_2 \star k} \left(k_1 \star k_2 \star f \star k \star g \right),$$

which is independent to the choice of the space of f.

Furthermore, let k_1, k_2 be as above and let $h := T^{-1}_{k_1 \star k_1 \star 1}(1 \star k_1)$. Then for all $f \in \overline{C[0,1]}^{T_{k_2}}$, $k_1 \star f \in \overline{C[0,1]}^{T_{k_2}}$ and

$$h \star (k_1 \star f) = T^{-1}_{k_1 \star k_1 \star 1 \star k_2}(1 \star k_1 \star k_1 \star k_2 \star f) = f. \tag{1}$$

Thus $T_h = T^{-1}_{k_1}$. This shows that the inverse convolution transform of functions k with $0 \in supp(k)$ is again a convolution, a convolution with a particular generalized function.

Through the additive component of the generalized function spaces, we will always have to deal with generalized functions that do not have zero in their support.(1) Thus in order to obtain a field with respect to addition and convolution we have to take a closer look at convolutions of functions that are zero on some interval $[0, a]$.

Let $k \in C[0, \infty)$ with $k = 0$ on $[0, a]$ for some $a > 0$. Then

$$\begin{aligned} k \star f(t) &= \int_0^t f(t-s)k(s)\,ds = \int_a^t f(t-s)k(s)\,ds \\ &= \int_0^{t-a} f(t-a-s)k(s+a)\,ds = (k_{-a} \star f)(t-a), \end{aligned}$$

where $k_{-a}(t) := k(t+a)$; i.e., the graph of k_{-a} is the graph of k shifted by a to the left. Thus, convoluting with a function k that is zero on some interval $[0, a]$ is the same as convoluting with the shifted function k_{-a} and then shifting back the result.

Therefore if the "shift"-operations were invertible, we could define the inverse of the convolution transform with respect to functions that do not have zero in

(1) Let $f \in \overline{C([0,1]}^{T_k}$. Then $supp(f) := supp(y)$, where $T_k f = k \star f = y \in C_0[0,1]$.

their support. Unfortunately, only the right-shift is invertible on $[0,\infty)$ and neither shift is invertible on $[0,1]$. On $(-\infty,\infty)$, however, they are invertible. Thus, if we regard functions f in $C_0[0,\infty)$ as functions $f_\infty \in C(-\infty,\infty)$ with

$$f_\infty(t) := \begin{cases} f(t) & \text{for } 0 \leq t \\ 0 & \text{else,} \end{cases}$$

the shift-operator S_α with $S_\alpha f(t) = f(t-\alpha)$ for some $\alpha \in \mathbb{R}$ is an invertible operator. The resulting functions can be identified with functions in $C_0[\alpha,\infty)$ for some $\alpha \in \mathbb{R}$. With this embedding of the continuous functions $C_0[\alpha,\infty)$ into $C(-\infty,\infty)$ we have to adapt the definition of the convolution operator in order to keep it one-to-one.

For $k \in [a,\infty)$ and $f \in C([b,\infty);X)$ for some constants $a, b \in \mathbb{R}$, define

$$k \star f : t \mapsto \int_{-\infty}^{\infty} k_\infty(t-s) f_\infty(s)\, ds,$$

where k_∞ and f_∞ are the functions obtained by extending k and f with zero to the left. With this interpretation we follow J. Mikusinski's interpretation of the shift operator in the context of his operational calculus ([Mi2] §66). If $a = b = 0$, then obviously this convolution coincides with the convolution discussed so far. We cannot use the space $C(-\infty,\infty)$ as a seed for our generalized function spaces, since it does not have the property that the convolution transform is one-to-one. For example

$$1 \star \frac{x}{1+x^2} := \int_{-\infty}^{\infty} \frac{t}{1+t^2}\, dt = 0.$$

For functions $f, g \in C(-\infty, \infty)$ with $f = 0$ on $(-\infty, \alpha]$ and $g = 0$ on $(-\infty, \beta]$ we have that

$$\begin{aligned}
f \star g(t) = &\int_{-\infty}^{\infty} f(t-s)g(s)\,ds = \int_{\beta}^{t-\alpha} f(t-s)g(s)\,ds \\
= &\int_{0}^{t-\alpha-\beta} f(t-\beta-s)g(s+\beta)\,ds \\
= &\int_{0}^{t-\alpha-\beta} f(t-\alpha-\beta-s+\alpha)g(s+\beta)\,ds \\
= &\int_{0}^{t-\alpha-\beta} f_{-\alpha}(t-\alpha-\beta-s)g_{-\beta}(s)\,ds \\
= &f_{-\alpha} \star g_{-\beta}(t-\alpha-\beta) = S_{\alpha+\beta}\left(S_{-\alpha}f \star S_{-\beta}g\right)(t).
\end{aligned}$$

Since the right shift $S_{-\alpha}f$ and $S_{-\beta}g$ are zero on $(-\infty, 0)$ and the convolution transform is one-to-one on $C[0, \infty)$, we obtain that the convolution product on the space of continuous functions that are zero on the interval $(-\infty, \alpha]$ for some $\alpha \in \mathbb{R}$ has no zero divisors. Furthermore, if $f = 0$ on $(-\infty, \alpha]$ and $g = 0$ on $(-\infty, \beta]$, then $f \star g = 0$ on $(-\infty, \alpha + \beta]$.

We can extend function spaces $C_0[a, \infty)$ to $\overline{C_0[a,\infty)}^{T_k}$ for $T_k : C_0[a, \infty) \to C_0[a, \infty)$ with

$$T_k f : t \mapsto \int_a^t k(t-s)f(s)\,ds$$

for $k \in C[0, \infty)$ by taking the completion of the space $C[a, \infty)$ with respect to the seminorms $\|f\|_{\alpha,T_k} := \sup_{t\in[a,\alpha]} \|T_k f(t)\|$. Any generalized function $f \in \overline{C_0[b,\infty)}^{T_k}$ is naturally embedded in $\overline{C_0[a,\infty)}^{T_k}$ for $a \leq b$ by identifying $k \star f$ with its zero-extension to $[a, \infty)$ and then taking T_k^{-1}. Recall that we identify $f \in \overline{C_0[a,\infty)}^{T_{k_1}}$ with $g \in \overline{C_0[a,\infty)}^{T_{k_2}}$, if $k_2 \star k_1 \star f = k_1 \star k_2 \star g$. Thus we consider two generalized functions $f_1 \in \overline{C_0[a_1,\infty)}^{T_{k_1}}$ and $f_2 \in \overline{C_0[a_2,\infty)}^{T_{k_2}}$ to be the same if

$$k_2 \star k_1 \star f_1 = k_1 \star k_2 \star f_2,$$

where the functions $k_2 \star k_1 \star f_1$ and $k_1 \star k_2 \star f_2$ are considered as functions in $C(-\infty, \infty)$ by extending them with zero to the left.

Theorem 4.2 (The field of generalized scalar functions). *Let*

$$\mathcal{F} := \{f \in \overline{C_0[a,\infty)}^{T_k} : a \in \mathbb{R}, k \in C[0,\infty) \text{ with } 0 \in supp(k)\},$$

and define

$$f \star g := T^{-1}_{k_1 \star k_2}(k_1 \star f \star k_2 \star g)$$

for $f \in \overline{C_0[a,\infty)}^{T_{k_1}}$ *and* $g \in \overline{C_0[a,\infty)}^{T_{k_2}}$. *Then* $\mathcal{F}$ *is a field with respect to addition and convolution.*

Proof. Clearly, with the above identification, $\mathcal{F}$ is an additive group. By the natural embedding of functions with different domains, the convolution is defined for all $f, g \in \mathcal{F}$. Furthermore, $f \star g = 0$ implies that $k_1 \star f \star k_2 \star g = 0$. Since the convolution transform is one-to-one, we obtain that either $k_1 \star f$ or $k_2 \star g = 0$ and hence $f = 0$ or $g = 0$. Let $f \in \overline{C_0[a,\infty)}^{T_k}$ with $a \in supp(k \star f)$. With the above identification, such a constant a always exists for $f \neq 0$. Then $S_{-a}(f \star k) := h \in C[0,\infty)$ with $0 \in supp(h)$. Let $f^{-1} := T_h^{-1}(S_{-a}k)$. Then for all $g \in \mathcal{F}$ we have that

$$f \star f^{-1} \star g = T^{-1}_{k \star h \star k_1}(k \star f \star h \star T_h^{-1} S_{-a}k \star k_1 \star g) = T^{-1}_{k \star h \star k_1}(S_{-a}h \star S_{-a}k \star k_1 \star g) = g.$$

Clearly, $f \star (g+h) = f \star g + f \star h$, and hence, $\mathcal{F}$ is a field. $\diamondsuit$

Since every element of the field is a generalized function and is therefore equal to $T_k^{-1} f$ for some continuous f and k, the field is the same as the one obtained via a field extension by J. Mikusinski.

So far we have only considered scalar-valued functions. Clearly, the vector-valued generalized functions cannot form a field. The convolution between two vector-valued functions is not defined if the vector space has no algebra structure. However, we can consider the spaces $\overline{C_0([(a,\infty);X)}^{T_k}$. As above, we can define the convolution of a generalized scalar-valued function with a vector-valued function. The vector-valued generalized functions form a vector space over the field of generalized functions. Thus integration (convoluting with the Heaviside function H_0), for example, and differentiation (convoluting with $H_0^{-1} = \delta_0'$) are "scalar" multiplications in this vector space.

Corollary 4.3 (The vector space of generalized functions). *Let X be a Banach space and let $\mathcal{F}$ be the field of generalized functions. Then*

$$\mathcal{V} := \{f \in \overline{C_0([a,\infty);X)}^{T_k} : a \in \mathbb{R}, k \in C[0,\infty) \text{ with } 0 \in supp(k)\}$$

is a vector space over $\mathcal{F}$ where scalar multiplication of a vector $f \in V$ with some $h \in \mathcal{F}$ is defined by

$$h \star f := T_{k_1 \star k_2}^{-1}(k_1 \star h \star k_2 \star f),$$

where $h \in \overline{C_0[a,\infty]}_{k_1}^{T}$ and $f \in \overline{C_0([a_2,\infty);X)}_{k_2}^{T}$.

III. The Asymptotic Laplace Transform

> "The modern Laplace transform is relatively recent. It was first used by Bateman in 1910, explored and codified by Doetsch in the 1920s and was first the subject of a textbook as late as 1937. In the 1920s and 1930s it was seen as a topic of front-line research; the applications that call upon it today were then treated by an older technique – the Heaviside operational calculus. This, however, was rapidly displaced by the Laplace transform and by 1950 the exchange was virtually complete. No other recent development in mathematics has achieved such ready popularisation and acceptance among the users of mathematics and the designers of undergraduate curricula."
>
> M. A. B. Deakin [De2].

The asymptotic Laplace transform is an extension of the classical Laplace transform. It was introduced in 1939 by J. C. Vignaux [Vi] and further investigated by J. C. Vignaux and M. Cotlar [Vi-Co], [Vi2] in 1944. It allows one to transform functions and generalized functions of arbitrary growth at infinity, while preserving all operative features of the classical Laplace transform. It fills the gap between the operational calculus developed by J. Mikusinski and the classical Laplace transform method. J. Mikusinski noted in Part V, Chapter II, §11 in [Mi2]: " In spite of the formal resemblance of the Laplace transform method and the direct method, the two methods are not equivalent. [...] This (the Laplace transform) method restricts the range of applicability of the operational calculus to a class of functions for which the integral

$$\int_0^\infty e^{-ts} f(t)\, dt$$

is convergent. [...] The Laplace transform method does not provide the full solution of the problem since it is necessary to assume during the calculation that the function sought does not increase too fast, i.e., to be more exact, that it is transformable. Consequently we do not know if the solution obtained is unique."

It was observed by W. A. Ditkin and P. L. Kusnezow [Di-Ku], and L. Berg [Ber], that the multiplicative field obtained in the image space of the asymptotic Laplace transform corresponds to the quotient field of J. Mikusinski. The advantage of the transform approach, especially in the vector valued case, is that the emphasis is kept on the approximative nature of the generalized functions and that one associates via the asymptotic Laplace transform a class of analytic, vector-valued functions with any element of Mikusinski's abstract quotient space. This perception allows the study of vector-valued differential and integral equations involving unbounded linear operators.

The basic approach to asymptotic Laplace transforms presented and extended here is due to G. Lumer and F. Neubrander [Lu-Ne]. The main difference to their work is that we consider functions $f \in L^1_{loc}[0,T);X)$ for $0 < T \leq \infty$, whereas G. Lumer and F. Neubrander consider only functions in the spaces $L^1[0,T];X)$ and $L^1_{loc}([0,\infty);X)$. Moreover, we restrict the asymptotic Laplace transform from the beginning to equivalence classes of analytic functions, whereas G. Lumer and F. Neubrander allow arbitrary functions.

III.1 Asymptotic Laplace Transforms for Functions of Arbitrary Growth

The goal of this section is to extend the Laplace transform $\hat{f}(\lambda) := \int_0^\infty e^{-\lambda t} f(t)\, dt$ to functions of arbitrary growth at infinity in such a way that the main operative features of the Laplace transform are still valid. The asymptotic Laplace transform $\mathcal{L}$ is a set-valued operator that assigns to every $f \in L^1_{loc}([0,\infty);X)$ an equivalence class of certain analytic functions with the following properties.

(a) $\hat{f} \in \mathcal{L}(f)$, if f is Laplace transformable.

(b) $\mathcal{L}$ is linear.

(c) $\mathcal{L}(f) \cap \mathcal{L}(g) \neq \emptyset$ if and only if $f = g$.

(d) $\mathcal{L}(f \star g) = \mathcal{L}(f) \cdot \mathcal{L}(g) + \mathcal{L}(0)$.

(e) $\mathcal{L}(f') = \lambda \mathcal{L}(f) - f(0)$.

(f) $\mathcal{L}(-tf) = (\mathcal{L}(f))' + \mathcal{L}(0)$.

The asymptotic Laplace transform does not only exist for functions of arbitrary growth on $[0,\infty)$, but also for functions that have a finite time blowup; i.e., for functions $f \in L^1_{loc}([0,T);X)$ with arbitrary growth in $[0,T)$. The asymptotic Laplace transform of such functions will be denoted by $\mathcal{L}_T(f)$ and has the same operational features than $\mathcal{L}$. As mentioned above, the main difference between the Laplace transform $f \mapsto \hat{f}$ and its asymptotic versions $f \mapsto \mathcal{L}(f)$ or $f \mapsto \mathcal{L}_T(f)$, is that the latter are set-valued. They consist of equivalence classes of analytic functions $q : \Omega \to X$, where the complex domain Ω is a post-sectorial region containing some real halfline (ω,∞).

Definition 1.1. Let $\Omega \subset \mathbb{C}$ be an open set. We say that Ω is a *post-sectorial region* if for all $-\pi/2 < \alpha < \pi_2$ there exists a constant $\lambda_\alpha > 0$, such that

$$\{\lambda e^{i\beta} : -|\alpha| \leq \beta \leq |\alpha|, \lambda > \lambda_\alpha\} \subset \Omega.$$

We denote with $\mathcal{A}(P;X)$ the vector space of all analytic functions defined in some postsector Ω with values in a Banach space X.

On the space $\mathcal{A}(P;X)$ we define an equivalence class via the equivalence relation $\approx_T$, where $r \approx_T q$ if $r-q$ and its derivatives are of exponential decay T; i.e., for $T \in \mathbb{R}$ we define

$$r \approx_T q \text{ if } \limsup_{\lambda\to\infty} \frac{1}{\lambda} \ln \|r^{(n)}(\lambda) - q^{(n)}(\lambda)\| \leq -T$$

for all $n \in \mathbb{N}$.

Proposition 1.2. *The relation $r \approx_T q$ is an equivalence relation on $\mathcal{A}(\mathcal{P};X)$. The relation $r \approx_T 0$ defines a linear subspace $\{0\}_T$ of $\mathcal{A}(\mathcal{P},X)$, and the quotient space $\hat{\mathcal{A}}_T(\mathcal{P};X)/\{0\}_T$ is a vector space. Furthermore, if $q \in \mathcal{A}(\mathcal{P};\mathbb{C})$ with $q \approx_S 0$ and $r \in \mathcal{A}(\mathcal{P};X)$ with $r \approx_T 0$, then $qr \in \mathcal{A}(\mathcal{P};X)$ with $qr \approx_{T+S} 0$.*

Proof. Clearly, "$\approx_T$" is an equivalence relation, since 1) $r \approx_T r$, 2) $r \approx_T q$ if and only if $q \approx_T r$, and 3) $p \approx_T q$ and $q \approx_T r$ implies that for all $\epsilon > 0$ there exists $\lambda_\epsilon > 0$ such that

$$\|p^{(n)}(\lambda) - r^{(n)}(\lambda)\| \leq \|p^{(n)}(\lambda) - q^{(n)}(\lambda)\| + \|q^{(n)}(\lambda) - r^{(n)}(\lambda)\| \leq 2e^{\epsilon\lambda}$$

for all $\lambda > \lambda_\epsilon$. Hence $p \approx_T r$.

Clearly, $r \approx_T 0$, implies that $\alpha r \approx_T 0$. Let $r \approx_T 0$ and $q \approx_T 0$. Then $-q \approx_T 0$ and thus $r \approx_T -q$ which implies, by definition, that $r+q \approx_T 0$. Thus $r \approx_T 0$ defines a linear subspace in $\mathcal{A}(\mathcal{P};X)$. The last statement follows from the fact that

$$\| (q(\lambda)r(\lambda))^{(n)} \| \leq \sum_{j=0}^{n} \binom{n}{j} \|q(\lambda)^{(j)}\| \, \|r(\lambda)^{(n-j)}\| \leq 2^n e^{-\lambda(S-\epsilon)} e^{-\lambda(T-\epsilon)}$$
$$= 2^n e^{-\lambda(T+S-2\epsilon)}$$

for λ large enough. $\diamondsuit$

Definition 1.3. Let $0 < T \leq \infty$, let $f \in L^1_{loc}([0,T); X)$. A function $q \in \mathcal{A}(\mathcal{P}; X)$ is an asymptotic Laplace transform of f, i.e., $q \in \mathcal{L}_T(f)$, if and only if

$$q(\lambda) \approx_t \int_0^t e^{-\lambda s} f(s) \, ds$$

for all $t \in [0,T)$.[1]

We first show that for all $f \in L^1_{loc}([0,T); X)$ the asymptotic Laplace transform $\mathcal{L}_T$ exists.

Theorem 1.4 (Existence). *If $f \in L^1_{loc}([0,T); X)$, then*

$$\lambda \mapsto q(\lambda) := \lambda \int_0^T e^{-\lambda t}(1 - e^{-\frac{\lambda \ln \lambda}{G(t)}}) F(t) \, dt \in \mathcal{L}_T(f),$$

where $F(t) := \int_0^t f(s) \, ds$, and $G(t) := \max\{\|F(t)\|, 1\}$.

Proof. Notice that

$$|1 - e^{-\mu}| = |\mu \int_0^1 e^{-\mu t} \, dt| \leq |\mu|$$

for all $\mu \in \mathbb{C}$ with $Re(\mu) > 0$. Let

$$\lambda \in \Omega := \{\lambda \in \mathbb{C} : Re(\lambda) \geq 0 \text{ and } Re(\lambda \ln \lambda) > 0\}.$$

[1] To keep notation as simple as possible, we sometimes write $r(\lambda) \approx_t q(\lambda)$ instead of $(\lambda \mapsto r(\lambda)) \approx_t (\lambda \mapsto q(\lambda))$.

Then

$$\begin{aligned}\int_0^T \|e^{-\lambda s}(1-e^{-\frac{\lambda\ln\lambda}{G(s)}})F(s)\|\,ds =& \int_0^T e^{-\lambda s}|1-e^{-\frac{\lambda\ln\lambda}{G(s)}}|\|F(s)\|\,ds\\ \le& \int_0^T e^{-Re(\lambda)s}\frac{|\lambda\ln\lambda|}{G(s)}\|F(s)\|\,ds\\ \le& |\lambda\ln\lambda|\int_0^T e^{-Re(\lambda)s}\,ds \le \frac{|\lambda\ln\lambda|}{Re(\lambda)}.\end{aligned}$$

Thus $q(\lambda) = \lambda\int_0^T e^{-\lambda s}(1-e^{-\frac{\lambda\ln\lambda}{G(s)}})F(s)\,ds$ exists for all $\lambda \in \Omega > 0$. It is easy to see that q is an analytic function on Ω. Moreover, Ω is a post-sector since

$$Re(\lambda\ln\lambda) = Re(re^{i\alpha}(\ln r + i\alpha)) = r\ln r\cos\alpha - \alpha r\sin\alpha > 0$$

if and only if $r > e^{\alpha\tan\alpha}$, where $\lambda =: re^{i\alpha}$.

Now we show that $q \in \mathcal{L}_T(f)$. For all $\lambda \in \Omega$ and all $n \in \mathbb{N}_0$, we have that

$$\begin{aligned}&\|\left(\frac{d}{d\lambda}\right)^n\int_0^t e^{-\lambda s}f(s)\,ds - \left(\frac{d}{d\lambda}\right)^n q(\lambda)\|\\ =&\|\left(\frac{d}{d\lambda}\right)^n\left(e^{-\lambda t}F(t)+\lambda\int_0^t e^{-\lambda s}F(s)\,ds - \lambda\int_0^T e^{-\lambda s}(1-e^{-\frac{\lambda\ln\lambda}{G(s)}})F(s)\,ds\right)\|\\ \le&\|\left(\frac{d}{d\lambda}\right)^n\left(\lambda\int_0^t e^{-\lambda s}e^{-\frac{\lambda\ln\lambda}{G(s)}}F(s)\,ds - \lambda\int_t^T e^{-\lambda s}(1-e^{-\frac{\lambda\ln\lambda}{G(s)}})F(s)\,ds\right)\|\\ &+\|\left(\frac{d}{d\lambda}\right)^n e^{-\lambda t}F(t)\|\\ \le&\|\left(\frac{d}{d\lambda}\right)^n\int_0^t \lambda e^{-\frac{\lambda\ln\lambda}{G(s)}-\lambda s}F(s)\,ds\| + \|\left(\frac{d}{d\lambda}\right)^n\lambda\int_t^T e^{-\lambda s}(1-e^{-\frac{\lambda\ln\lambda}{G(s)}})F(s)\,ds\|\\ &+\|\left(\frac{d}{d\lambda}\right)^n e^{-\lambda t}F(t)\|.\end{aligned}$$

We want to show that the right hand side is of exponential decay t. Clearly, the third term decays at that rate for any $n \in \mathbb{N}_0$. For the first term, there exists a function p with $|p(\lambda,s)| \le C(s^n\lambda^{n+1}+1)$ for all $s \in [0,t]$, $\lambda > 1$ and some constant

C such that

$$\| \left(\frac{d}{d\lambda}\right)^n \lambda \int_0^t e^{-\frac{\lambda \ln \lambda}{G(s)} - \lambda s} F(s)\, ds \| = \| \int_0^t p(\lambda, s) e^{-\frac{\lambda \ln \lambda}{G(s)} - \lambda s} F(s)\, ds \|$$
$$\leq e^{-\frac{\lambda \ln \lambda}{M}} \int_0^t e^{-\lambda s} M |p(\lambda, s)|\, ds,$$

where $M = \sup_{s \in [0,t]} \|F(s)\|$. Thus, the first term is of exponential decay of order ∞. For the second term, there exist functions q_j and p which satisfy $|q_j(\lambda, s)| + |p(\lambda, s)| \leq C_j(s^n \lambda^{n+1} + 1)$ for all $s \in [t, T)$, $\lambda > 1$ and some constant C_j such that

$$\begin{aligned}
&\int_t^T \| \left(\frac{d}{d\lambda}\right)^n \lambda e^{-\lambda s}(1 - e^{-\frac{\lambda \ln \lambda}{G(s)}}) F(s) \|\, ds \\
&= \int_t^T \|p(\lambda, s) e^{-\lambda s}(1 - e^{-\frac{\lambda \ln \lambda}{G(s)}}) F(s) + \sum_{j=1}^n \frac{q_j(\lambda, s) e^{-\frac{\lambda \ln \lambda}{G(s)} - \lambda s}}{(G(s))^j} F(s) \|\, ds \\
&\leq \int_t^T |p(\lambda, s)| e^{-\lambda s} \frac{|\lambda \ln \lambda|}{G(s)} \|F(s)\|\, ds + \int_t^T \sum_{j=1}^n \frac{|q_j(\lambda, s)| e^{-\frac{\lambda \ln \lambda}{G(s)} - \lambda s}}{(G(s))^j} \|F(s)\|\, ds \\
&\leq |\lambda \ln \lambda| \int_t^T p(\lambda, s) e^{-\lambda s}\, ds + \int_t^T \sum_{j=1}^n |q_j(\lambda, s)| e^{-\lambda s}\, ds.
\end{aligned}$$

Hence the third term is of exponential decay t and therefore $\int_0^t e^{-\lambda s} f(s)\, ds - q(\lambda) \approx_t 0$ for all $0 < t < T$ and hence $q \in \mathcal{L}_T(f)$. $\diamondsuit$

We proved the existence of an analytic element of the asymptotic Laplace transform defined on a post-sectorial region. In the above proof, in case that $T < \infty$ we can replace $\ln \lambda$ by T. Then $q(\lambda) := \lambda \int_0^T e^{-\lambda t}(1 - e^{\frac{-\lambda T}{G(t)}}) F(t)\, dt$ is analytic in the right halfplane and $q \in \mathcal{L}_T(f)$. J. C. Vignaux and M. Cotlar ([Vi-Co]) proved the existence of an element of the asymptotic Laplace transform which is analytic in the right halfplane also for the case $T = \infty$. Since their proof is not constructive and since for our purposes the existence in a post-sectorial region is fully sufficient, we will not go into details.

Now that we know that the asymptotic Laplace transform always exists, we will check the operative rules $(a)-(f)$ mentioned above.

Proposition 1.5. *Let $f \in L^1_{loc}([0,\infty);X)$. If $\hat{f}$ exists, then $\hat{f} \in \mathcal{L}(f)$.*

Proof. We recall the following facts from Laplace-transform theory (see [BaNe1]): If $\hat{f} := \lim_{T\to\infty} \int_0^T e^{-\lambda t} f(t)\,dt$ exists for some $\lambda_0 \in \mathbb{C}$, then $\hat{f}(\lambda)$ exists for all $\lambda \in \mathbb{C}$ with $Re(\lambda) > Re\lambda_0$. Moreover, $\hat{f}$ exists if and only if the antiderivative $F(t) := \int_0^t f(s)\,ds$ of f is exponentially bounded. Since

$$\hat{f}(\lambda) = \int_0^t e^{-\lambda s} f(s)\,ds + r_t(\lambda),$$

where

$$\begin{aligned} r_t(\lambda) :=& \int_t^\infty e^{-\lambda s} f(s)\,ds = -e^{-\lambda t} F(t) + \lambda \int_t^\infty e^{-\lambda s} F(s)\,ds \\ =& e^{-\lambda t} \left(-F(t) + \lambda \int_0^\infty e^{-\lambda t} F(s+t)\,ds \right), \end{aligned}$$

it follows that $r_t \approx_t 0$. Thus $\hat{f} \in \mathcal{L}(f)$. $\diamondsuit$

Proposition 1.6 (Linearity). *$\mathcal{L}_T : L^1_{loc}([0,T);X) \to \mathcal{A}(\mathcal{P},X)/\{0\}_T$ is a linear operator.*

Proof. Let $f \in L^1_{loc}([0,T);X)$ and $p,q \in \mathcal{L}_T(f)$. Then $p \approx_t q$ for all $t \in [0,T)$; i.e., $\limsup_{\lambda\to\infty} \frac{1}{\lambda} \ln \|p^{(n)}(\lambda) - q^{(n)}(\lambda)\| \leq -t$ for all $0 \leq t < T$. Hence $p \approx_T q$. Thus, the asymptotic Laplace transform $\mathcal{L}_T$ is a singlevalued operator from $L^1_{loc}([0,T);X)$ into $\mathcal{A}(\mathcal{P},X)$. It is easy to see that $\mathcal{L}_T$ is linear; i.e., that $\alpha\mathcal{L}_T(f) + \mathcal{L}_T(g) = \mathcal{L}_T(\alpha f + g)$ for all $f,g \in L^1_{loc}[0,T);X)$ and all $\alpha \in \mathbb{C}$. $\diamondsuit$

It is often more convenient to work in Banach spaces $L^1([0,t];X)$ for $0 < t < T$ instead of $L^1_{loc}([0,T);X)$. The following proposition provides the framework that allows the deduction from properties of the Laplace transform on $L^1[0,t]$ for $0 < t < T$ to properties of the asymptotic Laplace transforms on $L^1_{loc}([0,T);X)$.

Proposition 1.7. *Let $f \in L^1_{loc}([0,T);X)$. Then*

$$\mathcal{L}_T(f) = \bigcap_{0 \le t < T} \mathcal{L}_t(f).$$

Proof. Let $f \in L^1_{loc}([0,T);X)$ and $q \in \mathcal{L}_T(f)$. Then clearly, $q \in \bigcap_{0\le t<T} \mathcal{L}_t(f)$. By definition, for $0 \le \epsilon < T - t$, any function $p \in \mathcal{L}_{t+\epsilon}(f)$ satisfies $p \approx_t \int_0^t e^{-\lambda s} f(s)\, ds$. Thus for $p \in \bigcap_{0\le t<T} \mathcal{L}_t(f)$ we know that $p \approx_t \int_0^t e^{-\lambda s} f(s)\, ds$ for all $0 \le t < T$. Hence, $p \in \mathcal{L}_T(f)$. $\diamondsuit$

The uniqueness property "$\mathcal{L}_T(f) \cap \mathcal{L}_T(g) \neq \emptyset <=> f = g$" was already shown by J. C. Vignaux and M. Cotlar [Vi-Co], L. Berg [Ber], and Y. Lyubich [Ly]. We give a new proof and add a new inversion formula.

Theorem 1.8 (Uniqueness and Inversion). *Let $f, g \in L^1_{loc}([0,T);X)$ for some $0 < T \le \infty$. Then $\mathcal{L}_T(f) \cap \mathcal{L}_T(g) \neq \emptyset$ if and only if $f = g$. Furthermore, for any Müntz sequence sequence $(\beta_n)_{n\in\mathbb{N}}$ and $q \in \mathcal{L}_T(f)$, we have that*

$$\int_0^t f(s)\, ds = \lim_{k\to\infty} \sum_{n=1}^{N_k} \alpha_{k,n} e^{t\beta_{kn}} q(\beta_{kn}),$$

converging uniformly on compact subsets of $[0,T)$, where the constants $\alpha_{k,n}$ and N_k are the same as in Theorem II.3.1.

Proof. The proof is based on the Phragmén-Mikusinski inversion formula for Laplace transforms (see Theorem II.3.1). Clearly, $f = g$ implies that $\mathcal{L}_T(f) = \mathcal{L}_T(g)$.

Let $q \in \mathcal{L}(f) \cap \mathcal{L}(g)$. Then for all $0 < t < T$ there exists $\omega > 0$ and $r_f, r_g \in \mathcal{A}(\mathcal{P}, X)$ with $r_f, r_g \in \{0\}_t$ such that

$$q(\lambda) = \int_0^t e^{-\lambda s} f(s)\, ds + r_f(\lambda) = \int_0^t e^{-\lambda s} g(s)\, ds + r_g(\lambda).$$

By Corollary II.3.2, the Phragmén-Mikusinski inversion formula applied to r_f and r_g yields zero on $[0,t)$, and thus, for all $0 < s < t$,

$$\lim_{k\to\infty} \sum_{n=1}^{N_k} \alpha_{k,n} e^{s\beta_{kn}} q(\beta_{kj}) = \int_0^s f(u)\, du = \int_0^s g(s)\, ds.$$

Since t was chosen arbitrarily, the above equation holds for all $0 \leq s < T$ and hence $f = g$ almost everywhere on $[0,T)$. $\diamondsuit$

In applications to differential and integral equations, the most essential operational property of the Laplace transform is that it transforms convolution (in particular, integration and differentiation) into multiplication. It therefore transforms linear differential and integral equations into algebraic equations. We will show next that this crucial feature extends to the asymptotic Laplace transform; i.e., the convolution $k \star f$ between an operator-valued function K and a vector-valued function f is mapped by $\mathcal{L}_T$ to the application of $\mathcal{L}_T(K)$ to $\mathcal{L}_T(f)$.

By the uniform boundedness principle, any strongly continuous operator family $(K(t))_{t\in[0,T)} \subset \mathcal{L}(X,Y)$ is uniformly bounded on compact subsets of $[0,T)$. Thus $\hat{K}_t(\lambda) := \int_0^t e^{-\lambda s} K(s)\, ds$ is a bounded linear operator and $\lambda \mapsto \hat{K}_t(\lambda)$ is an entire function. As in the vector-valued case we define

$$\mathcal{L}_T(K) := \{Q \in \mathcal{A}(\mathcal{P}; \mathcal{L}(X,Y)) : Q(\lambda) \approx_t \int_0^t e^{-(\cdot)s} K(s)\, ds \text{ for all } t \in [0,T)\}.$$

The proofs of Theorem 1.4 and Theorem 1.8 are easily extendable to asymptotic Laplace transforms of strongly continuous operator families.

Theorem 1.9 (Transform of convolutions). *Let X, Y be Banach spaces, $0 < T \le \infty$, let $f \in C([0,T);X)$, and let $(K(t))_{t\in[0,T)} \subset \mathcal{L}(X,Y)$ be a strongly continuous operator family. Then*

$$\mathcal{L}_T(K \star f) = \mathcal{L}_T(K)\mathcal{L}_T(f) + \{0\}_T.$$

Proof. Let $0 < t < T$ and let $Q(\lambda) := \int_0^t e^{-\lambda s} K(s)\,ds + R(\lambda) \in \mathcal{L}_T(K)$ and $q(\lambda) := \int_0^t e^{-\lambda s} f(s)\,ds + r(\lambda) \in \mathcal{L}_T(f)$, where $R \approx_t 0$ and $r \approx_t 0$. Define

$$m(\lambda) := R(\lambda)\int_0^t e^{-\lambda s} f(s)\,ds + \int_0^t e^{-\lambda s} K(s) r(\lambda)\,ds + R(\lambda)r(\lambda).$$

Then

$$\begin{aligned}
Q(\lambda)q(\lambda) &= \int_0^t \int_0^t e^{-\lambda(s+u)} K(s) f(u)\,ds\,du + m(\lambda)\\
&= \int_0^t \int_u^{t+u} e^{-\lambda s} K(s-u) f(u)\,ds\,du + m(\lambda)\\
&= \int_0^t \int_0^s e^{-\lambda s} K(s-u) f(u)\,du\,ds + \int_t^{2t} \int_{s-t}^t e^{-\lambda s} K(s-u) f(u)\,du\,ds + m(\lambda)\\
&= \int_0^t e^{-\lambda s} (K \star f)(s)\,ds + e^{-\lambda t}\int_0^t e^{-\lambda s} \int_s^t K(s+t-u) f(u)\,du\,ds + m(\lambda)\\
&=: \int_0^t e^{-\lambda s} (K \star f)(s)\,ds + c(\lambda),
\end{aligned}$$

where $c(\lambda) = e^{-\lambda t}\int_0^t e^{-\lambda s} \int_s^t K(s+t-u) f(u)\,du\,ds + m(\lambda)$. Since $c \approx_t 0$ it follows that $\mathcal{L}_T(K)\mathcal{L}_T(f) \subset \mathcal{L}_T(K \star f)$.

Let $g \in \mathcal{L}_T(K \star f)$. Then, for all $0 < t < T$ there exists $d \approx_t 0$ such that $g(\lambda) = \int_0^t e^{-\lambda s}(K \star f)(s)\,ds + d(\lambda)$. Let $Q \in \mathcal{L}_T(K)$, and let $q \in \mathcal{L}_T(f)$. As shown above,

$$\int_0^t e^{-\lambda s}(K \star f)(s)\,ds = Q(\lambda)q(\lambda) - c(\lambda)$$

for some $c \approx_t 0$. Thus $g(\lambda) \approx_t Q(\lambda)q(\lambda)$ for all $t \in [0,T)$, and therefore $g(\lambda) \approx_T Q(\lambda)q(\lambda)$. Hence $g(\lambda) \in \mathcal{L}_T(K)\mathcal{L}_T(f) + \{0\}_T$. $\diamondsuit$

Corollary 1.10. *Let $f \in L^1_{loc}([0,T);X)$ be such that f' exists a.e. and $f' \in L^1_{loc}([0,T);X)$. Then $\mathcal{L}_T(f') = \lambda\mathcal{L}_T(f) - f(0)$.*

Proof. By Proposition 1.2 we know that $\frac{1}{\lambda}\{0\}_T = \{0\}_T$. Since $(1 \star f')(t) = \int_0^t f'(s)\,ds = f(t) - f(0)$, we obtain by the previous theorem and Proposition 1.3 that

$$\begin{aligned}\frac{1}{\lambda}\mathcal{L}_T(f') =&(\frac{1}{\lambda} + \{0\}_T)\mathcal{L}_T(f') + \{0\}_T = \mathcal{L}_T(1)\mathcal{L}_T(f') + \{0\}_T \\ =&\mathcal{L}_T(1 \star f') = \mathcal{L}_T(f - f(0)) = \mathcal{L}_T(f) - \frac{f(0)}{\lambda} + \{0\}_T = \mathcal{L}_T(f) - \frac{f(0)}{\lambda}.\end{aligned}$$

Hence, $\mathcal{L}_T(f') = \lambda\mathcal{L}_T(f) - f(0)$. ♢

The main difference in our definition of asymptotic Laplace transforms to the literature is that we restrict the range of $\mathcal{L}_T$ to analytic functions on post-sectors. This yields the following operational property, corresponding to Mikusinski's algebraic derivative (see [Mi2] pp.294), which is crucial, for example, in dealing with Laplace's equation

$$(a_2 t + b_2)y''(t) + (a_1 t + b_1)y'(t) + (a_0 t + b_0)y(t) = g(t).$$

Theorem 1.11. *Let $f \in L^1_{loc}([0,T);X)$. Then $\mathcal{L}(-tf) = (\mathcal{L}(f))'$.*

Proof. Let $q \in \mathcal{L}_T(-tf)$. Then, for all $0 < t < T$, there exists $r_t \approx_t 0$ such that

$$\begin{aligned}q(\lambda) =&\int_0^t e^{-\lambda s}(-s)f(s)\,ds + r_t(\lambda) = \frac{d}{d\lambda}\left(\int_0^t e^{-\lambda s}f(s)\,ds\right) + r_t(\lambda) \\ =&\frac{d}{d\lambda}\left(\int_0^t e^{-\lambda s}f(s)\,ds + R_t(\lambda)\right),\end{aligned}$$

where $R_t(\lambda) := -\int_{-\lambda}^{\infty} r_t(\mu)\,d\mu$ satisfies $R_t \approx_t 0$. Thus $q \in (\mathcal{L}_T(f))'$.

Now let $p \in \mathcal{L}_T(f)$. Then, for all $0 < t < T$, there exists $r_t \approx_t 0$ such that $p(\lambda) = \int_0^t e^{-\lambda s} f(s)\, ds + r_t(\lambda)$. Thus $p'(\lambda) = \int_0^t e^{-\lambda s}(-s) f(s)\, ds + r_t'(\lambda)$. Since $r' \approx_t 0$ it follows that $p' \in \mathcal{L}(-tf)$. $\diamondsuit$

In Theorem 1.4 we have seen that the asymptotic Laplace transform exists for all $f \in L^1_{loc}([0,T);X)$ and maps to f to analytic functions $q : \Omega \to X$, where Ω is some post-sectorial region in $\mathbb{C}$. Now we ask the converse question. For which analytic functions $q : \Omega \to X$, defined on a post-sectorial region Ω, does there exist $T > 0$, and $f \in L^1_{loc}([0,T);X)$ such that $q \in \mathcal{L}_T(f)$? The following theorem, which is a slightly reformulated version of a result due to G. Lumer and F. Neubrander [Lu-Ne], gives a partial answer to this important representation problem.

Theorem 1.12 (Complex Representation). *Let $q : \Omega \to X$ be analytic in a region Ω containing*

$$\Omega_\Psi := \{\lambda : Re(\lambda) \geq \beta > 0, |Im(\lambda)| \leq \Psi(Re(\lambda))\},$$

where Ψ is a positive, strictly increasing C^1-function with $\Psi(r) \to \infty$ as $r \to \infty$ and $\sup_{r \geq \beta} \frac{\Psi'(r)}{r^\alpha \Psi(r)} < \infty$ for some $\alpha \geq 0$. Then the following statements hold.

(a) If there exists $T > 0$ such that

$$\|q(\lambda)\| \leq \frac{e^{-T(\Psi^{-1}(|\lambda|))}}{|\lambda|}$$

for all $\lambda \in \Omega_\Psi$, then there exists $f \in C([0,T);X)$ such that $r \in \mathcal{L}_T(f)$.

(b) If there exist constants $c > 0, d > 1$ such that

$$\|q(\lambda)\| \leq \frac{e^{-c(\Psi^{-1}(|\lambda|))^d}}{|\lambda|}$$

for all $\lambda \in \Omega_\Psi$, then there exists $f \in C([0,\infty);X)$ such that $r \in \mathcal{L}(f)$.

Proof. (a) Let Γ be the oriented boundary of the region Ω_Ψ; i.e., $\Gamma = \Gamma_- \cup \Gamma_\beta \cup \Gamma_+$, where

$$\Gamma_\pm := \{r \pm i\Psi(r);\ \beta \le r < \infty\} \text{ and } \Gamma_\beta := \{\beta + ir;\ -\Psi(\beta) \le r \le \Psi(r)\}.$$

Let $C_0, C_1 > 0$ be such that

$$\left| \frac{1 \pm i\Psi'(r)}{r \pm i\Psi(r)} \right| \le C_0 + C_1 r^\alpha$$

for all $\lambda \in \Omega_\Psi$ and all $r \ge \beta$. Since Ψ^{-1} is increasing it follows that

$$\Psi^{-1}(|r \pm i\Psi(r)|) \ge \Psi^{-1}(|\Psi(r)|) = r$$

for all $r \ge \beta$. This implies for $\lambda = r \pm i\Psi(r) \in \Gamma_\pm$ the estimate

$$\begin{aligned} \|e^{\lambda t} q(\lambda)\, d\lambda\| &\le e^{rt} e^{-T\Psi^{-1}(|r \pm i\Psi(r)|)} \left| \frac{1 \pm i\Psi'(r)}{r \pm i\Psi(r)} \right| dr \\ &\le (C_0 + C_1 r^\alpha) e^{-r(T-t)}\, dr. \end{aligned}$$

It follows that

$$f(t) := \frac{1}{2\pi i} \int_\Gamma e^{\lambda t} q(\lambda)\, d\lambda$$

is well defined and continuous for $0 \le t < T$. Next it will be shown that

$$\int_0^t e^{-\lambda s} f(s)\, ds = q(\lambda) - a_t(\lambda)$$

for all $0 < t < T$ and λ in the interior Ω_Ψ^0 of Ω_Ψ, where

$$a_t(\lambda) := \frac{e^{-\lambda t}}{2\pi i} \int_\Gamma \frac{1}{\mu - \lambda} e^{\mu t} q(\mu)\, d\mu.$$

To see this, let $\Gamma(n) := \Gamma \cap \{Re\lambda \le n\}$ and $\Pi_n := \{n + ir; -\Psi(n) \le r \le \Psi(r)\}$. Then, for all $\lambda \in \Omega_\Psi^0$,

$$\begin{aligned}
\int_0^t e^{-\lambda s} f(s)\, ds &= \int_0^t e^{-\lambda s} \frac{1}{2\pi i} \int_\Gamma e^{\mu s} q(\mu)\, d\mu ds = \frac{1}{2\pi i} \int_\Gamma \int_0^t e^{(\mu-\lambda)s}\, ds q(\mu)\, d\mu \\
&= \frac{1}{2\pi i} \int_\Gamma \frac{1}{\mu - \lambda} \left[e^{(\mu-\lambda)t} - 1 \right] q(\mu)\, d\mu \\
&= \lim_{n\to\infty} \frac{1}{2\pi i} \left[\int_{\Gamma(n)} \frac{1}{\mu-\lambda} e^{(\mu-\lambda)t} q(\mu)\, d\mu - \int_{\Gamma(n)} \frac{1}{\mu-\lambda} q(\mu)\, d\mu \right] \\
&= \lim_{n\to\infty} \frac{1}{2\pi i} [\int_{\Gamma(n)} \frac{1}{\mu-\lambda} e^{(\mu-\lambda)t} q(\mu)\, d\mu \\
&\qquad + \left(- \int_{\Gamma(n)} + \int_{\Pi_n} - \int_{\Pi_n} \right) \frac{1}{\mu-\lambda} q(\mu)\, d\mu] \\
&= q(\lambda) + \lim_{n\to\infty} \frac{1}{2\pi i} [\int_{\Gamma(n)} \frac{1}{\mu-\lambda} e^{(\mu-\lambda)t} q(\mu)\, d\mu \\
&\qquad - \int_{\Pi_n} \frac{1}{\mu-\lambda} q(\mu)\, d\mu] \\
&= q(\lambda) + \frac{1}{2\pi i} \int_\Gamma \frac{1}{\mu-\lambda} e^{(\mu-\lambda)t} q(\mu)\, d\mu,
\end{aligned}$$

since, by the residue theorem,

$$\frac{1}{2\pi i} \left(- \int_{\Gamma(n)} + \int_{\Pi_n} \right) \frac{1}{\mu-\lambda} q(\mu)\, d\mu = q(\lambda)$$

for $n > Re\lambda$, and $\lim_{n\to\infty} \frac{1}{2\pi i} \int_{\Pi_n} \frac{1}{\mu-\lambda} q(\mu)\, d\mu = 0$. The last statement follows from the estimate

$$\begin{aligned}
\| \int_{\Pi_n} \frac{1}{\mu-\lambda} q(\mu)\, d\mu \| &\le \int_{-\Psi(n)}^{\Psi(n)} \frac{1}{|n+ir-\lambda|} e^{-T\Psi^{-1}(|n+ir|)} \frac{1}{|n+ir|}\, dr \\
&\le e^{-T\Psi^{-1}(n)} \int_{-\infty}^{\infty} \frac{1}{|n+ir-\lambda|^2}\, dr.
\end{aligned}$$

To show that $a_t \approx_t 0$, it suffices to prove the boundedness of

$$\left(\frac{d}{d\lambda} \right)^n \int_\Gamma \frac{1}{\mu-\lambda} e^{\mu t} q(\mu)\, d\mu$$

for $\lambda \to \infty$. Since $\int_\Gamma \frac{n!}{(\mu-\lambda)^n} e^{\mu t} q(\mu)\, d\mu$ exists, we have that

$$\left(\frac{d}{d\lambda}\right)^n \int_{\Gamma_\beta} \frac{1}{\mu - \lambda} e^{\mu t} q(\mu)\, d\mu = \int_{\Gamma_\beta} \frac{n!}{(\mu - \lambda)^n} e^{\mu t} q(\mu)\, d\mu \to 0$$

as $\lambda \to \infty$. Moreover,

$$\begin{aligned} \| \left(\frac{d}{d\lambda}\right)^n \int_{\Gamma_\pm} \frac{1}{\mu - \lambda} e^{\mu t} q(\mu)\, d\mu\| &= \| \int_{\Gamma_\pm} \frac{n!}{(\mu - \lambda)^n} e^{\mu t} q(\mu)\, d\mu\| \\ &\leq \int_\beta^\infty \frac{n!}{dist(\lambda, \Gamma_\pm)^n} e^{rt} e^{-T\Psi^{-1}(|r \pm i\Psi(r)|)} \left| \frac{1 \pm i\Psi'(r)}{r \pm \Psi(r)} \right| dr \\ &\leq \frac{n!}{dist(\lambda, \Gamma_\pm)^n} \int_\beta^\infty (C_0 + C_1 r^\alpha) e^{-r(T-t)}\, dr. \end{aligned}$$

This shows that $a_t \approx_t 0$. Thus, $q \in \mathcal{L}_T(f)$.

(b) Let the path Γ and the constants C_0, C_1 be as above. Then, for $\lambda = r \pm i\Psi(r) \in \Gamma_\pm$,

$$\begin{aligned} \|e^{\lambda t} q(\lambda)\, d\lambda\| &\leq e^{rt} e^{-c(\Psi^{-1}(|r \pm i\Psi(r)|))^d} \left| \frac{1 \pm i\Psi'(r)}{r \pm i\Psi(r)} \right| dr \\ &\leq (C_0 + C_1 r^\alpha) e^{rt - cr^d}\, dr. \end{aligned}$$

It follows that

$$f(t) := \frac{1}{2\pi i} \int_\Gamma e^{\lambda t} q(\lambda)\, d\lambda$$

is well defined and continuous for $t \geq 0$. As in part (a) one shows that

$$\int_0^t e^{-\lambda s} f(s)\, ds = q(\lambda) + \frac{e^{-\lambda t}}{2\pi i} \int_\Gamma \frac{1}{\mu - \lambda} e^{\mu t} q(\mu)\, d\mu.$$

Since $\int_{\Gamma_\beta} \frac{n!}{(\mu-\lambda)^n} e^{\mu t} q(\mu)\, d\mu \to 0$ as $\lambda \to \infty$ and

$$\begin{aligned} &\| \int_{\Gamma_\pm} \frac{n!}{(\mu - \lambda)^n} e^{\mu t} q(\mu)\, d\mu\| \\ &\qquad \leq \int_\beta^\infty \frac{n!}{dist(\lambda, \Gamma_\pm)^n} e^{rt} e^{-c(\Psi^{-1}(|r \pm i\Psi(r)|))^d} \left| \frac{1 \pm i\Psi'(r)}{r \pm \Psi(r)} \right| dr \\ &\qquad \leq \frac{n!}{dist(\lambda, \Gamma_\pm)^n} \int_\beta^\infty (C_0 + C_1 r^\alpha) e^{rt - cr^d}\, dr, \end{aligned}$$

it follows that $q \in \mathcal{L}(f)$. $\Diamond$

In essence, the previous theorem tells that an analytic function q defined on some post-sectorial region has an asymptotic Laplace representation $q = \mathcal{L}_T f$ for some continuous function f, if q decays sufficiently fast at infinity. Analytic functions q defined on postsectors which grow at infinity can not have a Laplace representation $q = \mathcal{L}_T(f)$ for some $f \in L^1_{loc}([0,T);X)$. In fact, if $q \in \mathcal{L}_T(f)$ for some $f \in L^1_{loc}([0,T);X)$, then there exists $r_t \approx_t 0$ such that

$$q(\lambda) = \int_0^t e^{-\lambda s} f(s)\,ds + r_t(\lambda) = e^{-\lambda t}\int_0^t f(s)\,ds + \lambda \int_0^t e^{-\lambda s}\int_0^u f(u)\,du\,ds + r_t(\lambda).$$

Thus $\|q(\lambda)\| \leq M$ for all $\lambda \in \mathbb{R}$ sufficiently large. As we will see in the following section, such functions have an asymptotic Laplace representation $q \in \mathcal{L}_T(f)$ for some generalized function f, if $\limsup_{\lambda\to\infty} \frac{1}{\lambda} \ln \|q(\lambda)\| < \infty$.

III.2 Asymptotic Laplace Transforms of Generalized Functions

As it was shown in Section II.4, generalized functions $f \in \overline{C_0[a,\infty)}^{T_k}$ for some constant $a \in \mathbb{R}$ and some $k \in C[0,\infty)$ with $0 \in supp(k)$ form a field with respect to convolution and addition, and their vector-valued counterparts $f \in \overline{C_0([a,\infty);X)}^{T_k}$ form a vector space over that field. In the last section we gave a method that assigns to any function $f \in C[a,\infty)$, $a \geq 0$, an equivalence class of analytic functions defined on a post-sector, namely, the asymptotic Laplace transform. Moreover, the asymptotic Laplace transform converts convolution into multiplication. It is not surprising that we can extend the asymptotic Laplace transform to all elements of the convolution field, taking the multiplicative structure as a basis for the extension. It is easy to see that for $f \in L^1_{loc}[0,\infty)$, the asymptotic Laplace transform of the shifted function $f_\alpha : t \mapsto \begin{cases} 0 & \text{for } 0 \leq t \leq \alpha \\ f(t-\alpha) & \text{else} \end{cases}$, satisfies

$$\mathcal{L}(f_\alpha) = e^{-\lambda\alpha}\mathcal{L}(f).$$

Thus the asymptotic Laplace transform of the generalized function representing the right shift operator, denoted by δ_α, must be $\mathcal{L}(\delta_\alpha) = e^{-\lambda\alpha} + \{0\}$. That suggests the following definition.

Definition 2.1 Let $k \in C[0,\infty)$ with $0 \in supp(k)$ and $f \in \overline{C_0([a,\infty);X)}^{T_k}$ for some $a \in \mathbb{R}$. Then $k \star f_{-a} \in C_0([0,\infty;X)$ and we define the asymptotic Laplace transform of f to be

$$\mathcal{L}(f) := e^{-\lambda a}\frac{\mathcal{L}(k \star f_{-a})}{\mathcal{L}(k)}.$$

For functions $f \in \overline{C[0,T);X)}^{T_k}$ and $k \in C[0,T)$ with $0 \in supp(k)$, define

$$\mathcal{L}_T(f) := \frac{\mathcal{L}_T(k \star f)}{\mathcal{L}_T(k)}.$$

The resulting functions $q \in \mathcal{L}_T(f)$ are now meromorphic function defined in a post-sectorial region (except at the poles). We denote the vector space of meromorphic functions with $\mathcal{M}(\mathcal{P};X)$. Notice that this vector space is a vector space over $\mathbb{C}$ as well as over $\mathcal{M}(\mathcal{P};\mathbb{C})$.

We will show first that this extension of the asymptotic Laplace transform is well defined and that the operational properties of the asymptotic Laplace transform extend to the generalized function case.

Proposition 2.2 *The asymptotic Laplace transform of generalized functions is well defined; i.e., generalized functions that are identified via natural embeddings yield the same asymptotic Laplace transform.*

Proof. Recall from Section II.4, that we identify two generalized functions $f \in \overline{C_0[a,\infty);X)}^{T_{k_1}}$ and $g \in \overline{C_0[b,\infty);X)}^{T_{k_2}}$ if

$$k_2 \star k_1 \star f = k_1 \star k_2 \star g,$$

considered as functions on $C(-\infty,\infty)$ by extending them with zero to the left.

Suppose that $k_2 \star k_1 \star f = k_1 \star k_2 \star g$ for $f \in \overline{C_0[a,\infty);X)}^{T_{k_1}}$ and $g \in \overline{C_0[b,\infty);X)}^{T_{k_2}}$. Then

$$\begin{aligned}
\mathcal{L}(f) =& e^{-\lambda a}\frac{\mathcal{L}((k_1 \star f)_{-a})}{\mathcal{L}(k_1)} = e^{-\lambda a}\frac{\mathcal{L}((k_1 \star f)_{-a} \star k_2)}{\mathcal{L}(k_1 \star k_2)} \\
=& e^{-\lambda(a+b)}\frac{\mathcal{L}((k_1 \star f \star k_2)_{-a-b})}{\mathcal{L}(k_1 \star k_2)} = e^{-\lambda b}\frac{\mathcal{L}(k_1 \star (g \star k_2)_{-b})}{\mathcal{L}(k_1 \star k_2)} \\
=& e^{-\lambda b}\frac{(g \star k_2)_{-b})}{\mathcal{L}(k_2)} = \mathcal{L}(g).
\end{aligned}$$

Clearly, the same argument holds for generalized functions $f \in \overline{C[0,T);X}^{T_{k_1}}$ and $g \in \overline{C[0,T);X}^{T_{k_2}}$ with $0 \in supp(k_1) \cap supp(k_2)$. $\diamondsuit$

Next, we show that the operative features of the asymptotic Laplace transform extend to asymptotic Laplace transforms of generalized functions. Denoting with $\mathcal{L} : V_X(+,\star)$ the vector space of generalized functions with values in a Banach space X as described in Section II.4, we show that $\mathcal{L} : V_X(+,\star) \to \mathcal{M}(\mathcal{P};X)/\{0\}$ is linear; i.e.,

(a) $\mathcal{L}(f+g) = \mathcal{L}(f) + \mathcal{L}(g)$

(b) $\mathcal{L}(f \star g) = \mathcal{L}(f)\mathcal{L}(g) + \{0\}$,

Proposition 2.3. *The map $\mathcal{L} : V_X(+,\star) \to \mathcal{A}(\mathcal{P};X)/\{0\}$ is linear; i.e., $\mathcal{L}(f+g) = \mathcal{L}(f) + \mathcal{L}(g)$ and $\mathcal{L}(f \star g) = \mathcal{L}(f)\mathcal{L}(g)$.*

Proof. Let $f, g \in V_X$. Then $f \in \overline{C_0[a,\infty); X)}^{T_{k_1}}$ and $g \in \overline{C_0[b,\infty); X)}^{T_{k_2}}$ for some constants $a \leq b \in \mathbb{R}$ and functions $k_1, k_2 \in C[0,\infty)$ with $0 \in supp(k_1) \cap supp(k_2)$. Thus, by definition, $f + g = T^{-1}_{k_1 \star k_2}(k_2 \star k_1 \star f + k_1 \star k_2 \star g)$. Hence,

$$\begin{aligned}
\mathcal{L}(f+g) =& e^{-\lambda a} \frac{\mathcal{L}((k_2 \star k_1 \star f + k_1 \star k_2 \star g)_a)}{\mathcal{L}(k_1 \star k_2)} \\
=& e^{-\lambda a} \frac{\mathcal{L}(k_2)\mathcal{L}((k_1 \star f)_a) + \mathcal{L}(k_1)\mathcal{L}((k_2 \star g)_a)}{\mathcal{L}(k_1)\mathcal{L}(k_2)} \\
=& e^{-\lambda a} \frac{\mathcal{L}((k_1 \star f)_a)}{\mathcal{L}(k_1)} + e^{-\lambda b} \frac{\mathcal{L}((k_2 \star g)_b)}{\mathcal{L}(k_2)} = \mathcal{L}(f) + \mathcal{L}(g).
\end{aligned}$$

Furthermore,

$$\begin{aligned}
\mathcal{L}(f \star g) =& e^{-\lambda(a+b)} \frac{\mathcal{L}((k_2 \star k_1 \star f \star k_1 \star k_2 \star g)_{a+b})}{\mathcal{L}(k_1 \star k_2)} \\
=& e^{-\lambda a} \frac{\mathcal{L}(k_2)\mathcal{L}((k_1 \star f)_a)}{\mathcal{L}(k_2)\mathcal{L}(k_1)} e^{-\lambda b} \frac{\mathcal{L}(k_1)\mathcal{L}((k_2 \star g)_b)}{\mathcal{L}(k_1)\mathcal{L}(k_2)} = \mathcal{L}(f)\mathcal{L}(g).
\end{aligned}$$

$\diamondsuit$

Since integrating is the same as convoluting with the Heaviside function $\chi_{[0,\infty)}$ (which we continue to denote by 1), and since $\mathcal{L}(1 \star f) = \frac{1}{\lambda}\mathcal{L}(f)$, the inverse function is differentiation and thus

$$\mathcal{L}(f') = \lambda \mathcal{L}(f).$$

Notice the difference of the above equality to item (e) of Section III.1, where we show that $\mathcal{L}(f') = \lambda\mathcal{L}(f) - f(0)$. At first glance this looks like a discrepancy; however, the generalized derivative of a constant function in $C[0,\infty)$ is not zero, but a multiple of the δ-function (see Section II.1). This is not the case if we just look at the usual derivative in $C^1[0,1]$, and hence this is corrected by the term $-f(0)$.

Thus, multiplying λ with the Laplace transform of a generalized function $f \in \overline{C_0[0,\infty);X)}^{T_k}$ corresponds to differentiating, while dividing by λ is equivalent to integrating the generalized function.

Example 2.4 (Laplace transform of Dirac's δ-function). We showed in Section I.1 that the δ-function can be obtained as the second generalized derivative of f, where $f(t) = t$ on $[0,\infty)$. Thus, $\delta \star 1 \star 1 = f$ or

$$\hat{\delta}(\lambda) \cdot \frac{1}{\lambda}\frac{1}{\lambda} = \frac{1}{\lambda^2}.$$

Thus $\hat{\delta}(\lambda) = 1$. Moreover, $\mathcal{L}(\delta') = \lambda$.

Recall from Section II.1 that the definition of a product between a generalized function and a continuous function was somewhat tricky; for generalized functions in $C^{-1}[0,1]$ it was only well defined for differentiable functions. However, we will show that for a given generalized function f, the product $tf := [tf_n]$ for $f = [f_n]$ is always well defined and

$$\mathcal{L}(f)' = \mathcal{L}(-tf).$$

Since

$$(\mathcal{L}(f))' = \left(\frac{\mathcal{L}(k \star f)}{\mathcal{L}(k)}\right)' = \frac{\mathcal{L}(k \star f)'\mathcal{L}(k) - \mathcal{L}(k \star f)\mathcal{L}(k)'}{\mathcal{L}(k)^2},$$

we do not expect the sequence tf_n to converge in the T_k-topology, but in the $T_{k\star k}$ topology.

Theorem 2.5 *Let $f = k - \lim f_n \in \overline{C([a,\infty);X)}^{T_k}$ for some $k \in C[0,\infty)$ with $0 \in supp(k)$. Then the generalized function*

$$tf := T_{k\star k} - \lim tf_n \in \overline{C([a,\infty);X)}^{T_{k\star k}} \text{ and } \mathcal{L}(tf) = -\mathcal{L}(-tf).$$

Proof. Suppose $k \star f_n$ converges in $C_0([a,\infty);X)$. Then, identifying f_n and k with their zero extensions,

$$\begin{aligned}(k \star k \star (tf_n)) =& k \star (t \mapsto \int_{-\infty}^{\infty} (t-s) f_n(t-s) k(s)\, ds) \\ =& k \star (t \mapsto t \int_{-\infty}^{\infty} f_n(t-s)k(s)\, ds - \int_{-\infty}^{\infty} f_n(t-s) s k(s)\, ds) \\ =& k \star (t(f_n \star k)) - k \star f_n \star (tk) \\ \to& k \star (t(f \star k)) - k \star f \star (tk) \in C_0([a,\infty);X).\end{aligned}$$

Hence, by Theorem 1.9 and Theorem 1.11,

$$\begin{aligned}\mathcal{L} t f_n =& \frac{\mathcal{L}(k \star (t(f \star k)) - k \star f \star (tk)}{\mathcal{L}(k \star k)} = \frac{-\mathcal{L}(k)\mathcal{L}(f \star k)' + \mathcal{L}(k \star f)\mathcal{L}(k)'}{\mathcal{L}(k)^2} \\ =& -\mathcal{L}(f)'.\end{aligned}$$

$\diamondsuit$

Equipped with this operational property we consider the following example.

Example 2.6 (Singular ordinary differential equation). To demonstrate the operational method developed in the last sections we consider the problem

$$ty''(t) + (t+3)y'(t) + y(t) = 0; \quad y(0) = 0 \tag{1}$$

and determine all $\omega \in \mathcal{F}$ (the field of generalized functions) that satisfy (1). Suppose there exist $k \in C[0,\infty)$ with $0 \in supp(k)$ and $T > 0$ such that $\omega \in \overline{C[0,T)}^{T_k}$ satisfies (1); i.e., $z := k \star \omega \in C_0[0,T)$ is a classical solution of the convoluted problem

$$k \star (ty''(t) + (t+3)y'(t) + y(t)) = 0.$$

From the operational rules

(a) $\mathcal{L}_T(\omega)' = \mathcal{L}_T(-t\omega)$ and

(b) $\mathcal{L}_T(\omega') = \lambda\mathcal{L}_T(\omega)$,

we deduce the following familiar rules from Laplace transform theory:

(c) $\mathcal{L}_T(\omega) + \lambda\mathcal{L}_T(\omega)' = \mathcal{L}_T(\omega')' = \mathcal{L}_T(-t\omega')$

(d) $\mathcal{L}_T(\omega'') = \lambda\mathcal{L}_T(\omega') = \lambda^2\mathcal{L}_T(\omega)$

(e) $2\lambda\mathcal{L}_T(\omega) + \lambda^2\mathcal{L}(\omega)' = \mathcal{L}(\omega'')' = \mathcal{L}(-t\omega'')$.

Thus, if $0 = t\omega''(t) + (t+3)\omega'(t) + \omega(t)$, then

$$\begin{aligned}\{0\}_T &= \mathcal{L}(t\omega''(t)) + \mathcal{L}(t\omega'(t) + 3\mathcal{L}(\omega'(t)) + \mathcal{L}(\omega(t)) \\ &= -2\lambda(\mathcal{L}_T(\omega) - \lambda^2\mathcal{L}_T(\omega)' - \mathcal{L}_T(\omega) - \lambda\mathcal{L}(\omega)' + 3\lambda\mathcal{L}_T(\omega)' + \lambda\mathcal{L}_T(\omega)\,; \\ &= (-\lambda^2 - \lambda)\mathcal{L}_T(\omega)' + \lambda\mathcal{L}_T(\omega)\end{aligned}$$

or equivalently, $\mathcal{L}_T(\omega)' - \frac{1}{\lambda+1}\mathcal{L}_T(\omega) = \{0\}_T$. Thus, the asymptotic Laplace transform reduces the second order problem to a first order problem which can be solved explicitly. Let $q \in \mathcal{L}_T(\omega)$. Then

$$q'(\lambda) - \frac{1}{\lambda+1}q(\lambda) = r(\lambda),$$

where $r \approx_T 0$; so $\left(\frac{1}{\lambda}q(\lambda)\right)' = \frac{1}{\lambda+1}r(\lambda) =: \tilde{r}(\lambda)$, where $\tilde{r} \approx_T 0$. This shows that there exists a constant c such that $\frac{1}{\lambda}q(\lambda) = c + \tilde{R}(\lambda)$, where

$$\tilde{R}(\lambda) = -\int_\lambda^\infty \tilde{r}(\mu)\,d\mu \approx_T 0.$$

Then

$$\mathcal{L}_T(\omega) = c(\lambda+1) + \{0\}_T$$

is the operational solution of (1), and $\omega = c\mathcal{L}^{-1}(\lambda+1) = c(\delta' + \delta)$ for $c \in \mathbb{C}$ is the unique class of solutions in $\mathcal{F}$ that solves (1).

The next proposition follows immediately from the definition.

Proposition 2.7. *Let $q \in \mathcal{A}(\mathcal{P}; X)$, let $k \in C[0,\infty)$ with $0 \in supp(k)$. Then the following are equivalent.*

(i) There exists a generalized function $f \in \overline{C([0,\infty); X)}^{T_k}$ such that $q \in \mathcal{L}(f)$.

(ii) $\tilde{k} \cdot q \in \mathcal{L}(g)$ for some $g \in C_0([0,\infty); X)$ and $\tilde{k} \in \mathcal{L}(k)$.

Proposition 2.7 sets the stage for several representation theorems, characterizing (ii) in different ways. G. Lumer and F. Neubrander in [Lu-Ne] proved the following sufficient conditions (recall from Section III.1 that for all $f \in C([0,a); X)$ there exists $\hat{f} \in \mathcal{L}_A(f)$ such that $\hat{f}$ is analytic in a postsectorial region):

Theorem 2.8 (Complex Representation. *Let $r \in \mathcal{A}(\mathcal{P}; X)$ be a function. Suppose there exists $k \in C[0,\infty)$ with $0 \in supp(k)$ and $q \in \mathcal{A}(\mathcal{P}; X)$ such that $q = r \cdot \tilde{k} \in \mathcal{A}(\mathcal{P}; X)$ for $\tilde{k} \in \mathcal{L}(k)$. Suppose the post sector on which q is analytic contains*

$$\Omega_\Psi := \{\lambda : Re(\lambda) \geq \beta > 0, |Im(\lambda)| \leq \Psi(Re(\lambda))\},$$

where Ψ is a positive, strictly increasing C^1-function with $\Psi(\lambda) \to \infty$ as $\lambda \to \infty$, and $\sup_{\lambda \geq \beta} \frac{\Psi'(\lambda)}{\lambda^\alpha \Psi(\lambda)} < \infty$ for some $\alpha \geq 0$. Then the following hold:

(a) If there exists $T > 0$ such that

$$\|q(\lambda)\| \leq \frac{e^{-T(\Psi^{-1}(|\lambda|))}}{|\lambda|}$$

for all $\lambda \in \Omega_\Psi$, then there exists $f \in \overline{C([0,T); X)}^{k}$ such that $r \in \mathcal{L}_T(f)$.

(b) If there exist $c > 0$, $d > 1$ such that

$$\|q(\lambda)\| \leq \frac{e^{-c(\Psi^{-1}(|\lambda|))^d}}{|\lambda|}$$

for all $\lambda \in \Omega_\Psi$ and for some $c > 0, d > 1$, then there exists $f \in \overline{C([0,\infty);X)}^k$ such that $r \in \mathcal{L}(f)$.

Proof. The proof follows immediately from Theorem 1.6 and the definition of asymptotic Laplace transforms of generalized functions. $\diamondsuit$

The above results give us a gauge to measure the regularity of f. The less growth in a post-sectorial region, the more regularity we obtain for f. Especially, if $\hat{f}$ grows polynomially we can multiply by $\frac{1}{\lambda^n}$ in order to satisfy the growth requirement. But multiplying by $1/\lambda^n$ corresponds to integrating n-times, thus f would be in $C^{-n}([0,\infty);X)$.

The philosophy that the growth determines the regularity is the idea behind the next theorem. J. Prüss in [Pr] has a similar characterization, although we do not need any derivatives of the analytic function.

Theorem 2.9 (Complex Representation Theorem for differentiable functions). *Let $q : \{Re(\lambda) > \omega \geq 0\} \to X$ be analytic in a right halfplane, let $0 < \beta \leq 1$, let $n \in \mathbb{N}_0$, and assume there exists a polynomial $p(\lambda) := \sum_{i=0}^{n-1} \lambda^i x_i$ such that $\lambda^{\beta+1}(\lambda^n q(\lambda) - p(\lambda))$ is bounded for all λ with $Re(\lambda) > \omega$. Then there exists a function f such that $q(\lambda) = \int_0^\infty e^{-\lambda t} f(t)\, dt$ and $f \in C^{n,\alpha}([0,\infty);X)$ for all $0 \leq \alpha < \beta$.*(1)

(1) A function f is locally α-Hölder continuous, or $f \in C^{0,\alpha}([0,\infty);X)$, if for all

Proof. Let $r(\lambda) := \lambda^n q(\lambda) - p(\lambda)$. Then $\lambda^{\beta+1} r(\lambda) = \lambda\lambda^{\beta-\alpha}\lambda^\alpha r(\lambda)$ and hence, by the Complex Representation Theorem, for every α there exists a function $g_\alpha \in C_0([0,\infty);X)$ such that $\lambda^{(\beta-\alpha)}\lambda^\alpha r(\lambda) = \lambda^{\beta-\alpha}\int_0^\infty e^{-\lambda t} g_\alpha(t)\,dt$ and thus

$$r(\lambda) = \frac{1}{\lambda^\alpha}\int_0^\infty e^{-\lambda t} g_\alpha(t)\,dt = \int_0^\infty e^{-\lambda t}\frac{t^{\alpha-1}}{\Gamma(\alpha)} \star g_\alpha(t)\,dt =: \int_0^\infty e^{-\lambda t} h(t)\,dt$$

for $0 < \alpha < \beta$. By the uniqueness of the Laplace transform, h is independent of α; and hence $h \in C_0^{0,\alpha}([0,\infty);X)$ for all $0 \leq \alpha < \beta$. Since

$$\begin{aligned} q(\lambda) =& \frac{1}{\lambda^n}(r(\lambda) + p(\lambda)) \\ =& \int_0^\infty e^{-\lambda t}\left(\left(\frac{t^{n-1}}{n-1!} \star h\right)(t) + \sum_{i=0}^{n-1}\frac{t^i}{i!}x_{n-1-i}\right)dt =: \int_0^\infty e^{-\lambda t} f(t)\,dt \end{aligned}$$

we obtain that $f \in C^{n,\alpha}([0,\infty);X)$. ♢

N there exists a constant M such that $\sup_{t,s\in[0,N]}\|f(t) - f(s)\| \leq M|t-s|^\alpha$. A function f is said to be in the Hölder space $C^{n,\alpha}([0,\infty);X)$ if $f \in C^n([0,\infty);X)$ and $f^{(n)} \in C^{0,\alpha}([0,\infty);X)$.

IV. The Abstract Cauchy Problem

J. Hadamard [Ha], Chapter I.

"A differential equation – whether ordinary or partial – admits of an infinite number of solutions. The older and classic point of view, concerning its integration, consisted in finding the so-called 'general integral,' i.e. a solution of the equation containing as many arbitrary elements (arbitrary parameters or arbitrary functions) as are necessary to represent any solution, save some exceptional ones. But, in more recent research, especially as concerns partial differential equations, this point of view had to be given up, not only because of the difficulty or impossibility of obtaining this 'general integral,' but, above all, because the question does not by any means consist merely in its determination. The question, as set by most applications, does not consist in finding *any* solution u of the differential equation, but in choosing, amongst all those possible solutions, a particular one defined by properly given accessory conditions.[...]

The true questions which actually lie before us are, therefore, the 'boundary problems,' each of which consists in determining an unknown function u so as to satisfy:
(1) an 'indefinite' partial differential equation;
(2) some 'definite' boundary conditions.
Such a problem will be 'correctly set' if those accessory conditions are such as to determine one and only one solution of the indefinite equation. The simplest of boundary problems is Cauchy's problem. It represents, for partial differential equations, the exact analog of the well-known fundamental problem in ordinary differential equations. The theory of the latter was founded by Cauchy on the following theorem: Given an ordinary differential equation, say of the second order, [...] a solution of this equation is (under proper hypotheses) determined if, for $x = 0$, we know the numerical values y_0, y_0' of y and $\frac{dy}{dx}$.[...]

Strictly, mathematically speaking, we have seen (this is Holmgren's theorem) that one set of Cauchy's data u_0, u_1 corresponds (at most) to one solution u of $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$, so that, if these quantities u_0, u_1 were 'known,' u would be determined without any possible ambiguity.

But, in any concrete application, 'known,' of course, signifies 'known with a certain approximation,' all kinds of errors being possible, provided their magnitude remains smaller than a certain quantity; and, on the other hand, we have seen that the mere replacing of the value zero for u_1 by the (however small) value changes the solution not by very small but by very great quantities. Everything takes place, physically speaking, as if the knowledge of Cauchy's data would not determine the unknown function. This shows how very differently things behave in this

> case and in those which correspond to physical questions. If a physical phenomenon were to be dependant on such an analytical problem as Cauchy's for $\nabla^2 u = 0$, it would appear to us as being governed by pure chance (which, since Poincaré, has been known to consist precisely in such a discontinuity in determinism) and not obeying any law whatever."

One of the applications of the theory of Banach space valued generalized functions and their asymptotic Laplace transforms is the study of the abstract Cauchy problem

$$u'(t) = Au(t); \quad u(0) = x; \quad t \in [0, a)$$

where $x \in X$ for some Banach space X, A is a linear operator on X, and $a \in \mathbb{R}^+ \cup \{+\infty\}$. We will show the equivalence of the solvability of the above problem with the solvability of the asymptotic characteristic equation

$$(\lambda I - A)y(\lambda) = x + r_a(\lambda)$$

for some remainder term $r_a(\lambda)$ of exponential decay a. The growth in the region of analyticity of the asymptotic local resolvent y will determine the regularity and growth of the (generalized) solution u.

IV.1 The Notion of a Generalized Solution

Recall from Chapter II, that the generalized derivative coincides with the derivative of a differentiable function u, as long as $u(0) = 0$, and that the generalized derivative of a constant function is the Dirac δ-function at 0. This means that we have to modify slightly the above notion of the abstract Cauchy problem in order to fit the setup of generalized derivation. We obtain an equivalent formulation of the abstract Cauchy problem if we write

$$(u(t)-x)' = Au(t); \quad u(0)=x; \quad t\in[0,a).$$

But now the derivative symbol on the left can be understood as a generalized derivative and solving the abstract Cauchy problem in a generalized sense will mean that both sides represent the same generalized function. Convoluting both sides with 1 leads to an equivalent equation without derivation, namely

$$u - x = 1 \star Au. \tag{ACP}$$

In order to claim that a generalized function[(1)] $u \in \overline{C([0,a);X)}^{new}$ solves (ACP), we face the problem that $u(t)$ is no longer defined for a specific t, nor in an "almost everywhere" sense. Hence, what is $Au(t)$? This problem can be resolved by emphasizing the limiting processes involved.

Definition 1.1 (Generalized solutions). *We say that $u \in \overline{C([0,a);X)}^{new}$ is a generalized solution of (ACP), if and only if there exists a sequence $u_n \in C([0,a);X)$ such that*

(a) $u_n(t) \in \mathcal{D}(A)$ for all $t\in[0,a)$ and $Au_n \in C([0,a);X)$,

(b) $u = new - \lim u_n$,

(c) $u - x = new - \lim 1 \star Au_n$.

If A is closed and if we can choose $u_n \in C([0,a);X)$ satisfying $(a)-(c)$ such that $u_n \to u$ and $Au_n \to Au$ in $C([0,a);X)$, then clearly u is a classical solution to the

(1) Clearly, $\overline{C([0,a);X)}^{new}$ is the completion of the Frechét space $C([0,a);X)$ equipped with a new set of seminorms.

abstract Cauchy problem. If we omit the requirement that $Au_n \to Au$, we obtain a mild solution to the abstract Cauchy problem since $1 \star Au_n = A(1 \star u_n) \to A(1 \star u)$ and hence $u - x = A(1 \star u)$.

Next we want to investigate how this notion of solution correlates to the notions of integrated, k-regularized and C-regularized solutions to the abstract Cauchy problem. Recall that $v \in C([0,a);X)$ is a k-generalized solution of the abstract Cauchy problem if $\int_0^t v(s)\,ds \in \mathcal{D}(A)$ and

$$v(t) = A\int_0^t v(s)\,ds + (k \star x)(t) \qquad (ACP_k)$$

for all $t \in [0,a)$. In the case that $k = \frac{t^{n+1}}{n+1!}$ we speak of an n-times integrated solution. If we replace $k(t)$ with a bounded operator $C \in \mathcal{L}(X)$ we obtain the notion of C-regularized solutions to the abstract Cauchy problem. See, for example, R. deLaubenfels [deL].

The following proposition is crucial in the interplay between generalized limit solutions and regularized solutions. We say that a linear operator A on a Banach space X is $(X_A \hookrightarrow X)$-closed, if there exists an auxiliary Banach space X_A such that $\mathcal{D}(A) \subset X_A \hookrightarrow X$ and the graph of A is closed in $X_A \times X$ (see Appendix). We denote with $[\mathcal{D}(A)]$ the Banach space

$$\left(\mathcal{D}(A) : \|x\|_{[\mathcal{D}(A)]} := \|x\|_{X_A} + \|Ax\|_X\right).$$

Proposition 1.2. *Let A be a $(X_A \hookrightarrow X)$-closed, linear operator on a Banach space X and let $a > 0$. Then for all $v \in C([0,a);X)$ with $1 \star v \in C_0([0,a);[\mathcal{D}(A)])$, there exists $v_n \in C([0,a);[\mathcal{D}(A)])$ such that $v_n \to v$ in X and $1 \star v_n \to 1 \star v$ in $[\mathcal{D}(A)]$.*

Proof. Let $v \in C([0,a);X_A)$ with $1 \star v \in C_0([0,a);[\mathcal{D}(A)])$. For $0 < h < a/2$, define

$$v_h(t) := \begin{cases} \frac{1}{h}\int_t^{t+h} v(s)\,ds & \text{for } 0 \le t < a-2h \\ \frac{1}{h}\int_{a-2h}^{a-h} v(s)\,ds & \text{else.} \end{cases}$$

Then, for $0 \le t \le a-2h$, we have that $v_h(t) = \frac{1\star v(t+h) - 1\star v(t)}{h}$ and thus $v_h \in C([0,a);[\mathcal{D}(A)])$. Clearly, $v_h \to v$ in $C([0,a);X)$ as $h \to 0$. Since

$$\begin{aligned}(1 \star v_h)(t) &= \frac{1}{h}\int_0^t \int_s^{s+h} v(r)\,dr\,ds = \frac{1}{h}\int_0^t \int_0^h v(r+s)\,dr\,ds \\ &= \frac{1}{h}\int_0^h \int_0^t v(r+s)\,ds\,dr = \frac{1}{h}\int_t^{t+h} \int_0^t v(r-t+s)\,ds\,dr \\ &= \frac{1}{h}\int_t^{t+h} \int_{r-t}^r v(s)\,ds\,dr = \frac{1}{h}\int_t^{t+h} (1\star v)(r) - (1\star v)(r-t)\,dr\end{aligned}$$

for $0 \le t < a-2h$, we obtain that $1 \star v_h \to 1 \star v$ in $C([0,a);[\mathcal{D}(A)])$. $\diamondsuit$

Recall from Chapter *II* the following definition. For an injective linear operator $T \in \mathcal{L}(C([0,a);X))$, we define $\overline{C([0,a);X)}^T$ to be the completion of the space $C([0,a);X)$ equipped with the seminorms

$$\|f\|_{t,T} := \|Tf\|_t := \sup_{s\in[0,t]} \|Tf(s)\|$$

for $t \in (0,a)$.

Theorem 1.3. *Let A be an $(X_A \hookrightarrow X)$-closed linear operator on a Banach space X and $x \in X$. Let $a > 0$ and let $T \in \mathcal{L}(C([0,a);X))$ be injective with $T(1 \star v) = 1 \star T(v)$. Suppose that for all $f \in C([0,a);[\mathcal{D}(A)])$ we have that $Tf \in C([0,a);[\mathcal{D}(A)])$ and $ATf = TAf$. Suppose that the image of T as an operator on $C([0,a);[\mathcal{D}(A)])$ is dense in $C([0,a);[\mathcal{D}(A)])$.*[2] *Let $u \in \overline{C([0,a);X)}^T$ with $1 \star u \in \overline{C([0,a);X_A)}^T$. Then the following are equivalent:*

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(2) Clearly, the theorem also holds for $v \in C_0([0,a);X)$ with the image of T being dense in $C_0([0,a);X)$.

(i) u is a generalized limit solution of (ACP),

(ii) $Tu =: v$ is a continuous solution of

$$v(t) = A\int_0^t v(s)\,ds + T(1x)(t).$$

Proof. Let $[u_n] := u \in \overline{C([0,a);X)}^T$ be a generalized limit solution of $u - x = 1 \star Au$ with $u_n \in C([0,a);[\mathcal{D}(A)])$. Then

$$\begin{aligned} Tu - T(1x) =& T(T - \lim 1 \star Au_n) = T - \lim T(1 \star Au_n) \\ =& \lim 1 \star A(T(u_n)) = A(1 \star T(u)). \end{aligned}$$

Thus, for $v = Tu$, we obtain that $v(t) = A\int_0^t v(s)\,ds + T(1x)(t)$.

Suppose there exists $v \in C([0,a);X)$ with $1 \star v \in C([0,a);X_A)$ and

$$v(t) = A\int_0^t v(s)\,ds + T(1x)(t).$$

Then $1 \star v \in C_0([0,a);[\mathcal{D}(A)])$. By Proposition 2.2 we know that there exist $v_n \in C([0,a);[\mathcal{D}(A)])$ such that $v_n \to v$ in $C([0,a);X)$ and $1 \star v_n \to 1 \star v$ in $C_0([0,a);[\mathcal{D}(A)])$. Since the image of T as an operator on $C([0,a);[\mathcal{D}(A)])$ is dense in $C([0,a);[\mathcal{D}(A)])$, there exist $u_n \in C([0,a);[\mathcal{D}(A)])$ with $Tu_n - v_n \to 0$ in $C([0,a);[\mathcal{D}(A)])$. Thus $Tu_n \to v$ in $C([0,a);X)$ and $1 \star Tu_n \to 1 \star v$ in $C([0,a);[\mathcal{D}(A)])$. Hence

$$T(1 \star Au_n) = A(1 \star Tu_n) \to A(1 \star v) = v - T(1x).$$

Therefore $T^{-1}v =: u \in \overline{C([0,a);X)}^T$ is a generalized limit solution of ACP. $\diamondsuit$

In terms of k-regularized and C-regularized solutions, this translates into the following Corollary. For an introduction to the theory of C-regularized solutions, see, for example, R. deLaubenfels [deL], for k-regularized solutions, see, for example, I. Cioranescu and G. Lumer [Ci-Lu].

Corollary 1.4. *Let A be a closed linear operator on a Banach space X, let $k \in L^1_{loc}[0,a)$ with $0 \in supp(k)$, and let $C \in \mathcal{L}(X)$ be one-to-one. Suppose for all $x \in \mathcal{D}(A)$ we have that $Cx \in \mathcal{D}(A)$ and $CAx = ACx$. Suppose furthermore that C as an operator on $[\mathcal{D}(A)]$ has a dense image in $[\mathcal{D}(A)]$. Let $T_k : f \mapsto k \star f$ be the convolution operator, and define $T_C : f \mapsto Cf$. Then*

(a) $u \in \overline{C[0,a);X)}^{T_k}$ is a generalized limit solution if and only if $v := k \star u$ is a k-regularized solution of (ACP).

(b) $u \in \overline{C[0,a);X)}^{T_C}$ is a generalized limit solution if and only if $v := Cu$ is a C-regularized solution of (ACP).

Next we will show that existence and uniqueness of generalized limit solutions of (ACP) for all $x \in X$ implies continuous dependence of the generalized solutions of the initial data; i.e., that a Cauchy problem describes a well-posed physical phenomena if it admits unique solutions for all $x \in X$.

Proposition 1.5 *Let A be a closed operator on a Banach space X, $a > 0$ and $k \in L^1_{loc}[0,a)$ with $0 \in supp(k)$. If (ACP) has a unique generalized limit solution $u = u(\cdot, x) \in \overline{C([0,a);X)}^{T_k}$ for all $x \in X$, then (ACP) is well-posed; i.e., for all $0 < T < a$ there exists $M_T > 0$ such that $\|u(\cdot,x)\|_{T_k} \leq M_T\|x\|$ on the interval $[0,T]$.*[3]

(3) **Conjecture.** If for all $x \in X$ there exists a $k \in L^1_{loc}[0,a)$ with $0 \in supp(k)$ such that (ACP) has a unique generalized solution $u_x \in \overline{C([0,a);X)}^{T_k}$, then there exists $k_0 \in L^1_{loc}[0,a)$ such that $u_x \in \overline{C([0,a);X)}^{T_{k_0}}$ is a generalized solution and hence (ACP) is well-posed.

Proof. Let $S(\cdot)x := k \star u(\cdot, x)$. Thus, by Theorem 1.3,

$$S(t)x = A\int_0^t S(s)x\, ds + (k \star 1)(t)x.$$

By the existence and uniqueness property, we know that $S : X \to C([0,a);X)$ is a linear operator. Furthermore, S is a closed operator, since $x_n \to x$ and $Sx_n \to v$ implies that $1 \star Sx_n \to 1 \star v$ and, by the closedness of A, that

$$A(1 \star v) = v - k \star 1x.$$

By the uniqueness property, $Sx = v$, and hence S is a closed linear operator. By the closed graph theorem, S is bounded and hence for all $0 \leq T < a$ there exists $M_T > 0$ such that

$$\|u(\cdot, x)\|_{T_k} = \|Sx\| = \sup_{t\in[0,T]} \|S(t)x\| \leq M_T\|x\|.$$

$\diamondsuit$

IV.2 Existence and Uniqueness of Solutions

In this section we show the equivalence between the abstract Cauchy problem and its characteristic equation

$$(\lambda I - A)\tilde{u}(\lambda) \approx_T x, \tag{CE}$$

where $\tilde{u} \in \mathcal{L}_A(u)$. We show that u is a generalized solution of (ACP) if and only if there exists $\tilde{u}$ solving (CE). Thus existence and uniqueness of solutions of (ACP) is equivalent to existence and uniqueness of solutions to (CE). Furthermore, recall from Section III.2 that we can deduce the regularity of a generalized function; i.e., we can determine the regularizing functions K such that $u \in \overline{C([0,a);X)}^K$ by the growth behaviour of its asymptotic Laplace transform in the region of analyticity.

For the sake of clarity we introduce the following notation for the asymptotic Laplace transform:

$$\{f\}_T := \mathcal{L}_T(f).$$

In order to investigate the abstract Cauchy problem with our operational calculus, we have to establish how the asymptotic Laplace transform interacts with closed or relatively closed linear operators A. Recall that we define the graph space $[\mathcal{D}(A)]$ for a $(X_A \hookrightarrow X)$-closed operator A via

$$[\mathcal{D}(A)] := \left(\mathcal{D}(A) : \|x\|_{[\mathcal{D}(A)]} := \|x\|_{X_A} + \|Ax\|_X\right).$$

By the definition of a $(X_A \hookrightarrow X)$-closed operator, this is a Banach space (for closed operators, $X_A = X$). Clearly, we need some conditions on f such that $\{Af\} = A\{f\}$. It is well known (see, for example, Hille-Phillips [Hi-Ph], Theorem 3.7.12) that if A is a closed operator, $f \in L^1[(0,t];X)$ with $f(s) \in \mathcal{D}(A)$ almost everywhere, and if $Af \in L^1([0,t];X)$, that then $\int_0^t f(s)\,ds \in \mathcal{D}(A)$ and

$$\int_0^t Af(s)\,ds = A\int_0^t f(s)\,ds.$$

However, this does not imply that $\{Af\} = A\{f\}$, since the remainder term on the right hand side might not be of exponential decay anymore. However, $f \in L^1[(0,t];X)$ and $Af \in L^1([0,t];X)$ implies that $f \in L^1_{loc}([0,T);[\mathcal{D}(A)])$. Hence, we can consider the asymptotic Laplace transform

$$\{f\}_{[\mathcal{D}(A)]} := \{q \in \mathcal{A}(\mathcal{P}; [\mathcal{D}(A)]) : q \approx_t \int_0^t e^{-\lambda s} f(s)\, ds \text{ for all } 0 < t < T\}$$

for $f \in L^1_{loc}([0,T); [\mathcal{D}(A)])$ and, as we will see in Proposition 2.1., we can obtain

$$\{Af\} = \{Af\}_{[\mathcal{D}(A)]} + \{0\} = A\{f\}_{[\mathcal{D}(A)]} + \{0\}.$$

On the other hand, let $f \in L^1_{loc}([0,T); X)$. If $\tilde{f}(\lambda) \in \mathcal{D}(A)$ for $\tilde{f} \in \{f\}$ and $A\tilde{f} \in \{g\}$ for some $g \in L^1_{loc}([0,T); X)$, we can deduce by the Phragmén inversion formula and the closedness of A that $A1 \star f = 1 \star g$ and hence $Af = g$.

Reformulating these statements for the general setting of relatively closed operators yields the following proposition.

Proposition 2.1. *Let A be a $(X_A \hookrightarrow X)$-closed operator. Suppose $f \in L^1_{loc}([0,T); X_A)$ with $f(t) \in \mathcal{D}(A)$ almost everywhere. Let $g \in L^1_{loc}([0,T); X)$. Then*

(a) $\{f\}_{X_A} + \{0\} = \{f\}$.

(b) $Af = g$ *implies that* $f \in L^1_{loc}([0,T); [\mathcal{D}(A)])$, $\{f\} = \{f\}_{[\mathcal{D}(A)]} + \{0\}$, *and*

$$\{g\} = \{Af\} = A\{f\}_{[\mathcal{D}(A)]} + \{0\}.$$

(c) for $\tilde{f} \in \{f\}_{X_A}$ *and* $\tilde{f}(\lambda) \in \mathcal{D}(A)$, *the condition that* $A\tilde{f} \in \{g\}$ *implies that* $Af = g$.

Proof. (a) Let $\tilde{f} \in \{f\}_{X_A}$. Then for all $t \in [0,T)$ there exists a remainder term r_t with $r_t \approx_t 0$ in X_A and $\tilde{f}(\lambda) = \int_0^t e^{-\lambda t} f(t)\, dt + r_t(\lambda)$. Since $X_A \hookrightarrow X$ we know that $r_t \approx_t 0$ in X and thus $\tilde{f} \in \{f\}$. Thus for all $q \in \{f\}$, $\tilde{f} \approx_t q$ in X. Hence $\{f\}_{X_A} + \{0\} = \{f\}$.

(b) The proof follows Theorem 3.7.11 of [Hi-Ph]. Since f is locally integrable in X_A, for each $\epsilon > 0$ and $t \in [0, T)$, there exists a countable subdivision S_n of $[0, t]$ of disjoint sets S_n of positive measure, such that the measure of $\bigcup S_n$ is t, and $\|f(s) - f_\epsilon(s)\|_{X_A} \leq \epsilon$ for all $s \in \bigcup S_n$ and a countably valued step function f_ϵ that is constant on S_n. Since Af is locally integrable in X, we obtain another subdivision for Af. Take a common refinement $\tilde{S}_n$. Then $\|f(s) - f_\epsilon(s)\|_{X_A} + \|Af(s) - g(s)\| \leq 2\epsilon$ for all $s \in \bigcup \tilde{S}_n$ for countably valued step functions f_ϵ, g_ϵ that are constant on $\tilde{S}_n$. Without loss of generality we can assume that for all n, $f_\epsilon(s) = f(s)$ for some $s \in \tilde{S}_n$. Let $\tilde{g}(s) := Af_\epsilon(s)$. Then $\|\tilde{g}(s) - g(s)\| \leq 2\epsilon$ and $\|f(s) - f_\epsilon(s)\|_{[\mathcal{D}(A)]} \leq 3\epsilon$ for all $s \in \bigcup \tilde{S}_n$. Thus, the restriction of f onto $[0, t]$ is the uniform limit almost everywhere of countably valued functions, and is therefore strongly measurable in $[\mathcal{D}(A)]$. Since

$$\int_0^t \|f(s)\|_{[\mathcal{D}(A)]}\, ds = \int_0^t \|f(s)\|_{X_A}\, ds + \int_0^t \|Af(s)\|\, ds < \infty,$$

we obtain that $f \in L^1([0, t]; [\mathcal{D}(A)])$ and therefore $f \in L^1_{loc}([0, T); [\mathcal{D}(A)])$.

Suppose $Af = g$. Let $\tilde{f} \in \{f\}_{[\mathcal{D}(A)]}$. Then for all $t \in [0, T)$ there exists a function r_t such that $r_t \approx_t 0$ in $[\mathcal{D}(A)]$, and

$$A\tilde{f}(\lambda) = A\int_0^t e^{-\lambda t} f(s)\, ds + Ar_t(\lambda) = \int_0^t e^{-\lambda t} Af(s)\, ds + Ar_t(\lambda).$$

Note that $r_t \approx_t 0$ and $Ar_t \approx_t 0$ in X. Therefore $\tilde{f} \in \{f\}$, $A\tilde{f} \in \{Af\}$, and thus $\{f\}_{[\mathcal{D}(A)]} + \{0\} = \{f\}$ and $A\{f\}_{[\mathcal{D}(A)]} + \{0\} = \{Af\}$.

(c) Suppose that $\tilde{f} \in \{f\}_{X_A}$, $\tilde{f}(\lambda) \in \mathcal{D}(A)$, and $A\tilde{f} \in \{g\}$. Then

$$A\tilde{f}(\lambda) = A(\int_0^t e^{-\lambda s} f(s)\, ds + r_t(\lambda)) = \int_0^t e^{-\lambda s} g(s)\, ds + q_t(\lambda),$$

for some $r_t \approx_t q_t \approx_t 0$. Since by the generalized Phragmén inversion (Theorem II.3.1)

$$\int_0^s f(r)\, dr = X_A - \lim_{k\to\infty} \sum_{k=1}^{N_k} a_{k,n} e^{s\beta_{kn}} \tilde{f}(\beta_{kn})$$

and

$$\int_0^s g(r)\, dr = \lim_{k\to\infty} \sum_{k=1}^{N_k} a_{k,n} e^{s\beta_{kn}} \tilde{A}f(\beta_{kn}),$$

the relative closedness of A implies that $A\int_0^s f(r)\, dr = \int_0^s g(r)\, dr$. Again, by the relative closedness of A, $Af(s) = g(s)$ for all $s \in [0,t]$ and all $t \in [0,T)$. Thus $Af = g$. $\diamondsuit$

With these operational rules at hand, we can now easily prove the following theorem.

Theorem 2.2. *Let A be a $(X_A \hookrightarrow X)$-closed, linear operator on a Banach space X. Let $T > 0$, let $k \in C[0,T)$ with $0 \in supp(k)$, and let $v \in L^1_{loc}([0,T);X)$ with $1 \star v \in L^1_{loc}([0,T);X_A)$. Then the following are equivalent.*

(i) v is a solution of (ACP_k); i.e., $\int_0^t v(s)\, ds \in \mathcal{D}(A)$ and $v(t) = A\int_0^t v(s)\, ds + \int_0^t k(s)x\, ds$.

(ii) There exists $\tilde{v} \in \{v\}$ with $\lambda \mapsto \frac{\tilde{v}(\lambda)}{\lambda} \in \{1 \star v\}_{X_A}$ and $\tilde{k} \in \{k\}$ such that $\frac{\tilde{v}}{\tilde{k}}$ solves (CE); i.e., $\tilde{v}(\lambda) \in \mathcal{D}(A)$ and $(\lambda I - A)\frac{\tilde{v}(\lambda)}{\tilde{k}(\lambda)} \approx_T x$.

Proof. Suppose (i) holds. Then $v(t) = A\int_0^t v(s)\,ds + k\star x(t)$, and by Proposition 2.1, $1\star v \in L^1_{loc}([0,T);[\mathcal{D}(A)])$. Thus

$$\{v\} = \{A1\star v\} + \{k\star x\} = A\{1\star v\}_{[\mathcal{D}(A)]} + \{k\star x\} + \{0\}.$$

Let $q \in \{1\star v\}_{[\mathcal{D}(A)]} \subset \{1\star v\}_X = \frac{\{v\}}{\lambda}$. Hence $q(\lambda) = \frac{\tilde{v}(\lambda)}{\lambda}$ for some $\tilde{v} \in \{v\}$ and

$$A\frac{\tilde{v}(\lambda)}{\lambda} + \tilde{k}(\lambda)\frac{x}{\lambda} \in \{v\} = \tilde{v}(\lambda) + \{0\}.$$

Thus $(\lambda - A)\frac{\tilde{v}(\lambda)}{\tilde{k}(\lambda)} \approx_T x$ and thus (ii) holds.

Suppose (ii) holds. Then

$$A\frac{\tilde{v}(\lambda)}{\lambda} \in \tilde{v}(\lambda) + \tilde{k}(\lambda)\frac{x}{\lambda} + \{0\} \subset \{v - k\star x\}.$$

By Proposition 2.1 (c), $(1\star v)(t) \in \mathcal{D}(A)$ for all $0 \le t < T$ and

$$A\int_0^t v(s)\,ds = v(t) - \int_0^t k(s)x\,ds.$$

$\diamondsuit$

The above theorem yields the following extension of the celebrated Lyubich uniqueness theorem (see [Ly] or [Pa],Thm. IV.1.2).

Corollary 2.3 (Lyubich). *Let A be a closed linear operator on a Banach space X. Let $a > 0$ and $k \in L^1_{loc}[0,a)$ with $0 \in supp(k)$. Suppose there exists a Müntz sequence $(\beta_n)_{n\in\mathbb{N}} \subset \mathbb{R}^+$ such that β_n is in the resolvent set of A for all $n \in \mathbb{N}$ and*

$$\limsup_{n\to\infty} \frac{1}{\beta_n} \ln \|R(\beta_n, A)\| \le 0.$$

Then every k-generalized solution to (ACP) is unique.

Proof. Suppose $(\beta_n - A)\frac{\tilde{v}(\beta_n)}{\tilde{k}(\beta_n)} = q(\beta_n) \approx_a 0$ for some $\tilde{k} \in \{k\}$ and some $\tilde{v} \in \mathcal{A}(\mathcal{P}; X)$. Then

$$\|\frac{\tilde{v}(\beta_n)}{\tilde{k}(\beta_n)}\| \leq \|R(\beta_n, A)\|\|q(\beta_n)\| \approx_a 0.$$

Since $0 \in supp(k)$, there exists a Müntz subsequence β_{n_j}, such that

$$\lim_{j\to\infty} \frac{1}{\beta_{n_j}} \ln \|\tilde{k}(\beta_{n_j})\| \leq 0.$$

Hence $\|\tilde{v}(\beta_{n_j})\| \approx_a 0$, implying that any regularized solution v is zero. $\diamondsuit$

The condition that the resolvent exists at points of a Müntz sequence and is of subexponential growth along those points, is also a necessary condition if we require existence of solutions for all initial data.

Theorem 2.4. *Let A be a closed linear operator on a Banach space X. Let $k \in L^1_{loc}[0, a)$ with $0 \in supp(k)$. Suppose (ACP) has a unique generalized limit solution $u = u(\cdot, x) \in \overline{C([0, a); X)}^{T_k}$ for all $x \in X$. Then every Müntz sequence $(\beta_n)_{n\in\mathbb{N}}$ has a subsequence $\left(\beta_{n_j}\right)_{j\in\mathbb{N}}$ such that β_{n_j} is in the resolvent set of A and*

$$\limsup_{j\to\infty} \frac{1}{\beta_{n_j}} \ln \|R(\beta_{n_j}, A)\| \leq 0.$$

Proof. By Proposition 1.5 there exists a strongly continuous operator family $(S(t))_{t\in[0,a)}$ such that

$$S(t)x = A\int_0^t S(s)x\, ds + (k \star 1)(t)x.$$

Furthermore, let $0 < T < a$. Let

$$\tilde{S}(\lambda)x := \int_0^T e^{-\lambda t} S(t)x\, dt.$$

Since $(1 \star S)(t) \in \mathcal{L}(X, [\mathcal{D}(A)])$ for all $0 \leq t < a$, we obtain that $\tilde{S}(\lambda) \in \mathcal{L}(X, [\mathcal{D}(A)])$ and $\left(\tilde{S}(\lambda)\right)_{\lambda \in \mathbb{C}}$ is a strongly analytic operator family. Let $\tilde{k} \in \{k\}_T$. Then, by Theorem 2.2,

$$(\lambda - A)\tilde{S}(\lambda)x = \tilde{k}(\lambda)x + r_x(\lambda),$$

where $r_x \approx_T 0$ for all $x \in X$. Furthermore, $r_{(\cdot)}(\lambda) \in \mathcal{L}(X)$. Let $0 < \epsilon < T/2$. Since $r_x \approx_T 0$, there exists M_x such that

$$e^{\lambda(T-\epsilon)}\|r_x(\lambda)\| \leq M_x$$

for all $\lambda > 0$. By the uniform boundedness principle there exists a constant M such that $r(\lambda) : x \mapsto r_x(\lambda)$ satisfies

$$\|r(\lambda)\| \leq Me^{-\lambda(T-\epsilon)}.$$

Let $(\beta_n)_{n \in \mathbb{N}}$ be a Müntz sequence. By Theorem II.3.6 there exists a subsequence $\left(\beta_{n_j}\right)_{j \in \mathbb{N}}$ such that $\lim_{j \to \infty} \frac{1}{\beta_{n_j}} \ln |\tilde{k}(\beta_{n_j})| = 0$ and thus there exists a constant J such that

$$|\tilde{k}(\beta_{n_j})| \geq e^{-\beta_{n_j}\epsilon} > 2\|r(\beta_{n_j})\|$$

for all $j > J$. Then $\|\frac{r(\beta_{n_j})}{\tilde{k}(\beta_{n_j})}\| < 1/2$. Thus $Id + \frac{r(\beta_{n_j})}{\tilde{k}(\beta_{n_j})}$ is an invertible operator that maps onto X and

$$\| \left(Id + \frac{r(\beta_{n_j})}{\tilde{k}(\beta_{n_j})} \right)^{-1} \| = \| \sum_{i=0}^{\infty} (-1)^i \left(\frac{r(\beta_{n_j})}{\tilde{k}(\beta_{n_j})} \right)^i \| \leq 2.$$

Since $\tilde{k}(\beta_{n_j}) \neq 0$ is scalar-valued, the operator

$$(\beta_{n_j} - A)\tilde{S}(\beta_{n_j}) = \tilde{k}(\beta_{n_j})(Id + \frac{r(\beta_{n_j})}{\tilde{k}(\beta_{n_j})})$$

is a continuously invertible operator. Let

$$Q(\beta_{n_j}) := \left(\tilde{k}(\beta_{n_j})(Id + \frac{r(\beta_{n_j})}{\tilde{k}(\beta_{n_j})})\right)^{-1}$$

and define

$$R(\beta_{n_j}) := \tilde{S}(\beta_{n_j})Q(\beta_{n_j}).$$

Then clearly, $\|Q(\beta_{n_j})\| \leq \frac{2}{\tilde{k}(\beta_{n_j})}$ and $(\beta_{n_j} - A)R(\beta_{n_j}) = Id$.

We show next that $(\beta_{n_j} - A)$ is one-to-one. Suppose $\beta_{n_j}x = Ax$ for some $x \in \mathcal{D}(A)$. Then

$$(e^{\beta_{n_j}t} - 1)x = A\frac{1}{\beta_{n_j}}(e^{\beta_{n_j}t} - 1)x = A\int_0^t e^{\beta_{n_j}s}x\,ds.$$

Since the solutions to ACP are unique, we deduce that $S(t)x = (k \star e^{\beta_{n_j}(\cdot)})(t)x$. Thus $\tilde{S}(\lambda)x$ is a scalar multiple of x for all $\lambda \in \mathbb{C}$. Hence,

$$0 = (\beta_{n_j} - A)\tilde{S}(\beta_{n_j})x = \tilde{k}(\beta_{n_j})(x + \frac{r(\beta_{n_j})x}{\tilde{k}(\beta_{n_j})}).$$

Therefore, $\| - x\| = \|\frac{r(\beta_{n_j})x}{\tilde{k}(\beta_{n_j})}\| \leq \|x\|/2$, and thus $x = 0$.

Hence $(\beta_{n_j} - A)$ is one-to-one and thus $R(\beta_{n_j}) = R(\beta_{n_j}, A)$ is the resolvent of A at β_{n_j}.

Since $\|Q(\beta_{n_j})\| \leq 2/|\tilde{k}(\beta_{n_j})|$, we know that

$$\limsup_{j\to\infty} \frac{1}{\beta_{n_j}} \ln \|Q(\beta_{n_j})\| = 0$$

and hence the same holds for $R(\beta_{n_j}) = \tilde{S}(\beta_{n_j})Q(\beta_{n_j})$. $\diamondsuit$

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Appendix: Relatively Closed Operators

The appendix is part of a preprint of F. Neubrander and the author ([Ba-Ne2]), and modified only slightly.

Some initial and/or boundary value problems for partial differential equations in function spaces on $I \times \Omega$ (I interval, $\Omega \subset \mathbb{R}^N$) lead to an equation

$$u'(\xi) = Au(\xi), \quad u(0) = x, \quad \xi \in [0, \xi_0],$$

in which the operator A is not closed. Since much of the significant information on the problem is contained in the space X and in the domain $\mathcal{D}(A)$, it is not always desirable to change the norm (by changing X) or the domain (by switching to the closure of A, if possible). As already remarked by Agmon and Nirenberg, in these cases it is *'convenient'* to consider the graph of A not in $X \times X$ but in an auxiliary space $X_A \times X$, where $\mathcal{D}(A) \subset X_A \hookrightarrow X$. We will show in this section that for a large class of linear operators $A \subset X \times X$ there exists an auxiliary Banach space X_A with $\mathcal{D}(A) \subset X_A \hookrightarrow X$ such that the graph of A is closed in $X_A \times X$. Such operators will be called *relatively closed with respect to* X_A or also *($X_A \hookrightarrow X$)-closed*. An operator A is called *relatively closed* if it is relatively closed with respect to some X_A.

The main result is Theorem A.4 below. There it is shown that the class of relatively closed operators with respect to a fixed Banach space Y is invariant under compositions, additions, and limits. In particular, compositions, additions, and limits of closed operators are relatively closed.

It is important to notice that a relatively closed, linear operator commutes with the Bochner and Stieltjes integral for sufficiently regular functions. We recall the following classical result (see 3.3 and 3.7 in [Hi-Ph]): Let A be $(X_A \hookrightarrow X)$-closed, let $u : [a,b] \to \mathcal{D}(A)$ be Bochner integrable in X_A, and let $Au : [a,b] \to X$ be Bochner integrable in X. Then $\int_a^b u(t)\,dt \in \mathcal{D}(A)$ and $\int_a^b Au(t)\,dt = A\int_a^b u(t)dt$.

Examples. (1) A natural class of examples of relatively closed operators are sums and compositions of operators A, B which are closed in $X \times X$. In general, the sum $S := A + B$ with domain $\mathcal{D}(S) = \mathcal{D}(A) \cap \mathcal{D}(B)$ and the composition $C := BA$ with domain $\mathcal{D}(C) = \{x \in \mathcal{D}(A) : Ax \in \mathcal{D}(B)\}$ will not be closed or closable in $X \times X$. However, both S and C are closed in $[\mathcal{D}(A)] \times X$, where $[\mathcal{D}(A)]$ denotes the Banach space $D(A)$ endowed with the graph norm. Since $\mathcal{D}(A) \subset [\mathcal{D}(A)] \hookrightarrow X$, sums and compositions of closed operators are relatively closed.

(2) We remark that the sum S and composition C can be relatively closed even if the operators A, B are not closed themselves. As example take a pair of *jointly closed* operators A, B on a Banach space X; i.e., $\mathcal{D}(A) \cap \mathcal{D}(B) \ni x_n \to x$, $Ax_n \to y_1$, and $Bx_n \to y_2$ implies that $x \in \mathcal{D}(A) \cap \mathcal{D}(B)$, $Ax = y_1$, and $Bx = y_2$ (see also [Sa]). Then $X_S := [\mathcal{D}(A) \cap \mathcal{D}(B)]$ with norm $\|x\|_{X_S} := \|x\| + \|Ax\| + \|Bx\|$ is a Banach space and $\mathcal{D}(S) \subset X_S \hookrightarrow X$. It is easy to see that the sum S is closed in $X_S \times X$.

(3) There are operators which are not closable in $X \times X$, but are relatively closed. As example, let A be the first derivative on $X = C[0,1]$ with maximal

domain and let B be the bounded operator $Bf := f(0)g$, where $0 \neq g \in X$. As seen above, the composition $Cf := BAf = f'(0)g$ with domain $\mathcal{D}(C) = \mathcal{D}(A) = C^1[0,1]$ is relatively closed. However, since there is a sequence $f_n \in D(C)$ with $f_n \to 0$ and $f_n'(0) = 1$, it follows from $Cf_n = f_n'(0)g = g \neq 0$ that C is not closable; i.e., the closure $C[0,1] \times \mathbb{C}g$ of the graph of C in $C[0,1] \times C[0,1]$ is not the graph of a single-valued operator. Because the multivalued closure of C does not contain any information about the original operator and because closedness is absolutely necessary for most operations, it is necessary to consider the graph as a subset of $X_A \times X$.

(4) In the example above one might take the domain $D(C_{max}) := \{f \in X : f'(0) \text{ ex.}\}$ instead of $C^1[0,1]$ and define bounded operators on X by

$$A_t f := \frac{f(t) - f(0)}{t} g.$$

Then, for each $f \in D(C_{max})$ one has that $A_t f \to C_{max} f$ as $t \to 0$. Because C_{max} is the pointwise limit of bounded operators, it follows from Theorem 2.4 below that C_{max} is relatively closed with respect to

$$X_{C_{max}} := \{f \in C[0,1] : \|f\|_{C_{max}} := \|f\| + \sup_{t \in [0,1]} \|\frac{f(t) - f(0)}{t}\| < \infty\}.$$

(5) Operators with "small" domains might not be relatively closed. Consider the identity operator I on $X = C[0,1]$ with the polynomials $\mathcal{P}$ as its domain. Clearly, I is closable. Assume it would be relatively closed. Then there exists a Banach space X_I such that the graph $\mathcal{G} = \{(p,p) : p \in \mathcal{P}\}$ of I is closed in $X_I \times X$.

Thus $\mathcal{G}$ is a complete metric space. However, $\mathcal{G}$ is also the union of countably many finite-dimensional subspaces and is thus of first category. By Baire's theorem, complete metric spaces are of second category, which is a contradiction. Thus, the operator I with domain $\mathcal{P}$ is not relatively closed. $\diamondsuit$

In the following proposition we collect some continuity properties of relatively closed operators and their restrictions to some Banach space $Y \hookrightarrow X$. The straightforward proofs are ommitted.

Proposition A.1. *Let A be $(X_A \hookrightarrow X)$-closed and $Y \hookrightarrow X$. The following hold.*

(a) Equipped with the graph norm, the domain

$$[\mathcal{D}(A)] := (\mathcal{D}(A), \|x\|_A := \|x\|_{X_A} + \|Ax\|_X)$$

is a Banach space and $A \in \mathcal{L}([\mathcal{D}(A)], X)$.

(b) Let A_Y be the restriction of A to Y (i.e. $\mathcal{D}(A_Y) = \mathcal{D}(A) \cap Y$). Then

$$[\mathcal{D}(A_Y)] := (\mathcal{D}(A_Y), \|x\|_{A_Y} := \|x\|_{X_A} + \|x\|_Y + \|Ax\|_X)$$

is a Banach space and $A_Y \in \mathcal{L}([\mathcal{D}(A_Y)], X)$.

(c) Let $A|_Y$ be the restriction of A in Y; i.e., $\mathcal{D}(A|_Y) := \{x \in \mathcal{D}(A) : Ax \in Y\}$. Then

$$[\mathcal{D}(A|_Y)] := (\mathcal{D}(A|_Y), \|x\|_{A|_Y} := \|x\|_{X_A} + \|Ax\|_Y)$$

is a Banach space, $A|_Y$ is closed in $X_A \times Y$ and hence $A|_Y \in \mathcal{L}([\mathcal{D}(A|_Y)], Y)$.

The following statement about relatively closed operators is an immediate consequence of the proposition above. Further properties of this type can be found in [Ca].

Corollary A.2. *An operator $A \subset X \times X$ is relatively closed if and only if there exists a Banach space Y such that $\mathcal{D}(A) \subset Y \hookrightarrow X$ and $A \in \mathcal{L}(Y, X)$. Moreover, if A is closed in $X_A \times X$, where $\mathcal{D}(A) \subset X_A \hookrightarrow X$, then the restriction $A|_{X_A}$ is closed in $X_A \times X_A$.* $\diamondsuit$

In applications, the following weak formulation of relative closedness is helpful.

Proposition A.3. *Let $\mathcal{D}(A) \subset X_A \hookrightarrow X$. The following are equivalent.*

(i) A is closed in $X_A \times X$. If $x_n \in \mathcal{D}(A)$, $x_n \to x$ in X_A, and $Ax_n \to y$ in X, then $x \in \mathcal{D}(A)$ and $Ax = y$.

(ii) If $x_n \in \mathcal{D}(A)$, $x_n \to x$ in X_A and $Ax_n \to y$ weakly in X, then $x \in \mathcal{D}(A)$ and $Ax = y$.

Proof. Clearly, *(ii)* implies *(i)*. Assume that *(i)* holds and that $x_i \to x$ in X_A and $y_i := Ax_i \to y$ weakly in X. Let $V_n := \{x_i : \|x_i - x\|_{X_A} \leq \frac{1}{n}\}$ and consider $y_i = Ax_i$ for $x_i \in V_n$. By Mazur's Theorem there exists a finite convex combination $\tilde{y}_n := \sum(a_i y_i) = A \sum(a_i x_i)$ such that $\|\tilde{y}_n - y\| \leq \frac{1}{n}$. Define $\tilde{x}_n := \sum(a_i x_i)$. Then $\|\tilde{x}_n - x\|_{X_A} \leq \frac{1}{n}$ and $\|A\tilde{x}_n - y\| \leq \frac{1}{n}$. Thus there exists a sequence $(\tilde{x}_n) \subset \mathcal{D}(A)$ with $\tilde{x}_n \to x$ in X_A and $A\tilde{x}_n \to y$. By *(i)*, $x \in \mathcal{D}(A)$ and $Ax = y$. $\diamondsuit$

Most linear operators that appear in applications can be decomposed into sums, products and/or limits of relatively closed operators. As we will see in the following, such operators are always relatively closed.

Notice that if $\{A_\alpha\}_{\alpha\in I}$ is a family of $(X_{A_\alpha} \hookrightarrow X)$-closed operators such that $Y \hookrightarrow X_{A_\alpha}$ for all $\alpha \in I$, then the restrictions of A_α to Y are $(Y \hookrightarrow X)$-closed.

Theorem A.4. *Finite sums and compositions of relatively closed operators are relatively closed. Also, limits, infinite sums and integrals of operators which are all relatively closed with respect to some $Y \hookrightarrow X$ are relatively closed.*

Proof. 1) Finite Sums. Let $A := \sum_{n=0}^{N} A_n$ with $\mathcal{D}(A) := \bigcap_{0\le n\le N} \mathcal{D}(A_n)$. If the operators A_n are $(X_{A_n} \hookrightarrow X)$-closed, then A is $(X_A \hookrightarrow X)$-closed, where

$$X_A := X_{A_N} \cap \bigcap_{0\le n\le N-1} [\mathcal{D}(A_n)] \quad , \quad \|x\|_{X_A} := \sup_{0\le n\le N} \|x\|_{X_{A_n}} + \sup_{0\le n\le N-1} \|A_n x\|.$$

2) Finite Products. Let $C_n := A_n \cdots A_0$ for $n \in \mathbb{N}_0$ with $\mathcal{D}(C_0) := \mathcal{D}(A_0)$ and $\mathcal{D}(C_n) := \{x \in \mathcal{D}(C_{n-1}) : C_{n-1}x \in \mathcal{D}(A_n)\}$ for $n > 0$. If the operators A_n are $(X_n \hookrightarrow X)$-closed, then C_n is $(X_{C_n} \hookrightarrow X)$-closed, where

$$X_{C_n} := [\mathcal{D}(C_{n-1}|_{X_{A_n}})] \quad , \quad \|x\|_{C_n} := \|x\|_{X_{A_0}} + \sup_{1\le i\le n} \|C_{i-1}x\|_{X_{A_i}}.$$

3) Limits. Let $Ax := \lim_{t\to 0} A_t x$ with $\mathcal{D}(A) := \{x \in \bigcap_{t\in I} \mathcal{D}(A_t) : \lim_{t\to 0} A_t x$ exists $\}$. Assume that the operators A_t are $(Y \hookrightarrow X)$-closed for all $t \in (0, 1]$. Let

$$X_A := \{x \in \bigcap_{t\in I} [\mathcal{D}(A_t)] : \|x\|_{X_A} := \|x\|_Y + \sup_{t\in I} \|A_t x\| < \infty\}.$$

Then X_A is a Banach space and $\mathcal{D}(A) \subset X_A \hookrightarrow X$. Suppose $\mathcal{D}(A) \ni x_n \to x$ in X_A. We show first that $x \in \mathcal{D}(A)$. Let $\epsilon > 0$. Choose n_0 such that $\|A_t x - A_t x_n\| < \epsilon/3$ for all $n \geq n_0$ and all $t \in (0,1]$. Then

$$\|A_t x - A_s x\| \leq \|A_t x - A_t x_{n_0}\| + \|A_t x_{n_0} - A_s x_{n_0}\| + \|A_s x_{n_0} - A_s x\| < \epsilon$$

for all $t, s < t_0$ for some $t_0 \in (0,1]$. Hence $x \in \mathcal{D}(A)$ and

$$\|Ax - Ax_n\| = \lim_{t \to 0} \|A_t x - A_t x_n\| \leq \epsilon/3$$

for all $n > n_0$. Thus $\mathcal{D}(A)$ is a closed subspace of X_A and A is a bounded linear operator from $\mathcal{D}(A)$ into X.

4) Infinite Sums. Let $Ax := \sum_{n=0}^{\infty} A_n x$ with $\mathcal{D}(A) := \{x \in \bigcap_{n \in \mathbb{N}} \mathcal{D}(A_n) : \sum_{n=0}^{\infty} A_n x \text{ exists}\}$. Assume that the operators A_n are $(Y \hookrightarrow X)$-closed for all $n \in \mathbb{N}$. Applying the limit case to partial sums we obtain that A $(X_A \hookrightarrow X)$-closed, where

$$X_A := \{x \in \bigcap_{n \in \mathbb{N}} \mathcal{D}(A_n) : \|x\|_{X_A} := \|x\|_Y + \sup_{n \in \mathbb{N}} \|\sum_{i=1}^{n} A_i x\| < \infty\}.$$

5) Integrals. Let $Ax := \int A_t x \, dt$ with $\mathcal{D}(A) := \{x \in \mathcal{D}(A_t)$ for almost all t and $A_{(\cdot)} x \in L^1(I, X)\}$. Assume that the operators A_t are $(Y \hookrightarrow X)$-closed for almost all $t \in I$. Let

$$X_A := \{x \in \mathcal{D}(A) : \|x\|_{X_A} := \|x\|_Y + \int \|A_t x\| \, dt\}.$$

To see that X_A is a Banach space, let $(x_i)_{i \in \mathbb{N}}$ be a Cauchy sequence in X_A. Then $(A_{(\cdot)} x_i)_{i \in \mathbb{N}}$ is a Cauchy sequence in $L^1(I, X)$. Hence there exists a function

$y_{(\cdot)} \in L^1(I, X)$ such that $A_{(\cdot)} x_i \to y_{(\cdot)}$ in $L^1(I, X)$. It follows that there exists a subsequence (x_{i_k}) for which $A_t x_{i_k} \to y_t$ pointwise for almost all t. Since (x_i) is also a Cauchy sequence in Y, there exists x such that $x_i \to x$ in Y. The relative closedness of A_t implies that $x \in \mathcal{D}(A_t)$ and $A_t x = y_t$ for almost all t. Since $y_{(\cdot)} \in L^1(I, X)$ we obtain that $A_{(\cdot)} x \in L^1(I, X)$ and therefore $x \in X_A$. It follows from

$$\|x_i - x\|_{X_A} = \|x_i - x\|_Y + \int \|A_t x_i - y_t\| \, dt$$

that X_A is a Banach space. Since

$$\|Ax - Ax_i\| = \| \int A_t x - A_t x_i \, dt\| \le \int \|A_t(x - x_i)\| \, dt \le \|x - x_i\|_{X_A}$$

we obtain that A is a bounded operator from X_A into X. $\diamondsuit$

Considering infinite compositions of relatively closed operators, one has to make sure that there exists a Banach space Y such that the finite composition operators are all $(Y \hookrightarrow X)$-closed. In the special case of the infinite composition of closed operators, Y can be chosen as follows.

Corollary A.5. *Compositions of closed operators are relatively closed.*

Proof. Let A_n be closed operators in $X \times X$ and let $C_n := A_n \cdots A_0$ for $n \in \mathbb{N}$. Then the restriction $C_{n\,Y}$ of C_n to Y is $(Y \hookrightarrow X)$-closed, where

$$Y := \{x \in \bigcap_{n \in \mathbb{N}} \mathcal{D}(C_n) : \|x\|_Y := \|x\| + \sup_{n \in \mathbb{N}} \|C_n x\| < \infty\}.$$

Now it follows that $C := \lim_{n \to \infty} C_n$ is $(Y \hookrightarrow X)$-closed. $\diamondsuit$

Vita

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