Exponential stability, exponential expansiveness, and exponential dichotomy of evolution equations on the half-line

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Abstract

For an evolution family on the half-line $\mathcal{U} = (U(t,s))_{t\geq s\geq 0}$ of bounded linear operators on a Banach space X we introduce operators G_0, G_X and I_X on certain spaces of X-valued continuous functions connected with the integral equation $u(t) = U(t,s)u(s) + \int_s^t U(t,\xi)f(\xi)d\xi$. We characterize exponential stability, exponential expansiveness and exponential dichotomy of $\mathcal{U}$ by properties of G_0, G_X and I_X, respectively. This extends related results known for finite dimensional spaces and for evolution families on the whole line, respectively.

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Introduction

We consider the non-autonomous linear evolution equation

$$\frac{d}{dt}u(t) = A(t)u(t),\ t \in J = \mathbb{R}_+ \text{ or } \mathbb{R}, \qquad (NCP)$$

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on a Banach space X. If the operators $A(t),\ t \in J$, are bounded or if X is finite dimensional there is an extensive literature initiated by the work of Perron which connects asymptotic properties (of the solutions) of (NCP) with specific properties of the operator L defined by

$$Lu(t) = \frac{d}{dt}u(t) - A(t)u(t),\ t \in J,$$

on a space of X-valued functions. For example, the existence of an exponential dichotomy of (NCP) is related to the invertibility of L if $J = \mathbb{R}$ and to Fredholm properties of L if $J = \mathbb{R}_+$ (see [BeG], [BGK], [Cop], [DaK], [MaS], [Pal] and the references therein). If the operators $A(t),\ t \in J$, are unbounded the situation becomes more delicate and usually one has to impose rather restrictive conditions in order to obtain similar results referring to the operator L (e.g. see [Hen], [Zha]).

A less restrictive assumption is the well-posedness of (NCP) in the sense that the solutions of (NCP) yield an evolution family $\mathcal{U} = (U(t,s))_{t\geq s,\, t,s\in J}$ of bounded linear operators on X (see [Nic], [Paz], [RRS], [RS2], [Sch] and [Tan] for sufficient conditions and examples). Note that solution does not necessarily mean 'classical' solution. Then instead of the operator L one investigates the integral equation

$$u(t) = U(t,s)u(s) + \int_s^t U(t,\xi)f(\xi)d\xi,\ t \geq s,\ t,s \in J, \tag{IE}$$

and its connection with asymptotic properties of the evolution family $\mathcal{U}$ (see [Bus], [DaK], [Dat], [LRS], [LeZ], [Zhi]). For instance, if $J = \mathbb{R}$ then it is shown in [LRS] that $\mathcal{U}$ has an exponential dichotomy if and only if for every $f \in C^b(\mathbb{R}, X)$ there is a unique solution $u \in C^b(\mathbb{R}, X)$ of (IE).

Another approach uses the so-called evolution semigroup $\mathcal{T} = (T(t))_{t\geq 0}$ on a space of X-valued functions induced by the evolution family $\mathcal{U}$ (see [AuM], [LaM], [LaR], [LMR1], [LMR2], [Mi1], [Mi2], [Nic], [RRS], [RS1], [RS2], [Rau], [Sch] and the references therein for this concept). Roughly speaking (if $J = \mathbb{R}$) the evolution semigroup $\mathcal{T}$ is given by

$$T(t)f(\xi) = U(\xi, \xi - t)f(\xi - t),\ t \geq 0,$$

(see (1.1) for the exact definition in the present context) and in certain situations its generator G is an extension of $-L = -\frac{d}{dt} + A(\cdot)$ (see [LMR1, Prop. 2.9], [Sch, Prop. 1.13]). It turns out that in the spirit of Perron's observations asymptotic properties of $\mathcal{U}$ like exponential stability or exponential dichotomy can be described by spectral properties of the operator G (see [AuM], [LaM], [LaR], [LMR1], [LMR2], [Mi1], [Mi2], [RS1], [Rau], [Sch]). However, we have to point out that most results in this direction are restricted to the line case $J = \mathbb{R}$

In the present paper we characterize exponential stability, exponential expansiveness and exponential dichotomy of an evolution family $\mathcal{U}$ on the half-line $J = \mathbb{R}_+$. Our approach is based on the use of (generators of) evolution semigroups and their connection with the integral equation (IE) (see Section 1). Exponential stability and exponential expansiveness of $\mathcal{U}$ is characterized in Theorem 2.2 and Theorem 2.5, respectively. Section 3 deals with exponential stability of individual orbits. Our main interest, however, is directed to the exponential dichotomy of $\mathcal{U}$ which will be characterized in Theorem 4.3 and Theorem 4.5. Concerning the relevance of exponential dichotomy and its far-reaching applications we refer to [SaS] and the references therein.

1. Preliminaries

Recall that a family of operators $\mathcal{U} = (U(t,s))_{t\geq s\geq 0}$ of bounded linear operators on a Banach space X is a *(strongly continuous, exponentially bounded) evolution family* on the half-line if

1. $U(t,t) = Id$ and $U(t,r)U(r,s) = U(t,s)$ for $t \geq r \geq s \geq 0$,
2. $(t,s) \mapsto U(t,s)x$ is continuous for every $x \in X$,
3. there are constants $K \geq 0$ and $\alpha \in \mathbb{R}$ such that $\|U(t,s)\| \leq Ke^{\alpha(t-s)}$ for $t \geq s \geq 0$.

Then $\omega(\mathcal{U}) := \inf\{\alpha \in \mathbb{R} \colon \text{there is } K \geq 0 \text{ such that } \|U(t,s)\| \leq Ke^{\alpha(t-s)},\ t \geq s \geq 0\}$ is called the *growth bound* of $\mathcal{U}$.

Throughout the whole paper the following function spaces (endowed with the sup-norm) play an important role:

$$\begin{aligned}
C_0 &:= \{v : [0,\infty) \to X :\ v \text{ is continuous}, v(0) = 0 = \lim_{t\to\infty} v(t)\},\\
C_X &:= \{v : [0,\infty) \to X :\ v \text{ is continuous},\ \lim_{t\to\infty} v(t) = 0\},\\
C_X(t_0) &:= \{v|_{[t_0,\infty)} :\ v \in C_X\},\ t_0 \geq 0,\\
C^b &:= \{v : [0,\infty) \to X :\ v \text{ is continuous and bounded}\}.
\end{aligned}$$

An evolution family $\mathcal{U}$ on X always defines an *evolution semigroup* $\mathcal{T} = (T(t))_{t\geq 0}$ of bounded linear operators on C_0 and C_X by setting

$$[T(t)v](s) = \begin{cases} U(s,s-t)v(s-t), & s \geq t,\\ U(s,0)v(0), & 0 \leq s \leq t, \end{cases} \tag{1.1}$$

for v in C_0 and C_X, respectively. It can be easily seen that $\mathcal{T}$ is strongly continuous. We denote the infinitesimal generator of $\mathcal{T}$ on C_0 and C_X by G_0 and G_X, respectively. Note that $D(G_0) = D(G_X) \cap C_0$ and $G_0 v = G_X v$ for $v \in D(G_0)$, i.e. G_0 is the part of G_X in C_0.

The aim of this paper is to characterize asymptotic properties of a given evolution family $\mathcal{U}$ by (spectral) properties of the generators G_0 and G_X. The following lemma is the key tool in our strategy. It connects the operators G_0 and G_X with the following inhomogeneous integral equation

$$u(t) = U(t,s)u(s) + \int_s^t U(t,\xi)f(\xi)d\xi,\ t \geq s \geq 0. \tag{1.2}$$

1.1 Lemma. *a) Let $u, f \in C_0$. Then $u \in D(G_0)$ and $G_0 u = -f$ if and only if*

$$u(t) = \int_0^t U(t,\xi)f(\xi)d\xi,\ t \geq 0. \tag{1.3}$$

b) Let $u \in C_X$ and $f \in C_0$. Then $u \in D(G_X)$ and $G_X u = -f$ if and only if (1.2) holds.

Proof. We only prove the first assertion. The proof of the second one follows the same lines.
Let $G_0 u = -f$. The general theory of linear semigroups (see e.g. [Paz, p.4-5]) yields

$$T(t)u - u = \int_0^t T(\xi) G_0 u \, d\xi = -\int_0^t T(\xi) f \, d\xi \text{ for } t \geq 0.$$

Thus,

$$u = T(t)u + \int_0^t T(\xi) f \, d\xi.$$

From the definition of $\mathcal{T}$ we easily obtain that u is a solution of the integral equation (1.2) and has the form (1.3).
Conversely, if $u, f \in C_0$ and u satisfies (1.3), then by reversing the above argument we obtain

$$T(t)u - u = -\int_0^t T(\xi) f \, d\xi \text{ for } t \geq 0.$$

In particular, this implies $u \in D(G_0)$ and $G_0 u = -f$. $\square$

In the following remark we collect some additional properties of the operators G_0 and G_X. For sake of convenience we set $U(t,s) = 0$ for $0 \leq t < s$.

1.2 Remarks. a) From Lemma 1.1 we immediately obtain that

- G_0 is injective
- $\ker G_X = \{u \in C_X : \; u(t) = U(t,0)u(0), \; t \geq 0\}$.

b) The range $R(G_X)$ of G_X is always contained in C_0 (in particular G_X is never invertible). In fact, if $u \in D(G_X)$ then

$$[G_X u](0) = \lim_{t \downarrow 0} \frac{T(t)u(0) - u(0)}{t} = \lim_{t \downarrow 0} \frac{u(0) - u(0)}{t} = 0.$$

c) Let $t_0 \geq 0$, $x \in X$ and $\varphi : [0, \infty) \to \mathbb{R}$ be continuously differentiable such that $\varphi_{|[0,t_0)} = 0$. Set $u(t) = \varphi(t) U(t,t_0)x$, $t \geq 0$, and $f(t) = \varphi'(t) U(t,t_0)x$, $t \geq 0$. If $u \in C_X$ and $f \in C_0$ then an immediate application of Lemma 1.1 yields $u \in D(G_X)$ and $G_X u = -f$.

Next we define an operator I_X on C_X connected with the integral equation (1.2). If $u, f \in C_X$ satisfy (1.2) we set

$$\begin{aligned} I_X u &:= f, \\ D(I_X) &:= \{u \in C_X : \text{ there is } f \in C_X \text{ such that } u \text{ and } f \text{ satisfy (1.2)}\}. \end{aligned}$$

1.3 Lemma. *$(I_X, D(I_X))$ is a well-defined closed linear operator on C_X and an extension of $(-G_X, D(G_X))$.*

Proof. Let $I_X u = f$ and $I_X u = g$ for $u, f, g \in C_X$. Then

$$\int_s^t U(t,\xi)(f(\xi) - g(\xi))d\xi = 0 \text{ for } t \geq s \geq 0.$$

By the continuity of the integrand we obtain $0 = U(t,t)(f(t)-g(t)) = f(t)-g(t),\ t \geq 0$, i.e. $f = g$.
The closedness of I_X follows immediately from the definition of I_X by the integral equation (1.2). Finally, Lemma 1.1 shows that I_X is an extension of $-G_X$. □

1.4 Remarks. a) In general, $-I_X$ is a proper extension of G_X, in particular, I_X can be surjective (cf. Theorem 4.3). Moreover, the spectra of $-I_X$ and G_X are different. In fact, for $\lambda \in \mathbb{C}$ with $\mathrm{Re}\lambda < -\omega(\mathcal{U})$ the function $u_\lambda : t \mapsto e^{\lambda t}U(t,0)x,\ t \geq 0,\ x \in X$, belongs to C_X and $I_X u_\lambda = \lambda u_\lambda$. Thus $\{\lambda \in \mathbb{C} : \mathrm{Re}\lambda > \omega(\mathcal{U})\}$ is contained in the (point) spectrum of $-I_X$ whereas the spectrum of G_X is contained in a left half-plane (see [Paz, Remark 1.5.4]). Note, however, that we always have $\ker I_X = \ker G_X = \{u \in C_X :\ u(t) = U(t,0)u(0),\ t \geq 0\}$.
b) From Lemma 1.1 it easily follows that G_0 is the part of $-I_X$ in C_0, i.e. $D(G_0) = \{u \in D(I_X) \cap C_0 : I_X u \in C_0\}$ and $G_0 u = -I_X u$ for $u \in D(G_0)$.

2. Exponential stability and exponential expansiveness

In this section we characterize exponentially stable and exponentially expansive evolution families $\mathcal{U} = (U(t,s))_{t \geq s \geq 0}$ by means of the operators G_0 and I_X and the integral equation (1.2), respectively. First we give the definition of exponential stability and exponential expansiveness of an evolution family.

2.1 Definition. The evolution family $\mathcal{U}$ on the Banach space X is said to be
a) *exponentially stable* if there are constants $N, \nu > 0$ such that $\|U(t,s)\| \leq Ne^{-\nu(t-s)}$ for all $t \geq s \geq 0$ (or, in other words, if $\omega(\mathcal{U}) < 0$),
b) *exponentially expansive* if $U(t,s)$ is invertible and there are constants $N, \nu > 0$ such that $\|U(t,s)x\| \geq Ne^{\nu(t-s)}\|x\|$ for $x \in X$ and $t \geq s \geq 0$.

We have the following characterization of exponentially stable evolution families.

2.2 Theorem. *Let $\mathcal{U} = (U(t,s))_{t \geq s \geq 0}$ be an evolution family on the Banach space X. Then the following assertions are equivalent:*
(i) $\mathcal{U}$ is exponentially stable.
(ii) G_0 is invertible.
(iii) For every $f \in C_0$ the function $t \mapsto u_f(t) = \int_0^t U(t,\xi)f(\xi)d\xi$ belongs to C_0.
(iv) For every $f \in C_0$ the function $t \mapsto u_f(t) = \int_0^t U(t,\xi)f(\xi)d\xi$ belongs to C^b.

From the proof of the theorem we separate the following lemma for later use.

2.3 Lemma. *Let $\chi : [t_0, t_1) \to (0, \infty)$ be a continuous function and let $c > 0$ and $K, \alpha \geq 0$ be constants such that $\chi(t) \leq K e^{\alpha(t-t_0)}$ and $\int_{t_0}^{t} \chi(t)\chi(\xi)^{-1} d\xi \leq c$ for $t \in [t_0, t_1)$. Then $\chi(t) \leq \max(cK, K) e^{\alpha + \frac{1}{c}} e^{-\frac{1}{c}(t-t_0)}$.*

Proof of Lemma 2.3. If $t \in [t_0, t_1)$ and $t \leq t_0+1$, then $\chi(t) \leq K e^{\alpha} \leq K e^{\alpha+\frac{1}{c}} e^{-\frac{1}{c}(t-t_0)}$. Now let $t \in [t_0, t_1)$ and $t > t_0 + 1$. Set $\phi(s) := \int_{t_0}^{s} \chi(\xi)^{-1} d\xi$ for $s \in [t_0, t_1)$. By our assumption $\phi(s) \leq c\phi'(s)$, and hence $\phi(s_1) \geq \phi(s_0) e^{\frac{1}{c}(s_1 - s_0)}$ for $t_1 > s_1 \geq s_0 > t_0$. Thus

$$\chi(t) = \frac{1}{\phi'(t)} \leq \frac{c}{\phi(t_0+1)} e^{-\frac{1}{c}(t-t_0-1)} \leq cK e^{\alpha+\frac{1}{c}} e^{-\frac{1}{c}(t-t_0)}. \quad \square$$

Proof of Theorem 2.2. The implications $(i) \Rightarrow (iii) \Rightarrow (iv)$ are obvious. The equivalence of (ii) and (iii) follows from Lemma 1.1.
$(iv) \Rightarrow (i)$: By assumption $B : C_0 \to C^b : f \mapsto u_f$ is linear and everywhere defined. Moreover, B is closed, and hence B is bounded. Let $c := \|B\|$.
Now fix $t_0 \geq 0$ and $0 \neq y \in X$. Let $t_1 := \sup\{t \geq t_0 : U(t,t_0)y \neq 0\}$. Then $U(t,t_0)y \neq 0$ for $t \in [t_0, t_1)$ and $U(t,t_0)y = 0$ for $t \geq t_1$. Set $\chi(t) := \|U(t,t_0)y\|$, $t \in [t_0, t_1)$. For $n \in \mathbb{N}$ sufficiently large choose a real continuous function φ_n on $[0,\infty)$ such that φ_n has compact support contained in (t_0, t_1), $0 \leq \varphi_n \leq 1$ and

$$\varphi_n(t) = 1 \quad \text{for } t \in [t_0 + \frac{1}{n}, \min\{n, t_1 - \frac{1}{n}\}].$$

Let

$$f_n(t) = \begin{cases} \varphi_n(t)\chi(t)^{-1} U(t,t_0)y & \text{for } t \in [t_0, t_1), \\ 0 & \text{for } t \in [0, t_0] \cup [t_1, \infty). \end{cases}$$

Then $f_n \in C_0$ and

$$\int_{t_0}^{t} \frac{\varphi_n(\xi)}{\chi(\xi)} d\xi \, \chi(t) = \int_{t_0}^{t} \frac{\varphi_n(\xi)}{\chi(\xi)} d\xi \, \|U(t,t_0)y\| = \| \int_{t_0}^{t} U(t,\xi) f_n(\xi) d\xi \| \leq c\|f_n\| = c$$

for $t \in [t_0, t_1)$. By letting $n \to \infty$ we obtain $\int_{t_0}^{t} \chi(t)\chi(\xi)^{-1} d\xi \leq c$ for $t \in [t_0, t_1)$. Moreover, by the exponential boundedness of $\mathcal{U}$ we have $\chi(t) \leq K e^{\alpha(t-t_0)} \|y\|$ on $[t_0, t_1)$. An application of Lemma 2.3 yields

$$\chi(t) \leq \max\{cK, K\} e^{\alpha+\frac{1}{c}} e^{-\frac{1}{c}(t-t_0)} \|y\|$$

for $t \in [t_0, t_1)$. Thus

$$\|U(t,t_0)y\| \leq \max\{cK, K\} e^{\alpha+\frac{1}{c}} e^{-\frac{1}{c}(t-t_0)} \|y\|$$

for $t \geq t_0$. Since the constants K, α and c are independent of t_0 and y the exponential stability of $\mathcal{U}$ follows. $\square$

Remark. The proof of (iv) $\Rightarrow$ (i) follows ideas used in [DaK, Proof of Theorem IV.3.3]. By different methods the equivalence of (i) and (iv) has been shown by C. Buşe [Bus, Theorem 1] (see also [Dat, Theorem 8] for a related result) and the equivalence of (i), (ii) and (iii) is proved by Y. Latushkin, S. Montgomery-Smith and T. Randolph [LMR2, Theorem 2.2, Corollary 2.3]. Note that Theorem 2.2 also holds if we replace C_0 and C^b by $L^p(\mathbb{R}_+, X)$, $1 \leq p < \infty$, and consider the evolution semigroup $\mathcal{T}$ on $L^p(\mathbb{R}_+, X)$ induced by $\mathcal{U}$ (see [Dat, Theorem 6] and [LMR2, Theorem 2.2, Corollary 2.3]).

An immediate consequence of Theorem 2.2 is the spectral mapping theorem for the evolution semigroup on C_0. Recall that $\sigma(B)$ denotes the *spectrum* of a linear operator B and $\rho(B) := \mathbb{C} \backslash \sigma(B)$ is the *resolvent set.* Moreover, $s(B) := \sup\{\mathrm{Re}\lambda : \lambda \in \sigma(B)\}$ is the *spectral bound* and $r(B) := \sup\{|\lambda| : \lambda \in \sigma(B)\}$ is the *spectral radius* of B.

2.4 Corollary. *Let $\mathcal{U}$ be an evolution family on the Banach space X. Then the evolution semigroup $\mathcal{T}$ on C_0 satisfies the spectral mapping theorem*

$$e^{t\sigma(G_0)} = \sigma(T(t)) \setminus \{0\}, \; t \geq 0.$$

Furthermore, $\sigma(G_0) = \{\lambda \in \mathbb{C} : \mathrm{Re}\lambda \leq s(G_0)\}$ is a left half-plane and $\sigma(T(t)) = \{\lambda \in \mathbb{C} : |\lambda| \leq r(T(t))\}$, $t > 0$, is a disc.

Proof. Let $\lambda \in \rho(G_0)$ and $\mu \in \mathbb{C}$ such that $\mathrm{Re}\mu \geq \mathrm{Re}\lambda$. Note that $G_0 - \lambda Id$ is the generator of the evolution semigroup $\mathcal{T}_\lambda = (e^{-\lambda t}T(t))_{t \geq 0}$ on C_0 induced by the evolution family $\mathcal{U}_\lambda = (e^{-\lambda(t-s)}U(t,s))_{t \geq s \geq 0}$. Theorem 2.2 implies that $\mathcal{U}_\lambda$ and hence $\mathcal{U}_\mu$ is exponentially stable and, as a consequence, that $G_0 - \mu Id$ is invertible, i.e. $\mu \in \rho(G_0)$. Thus $\sigma(G_0) = \{\lambda \in \mathbb{C} : \mathrm{Re}\lambda \leq s(G_0)\}$ is a left half-plane.
If $\mathrm{Re}\lambda > s(G_0)$, then again by Theorem 2.2 the evolution family $\mathcal{U}_\lambda$ is exponentially stable, and hence the induced evolution semigroup $\mathcal{T}_\lambda$ is exponentially stable as well. In particular $r(e^{-\lambda t}T(t)) < 1$, i.e. $r(T(t)) < e^{\mathrm{Re}\lambda \cdot t}$ for $t > 0$. Thus $r(T(t)) \leq e^{s(G_0) \cdot t}, t \geq 0$. Together with the spectral inclusion theorem $e^{t\sigma(G_0)} \subseteq \sigma(T(t))$, $t \geq 0$ (see [Paz, 2.2.3]), it follows that $\sigma(T(t))$, $t > 0$, is a disc and that the spectral mapping theorem holds. $\square$

Remark. The same result (with a different proof) can be found in [LMR2, Theorem 1.2, Theorem 2.2, Corollary 2.3] and [Sch, Theorem 5.3]. The observation that on C_0 the spectrum of $T(t)$, $t \geq 0$, is always a disc goes back to [Rau, Proposition 2].

We now come to the characterization of exponentially expansive evolution families.

2.5 Theorem. *Let $\mathcal{U} = (U(t,s))_{t \geq s \geq 0}$ be an evolution family on the Banach space X. Then the following assertions are equivalent:*

(i) $\mathcal{U}$ is exponentially expansive.

(ii) For every $t_0 \geq 0$ and every $f \in C_X(t_0)$ there exists a unique $u_f \in C_X(t_0)$ such that

$$u_f(t) = U(t,s)u_f(s) + \int_s^t U(t,\xi)f(\xi)d\xi, \; t \geq s \geq t_0. \tag{2.1}$$

Remark. Condition (ii) implies that the operator I_X is invertible. However, Example 4.6 shows that the exponential expansiveness of $\mathcal{U}$ (and hence (ii)) is not equivalent to the invertibility of I_X. On the contrary, if $U(t,s)$ is surjective for all $t \geq s \geq 0$, then from the proof of Theorem 2.5 it follows that $\mathcal{U}$ is exponentially expansive if and only if I_X is invertible.

As above we separate from the proof of the theorem the following lemma which is essentially contained in [DaK, Proof of Theorem IV.3.3].

2.6 Lemma. *Let $\chi : [0,\infty) \to (0,\infty)$ be a continuous function and let $c > 0$ and $K, \alpha > 0$ be constants such that $\chi(\tau) \leq K e^{\alpha(\tau - t)} \chi(t)$, $\tau \geq t \geq 0$, and $\int_t^\infty \chi(t)\chi(\tau)^{-1} d\tau \leq c$, $t \geq 0$. Then there exists $N \geq 0$ only dependent on K, α and c such that $\chi(t) \geq N e^{\frac{1}{c}(t-s)} \chi(s)$ for $t \geq s \geq 0$.*

Proof. Let $\phi(t) := \int_t^\infty \chi(\tau)^{-1} d\tau$, $t \geq 0$. By our assumption $\phi(t) \leq -c\phi'(t)$. Thus $\phi(t) \leq \phi(s) e^{-\frac{1}{c}(t-s)}$ for $t \geq s \geq 0$. On the other hand the exponential estimate of χ yields

$$\begin{aligned}
\chi(t)\phi(t) = \chi(t) \int_t^\infty \frac{1}{\chi(\tau)} d\tau &\geq K^{-1} \int_t^\infty e^{-\alpha(\tau - t)} d\tau \\
&\geq K^{-1} \sum_{k=1}^\infty \int_{t+(k-1)}^{t+k} e^{-\alpha(t+k-t)} d\tau = K^{-1} \sum_{k=1}^\infty e^{-\alpha k} =: \tilde{N}.
\end{aligned}$$

Moreover, by our assumption, $\chi(s)\phi(s) \leq c$, $t \geq s \geq 0$. Thus

$$\chi(t) \geq \frac{\tilde{N}}{\phi(t)} \geq \frac{\tilde{N}}{\phi(s)} e^{\frac{1}{c}(t-s)} \geq \frac{\tilde{N}}{c} e^{\frac{1}{c}(t-s)} \chi(s) \quad \text{for } t \geq s \geq 0. \qquad \square$$

Now we are in a position to give the proof of Theorem 2.5.

Proof of Theorem 2.5. *(i)* $\Rightarrow$ *(ii)*: Let $\mathcal{U}$ be exponentially expansive and $t_0 \geq 0$. If $f \in C_X(t_0)$ then the function $t \mapsto u_f(t) := -\int_t^\infty U(\xi,t)^{-1} f(\xi) d\xi$ belongs to $C_X(t_0)$. It is a straightforward computation to show that u_f satisfies (2.1). If $u \in C_X(t_0)$ satisfies (2.1) with $f = 0$, then $u(t) = U(t,t_0)u(t_0)$, $t \geq t_0$. Thus $\|u(t)\| \geq N e^{\nu(t-t_0)} \|u(t_0)\|$ for constants $N, \nu > 0$. This yields $u(t_0) = 0$, and hence $u = 0$.

(ii) $\Rightarrow$ *(i)*: **A)** The assumption implies that I_X is invertible. Since I_X is closed its inverse I_X^{-1} is bounded and $c := \|I_X^{-1}\| > 0$.

B) Fix $0 \neq x \in X$. The unique solvability of (2.1) for $f = 0$ in C_X yields $u(t) := U(t,0)x \neq 0$ for $t \geq 0$. For each $n \in \mathbb{N}$ choose a real continuous function φ_n on $[0,\infty)$ such that $0 \leq \varphi_n \leq 1$, $\varphi_n = 1$ on $[0,n]$ and $\varphi_n = 0$ on $[n+1,\infty)$. Let $f_n(t) := -\varphi_n(t)\|u(t)\|^{-1} u(t)$ for $t \geq 0$. An easy computation shows that

$$u_n(t) := \int_t^\infty \frac{\varphi_n(\xi)}{\|u(\xi)\|} d\xi \, u(t), \ t \geq 0,$$

and f_n solve (1.2), i.e. $I_X u_n = f_n$. Hence, $\|u_n\| \leq c\|f_n\| = c$. Letting $n \to \infty$ we obtain

$$\int_t^\infty \frac{1}{\|u(\xi)\|} d\xi \leq \frac{c}{\|u(t)\|}, \ t \geq 0.$$

The exponential boundedness of $\mathcal{U}$ yields $\|u(\tau)\| \leq Ke^{\alpha(\tau-t)}\|u(t)\|, \tau \geq t \geq 0$, for constants $K, \alpha > 0$ independent of x. By Lemma 2.6 there is a constant $N \geq 0$ independent of x such that $\|u(t)\| \geq Ne^{\frac{1}{c}(t-s)}\|u(s)\|$ for $t \geq s \geq 0$.
C) It remains to show that $U(t,s)$ is surjective for $t \geq s \geq 0$. Fix $s \geq 0$. Let $y \in X$ and set $v(t) := U(t,s)y,\ t \geq s$. We choose a real continuous function φ on $[0,\infty)$ with compact support such that $|\varphi(t)| \leq 1, \operatorname{supp}\varphi \subset (s,\infty)$ and

$$\int_s^\infty \varphi(\xi)d\xi = 1.$$

Now consider the function w defined by

$$w(t) := v(t)\int_t^\infty \varphi(\xi)d\xi,\ t \geq s.$$

Then $w \in C_X(s)$ and w is a solution of equation (2.1) with $f(t) := -\varphi(t)v(t),\ t \geq s$. Extend f continuously to $[0,\infty)$ by setting $f(t) = 0$ on $[0,s]$. Then there is a solution $z \in C_X$ of equation (2.1) on $[0,\infty)$. By our assumption $z|_{[s,\infty)} = w$. Thus

$$y = w(s) = z(s) = U(s,0)z(0) + \int_0^s U(s,\xi)f(\xi)d\xi = U(s,0)z(0).$$

This proves the surjectivity of $U(s,0)$ for $s \geq 0$. Together with $U(t,0) = U(t,s)U(s,0)$ this yields the surjectivity of $U(t,s)$ for $t \geq s \geq 0$. $\square$

3. Exponential stability of bounded orbits

In Theorem 2.2 we saw that exponential stability of an evolution family $\mathcal{U} = (U(t,s))_{t\geq s\geq 0}$ is characterized by the invertibility of the generator G_0 of the evolution semigroup $\mathcal{T}$ on C_0. We now show that a condition on the approximate point spectrum of G_0 implies exponential stability of all bounded orbits $U(t,t_0)x,\ t \geq t_0 \geq 0$. Recall that for an operator B on a Banach space Y the *approximate point spectrum* $A\sigma(B)$ of B is the set of all complex numbers λ such that for every $\epsilon > 0$ there exists $y \in D(B)$ with $\|y\| = 1$ and $\|(\lambda - B)y\| \leq \epsilon$.

As the whole spectrum $\sigma(G_0)$ (see Corollary 2.4) the approximate point spectrum $A\sigma(G_0)$ is invariant under translations along the imaginary axis.

3.1 Lemma. *$A\sigma(G_0) = i\mu + A\sigma(G_0)$ for all $\mu \in \mathbb{R}$*

Proof. Consider the multiplication operator M_μ on C_0 defined by $(M_\mu v)(t) := e^{i\mu t}v(t),\ t \geq 0,\ v \in C_0$. Then M_μ is an isometry and

$$\begin{aligned}(M_\mu T(t)M_{-\mu}v)(s) &= \begin{cases} e^{i\mu s}U(s,s-t)e^{-i\mu(s-t)}v(s-t), & s \geq t \geq 0,\\ 0, & 0 \leq s \leq t,\end{cases}\\ &= e^{i\mu t}(T(t)v)(s),\end{aligned}$$

$v \in C_0,\ s \geq 0$. Thus $M_\mu T(t)M_{-\mu} = e^{i\mu t}T(t),\ t \geq 0$, and hence $M_\mu G_0 M_{-\mu} = i\mu + G_0$. From this the assertion immediately follows. $\square$

We now come to the main result of this section. A special case of it has been shown in [AuM, Theorems 7 and 7']. Our proof follows the techniques of [AuM].

3.2 Theorem. *Let* $\mathcal{U} = (U(t,s))_{t\geq s\geq 0}$ *be an evolution family on the Banach space* X *such that* $A\sigma(G_0)\cap i\mathbb{R}\neq i\mathbb{R}$. *Then every bounded orbit of* $\mathcal{U}$ *is exponentially stable. Precisely, if* $\sup_{t\geq t_0}\|U(t,t_0)x\| < \infty$ *for fixed* $x\in X$ *and* $t_0\geq 0$, *then there exist constants* $N,\nu > 0$ *independent of* x *and* t_0 *such that*

$$\|U(t,t_0)x\| \leq Ne^{-\nu(t-s)}\|U(s,t_0)x\|,\ \ t\geq s\geq t_0.$$

Proof. Let $x\in X$ and $t_0\geq 0$ such that $\sup_{t\geq t_0}\|U(t,t_0)x\| < \infty$.
A) $\lim_{t\to\infty}\|U(t,t_0)x\| = 0$:
Choose a real continuous function φ on $[0,\infty)$ with compact support such that $\varphi(t_0+1) = 1$ and $\varphi|_{[0,t_0]} = 0$. Define

$$v(t) := \begin{cases} \varphi(t)U(t,t_0)x & \text{for } t > t_0,\\ 0 & \text{for } 0\leq t\leq t_0.\end{cases}$$

Then $v\in C_0$ and

$$\begin{aligned}&\|T(t+h)v - T(t)v\|\\ &\quad = \max\{\sup_{s\geq t+h}\|U(s,s-t-h)v(s-t-h) - U(s,s-t)v(s-t)\|,\\ &\qquad\quad \sup_{t+h\geq s\geq t}\|U(s,s-t)v(s-t)\|\}\end{aligned}$$

for $t\geq 0$ and $h > 0$. From

$$U(s,s-t)v(s-t) = \begin{cases}\varphi(s-t)U(s,t_0)x & \text{for } s-t > t_0,\\ 0 & \text{for } 0\leq s-t\leq t_0,\end{cases}$$

$\varphi(0) = 0$ and the uniform boundedness of $U(t,t_0)x,\ t\geq t_0$, we obtain

$$\lim_{h\downarrow 0}\sup_{t\geq 0}\sup_{t+h\geq s\geq t}\|U(s,s-t)v(s-t)\| = 0.$$

On the other hand

$$\begin{aligned}&\sup_{s\geq t+h}\|U(s,s-t-h)v(s-t-h) - U(s,s-t)v(s-t)\|\\ &\qquad = \sup_{s\geq t+h}\|\varphi(s-t-h)U(s,t_0)x - \varphi(s-t)U(s,t_0)x\|\\ &\qquad \leq \sup_{s\geq t+h}|\varphi(s-t-h) - \varphi(s-t)|\sup_{s\geq t_0}\|U(s,t_0)x\|.\end{aligned}$$

By the uniform continuity of φ we have

$$\lim_{h\downarrow 0}\sup_{t\geq 0}\sup_{s\geq t+h}\|U(s,s-t-h)v(s-t-h) - U(s,s-t)v(s-t)\| = 0.$$

Thus $t \mapsto T(t)v$ is a uniformly continuous C_0-valued function on $[0, \infty)$. Our assumption on G_0 and Lemma 3.1 imply $A\sigma(G_0) \cap i\mathbb{R} = \emptyset$. An application of a result of Batty and Vũ [BaV, Theorem 1] yields $\lim_{t\to\infty} T(t)v = 0$. In particular,

$$\begin{aligned} 0 &= \lim_{t\to\infty} \|(T(t)v)(t + t_0 + 1)\| = \lim_{t\to\infty} \|U(t + t_0 + 1, t_0 + 1)v(t_0 + 1)\| \\ &= \lim_{t\to\infty} \|U(t + t_0 + 1, t_0)x\|. \end{aligned}$$

B) $\|U(t, t_0)x\| \le Ne^{-\nu(t-t_0)}\|x\|, \; t \ge t_0$:
Without loss of generality we may assume $\|x\| = 1$. Since $0 \notin A\sigma(G_0)$ there is a constant $\nu > 0$ such that

$$\|G_0 v\| \ge \nu\|v\| \text{ for } v \in D(G_0). \tag{3.1}$$

Let $u(t) := U(t, t_0)x, \; t \ge t_0$, and $t_1 := \sup\{t \ge t_0 : \; U(t, t_0)x \ne 0\}$. The exponential boundedness of $\mathcal{U}$ implies that there are constants $K, \alpha > 0$ (independent of x and t_0) such that $\|u(t)\| \le Ke^{\alpha(t-t_0)}, \; t \ge t_0$. For $n \in \mathbb{N}$ sufficiently large we choose a real continuous function ψ_n on $[0, \infty)$ such that ψ_n has compact support contained in $(t_0, t_1), \; 0 \le \psi_n \le 1$ and

$$\psi_n(t) = 1 \quad \text{for } t \in [t_0 + \frac{1}{n}, \min\{n, t_1 - \frac{1}{n}\}].$$

Let

$$\varphi_n(t) := \begin{cases} 0 & \text{for } t \in [0, t_0], \\ \int_{t_0}^{t} \psi_n(\xi)\|u(\xi)\|^{-1} d\xi & \text{for } t \in (t_0, t_1), \\ \int_{t_0}^{t_1} \psi_n(\xi)\|u(\xi)\|^{-1} d\xi & \text{for } t \in [t_1, \infty), \end{cases}$$

$u_n(t) := \varphi_n(t)U(t, t_0)x, \; t \ge 0$, and $f_n(t) := \varphi_n'(t)U(t, t_0)x, \; t \ge 0$, where we set $U(t, t_0) = 0$ for $0 \le t < t_0$. From A) and the definition of φ_n it follows that u_n and f_n satisfy the assumptions of Remark 1.2 c). Hence $u_n \in D(G_0)$ and $G_0 u_n = -f_n$. Thus, by (3.1),

$$1 = \sup_{t \ge t_0} \|f_n(t)\| = \|f_n\| \ge \nu\|u_n\| = \nu \sup_{t \ge t_0} \|u_n(t)\|.$$

In particular,

$$\nu^{-1} \ge \|u_n(t)\| = |\varphi_n(t)|\|u(t)\| = \int_{t_0}^{t} \psi_n(\xi)\|u(\xi)\|^{-1} d\xi \; \|u(t)\| \quad \text{for } t \in [t_0, t_1).$$

Letting $n \to \infty$ we obtain

$$\int_{t_0}^{t} \|u(t)\|\|u(\xi)\|^{-1} d\xi \le \nu^{-1}.$$

By Lemma 2.3 there is a constant N only dependent on ν, K and α such that

$$\|U(t, t_0)x\| = \|u(t)\| \le Ne^{-\nu(t-t_0)}, \; t \ge t_0.$$

C) Fix $s \geq t_0$ and set $y := U(s,t_0)x$. Then $\sup_{t\geq s} \|U(t,s)y\| < \infty$ and from A) and B) we obtain

$$\|U(t,t_0)x\| = \|U(t,s)y\| \leq Ne^{-\nu(t-s)}\|y\| = Ne^{-\nu(t-s)}\|U(s,t_0)x\|, \ t \geq s. \quad \square$$

Denote by $X_0(t_0) := \{x \in X : \lim_{t\to\infty} U(t,t_0)x = 0\}$ the *stable subspace* corresponding to $\mathcal{U}$ and $t_0 \geq 0$. From Theorem 3.2 we obtain the following properties of $X_0(t_0)$.

3.3 Corollary. *Under the conditions of Theorem 3.2 we have*

$$\begin{aligned} X_0(t_0) &= \{x \in X : \sup_{t\geq t_0} \|U(t,t_0)x\| < \infty\} \\ &= \{x \in X : \|U(t,t_0)x\| \leq Ne^{-\nu(t-t_0)}\|x\|\}, \ t_0 \geq 0, \end{aligned}$$

for fixed constants $N, \nu > 0$, and $X_0(t_0)$ is a closed linear subspace of X.

Remarks. a) Lemma 3.1 and Lemma 1.1 imply that the following assertions are equivalent:

(i) $A\sigma(G_0) \cap i\mathbb{R} \neq i\mathbb{R}$
(ii) $0 \notin A\sigma(G_0)$.
(iii) $A\sigma(G_0) \cap i\mathbb{R} = \emptyset$.
(iv) There is a constant $\nu > 0$ such that for any pair $u, f \in C_0$ with $u(t) = \int_0^t U(t,\xi)f(\xi)d\xi$, $t \geq 0$, one has $\|f\| \geq \nu\|u\|$.

b) An evolution family with an exponential dichotomy (see Definition 4.1) always satisfies $A\sigma(G_0) \cap i\mathbb{R} = \emptyset$ (see the Remark after Theorem 4.3).

c) If B is strongly continuous and uniformly bounded function from $[0,\infty)$ into the space of bounded linear operator on X, then there is a unique evolution family $\mathcal{U}_B = (U_B(t,s))_{t\geq s\geq 0}$ satisfying the variation of constants formula

$$U_B(t,s)x = U(t,s)x + \int_s^t U(t,\xi)B(\xi)U_B(\xi,s)x d\xi, \ t \geq s \geq 0, \ x \in X$$

(see [RS2], [Sch] and the references therein). The generator of the evolution semigroup $\mathcal{T}_B$ on C_0 induced by $\mathcal{U}_B$ is given by $G_0 + B(\cdot)$. Suppose that G_0 is invertible resp. $0 \notin A\sigma(G_0)$. Then $G_0 + B(\cdot)$ has the same property for every perturbation $B(\cdot)$ such that $\|B(\cdot)\| = \sup_{t\geq 0} \|B(t)\|$ is sufficiently small (cf. [Kat, IV.3.1]). In other words exponential stability of $\mathcal{U}$ resp. of all bounded orbits of $\mathcal{U}$ is robust with respect to small perturbations $B(\cdot)$ of $\mathcal{U}$ (see also [LMR2, Section 3.1], [Sch, Section 5.4]).

4. Exponential dichotomy

In this section we characterize the exponential dichotomy of an evolution family $\mathcal{U}$ by properties of the operators G_0, G_X and I_X, and of the operators G_Z and I_Z to be defined below. At first we recall the following definition.

4.1 Definition. An evolution family $\mathcal{U} = (U(t,s))_{t\geq s\geq 0}$ on the Banach space X is said to have an *exponential dichotomy* if there exist bounded linear projections $P(t),\ t \geq 0$, on X and constants $N, \nu > 0$ such that

a) $U(t,s)P(s) = P(t)U(t,s),\ t \geq s \geq 0$,

b) the restriction $U(t,s)_| : \ker P(s) \to \ker P(t),\ t \geq s \geq 0$, is an isomorphism (and we denote its inverse by $U(s,t)_| : \ker P(t) \to \ker P(s)$),

c) $\|U(t,s)x\| \leq Ne^{-\nu(t-s)}\|x\|$ for $x \in P(s)X,\ t \geq s \geq 0$,

d) $\|U(s,t)_| x\| \leq Ne^{-\nu(t-s)}\|x\|$ for $x \in \ker P(t),\ t \geq s \geq 0$.

In the following lemma we collect some properties of the family $P(t), t \geq 0$. By $\mathcal{L}(Y, X)$ we denote the space of bounded linear operators between the Banach spaces Y and X.

4.2 Lemma. *Let $\mathcal{U}$ be an evolution family having an exponential dichotomy with corresponding family of projections $P(t),\ t \geq 0$, and constants $N, \nu > 0$. Then the following holds:*

a) $M := \sup_{t\geq 0} \|P(t)\| < \infty$,

b) $[0,t] \ni s \mapsto U(s,t)_| \in \mathcal{L}(\ker P(t), X)$ *is strongly continuous for* $t > 0$.

c) $t \mapsto P(t)$ *is strongly continuous,*

d) $\|U(t,s)P(s)\| \leq MNe^{-\nu(t-s)}$ *for* $t \geq s \geq 0$,

e) $\|U(s,t)_|(I - P(t))\| \leq MNe^{-\nu(t-s)}$ *for* $t \geq s \geq 0$.

Proof. a) can be shown as in [DaK, Lemma IV.1.1 and Lemma IV.3.2]. For sake of completeness we present the details. Fix $t_0 \geq 0$. Let $P_0 := P(t_0),\ P_1 := Id - P(t_0)$ and $X_k := P_k X,\ k = 0, 1$. Set $\gamma_{t_0} := \inf\{\|x_0 + x_1\| : x_k \in X_k,\ \|x_0\| = \|x_1\| = 1\}$. If $x \in X$ and $P_k x \neq 0,\ k = 0, 1$, then

$$\begin{aligned}\gamma_{t_0} &\leq \|\frac{P_0 x}{\|P_0 x\|} + \frac{P_1 x}{\|P_1 x\|}\| = \frac{1}{\|P_0 x\|}\|P_0 x + \frac{\|P_0 x\|}{\|P_1 x\|}P_1 x\| \\ &= \frac{1}{\|P_0 x\|}\|x + \frac{\|P_0 x\| - \|P_1 x\|}{\|P_1 x\|}P_1 x\| \leq \frac{2\|x\|}{\|P_0 x\|}.\end{aligned}$$

As a consequence $\|P_0\| \leq 2\gamma_{t_0}^{-1}$. It remains to show that there is a constant $c > 0$ (independent of t_0) such that $\gamma_{t_0} \geq c$. For this fix $x_k \in X_k,\ k = 0, 1$, with $\|x_0\| = \|x_1\| = 1$. By the exponential boundedness of $\mathcal{U}$ we have $\|U(t,t_0)(x_0 + x_1)\| \leq Ke^{\alpha(t-t_0)}\|x_0 + x_1\|$ for $t \geq t_0$ and constants $K, \alpha \geq 0$. Thus

$$\begin{aligned}\|x_0 + x_1\| &\geq K^{-1}e^{-\alpha(t-t_0)}\|U(t,t_0)x_0 + U(t,t_0)x_1\| \\ &\geq K^{-1}e^{-\alpha(t-t_0)}(N^{-1}e^{\nu(t-t_0)} - Ne^{-\nu(t-t_0)}) \\ &=: c_{t-t_0},\ t \geq t_0,\end{aligned}$$

and hence $\gamma_{t_0} \geq c_{t-t_0}$. Obviously $c_m > 0$ for m sufficiently large. Thus $0 < c_m \leq \gamma_{t_0}$.
b) Fix $t > 0,\ 0 \leq s_0 \leq t$ and $x \in \ker P(t)$, and let (s_n) be a sequence in $[0,t]$ converging to s_0. There is $y \in \ker P(0)$ such that $U(t,0)y = x$. By the strong continuity of $\mathcal{U}$ we have

$$\lim_n \|U(s_n,t)_| x - U(s_0,t)_| x\| = \lim_n \|U(s_n,0)_| y - U(s_0,0)_| y\| = 0.$$

c) Note that

$$\begin{aligned}\|P(t)x - P(s)x\| &\le \|P(t)x - P(t)U(t,s)x\| + \|U(t,s)P(s)x - P(s)x\| \\ &\le (\sup_{r\ge 0}\|P(r)\|)\,\|x - U(t,s)x\| + \|U(t,s)P(s)x - P(s)x\|\end{aligned}$$

for $x \in X$ and $t \ge s \ge 0$. By the strong continuity of $\mathcal{U}$ we obtain that $P(\cdot)$ is strongly continuous from the right. In order to show strong continuity from the left set $Q(\cdot) := Id - P(\cdot)$ and fix $t > 0$ and $x \in X$. For $0 \le s \le t$ we have $Q(s)x = U(s,t)_{|}U(t,s)Q(s)x = U(s,t)_{|}Q(t)U(t,s)x$. By b) the family $(U(s,t)_{|})_{s\in[0,t]} \subseteq \mathcal{L}(\ker P(t), X)$ is strongly continuous and uniformly bounded. The strong continuity of $\mathcal{U}$ yields $\lim_{s\uparrow t} Q(t)U(t,s)x = Q(t)x$. Thus

$$\lim_{s\uparrow t} Q(s)x = \lim_{s\uparrow t} U(s,t)_{|}Q(t)U(t,s)x = Q(t)x,$$

i.e. $Q(\cdot)x$ and hence $P(\cdot)x$ is continuous from the left.
Assertions d) and e) are immediate consequences of a). $\square$

We come to our first main result. It characterizes evolution families with an exponential dichotomy by conditions on the operators G_0, G_X and I_X, respectively.

4.3 Theorem. *Let $\mathcal{U} = (U(t,s))_{t\ge s\ge 0}$ be an evolution family on the Banach space X. Then the following assertions are equivalent:*
(i) $\mathcal{U}$ has an exponential dichotomy.
(ii) The range $R(G_X)$ of G_X coincides with C_0 and $X_0(0)$ is complemented in X.
(iii) I_X is surjective and $X_0(0)$ is complemented in X.

Proof. $(i) \Rightarrow (ii)$: Let $P(t)$, $t \ge 0$, be the family of projections given by the exponential dichotomy. Then $X_0(0) = P(0)X$, and hence $X_0(0)$ is complemented. If $f \in C_0$ define $v : [0,\infty) \to X$ by

$$v(t) = \int_0^t U(t,\xi)P(\xi)f(\xi)d\xi - \int_t^\infty U(t,\xi)_{|}(Id - P(\xi))f(\xi)d\xi. \tag{4.1}$$

An easy computation shows that $v \in C_X$ and v is a solution of equation (1.2). By Lemma 1.1 we have $G_X v = -f$. Together with Remark 1.2 b) this implies $R(G_X) = C_0$.
$(i) \Rightarrow (iii)$: We can use the same arguments as in the proof of $(i) \Rightarrow (ii)$. Note that for $f \in C_X$ the function v defined by (4.1) is also in C_X and satisfies (1.2). Hence by the definition of I_X we have $v \in D(I_X)$ and $I_X v = f$, i.e. I_X is surjective.
$(ii) \Rightarrow (i)$: **A)** Let $Z \subseteq X$ be a complement of $X_0(0)$ in X, i.e. $X = X_0(0) \oplus Z$. Set $X_1(t) := U(t,0)Z$, $t \ge 0$. Clearly,

$$\begin{aligned} U(t,s)X_0(s) &\subseteq X_0(t),\ t \ge s, \\ U(t,s)X_1(s) &= X_1(t),\ t \ge s. \end{aligned} \tag{4.2}$$

B) There are constants $N, \nu > 0$ such that

$$\|U(t,0)x\| \ge Ne^{\nu(t-s)}\|U(s,0)x\| \text{ for } x \in X_1(0) \text{ and } t \ge s \ge 0. \tag{4.3}$$

In fact, let $Y := \{v \in D(G_X) : \ v(0) \in X_1(0)\}$ endowed with the graph norm $\|v\|_{G_X} := \|v\| + \|G_X v\|$. Then Y is a closed subspace of the Banach space $(D(G_X), \|\cdot\|_{G_X})$, and hence Y is complete. By Remark 1.2 a) we have $\ker G_X = \{v \in C_X : v(t) = U(t,0)x$ for some $x \in X_0(0)\}$. Since $X = X_0(0) \oplus X_1(0)$ and $R(G_X) = C_0$ we obtain that $G_X : Y \to C_0$ is bijective and hence an isomorphism. Thus there is a constant $\nu > 0$ such that

$$\|G_X v\| \geq \nu \|v\|_{G_X} \geq \nu \|v\| \text{ for } v \in Y. \tag{4.4}$$

Let now $0 \neq x \in X_1(0)$ and set $u(t) := U(t,0)x, \ t \geq 0$. By Remark 1.2 a) we have $u(t) \neq 0$ for all $t \geq 0$. For each $n \in \mathbb{N}$ choose a real continuous function φ_n on $[0,\infty)$ such that $0 \leq \varphi_n \leq 1, \ \varphi_n = 1$ on $[\frac{1}{n}, n]$ and $\varphi_n = 0$ on $\{0\} \cup [n+1, \infty)$. Set $f_n(t) := -\varphi_n(t)\|u(t)\|^{-1}u(t), \ t \geq 0$, and

$$u_n(t) := \int_t^\infty \frac{\varphi_n(\xi)}{\|u(\xi)\|} d\xi \, u(t), \ t \geq 0.$$

Then $f_n \in C_0$ and $u_n \in C_X$. An easy computation shows that f_n and u_n satisfy (1.2). By Lemma 1.1 we have $u_n \in D(G_X)$ and $G_X u_n = -f_n$. From (4.4) we obtain $\|f_n\| \geq \nu \|u_n\|$. Letting $n \to \infty$ this yields

$$\int_t^\infty \frac{1}{\|u(\xi)\|} d\xi \leq \nu^{-1} \|u(t)\|^{-1}, \ t \geq 0.$$

Now the exponential boundedness of $\mathcal{U}$ and Lemma 2.6 imply that there is a constant $N \geq 0$ independent of x such that $\|u(t)\| \geq N e^{\nu(t-s)} \|u(s)\|, \ t \geq s \geq 0$.
C) $X = X_0(t) \oplus X_1(t), \ t \geq 0$:
Let $Y \subseteq C_X$ be as in B). Then $D(G_0) = D(G_X) \cap C_0 \subseteq Y$ and (4.4) yields $\|G_0 v\| \geq \nu \|v\|$ for $v \in D(G_0)$. Thus $0 \notin A\sigma(G_0) \cap i\mathbb{R}$ and Corollary 3.3 implies that $X_0(t)$ is closed for $t \geq 0$.
From (4.2), (4.3) and the closedness of $X_1(0)$ we derive that $X_1(t)$ is closed and $X_0(t) \cap X_1(t) = \{0\}$ for $t \geq 0$.
Finally, fix $t_0 > 0$ and $x \in X$. Choose a real continuous function φ on $[0,\infty)$ such that φ has compact support contained in $[t_0, \infty)$ and $\int_{t_0}^\infty \varphi(\xi) d\xi = 1$. Set $v(t) := \int_t^\infty \varphi(\xi) d\xi \, U(t,t_0)x$ and $f(t) := -\varphi(t) U(t,t_0)x, \ t \geq t_0$. Then v is a solution of equation (2.1), and $v \in C_X(t_0)$. Extend f continuously to $[0,\infty)$ by setting $f_{|[0,t_0)} = 0$. Then $f \in C_0$ and by assumption there exists $w \in C_X$ such that $G_X w = -f$. In view of Lemma 1.1 w is a solution of equation (1.2). In particular, $w_{|[t_0,\infty)}$ satisfies (2.1). Thus

$$v(t) - w(t) = U(t,t_0)(v(t_0) - w(t_0)) = U(t,t_0)(x - w(t_0)), \ t \geq t_0.$$

Since $v - w_{|[t_0,\infty)} \in C_X(t_0)$ this implies $x - w(t_0) \in X_0(t_0)$. On the other hand $w(0) = w_0 + w_1$ with $w_k \in X_k(0), \ k = 0,1$. Then $w(t_0) = U(t_0,0)w_0 + U(t_0,0)w_1$ and by (4.2) we have $U(t_0,0)w_k \in X_k(t_0), \ k = 0,1$. Hence $x = x - w(t_0) + w(t_0) \in X_0(t_0) + X_1(t_0)$. This proves C).
D) Let $P(t)$ be the projection from X onto $X_0(t)$ with kernel $X_1(t), \ t \geq 0$. Then (4.2) implies $P(t)U(t,s) = U(t,s)P(s), \ t \geq s \geq 0$. From (4.2) and (4.3) we obtain

that $U(t,s)_| : \ker P(s) \to \ker P(t),\ t \geq s \geq 0$ is an isomorphism. Finally, by (4.3), Theorem 3.2 and our assumption $A\sigma(G_0) \cap i\mathbb{R} \neq i\mathbb{R}$ there exist constants $N, \nu > 0$ such that

$$\|U(t,s)x\| \leq Ne^{-\nu(t-s)}\|x\| \text{ for } x \in P(s)X,\ t \geq s \geq 0,$$

$$\|U(s,t)_| x\| \leq Ne^{-\nu(t-s)}\|x\| \text{ for } x \in \ker P(t),\ t \geq s \geq 0.$$

Thus $\mathcal{U}$ has an exponential dichotomy.
(iii) $\Rightarrow$ *(i)*: We use exactly the same arguments as in (ii) $\Rightarrow$ (i). We only have to replace the operator G_X by I_X. $\square$

Remark. Part C) of the proof shows that for an evolution family $\mathcal{U}$ with an exponential dichotomy the generator G_0 of the evolution semigroup on C_0 always satisfies $0 \notin A\sigma(G_0)$, and hence $A\sigma(G_0) \cap i\mathbb{R} = \emptyset$ (see Section 3, Remark a)).

If X is a Hilbert space we only have to assume the closedness of the stable subspace.

4.4 Corollary. *Let $\mathcal{U} = (U(t,s))_{t \geq s \geq 0}$ be an evolution family on the Hilbert space H. Then the following assertions are equivalent:*
(i) $\mathcal{U}$ has an exponential dichotomy.
(ii) $R(G_H) = C_0$ and $H_0(0)$ is closed.
(iii) I_H is surjective and $H_0(0)$ is closed.

In the proof of Theorem 4.3, (ii) $\Rightarrow$ (i), we saw that the invertibility of G_X restricted to a certain subspace of C_X plays a crucial role in order to prove the existence of an exponential dichotomy for a given evolution family $\mathcal{U}$. In our next result we show that the existence of an exponential dichotomy can be even characterized by such an invertibility condition. Let us introduce the following notion. For a closed linear subspace Z of X let

$$C_Z := \{f \in C_X : \ f(0) \in Z\}.$$

Denote by G_Z the part of G_X in C_Z, i.e. $D(G_Z) = D(G_X) \cap C_Z$ and $G_Z v = G_X v$ for $v \in D(G_Z)$. Then G_0 is the part of G_Z in C_0. In the same way let I_Z be the restriction of I_X to C_Z, i.e. $D(I_Z) = D(I_X) \cap C_Z$ and $I_Z v = I_X v$ for $v \in D(I_Z)$. Notice that the evolution semigroup $\mathcal{T}$ leaves C_Z invariant and that G_Z is the generator of the restriction of $\mathcal{T}$ to C_Z.

With this notation we obtain the following characterization of evolution families with exponential dichotomy. A similar result is shown in [BGK, Theorem 1.1] for the finite dimensional case (see also [Pal]). We point out that in contrast to Theorem 4.3 we do not have to assume that $X_0(0)$ is complemented.

4.5 Theorem. *Let $\mathcal{U} = (U(t,s))_{t \geq s \geq 0}$ be an evolution family on the Banach space X and let Z be a closed linear subspace of X. Then the following assertions are equivalent:*
(i) $\mathcal{U}$ has an exponential dichotomy with $\ker P(0) = Z$.
(ii) $G_Z : D(G_Z) \subseteq C_Z \to C_0$ is invertible.
(iii) $I_Z : D(I_Z) \subseteq C_Z \to C_X$ is invertible.

Proof. *(i)* $\Rightarrow$ *(ii)* : Let $P(t),\ t \geq 0$, be a family of projections given by the exponential dichotomy such that $\ker P(0) = Z$. Then $P(0)X = X_0(0)$ and $X = X_0(0) \oplus Z$. Fix

$f \in C_0$. By Theorem 4.3 there is $v \in D(G_X)$ such that $G_X v = f$. On the other hand $u : [0,\infty) \to X : \ t \mapsto U(t,0)P(0)v(0)$ belongs to C_X and $G_X u = 0$ (cf. Remark 1.2 a)). Then $v - u \in D(G_Z)$ and $G_Z(v-u) = G_X v = f$. Hence $G_Z : D(G_Z) \to C_0$ is surjective. If $w \in \ker G_Z$ then $w(t) = U(t,0)w(0), \ t \geq 0$ (cf. Remark 1.2 a)). Since $w \in C_X$ we have $w(0) \in Z \cap X_0(0) = \{0\}$ and hence $w = 0$, i.e. G_Z is injective.
$(i) \Rightarrow (iii)$ can be shown by the same arguments. One only has to use Remark 1.4 a) instead of Remark 1.2 a).
$(ii) \Rightarrow (i)$: Let $G_Z : D(G_Z) \to C_0$ be invertible. By Remark 1.2 b) we have $C_0 = R(G_Z) = R(G_X)$. The closedness of G_Z implies that G_Z^{-1} is bounded, and hence there exists $\nu > 0$ such that $\|G_Z v\| \geq \nu \|v\|$ for all $v \in D(G_Z)$. Since G_0 is the part of G_Z in C_0 we obtain $0 \notin A\sigma(G_0)$. Finally we show that $X = X_0(0) \oplus Z$. Corollary 3.3 implies that $X_0(0)$ is closed. Now let $x \in X$. If $U(t,0)x = 0$ for some $t > 0$, then $x \in X_0(0)$. Otherwise $U(t,0)x \neq 0$ for all $t \geq 0$. Let $\varphi : [0,\infty) \to \mathbb{R}$ be continuously differentiable such that $\varphi_{|[0,1]} = 1$ and $\varphi_{|[2,\infty)} = 0$. Set $u(t) := \varphi(t)U(t,0)x, \ t \geq 0$, and $f(t) := \varphi'(t)U(t,0)x, \ t \geq 0$. By Remark 1.2 c) we have $u \in D(G_X)$ and $G_X u = -f$. On the other hand there exists $v \in D(G_Z)$ such that $G_Z v = -f$. Thus $u - v \in \ker G_X$ and hence

$$(u-v)(t) = U(t,0)(u(0) - v(0)) = U(t,0)(x - v(0)), \ t \geq 0.$$

Since $u - v \in C_X$ this implies $x - v(0) \in X_0(0)$. Thus $x = x - v(0) + v(0) \in X_0(0) + Z$. If $y \in Z \cap X_0(0)$, then w defined by $w(t) := U(t,0)y, \ t \geq 0$, belongs to $C_Z \cap \ker G_X$ (see Remark 1.2 a)). Hence $G_Z w = 0$ and by the invertibility of G_Z we have $w = 0$. Thus $0 = w(0) = y$, i.e. $X_0(0) \cap Z = \{0\}$. This shows $X = X_0(0) \oplus Z$. The assertion now follows from Theorem 4.3.
$(iii) \Rightarrow (i)$ is shown by the same arguments as (ii) $\Rightarrow$ (i). $\square$

Remark. As a special case of Theorem 4.5 the invertibility of G_X resp. I_X characterizes evolution families with an exponential dichotomy such that $X_0(0) = P(0)X = \{0\}$.

We conclude with an example of an evolution family $\mathcal{U}$ with non-trivial exponential dichotomy such that I_X is invertible. In particular this shows that invertibility of I_X is not equivalent to the exponential expansiveness of $\mathcal{U}$ (cf. Theorem 2.5).

4.6 Example. Let $X = L^1[0,\infty)$. For $0 \leq s \leq t \leq 1$ set

$$U_1(t,s)f(\xi) := \begin{cases} 0, & 0 \leq \xi \leq t-s, \\ f(\xi - t + s), & t - s < \xi, \end{cases}$$

and for $1 \leq s \leq t$ set

$$U_2(t,s)f(\xi) := \begin{cases} e^{-(t-s)}f(\xi), & 0 \leq \xi \leq 1, \\ e^{(t-s)}f(\xi), & \xi > 1, \end{cases}$$

$f \in X$. Then $\mathcal{U} = (U(t,s))_{t \geq s \geq 0}$ defined by

$$U(t,s) := \begin{cases} U_1(t,s), & 0 \leq s \leq t \leq 1, \\ U_2(t,1)U_1(1,s), & s \leq 1 \leq t, \\ U_2(t,s), & 1 \leq s \leq t, \end{cases}$$

is an evolution family on X with an exponential dichotomy. The corresponding family of projections $P(t),\ t \geq 0$, is given by

$$P(t)f = \chi_{[0,\min\{1,t\}]}f,\ f \in X,$$

where χ_C denotes the characteristic function of a set C. In particular, $\{0\} = P(0)X = X_0(0)$. Theorem 4.5 implies that I_X is invertible. Since $\|U(t,1)P(1)\| \leq e^{-(t-1)},\ t \geq 1$, the evolution family $\mathcal{U}$ cannot be exponentially expansive.

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