Non-autonomous Miyadera perturbation

Frank Räbiger[1], Abdelaziz Rhandi[2,*], Roland Schnaubelt[1,†], and Jürgen Voigt[3,‡]
[1]Universität Tübingen, Mathematisches Institut, Auf der Morgenstelle 10, 72076 Tübingen, Germany
[2]Département de Mathématiques, Faculté des Sciences Semlalia, B.P.: S.15, 40000 Marrakech, Maroc
[3]Technische Universität Dresden, Fachrichtung Mathematik, 01062 Dresden, Germany

Abstract: We consider time dependent perturbations $B(t)$ of a non–autonomous Cauchy problem $\dot{v}(t) = A(t)v(t)$ (CP) on a Banach space X. The existence of a mild solution u of the perturbed problem is proved under Miyadera type conditions on $B(\cdot)$. In the parabolic case and $X = L^d(\Omega)$, $1 < d < \infty$, we show that u is differentiable a.e. and satisfies $\dot{u}(t) = (A(t)+B(t))u(t)$ for a.e. t. Our approach uses perturbation results due to one of the authors, [29], and S. Monniaux and J. Prüß, [14], which are applied to the evolution semigroup induced by the evolution family related to (CP). As an application we obtain solutions of a second order parabolic equation with singular lower order coefficients.

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1. Introduction

Consider the non–autonomous linear abstract Cauchy problems

$$(CP)\begin{cases} \frac{d}{dt}u(t) &= A(t)u(t), \\ u(s) &= x, \quad t \geq s, \end{cases} \qquad \text{and} \qquad (pCP)\begin{cases} \frac{d}{dt}u(t) &= (A(t)+B(t))\,u(t), \\ u(s) &= x, \quad t \geq s, \end{cases}$$

on a Banach space X. In order to solve (pCP) we assume that (CP) is "well–posed" and that the operators $B(t)$ are "small with respect to $A(t)$". In the autonomous case, i.e., $A(t) \equiv A$, $B(t) \equiv B$, and A generates a $\mathcal{C}_0$–semigroup $(T(t))_{t\geq 0}$, there are various conditions on A and B ensuring that the operator $A+B$ is also a generator, and hence (pCP) is well–posed. For instance, if B is A–bounded and satisfies the *Miyadera condition*

$$\exists\, \alpha > 0,\ \beta \in [0,1) \quad \text{such that} \qquad \int_0^\alpha \|BT(t)x\|\,dt \leq \beta\,\|x\|\,, \quad x \in D(A), \tag{1.1}$$

then $(A+B, D(A))$ generates a $\mathcal{C}_0$–semigroup $(T_B(t))_{t\geq 0}$, [13], [29], where $D(A)$ is the domain of A.

In the present paper we prove a corresponding perturbation theorem in the non–autonomous situation extending a previous result by part of the authors, [24]. Here, instead of a $\mathcal{C}_0$–semigroup $(T(t))_{t\geq 0}$, we consider an *evolution family* $\mathcal{U} = (U(t,s))_{(t,s)\in D}$ in the space $\mathcal{L}(X)$ of bounded linear operators on X, that is,

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(E1) $U(s,s) = Id$, $U(t,s) = U(t,r)U(r,s)$ for $t \geq r \geq s$ in I and

(E2) the mapping $D \ni (t,s) \mapsto U(t,s)$ is strongly continuous,

where I is an interval in $\mathbb{R}$ and $D = D_I := \{(t,s) \in I^2 : t \geq s\}$. Observe that for a $\mathcal{C}_0$–semigroup $(T(t))_{t\geq 0}$ the operators $U(t,s) := T(t-s)$, $t \geq s$, define an evolution family with index set $D_{\mathbb{R}}$. Further, for $I \in \{[a,b], [a,\infty), \mathbb{R}\}$ the Cauchy problem (CP) is called *well–posed* (on spaces Y_s) if Y_s, $s \in I$, is dense in X and for $x \in Y_s$ there is a unique solution $u \in C^1(I \cap [s,\infty), X)$ of (CP) depending continuously on initial data. In this case there exists an evolution family $\mathcal{U}$ such that $u(t) = U(t,s)x$, that is, $\mathcal{U}$ *solves* (CP), see [19].

In contrast to the autonomous case, it can happen that the mapping $t \mapsto U(t,s)x$ is differentiable only if $x = 0$ (e.g., take $X = \mathbb{C}$ and $U(t,s) = p(t)/p(s)$, where $p, 1/p \in C_b(\mathbb{R})$ and p is nowhere differentiable). Moreover, regularity properties of $\mathcal{U}$ are not preserved even for bounded perturbations $B(\cdot)$, see [23, Ex. 6.4]. Thus, at first, we will not assume that the evolution family $\mathcal{U}$ is differentiable in any sense and look for an evolution family $\mathcal{U}_B$ such that the *variation of constants formula*

$$U_B(t,s)x = U(t,s)x + \int_s^t U(t,\tau)B(\tau)U_B(\tau,s)x\,d\tau \tag{1.2}$$

holds. If $\mathcal{U}$ is related to (CP) then we say that $t \mapsto u(t) = U_B(t,s)x$ is a *mild solution* of (pCP). Note that if $\mathcal{U}$ solves (CP) on Y_s and $\tau \mapsto B(\tau)U_B(\tau,s)x$, $x \in Y_s$, is sufficiently smooth, then u satisfies (pCP), see e.g. [22, §5.5, 5.7].

In Theorem 3.4 we show the existence of $\mathcal{U}_B$ by assuming a non–autonomous version of the Miyadera condition (1.1). Our approach is based on the so–called *evolution semigroup* $\mathcal{T} = (T(t))_{t\geq 0}$ on $E = L^p(I,X)$ given by $(T(t)f)(s) := \chi_I(s-t)\,U(s,s-t)f(s-t)$, $f \in E$, see Section 2. We perturb $\mathcal{T}$ by the multiplication operator $\mathcal{B} = B(\cdot)$ on E. By applying a Miyadera type perturbation theorem for semigroups, [29], and an abstract characterization of evolution semigroups (see Theorem 2.4) we then obtain a perturbed evolution semigroup $\mathcal{T}_B$. Finally, the associated evolution family $\mathcal{U}_B$ satisfies (1.2).

In Section 4 we consider evolution families $\mathcal{U}$ solving (CP) where the operators $A(t)$ generate analytic semigroups and perturbations $B(t)$ which are bounded operators from the domain of a fractional power of $A(t)$ to X. For $X = L^d(\Omega)$, $1 < d < \infty$, we can show that $t \mapsto U_B(t,s)x$, $x \in X$, is differentiable for a.e. $t > s$ and satisfies (pCP) for a.e. $t > s$. Our proof relies on a recent perturbation result of Dore–Venni type by S. Monniaux and J. Prüß, [14], which we apply to the evolution semigroup $\mathcal{T}$. Finally, in the last section, we use the above results to solve a second order parabolic equation with singular lower order coefficients.

2. A characterization of evolution semigroups

In this section we consider an evolution family $\mathcal{U} = (U(t,s))_{(t,s)\in D}$ on a Banach space X defined for a right–closed interval $I \subseteq \mathbb{R}$. We call $\mathcal{U}$ *exponentially bounded* if there are constants $M \geq 1$ and $w \in \mathbb{R}$ such that $\|U(t,s)\| \leq Me^{w(t-s)}$ for $(t,s) \in D$. For compact intervals I this estimate always holds with $w = 0$ due to the principle of uniform boundedness. For exponentially bounded $\mathcal{U}$,

$$(T(t)f)(s) := \chi_I(s-t)\, U(s,s-t)f(s-t) = \begin{cases} U(s,s-t)f(s-t), & \text{if } s-t \in I, \\ 0, & \text{if } s-t \notin I, \end{cases}$$

defines a bounded operator $T(t)$ on the Bochner–Lebesgue space $E = L^p(I,X)$, $1 \le p < \infty$, where χ_I is the characteristic function of I. It is easily seen that $\mathcal{T} = (T(t))_{t\ge 0}$ is a strongly continuous semigroup on E, see e.g. [27], [28]. We call $\mathcal{T}$ the *evolution semigroup* associated with $\mathcal{U}$ and denote its generator by $(G, D(G))$. For further information concerning this approach we refer to [5], [9], [11], [12], [16]–[19], [21], [24]–[28], and the references therein.

The resolvent $R(\lambda, G) := (\lambda - G)^{-1}$, $\lambda \in \rho(G)$, has an important smoothing property which is stated below. For $I = \mathbb{R}$ this result is contained in [25, Prop. 3.3]. The general case can be shown analogously and the proof is therefore omitted, see also [28, Prop. 2.2]. For a similar result in the case of bounded I we refer to [17, Cor. 4.7]. By $C_{00}(I,X)$ we denote the space of continuous functions $f : I \to X$ vanishing at $a := \inf I$ and at infinity (if I is unbounded) endowed with the sup–norm $\|\cdot\|_\infty$. (Notice that $C_{00}(\mathbb{R},X)$ is usually denoted by $C_0(\mathbb{R},X)$.)

Proposition 2.1. *Let I be a right–closed interval in $\mathbb{R}$ and $(U(t,s))_{(t,s)\in D}$ an exponentially bounded evolution family on X. Let $(G, D(G))$ be the generator of the associated evolution semigroup on $L^p(I,X)$, $1 \le p < \infty$. Consider $\lambda \in \rho(G)$. Then $R(\lambda, G) : L^p(I,X) \to C_{00}(I,X)$ is continuous with dense image. In particular, $D(G)$ is dense in $C_{00}(I,X)$.*

Following an approach due to J. Howland, [9], we reduce the characterization of evolution semigroups to the characterization of multiplication operators on E. We consider the space $C_b(I, \mathcal{L}_s(X))$ of bounded, strongly continuous functions $M(\cdot) : I \to \mathcal{L}(X)$, where $I \subseteq \mathbb{R}$ is an interval. Clearly, $M(\cdot) \in C_b(I, \mathcal{L}_s(X))$ induces a bounded *multiplication operator* $\mathcal{M}$ on $L^p(I,X)$ by setting $\mathcal{M}f := M(\cdot)f(\cdot)$, and $\|\mathcal{M}\| = \|M(\cdot)\|_\infty$. We need the following notion (see also [17, Def. 4.4], [24, Def. 3.1]).

Definition 2.2. *Let $\mathcal{M} \in \mathcal{L}(L^p(I,X))$, $1 \le p < \infty$. A subspace F of $L^p(I,X)$ is called* $\mathcal{M}$–determining *if*

(D1) F and $\mathcal{M}F$ consist of continuous functions;

(D2) $F^s := \{f(s) : f \in F\}$ is dense in X for all $s \in I$;

(D3) F is dense in $L^p(I,X)$.

For $I = \mathbb{R}$ the next result was shown in [24, Prop. 3.3]. The proof carries over to the case $I \neq \mathbb{R}$ with obvious changes. For related characterizations we refer to [5] and [9].

Proposition 2.3. *Let I be an interval in $\mathbb{R}$ and X a Banach space. Consider $\mathcal{M} \in \mathcal{L}(L^p(I,X))$, $1 \le p < \infty$. Then $\mathcal{M} = M(\cdot) \in C_b(I, \mathcal{L}_s(X))$ if and only if there is an $\mathcal{M}$–determing subspace F of $L^p(I,X)$ such that $\mathcal{M}(\varphi f) = \varphi\, \mathcal{M}f$ for $f \in F$ and $\varphi \in L^\infty(I)$.*

The right translations $\mathcal{R}(t)$, $t \in \mathbb{R}$, defined by $(\mathcal{R}(t)f)(s) := \chi_I(s-t)\, f(s-t)$ for $s \in I$ and $f \in E$, yield a $\mathcal{C}_0$–semigroup $(\mathcal{R}(t))_{t\ge 0}$ on $E = L^p(I,X)$. For $X = \mathbb{C}$ we write $R(t)$ instead of $\mathcal{R}(t)$. Let $I_t := I \cap (I+t)$ for $t \in \mathbb{R}$. We identify the space $L^p(I_t,X)$ with a subspace of $E = L^p(I,X)$, $1 \le p < \infty$, by extending functions by 0. Recall that a *core* of a closed operator $(A, D(A))$ is a subspace of $D(A)$ which is dense in the graph norm $\|x\|_A := \|x\| + \|Ax\|$. Further, $C_c^1(I)$ is the space of continuously differentiable functions with compact support. Finally, we make use of the following condition, cf. Proposition 2.1:

(R) There is $\lambda \in \rho(G)$ such that $R(\lambda, G) : L^p(I,X) \to C_{00}(I,X)$ is continuous with dense image $D(G)$.

If I contains its left endpoint a and $\mathcal{U}$ is an exponentially bounded evolution family with index set D_I, then the restriction of $\mathcal{U}$ to the index set $D_{I'}$, $I' := I \setminus \{a\}$, induces the same evolution semigroup as $\mathcal{U}$ on $L^p(I,X)$. So, as in [17], in our characterization of evolution semigroups it suffices to consider a left half–open interval I, that is, I is left–open and right–closed. Observe that, in general, an evolution family defined for a left–open interval has no continuous extension to $D_{\bar{I}}$. For instance, let $X = \mathbb{C}$, $I = (0, \infty)$, and $U(t,s) = p(t)/p(s)$ for $p(t) = 2 + \sin \frac{1}{t}$.

We now formulate the main result of this section. It extends [24, Thm. 3.4] where the case $I = \mathbb{R}$ was considered. A closely related result for bounded intervals I is due to H. Neidhardt, [17, Thm. 4.12]. Evolution semigroups on $C_{00}(I,X)$ were characterized in [5], [12], [17], [21].

Theorem 2.4. *Let I be a left half–open interval in $\mathbb{R}$. Consider a $\mathcal{C}_0$–semigroup $\mathcal{T} = (T(t))_{t\ge 0}$ on $E = L^p(I,X)$, $1 \le p < \infty$, with generator $(G, D(G))$. Let $\mathcal{D}$ be a core of G. Then the following assertions are equivalent:*

(a) $\mathcal{T}$ is an evolution semigroup induced by an exponentially bounded evolution family $(U(t,s))_{(t,s)\in D}$.

(b) Condition (R) holds, and we have $T(t)\,(\varphi f) = (R(t)\varphi)\,T(t)f$ for $\varphi \in L^\infty(I)$, $f \in E$, and $t \ge 0$.

(c) Condition (R) holds, and for $f \in \mathcal{D}$ and $\varphi \in C_c^1(I)$ we have $\varphi f \in D(G)$ and

$$G\,(\varphi f) = -\varphi' f + \varphi\, G f. \tag{2.1}$$

(d) Condition (R) holds, and there is $\mu \in \rho(G)$ such that $R(\mu, G)\,(\varphi' R(\mu,G) f) = \varphi\, R(\mu, G) f - R(\mu, G)\,(\varphi f)$ for $f \in E$ and $\varphi \in C_c^1(I)$.

Proof. By Proposition 2.1, assertion (a) implies (R). The implications (a) $\Rightarrow$ (c) $\Rightarrow$ (b) and (c) $\Leftrightarrow$ (d) can be shown as in [24, Thm. 3.4], cf. [28, Thm. 2.6].

(b) $\Rightarrow$ (a): Observe that the assumption implies $T(t)f \in L^p(I_t, X)$ for $f \in E$ and $t \ge 0$. Set $\mathcal{M}_t := T(t)\mathcal{R}(-t)$ and $F_t := \mathcal{R}(t)D(G)$ for $t \ge 0$. By (R) the space F_t consists of continuous functions vanishing on $I \setminus I_t$. We have

$$\mathcal{M}_t\,(\varphi f) = (R(t)R(-t)\varphi)\,T(t)\mathcal{R}(-t)f = \varphi\,\mathcal{M}_t f$$

for $f \in E$ and $\varphi \in L^\infty(I)$ with support in I_t. In particular, $\mathcal{M}_t$ leaves $L^p(I_t, X)$ invariant. From (R) we derive that F_t is dense in $L^p(I_t, X)$ and $C_{00}(I_t, X)$. Thus, F_t satisfies (D2) and (D3) for the interval I_t. Further, let $f \in F_t$. Then $f = \mathcal{R}(t)g$ for some $g \in D(G)$, and

$$\mathcal{M}_t f = T(t)\mathcal{R}(-t)\mathcal{R}(t)g = T(t)\,(\chi_{I_{-t}}\, g) = \chi_{I_t}\, T(t)g = T(t)g \in D(G) \subseteq C_{00}(I,X).$$

Hence, F_t is $\mathcal{M}_t$–determining on $L^p(I_t, X)$, $t \geq 0$. By Proposition 2.3 there are bounded operators $M(t,s)$, $s \in I_t$, such that $\mathcal{M}_t = M(t,\cdot) \in C_b(I_t, \mathcal{L}_s(X))$, $t \geq 0$. Moreover, $\|M(t,s)\| \leq \|\mathcal{M}_t\| \leq Me^{wt}$ for some constants $M \geq 1$ and $w \in \mathbb{R}$. Notice that $T(t) = M(t,\cdot)\mathcal{R}(t)$. We set $U(t,s) := M(t-s,t)$ for $(t,s) \in D_I$. Then

$$T(t)f\,(s) = \chi_I(s-t)U(s,s-t)f(s-t)$$

for $f \in E$, $s \in I$, and $t \geq 0$. From the semigroup law for $\mathcal{T}$ one easily derives (E1) for $\mathcal{U}$, see e.g. [17, p.291] or [24, p.524]. It remains to show the strong continuity of $M(\cdot,\cdot)$. Fix $t \geq 0$ and $s \in I_t$. By (D2) and the exponential bound for $M(t,s)$ it suffices to consider $x \in F_t^s$. So let $f \in D(G)$ with $f(s-t) = x$. Set $g = (\lambda - G)f$. Then, $T(t)f = R(\lambda, G)T(t)g \in C_{00}(I, X)$ by (R). Thus we obtain

$$\begin{aligned}
&\|M(t',s')x - M(t,s)x\| \\
&\quad = \|M(t',s')\left(f(s-t) - f(s'-t)\right) + M(t',s')\left(f(s'-t) - f(s'-t')\right) \\
&\qquad + M(t',s')f(s'-t') - M(t,s')f(s'-t) + M(t,s')f(s'-t) - M(t,s)f(s-t)\| \\
&\quad \leq Me^{wt'}\left(\|\mathcal{R}(t)f\,(s) - \mathcal{R}(t)f\,(s')\| + \|\mathcal{R}(t)f - \mathcal{R}(t')f\|_\infty\right) \\
&\qquad + \|T(t')f - T(t)f\|_\infty + \|T(t)f\,(s') - T(t)f\,(s)\|
\end{aligned}$$

for $t' \geq 0$ and $s' \in I_t \cap I_{t'}$. The first, second, and fourth summand converge to 0 as $(t', s') \to (t, s)$ since $f, T(t)f \in C_{00}(I, X)$. For the third summand we use

$$\|T(t')f - T(t)f\|_\infty \leq \|R(\lambda, G)\|_{\mathcal{L}(L^p, C_{00})}\,\|T(t')g - T(t)g\|_p$$

which yields the assertion. □

Remark 2.5.

(a) *Implication "(b) ⇒ (a)" is false if we drop condition (R). Consider for instance $X = \mathbb{C}$, $E = L^1(\mathbb{R})$, and $U(t,s) = p(t)/p(s)$, where $p, 1/p \in L^\infty(\mathbb{R})$ are discontinuous, but $p(r) \to p(t)$ as $r \nearrow t$ for a.e. $t \in \mathbb{R}$. See also [5, Thm. 6.4] and [9, Thm. 1].*

(b) *The results of this section remain valid if we replace $L^p(I, X)$ by an X–valued Banach function space $E(X)$ such that the norm on E is order continuous and the translations $R(t)$, $t \in \mathbb{R}$, are uniformly bounded on E, see [24] and [28].*

3. Miyadera perturbations

In this section we show a perturbation result for evolution families by applying Theorem 2.4 and the following Miyadera type perturbation theorem proved in [29, Thm. 1]. The proof of part (d) of the theorem uses an idea contained in the proof of [30, Prop. 1.3]. Let $(A, D(A))$ and $(B, D(B))$ be linear operators. Recall that B is A–*bounded* if $D(A) \subseteq D(B)$ and $\|Bx\| \leq a\|x\| + b\|Ax\|$ for $x \in D(A)$ and constants $a, b \geq 0$. Observe that if there exists $\lambda \in \rho(A)$ then B is A–bounded if and only if $D(A) \subseteq D(B)$ and $BR(\lambda, A)$ is closed (and hence bounded).

Theorem 3.1. *Let $\mathcal{T} = (T(t))_{t\geq 0}$ be a $\mathcal{C}_0$–semigroup on a Banach space F with generator $(G, D(G))$. Consider a dense subspace $\mathcal{D}$ of F and a linear operator $(B, D(B))$ such that*

(i) $T(t)\mathcal{D} \subseteq \mathcal{D} \subseteq D(G) \cap D(B)$ and $[0,\infty) \ni t \mapsto BT(t)f \in F$ is continuous for $f \in \mathcal{D}$;

(ii) there are constants $\alpha > 0$ and $\gamma \in [0,1)$ such that $\int_0^\alpha \|BT(t)f\|\,dt \leq \gamma\,\|f\|$ for $f \in \mathcal{D}$.

Then the following assertions hold:

(a) The operator $(G+B, \mathcal{D})$ is closable. Its closure $(G_B, D(G_B))$ generates a $\mathcal{C}_0$–semigroup $\mathcal{T}_B = (T_B(t))_{t\geq 0}$ and satisfies $D(G_B) = D(G)$. The operator $(B, \mathcal{D})$ possesses a unique G–bounded extension $\hat{B}$ to $D(G)$ and we have $G_B = G + \hat{B}$. Further, $\mathcal{D}$ is a core of G and G_B.

(b) For sufficiently large real $\lambda \in \rho(G) \cap \rho(G_B)$ there exists an invertible operator $C_\lambda \in \mathcal{L}(F)$ such that $R(\lambda, G_B) = R(\lambda, G)C_\lambda$.

(c) For $f \in D(G)$ and $t \geq 0$ we have

$$T_B(t)f = T(t)f + \int_0^t T_B(t-\tau)\hat{B}T(\tau)f\,d\tau, \tag{3.1}$$
$$T_B(t)f = T(t)f + \int_0^t T(t-\tau)\hat{B}T_B(\tau)f\,d\tau. \tag{3.2}$$

Moreover, $\mathcal{T}_B$ is the only $\mathcal{C}_0$–semigroup satisfying (3.1) for $f \in \mathcal{D}$.

(d) In addition, assume that B is closed. Then $\hat{B}f = Bf$ for $f \in D(G)$. Moreover, for all $f \in F$ we have $T(t)f, T_B(t)f \in D(B)$ for a.e. $t \geq 0$, the functions $BT(\cdot)f$ and $BT_B(\cdot)f$ are locally integrable on $\mathbb{R}_+$, and (3.1) and (3.2) hold for all $t \geq 0$ and $f \in F$ with $\hat{B}$ replaced by B.

Proof. By [15, A-I.1.9] $\mathcal{D}$ is a core of G. Assertion (a) and (b) and uniqueness follow from [29], Thm. 1, Remark 2, and (1.7). Equation (3.1) is shown in [29, Thm. 1] for $f \in \mathcal{D}$. Since $\mathcal{D}$ is a core of G and $\hat{B}$ is G–bounded, (3.1) holds for $f \in D(G)$ by the dominated convergence theorem. For $0 \leq \tau \leq t$ and $f \in D(G) = D(G_B)$ we have

$$\begin{aligned}\frac{d}{d\tau}T(t-\tau)T_B(\tau)f &= -T(t-\tau)GT_B(\tau)f + T(t-\tau)G_BT_B(\tau)f \\ &= T(t-\tau)\hat{B}T_B(\tau)f.\end{aligned}$$

Integration from 0 to t yields (3.2).

Now assume that B is closed. Then $\hat{B}$ is the restriction of B to $D(G)$, [29, Cor. 4]. Further, (ii) holds for all $f \in D(G)$, [30, Thm. 1.1], and [20, Lemma 1.1] yields $\int_0^\alpha \|BT_B(t)f\|\,dt \leq \frac{\gamma}{1-\gamma}\,\|f\|$ for every $f \in D(G)$. One easily sees that this implies

$$\int_0^t \|BT(\tau)f\|\,d\tau \leq c\,\|f\| \qquad \text{and} \qquad \int_0^t \|BT_B(\tau)f\|\,d\tau \leq \tilde{c}\,\|f\| \tag{3.3}$$

for $f \in D(G)$, $t \geq 0$, and constants $c, \tilde{c}$ depending only on t. Next, fix $f \in F$. Choose $f_n \in D(G)$ converging to f. By (3.3) the functions $BT(\cdot)f_n$ and $BT_B(\cdot)f_n$ converge in $L^1([0,t], F)$, and thus (by passing to subsequences) they converge pointwise a.e.. Since B is closed, we derive $T(t)f, T_B(t)f \in D(B)$ for a.e. $t \geq 0$ and $BT(\cdot)f$ and $BT_B(\cdot)f$ are locally integrable on $\mathbb{R}_+$. Moreover, the estimates (3.3) hold for all $f \in F$ and $t \geq 0$. Observe that (3.1) and (3.2) are satisfied for f_n with $\hat{B}$ replaced by B. Hence, due to (3.3) the last assertion in (d) follows by approximation. □

In the sequel we need the following measure theoretical lemma which is essentially shown in [17, Thm. 4.2], see also [26, Lemma 2.2]. Throughout "measurable" means "strongly measurable" and the integrals are Bochner integrals.

Lemma 3.2. *Let X be a Banach space and let I be an interval in $\mathbb{R}$. Assume that $u : [c,d] \to L^p(I,X)$, $1 \leq p < \infty$, is integrable. Then there is a measurable function $\Phi : [c,d] \times I \to X$ such that for a.e. $t \in [c,d]$ we have $u(t)\,(s) = \Phi(t,s)$ for a.e. $s \in I$. Moreover,*

$$\Phi(\cdot, s) \text{ is integrable and } \left(\int_c^d u(t)\,dt\right)(s) = \int_c^d \Phi(t,s)\,dt \tag{3.4}$$

for a.e. $s \in I$. If, in addition, there is a measurable function $\tilde{u} : [c,d] \times I \to X$ such that for a.e. $t \in [c,d]$ we have $u(t)\,(s) = \tilde{u}(t,s)$ for a.e. $s \in I$, then (3.4) holds with Φ replaced by $\tilde{u}$.

We often use the fact that for a measurable function $u : I \times J \to X$ and intervals $I, J \subseteq \mathbb{R}$ we have $u(t,s) = 0$ for a.e. $(t,s) \in I \times J$ if and only if for a.e. $t \in I$ we have $u(t,s) = 0$ for a.e. $s \in J$. This is an easy consequence of Fubini's theorem.

Let I be a left half–open interval in $\mathbb{R}$. Consider an exponentially bounded evolution family $\mathcal{U} = (U(t,s))_{(t,s)\in D}$ on a Banach space X. Let $\mathcal{T} = (T(t))_{t\geq 0}$ be the associated evolution semigroup on $E = L^p(I,X)$, $1 \leq p < \infty$, with generator $(G, D(G))$. In the next lemma we describe a certain class of functions contained in $D(G)$. To simplify notation we set $U(t,s) := 0$ for $t < s$.

Lemma 3.3. *Let $x \in X$, $r \in I$, and $\varphi \in C_c^1(I)$ with $\varphi(s) = 0$ for $s \leq r$. Set $f(\cdot) := \varphi(\cdot)U(\cdot, r)x$. Then $T(t)f = (R(t)\varphi)\,U(\cdot, r)x$ and $f \in D(G)$.*

For the straightforward proof we refer to [28, Lemma 1.12], see also [25, Lemma 2.1]

We now introduce the assumptions on the operators $(B(t), D(B(t)))$, $t \in I$, on X which are needed for our perturbation result. For $q \geq 1$ we denote by q' the conjugate exponent satisfying $\frac{1}{q} + \frac{1}{q'} = 1$.

(M) There are dense subspaces Y_t, $t \in I$, of X such that $U(t,s)Y_s \subseteq Y_t$ for $(t,s) \in D$. Moreover, for all $s \in I$ and $y \in Y_s$ we have $U(t,s)y \in D(B(t))$ for a.e. $t \in I \cap [s,\infty)$, $B(\cdot)U(\cdot,s)y$ is measurable, and

$$\int_0^\alpha \chi_I(s+t)\,\|B(s+t)U(s+t,s)y\|^p\,dt \leq \beta^p\,\|y\|^p \tag{3.5}$$

for constants $p \geq 1$, $\alpha > 0$ and $\beta \geq 0$ such that $\gamma := \alpha^{1/p'}\beta < 1$.

It is straightforward to see that (M) implies

$$\int_0^d \chi_I(s+t)\,\|B(s+t)U(s+t,s)y\|^p\,dt \leq c\,\|y\|^p \tag{3.6}$$

for $s \in I$, $d \geq 0$, and $y \in Y_s$, where the constant c is independent of s and y. Observe that if (M) is satisfied for $p > 1$, $\alpha > 0$, and $\beta \geq 0$ then, by Hölder's inequality, (M) is also true for $p = 1$, $\alpha > 0$, and $\gamma \in [0, 1)$. Clearly, (M) holds if $Y_t \equiv X$ and $B(\cdot) \subseteq \mathcal{L}(X)$ such that $\|B(t)\| \leq C$ for a.e. $t \in I$ and $B(\cdot)x$ is measurable for $x \in X$. Other examples will be studied in Section 4 and 5.

On $E = L^p(I, X)$ we define the linear operator $\mathcal{B}$ by setting $\mathcal{B}f := B(\cdot)f(\cdot)$ and $D(\mathcal{B}) := \{f \in E : f(s) \in D(B(s))$ for a.e. $s \in I,\ B(\cdot)f(\cdot) \in E\}$. Further, we consider the space

$$\mathcal{D} := \operatorname{lin}\{f \in E : f(\cdot) = \varphi(\cdot)\, U(\cdot, r)y,\ r \in I, y \in Y_r,\ \varphi \in C_c^1(I) \text{ with } \varphi(s) = 0,\ s \leq r\}.$$

We now state the main result of this section. It generalizes [24, Thm. 4.2] where the case $U(t, s) = e^{(t-s)A}$ was considered; cf. [18, Thm 4.3] for the case $U(t, s) = Id$. Bounded perturbations $B(\cdot)$ were treated in [26, Thm. 2.3] with the same approach; see also the references therein.

Theorem 3.4. *Let I be a left half–open interval in $\mathbb{R}$. Let $\mathcal{U} = (U(t, s))_{(t,s)\in D}$ be an exponentially bounded evolution family on X and $(B(t), D(B(t)))$, $t \in I$, linear operators on X satisfying (M). Denote by $(G, D(G))$ the generator of the associated evolution semigroup $\mathcal{T}$ on $E = L^p(I, X)$. Then the following assertions hold:*

(a) There is a unique exponentially bounded evolution family $\mathcal{U}_B = (U_B(t, s))_{(t,s)\in D}$ satisfying

$$U_B(t, s)f(s) = U(t, s)f(s) + \int_s^t U_B(t, \tau)B(\tau)U(\tau, s)f(s)\, d\tau \tag{3.7}$$

for $f \in \mathcal{D}$ and $(t, s) \in D$.

(b) Assume, in addition, that $(\mathcal{B}, D(\mathcal{B}))$ is G–bounded and denote by $(G_B, D(G_B))$ the generator of the evolution semigroup $\mathcal{T}_B$ on E associated with $\mathcal{U}_B$. Then $D(G) = D(G_B) \subseteq D(\mathcal{B})$ and $G_B f = Gf + \mathcal{B}f$ for $f \in D(G)$. Moreover, $(\mathcal{B}, D(\mathcal{B}))$ is G_B–bounded.

(c) Suppose now that $(B(t), D(B(t)))$ is closed for a.e. $t \in I$. Then $\mathcal{B}$ is G-bounded, hypothesis (M) is valid with $Y_t = X$ for all $t \in I$, and

$$U_B(t, s)x = U(t, s)x + \int_s^t U_B(t, \tau)B(\tau)U(\tau, s)x\, d\tau \tag{3.8}$$

holds for all $x \in X$ and $(t, s) \in D$. Further, for $x \in X$ and $s \in I$ we have $U_B(t, s)x \in D(B(t))$ for a.e. $t \in I \cap [s, \infty)$, the function $B(\cdot)U_B(\cdot, s)x$ is locally integrable and

$$U_B(t, s)x = U(t, s)x + \int_s^t U(t, \tau)B(\tau)U_B(\tau, s)x\, d\tau \tag{3.9}$$

holds for all $x \in X$ and $(t, s) \in D$.

Proof. (a) At first we verify conditions (i) and (ii) in Theorem 3.1 for $\mathcal{T}$, G, $\mathcal{B}$, and $\mathcal{D}$ as defined above. By Lemma 3.3 we have $T(t)\mathcal{D} \subseteq \mathcal{D} \subseteq D(G)$. Using the arguments of the proof of [11, Prop. 2.9] it is easy to show that $\mathcal{D}$ is dense in E, cf. [28, Prop. 1.13].

Moreover, (3.6) yields $\mathcal{D} \subseteq D(\mathcal{B})$. Let $f(\cdot) = \varphi(\cdot)\, U(\cdot, r)y$ for $r \in I$, $y \in Y_r$, and $\varphi \in C_c^1(I)$ with $\varphi(s) = 0$ for $s \leq r$. Then for $0 \leq t, t' \leq d$ we have

$$\begin{aligned}
\|\mathcal{B}T(t')f - \mathcal{B}T(t)f\|_p^p &= \int_I \|(R(t')\varphi\,(s) - R(t)\varphi\,(s))\, B(s)U(s,r)y\|^p \, ds \\
&\leq \|R(t')\varphi - R(t)\varphi\|_\infty \int_r^c \|B(s)U(s,r)y\|^p \, ds,
\end{aligned}$$

where c only depends on d and φ. By (3.6) this yields (i). In order to check (ii) we consider a function $f \in \mathcal{D}$ given by

$$f(s) = \sum_{k=1}^{n} \varphi_k(s)\, U(s, s_k) y_k \, .$$

Notice that due to (M) the mapping $(t, s) \mapsto \chi_I(s{+}t)\, \|B(s{+}t)U(s{+}t, s)f(s)\|^p$ is measurable. We compute

$$\begin{aligned}
\int_0^\alpha \|\mathcal{B}T(t)f\|_p \, dt &= \int_0^\alpha \left(\int_I \left\| \sum_{k=1}^n \chi_I(s-t)\varphi_k(s-t)\, B(s)U(s,s_k)y_k \right\|^p ds \right)^{1/p} dt \\
&= \int_0^\alpha \left(\int_{I_{-t}} \left\| \sum_{k=1}^n \varphi_k(s) B(s+t)U(s+t,s)U(s,s_k)y_k \right\|^p ds \right)^{1/p} dt \\
&= \int_0^\alpha \left(\int_I \chi_I(s+t)\, \|B(s+t)U(s+t,s)f(s)\|^p \, ds \right)^{1/p} dt \\
&\leq \alpha^{1/p'} \left(\int_0^\alpha \int_I \chi_I(s+t)\, \|B(s+t)U(s+t,s)f(s)\|^p \, ds\, dt \right)^{1/p} \\
&= \alpha^{1/p'} \left(\int_I \int_0^\alpha \chi_I(s+t)\, \|B(s+t)U(s+t,s)f(s)\|^p \, dt\, ds \right)^{1/p} \\
&\leq \alpha^{1/p'}\, \beta \left(\int_I \|f(s)\|^p \, ds \right)^{1/p} \\
&= \alpha^{1/p'} \beta\, \|f\|_p \, ,
\end{aligned}$$

where $\frac{1}{p} + \frac{1}{p'} = 1$. (Here we have used Hölder's inequality, Fubini's theorem, and (3.5).) By setting $\gamma := \alpha^{1/p'}\beta < 1$, we obtain (ii).

Hence, by Theorem 3.1, the operator $(\mathcal{B}, \mathcal{D})$ can be extended to a G–bounded operator $\hat{\mathcal{B}}$ on $D(G)$, and $G_B := G + \hat{\mathcal{B}}$ with $D(G_B) := D(G)$ generates a $\mathcal{C}_0$–semigroup $\mathcal{T}_B$ on E. Also, $\mathcal{D}$ is a core for G_B. Since G generates an evolution semigroup, G_B satisfies (2.1) in Theorem 2.4 for $f \in \mathcal{D}$. Further, (R) holds for $R(\lambda, G)$, $\lambda \in \rho(G)$, due to Proposition 2.1. So, by Theorem 3.1(b), $R(\lambda, G_B)$ satisfies (R) for $\lambda \in \mathbb{R}$ sufficiently large. Thus Theorem 2.4 shows that $\mathcal{T}_B$ is an evolution semigroup induced by an exponentially bounded evolution family $\mathcal{U}_B$.

Fix $f \in \mathcal{D}$ and $t \geq 0$. Equation (3.1) implies

$$\mathcal{R}(-t)T_B(t)f = \mathcal{R}(-t)T(t)f + \int_0^t u(\tau) d\tau,$$

where $u(\tau) := \mathcal{R}(-t)T_B(t-\tau)\mathcal{B}T(\tau)f$, $0 \leq \tau \leq t$. The function $u : [0,t] \to E$ is continuous. For each $\tau \in [0,t]$ we have

$$(u(\tau))\,(s) = \chi_I(s+t)\, U_B(s+t, s+\tau)B(s+\tau)U(s+\tau, s)f(s) =: \tilde{u}(\tau, s)$$

for a.e. $s \in I$. Since the mapping $\tilde{u} : [0,t] \times I \to X$ is measurable, Lemma 3.2 implies

$$U_B(s+t,s)f(s) = U(s+t,s)f(s) + \int_s^{s+t} U_B(s+t,\tau)B(\tau)U(\tau,s)f(s)\,d\tau \tag{3.10}$$

for $f \in \mathcal{D}$, $t \geq 0$, and a.e. $s \in I$ with $s+t \in I$. Next we show that in fact (3.10) holds for all $s \in I$ such that $s+t \in I$.

For this it suffices to prove that the integral in (3.10) is continuous from the left with respect to s. First notice that $f(s) \in Y_s$ for $s \in I$ and $f \in \mathcal{D}$ so that due to (3.6) the integral in (3.10) is defined for all $(s+t,s) \in D$. Clearly, we only have to consider functions of the form $f(\cdot) = \varphi(\cdot)U(\cdot,r_0)y$ for $r_0 \in I$, $y \in Y_{r_0}$, and $\varphi \in C_c^1(I)$ with $\varphi(r) = 0$ for $r \leq r_0$. Further we may assume $t > 0$ and $s > r_0$. Let $s' \in [r_0,s]$ with $s'+t \geq s$. Then we obtain

$$\begin{aligned}
&\int_s^{s+t} U_B(s+t,\tau)B(\tau)U(\tau,s)f(s)\,d\tau - \int_{s'}^{s'+t} U_B(s'+t,\tau)B(\tau)U(\tau,s')f(s')\,d\tau \\
&= \int_s^{s+t} \chi_{[s'+t,s+t]}(\tau)\,U_B(s+t,\tau)B(\tau)U(\tau,s)f(s)\,d\tau \\
&\quad + \int_s^{s+t} \chi_{[s,s'+t]}(\tau)\,(U_B(s+t,\tau) - U_B(s'+t,\tau))\,B(\tau)U(\tau,s)f(s)\,d\tau \\
&\quad + \int_s^{s'+t} U_B(s'+t,\tau)B(\tau)U(\tau,s)\,(f(s) - U(s,s')f(s'))\,d\tau \\
&\quad - \varphi(s')\int_{r_0}^{s} \chi_{[s',s]}(\tau)\,U_B(s'+t,\tau)B(\tau)U(\tau,r_0)y\,d\tau.
\end{aligned}$$

The integrands of the first, second, and fourth term tend to 0 for a.e. τ as $s' \nearrow s$, and they are bounded by $C_1\,\|B(\tau)U(\tau,s)f(s)\|$ and $C_2\,\|B(\tau)U(\tau,r_0)y\|$, respectively, where C_k are constants independent of s' and τ. So the first, second, and fourth integral converge to 0 as $s' \nearrow s$ due to (3.6) and the dominated convergence theorem. Since $f(s) - U(s,s')f(s') \in Y_s$, by (3.6) the third summand can be estimated from above by $C_3\,\|f(s) - U(s,s')f(s')\|$, where C_3 does not depend on s'. Since f is continuous and $\mathcal{U}$ strongly continuous this term also tends to 0 as $s' \nearrow s$.

In order to show uniqueness of $\mathcal{U}_B$, let $\mathcal{V}$ be an exponentially bounded evolution family satisfying (3.7) for $f \in \mathcal{D}$. Let $(S(t))_{t\geq 0}$ be the associated evolution semigroup on E. As above, Lemma 3.2 implies that $(S(t))_{t\geq 0}$ satisfies (3.1) for $f \in \mathcal{D}$ and $t \geq 0$. Consequently, by Theorem 3.1, $S(t) = T_B(t)$, and thus $\mathcal{V} = \mathcal{U}_B$. This establishes (a).

(b) Since $(\mathcal{B}, D(\mathcal{B}))$ is G–bounded, we have $\hat{\mathcal{B}}f = \mathcal{B}f$ for $f \in D(G)$ by Theorem 3.1(a). Also, $\mathcal{B}$ is G_B–bounded by Theorem 3.1(b). The other assertions in (b) follow from Theorem 3.1(a).

(c) Observe that $(\mathcal{B}, D(\mathcal{B}))$ is closed due to our additional assumption. Hence, the first assertion in (c) holds by Theorem 3.1(d),(a). The remaining statements are shown for $\mathcal{U}$ and $\mathcal{U}_B$, separately.

(i) Let $s \in I$, $d \geq 0$, and c as in (3.6). Since Y_s is dense and $B(t)$ is closed for a.e. $t \in I$ one concludes that $B(\cdot)U(\cdot,s)y \in L^p([s,s+d],X)$ and (3.6) holds for all $y \in X$, cf. the proof of Theorem 3.1(d). This shows (M) with $Y_s = X$. Let $(t,s) \in D$. Since $\{f(s) : f \in \mathcal{D}\}$ is dense in X and all terms in (3.7) are continuous with respect to $x = f(s)$, this equation carries over to all $x \in X$.

(ii) Next, we treat the assertions concerning $\mathcal{U}_B$. Fix $f \in D(G)$ and $t \geq 0$. We claim that

$$\tilde{h} : [0,t] \times I;\ (\tau, s) \mapsto \chi_I(s-\tau)\, B(s) U_B(s, s-\tau) f(s-\tau) \quad \text{is measurable.} \tag{3.11}$$

We define $g(\tau) := T_B(\tau) f$ for $\tau \in [0,t]$ and $\tilde{g}(\tau, s) := \chi_I(s-\tau)\, U_B(s, s-\tau) f(s-\tau)$ for $(\tau, s) \in [0,t] \times I$. Theorem 3.1(b) implies that $\tau \mapsto g(\tau)$ is continuous with respect to the graph norm of G. Fix $n \in \mathbb{N}$. There exist disjoint intervals $J_{n,k} = [t_{n,k}, t_{n,k+1})$ and functions $h_{n,k} \in \mathcal{D}$, $k = 1, \cdots, m(n)$, such that $[0,t] = \bigcup_k \overline{J_{n,k}}$ and

$$\|h_{n,k} - g(\tau)\|_p + \|G(h_{n,k} - g(\tau))\|_p \leq \frac{1}{n} \qquad \text{for } \ \tau \in J_{n,k}. \tag{3.12}$$

Let $h_n(\tau) := h_{n,k}$ and $\tilde{h}_n(\tau, s) := h_{n,k}(s)$ for $\tau \in J_{n,k}$, $s \in I$, and $n \in \mathbb{N}$. From (3.12) we infer that $\tilde{h}_n$ converges to $\tilde{g}$ in $L^p([0,t] \times I, X)$. Further, (M) implies that $[0,t] \times I \ni (\tau, s) \mapsto B(s)\tilde{h}_n(\tau, s)$ is measurable. Since $\mathcal{B}$ is G–bounded, inequality (3.12) yields

$$\|\mathcal{B}\, h_n(\tau) - \mathcal{B}\, g(\tau)\|_p \leq c\, \|h_{n,k} - g(\tau)\|_p + c\, \|G(h_{n,k} - g(\tau))\|_p \leq \frac{c}{n}$$

for each $\tau \in J_{n,k}$ and a constant c. Hence, the function $(\tau, s) \mapsto B(s)\tilde{h}_n(\tau, s)$ converges in $L^p([0,t] \times I, X)$ to a function Φ. By passing to subsequences, we can assume that $\tilde{h}_n(\tau, s) \to \tilde{g}(\tau, s)$ and $B(s)\tilde{h}_n(\tau, s) \to \Phi(\tau, s)$ for a.e. (τ, s) as $n \to \infty$. Since $B(s)$ is closed for a.e. $s \in I$, we obtain that $\tilde{g}(\tau, s) \in D(B(s))$ and $B(s)\tilde{h}_n(\tau, s) \to B(s)\tilde{g}(\tau, s)$ for a.e. (τ, s). This proves the claim.

If $f \in D(G)$ and $t \geq 0$, then Theorem 3.1(d) yields

$$\begin{aligned} \mathcal{R}(-t) T_B(t) f &= \mathcal{R}(-t) T(t) f + \int_0^t v(\tau)\, d\tau, \quad \text{where} \\ v(\tau) &:= \mathcal{R}(-t) T(t-\tau) \mathcal{B} T_B(\tau) f \end{aligned}$$

for $\tau \in [0,t]$. Moreover, $v : [0,t] \to E$ is integrable and for each $\tau \in [0,t]$ we have

$$v(\tau)\,(s) = \chi_I(s+t)\, U(s+t, s+\tau) B(s+\tau) U_B(s+\tau, s) f(s) =: \tilde{v}(\tau, s)$$

for a.e. $s \in I$. By (3.11) the function $\tilde{v} : [0,t] \times I \to X$ is measurable. Thus,

$$U_B(s+t, s) f(s) = U(s+t, s) f(s) + \int_s^{s+t} U(s+t, \tau) B(\tau) U_B(\tau, s) f(s)\, d\tau \tag{3.13}$$

for a.e. $s \in I$ with $s + t \in I$ due to Lemma 3.2.

As mentioned above, (M) also holds for $p = 1$ and with β replaced by $\gamma \in [0,1)$. Therefore, for the remainder of the proof, we may and shall assume $p = 1$. For $f \in D(G)$, the function $\tilde{h}$ defined in (3.11) is measurable, and from (3.3) we derive

$$\int_0^t \int_I \|\tilde{h}(\tau, s)\|\, ds\, d\tau = \int_0^t \|\mathcal{B} T_B(\tau) f\|_{L^1(I,X)}\, d\tau \leq \tilde{c}\, \|f\|_{L^1(I,X)}\,. \tag{3.14}$$

Now, fix $f \in L^1(I, X)$. Choose $f_n \in D(G)$ such that $f_n \to f$ in $L^1(I, X)$, and denote by $\tilde{h}_n$ the measurable function corresponding to f_n. Then (3.14), applied to the difference $f_n - f_m$, shows that the functions $\tilde{h}_n$ converge in $L^1([0,t] \times I, X)$; in particular, a subsequence converges for a.e. (τ, s). For a subsubsequence, $U_B(s, s-\tau) f_n(s-\tau)$ tends to $U_B(s, s-$

$\tau)f(s-\tau)$ for a.e. (τ,s). Since $B(s)$ is closed for a.e. $s\in I$ we conclude that $\chi_I(s-\tau)\,U_B(s,s-\tau)f(s-\tau)\in D(B(s))$ for a.e. (τ,s) and that (3.11) and (3.14) hold for all $f\in L^1(I,X)$. Note that, using Fubini's theorem, equation (3.14) can be rewritten as

$$\int_I\int_0^t \chi_I(s+\tau)\,\|B(s+\tau)U_B(s+\tau,s)f(s)\|\,d\tau\,ds \le \tilde c\,\|f\|_{L^1(I,X)}\,. \tag{3.15}$$

Fix $x\in X$ and $s_0\in I$. For $0<r<\frac{t}{2}$ with $s_0+r\in I$ we define $f(s)=\chi_{[s_0,s_0+r]}(s)\,U_B(s,s_0)x$. With this f, we have $B(s+\tau)U_B(s+\tau,s)f(s)=B(s+\tau)U_B(s+\tau,s_0)x$ for $s\in[s_0,s_0+r]$ and $s+\tau\in I$. Hence, $B(\cdot)U_B(\cdot,s_0)x$ is measurable and estimate (3.15) implies

$$\begin{aligned}
\tilde c\int_{s_0}^{s_0+r}\|U_B(s,s_0)x\|\,ds &\ge \int_{s_0}^{s_0+r}\int_0^t \chi_I(s+\tau)\,\|B(s+\tau)U_B(s+\tau,s_0)x\|\,d\tau\,ds\\
&\ge \int_{s_0}^{s_0+r}\int_r^{t-r} \chi_I(s_0+\tau)\,\|B(s_0+\tau)U_B(s_0+\tau,s_0)x\|\,d\tau\,ds\\
&= r\int_r^{t-r} \chi_I(s_0+\tau)\,\|B(s_0+\tau)U_B(s_0+\tau,s_0)x\|\,d\tau.
\end{aligned}$$

Dividing by r and letting $r\to 0$ we obtain

$$\int_0^t \chi_I(s_0+\tau)\,\|B(s_0+\tau)U_B(s_0+\tau,s_0)x\|\,d\tau \le \tilde c\,\|x\|. \tag{3.16}$$

Applying (3.13) and (3.16) we can now conclude the proof in the same way as we showed that (3.8) holds for all $t\ge s$ and $x\in X$: First, instead of $f\in\mathcal{D}$ we take functions of the form $g=\varphi(\cdot)U_B(\cdot,s)x$, where $x\in X$, $s\in I$, and $\varphi\in C_c^1(I)$ with $\varphi(r)=0$ for $r\le s$. By Lemma 3.3 we have $g\in D(G_B)=D(G)$. Further, we only used (3.10), strong continuity of $\mathcal{U}$, and estimates as in (M) in the proof of (3.8). Since the corresponding hypotheses are satisfied this proof implies (3.9). □

The above result can be extended to closed intervals $I\subseteq\mathbb{R}$ with $a=\min I\in\mathbb{R}$. Assume that (M) holds for an exponentially bounded evolution family $\mathcal{U}=(U(t,s))_{(t,s)\in D_I}$ on X and linear operators $(B(t),D(B(t)))$, $t\in I$. Set $J:=(a-1,\infty)\cap I$. Define $\tilde B(t):=B(t)$ and $\tilde Y_t:=Y_t$ for $t\in I$ and $\tilde B(t):=0$ and $\tilde Y_t:=Y_a$ for $t\in(a-1,a)$. Define an evolution family $\tilde{\mathcal{U}}$ with index set D_J by setting

$$\tilde U(t,s):=\begin{cases} Id, & a\ge t\ge s>a-1,\\ U(t,\max\{s,a\}), & \text{otherwise.}\end{cases}$$

(See Section 4 for a different extension of $\mathcal{U}$.) Then (M) also holds for $\tilde{\mathcal{U}}$, $\tilde B(\cdot)$, and $\tilde Y_t$. Let $(\tilde G,D(\tilde G))$ be the generator of the evolution semigroup on $L^p(J,X)$ associated with $\tilde{\mathcal{U}}$.

Corollary 3.5. *Let $\mathcal{U}$, $\tilde{\mathcal{U}}$, $B(\cdot)$, and $\tilde B(\cdot)$ be as considered above. Then there is an evolution family $\mathcal{U}_B$ with index set D_J satisfying the conclusions of Theorem 3.4 for $\tilde{\mathcal{U}}$, $\tilde B(\cdot)$, and $\tilde G$ on D_J. In particular, if $B(t)$ is closed for a.e. $t\in I$, then $\mathcal{U}_B$ satisfies (3.8) and (3.9) for $x\in X$ and $(t,s)\in D_I$.*

Remark 3.6. *Let $I_n:=I\cap(-n,n]$, $n\in\mathbb{N}$. We weaken the hypotheses of Theorem 3.4 by only assuming that $\mathcal{U}$ is bounded on D_{I_n} and (M) holds on I_n for all $n\in\mathbb{N}$. In addition, let $B(t)$ be closed for a.e. $t\in I$. Then there exists an evolution family $\mathcal{U}_{n,B}$ with index set D_{I_n} fulfilling Theorem 3.4(a)–(c) for I_n. By uniqueness, $U_B(t,s):=U_{n,B}(t,s)$ for $(t,s)\in D_{I_n}$ is a (well–defined) evolution family with index set D_I which satisfies (3.8) and (3.9) for $x\in X$ and $(t,s)\in D_I$.*

4. Differentiability in the parabolic case

In this section we suppose that the evolution family $\mathcal{U}$ solves the Cauchy problem related to operators $(A(t), D(A(t)))$, $t \geq 0$, which satisfy the following assumption. We set $\Sigma = \Sigma_\phi := \{0\} \cup \{\lambda \in \mathbb{C} \setminus \{0\} : |\arg \lambda| < \phi\}$ for $0 < \phi < \pi$.

(P1) $D(A(t))$ is dense, $\rho(A(t)) \supseteq \Sigma_\phi$ for some $\frac{\pi}{2} < \phi < \pi$, $\|R(\lambda, A(t))\| \leq \frac{M}{1+|\lambda|}$, and

$$\|A(t)R(\lambda, A(t))\,(A(t)^{-1} - A(s)^{-1})\| \leq L\,\frac{|t-s|^\mu}{1+|\lambda|^\nu}$$

for $\lambda \in \Sigma$, $t, s \in I$, and constants $M, L \geq 0$ and $\mu, \nu \in (0,1]$ with $\mu + \nu > 1$.

In view of applications, cf. Section 5, this situation is referred to as the *parabolic case*. Observe that $A(\cdot)^{-1}$ is Hölder continuous and $A(t)$ generates a bounded analytic semigroup. Condition (P1) was introduced (in a somewhat weaker form) by P. Acquistapace and B. Terreni, [2]. Assuming (P1), there is a unique evolution family $\mathcal{U}$ solving (CP) on the spaces $D(A(t))$, [1, Thm. 2.3], see also [3], [31], and the references therein. (To be precise, strong continuity of $\mathcal{U}$ at (s,s) follows from the proof of [1, Thm. 2.3(ii)].) Moreover, $\mathcal{U}$ is exponentially bounded, see (4.1) below.

We extend $A(\cdot)$ to the interval $I := (-1, \infty)$ by setting $A(t) := A(0)$ for $t \in (-1, 0)$. Clearly, the extension still satisfies (P1) with the same constants. Further, we define

$$U(t,s) := \begin{cases} e^{(t-s)A(0)}, & 0 \geq t \geq s > -1, \\ U(t,0)e^{-sA(0)}, & t > 0 > s > -1. \end{cases}$$

Then $\mathcal{U} = (U(t,s))_{t \geq s > -1}$ is an exponentially bounded evolution family solving (CP) on I. Let $(G, D(G))$ be the generator of the evolution semigroup $\mathcal{T}$ on $E = L^p(I, X)$ associated with $\mathcal{U}$ (where $p \in [1, \infty)$ is specified below in condition (B)).

For $\theta \in [0,1]$ and $t > -1$, let Y_t be the domain of the fractional power $(-A(t))^\theta$ endowed with the norm $\|x\|_{Y_t} := \|(-A(t))^\theta x\|$. (See, e.g., [22, §2.6] for the basic theory of fractional powers.) By (P1) we obtain $\sup_{t \in I} \|(-A(t))^{-\theta}\| < \infty$ and $\sup_{t \in I} \|(-A(t))^\theta A(t)^{-1}\| < \infty$ (see [22], proof of Lemma 2.6.3). It is shown in [6, Thm. 2.3] that $U(t,s)X \subseteq Y_t$ for $t > s > -1$ and

$$\|(-A(t))^\theta U(t,s)\| \leq C\, e^{w(t-s)}\,(t-s)^{-\theta} \tag{4.1}$$

for $(t,s) \in D' = D'_I := \{(t,s) \in D_I : t > s\}$ and constants $C \geq 1$ and $w \in \mathbb{R}$.

We denote by $L^q_{loc,u}(I)$ the space of locally uniformly q-integrable functions on I endowed with the norm

$$\|\varphi\|_{L^q_{loc,u}} := \sup_{s \in I} \left(\int_s^{s+1} |\varphi(\tau)|^q \, d\tau \right)^{\frac{1}{q}}.$$

On the perturbations $B(t)$, $t \geq 0$, we impose the following condition.

(B) Let $p \geq 1$ and $0 < \theta < 1$ with $p\theta < 1$. Let $q = \left(\frac{1}{p} - \frac{\theta}{\rho}\right)^{-1}$ for some $\rho \in (p\theta, 1)$. Let $B(t) \in \mathcal{L}(Y_t, X)$ and $\|B(t)\|_{\mathcal{L}(Y_t,X)} \leq \psi(t)$ for a.e. $t \geq 0$, where $\psi \in L^q_{loc,u}(\mathbb{R}_+)$. Finally, let $B(\cdot)U(\cdot, s)y$ be measurable for $s \geq 0$ and $y \in Y_s$.

Notice that $q > p(1-p\theta)^{-1}$ and $B(t)A(t)^{-1} \in \mathcal{L}(X)$. We remark that in our application $B(t)$ is a differential operator and the assumptions on q, θ, p, ρ lead to an integrability condition for the coefficients of this operator. Let $B(t) := 0$ and $\psi(t) := 0$ for $t \in (-1, 0)$. On $E = L^p(I, X)$ we define a linear operator by setting $\mathcal{B}f := B(\cdot)f(\cdot)$ for $f \in D(\mathcal{B}) := \{f \in E : f(t) \in Y_t$ for a.e. $t \in I,\ B(\cdot)f(\cdot) \in E\}$. In the next lemma we check the assumptions of Theorem 3.4(a),(b) for $\mathcal{U}$ and $B(\cdot)$.

Lemma 4.1. *Assume (P1) and (B). Then (M) holds for $\mathcal{U}$, $B(\cdot)$, and Y_t on I. Assume, in addition, that $B(\cdot)A(\cdot)^{-1}x$ is measurable for all $x \in X$. Then $(\mathcal{B}, D(\mathcal{B}))$ is G–bounded.*

Proof. (i) For the first assertion we only have to verify (3.5). Consider $x \in X$, $s > -1$, $t, r > 0$, and $\mu \in \mathbb{R}$. Notice that by (B) and (4.1) we obtain

$$\|B(s+t)U(s+t,s)x\| \le Ct^{-\theta}e^{wt}\,\psi(s+t)\,\|x\| \tag{4.2}$$

for a.e. $t > 0$ and all $x \in X$. Set $\varepsilon := (\mu - w)p$. Then (4.2) and Hölder's inequality imply

$$\begin{aligned}
&\int_0^r \|e^{-\mu t}B(s+t)U(s+t,s)x\|^p\,dt \\
&\le\ C^p\,\|x\|^p \int_0^r e^{(w-\mu)pt}\,t^{-p\theta}\,\psi(s+t)^p\,dt \\
&\le\ C^p\,\|x\|^p \left(\int_0^r \left(e^{-\frac{\varepsilon t}{2}}t^{-p\theta}\right)^{\frac{\rho}{p\theta}}\,dt\right)^{\frac{p\theta}{\rho}} \left(\int_0^r \left(e^{-\frac{\varepsilon t}{2}}\psi(s+t)^p\right)^{(1-\frac{p\theta}{\rho})^{-1}}\,dt\right)^{1-\frac{p\theta}{\rho}} \\
&=\ C^p\,\|x\|^p \left(\int_0^r e^{-\varepsilon_1 t}\,t^{-\rho}\,dt\right)^{\frac{p\theta}{\rho}} \left(\int_0^r e^{-\varepsilon_2 t}\,\psi(s+t)^q\,dt\right)^{\frac{p}{q}}
\end{aligned} \tag{4.3}$$

for some constants $\varepsilon_k \in \mathbb{R}$. For $r \le r_0$ and $\mu = 0$, estimate (4.3) yields

$$\begin{aligned}
\int_0^r \|B(s+t)U(s+t,s)x\|^p\,dt\ &\le\ C_1\,\|x\|^p \left(\int_0^r t^{-\rho}\,dt\right)^{\frac{p\theta}{\rho}} \left(\int_s^{s+r} \psi(t)^q\,dt\right)^{\frac{p}{q}} \\
&\le\ C_2\,\|\psi\|^p_{L^q_{loc,u}}\,r^{\kappa}\,\|x\|^p
\end{aligned}$$

for some constants $C_k, \kappa > 0$. Letting $r \to 0$ establishes (3.5). Further, let $\mu > w$. Hence, $\varepsilon_k > 0$. We derive from (4.3) that

$$\begin{aligned}
&\int_0^\infty \|e^{-\mu t}B(s+t)U(s+t,s)x\|^p\,dt \\
&\le\ C^p\,\|x\|^p \left(\int_0^\infty \frac{e^{-\varepsilon_1 t}}{t^\rho}\,dt\right)^{\frac{p\theta}{\rho}} \left(\sum_{k=0}^{\infty} e^{-\varepsilon_2 k}\int_{s+k}^{s+k+1} \psi(t)^q\,dt\right)^{\frac{p}{q}} \le C_3\,\|\psi\|^p_{L^q_{loc,u}}\,\|x\|^p
\end{aligned} \tag{4.4}$$

for a constant $C_3 > 0$.

(ii) Next assume that $B(\cdot)A(\cdot)^{-1}x$, $x \in X$, is measurable. Consider the space

$$F := \{f \in E : f(t) \in Y_t \text{ for a.e. } t \in I,\ (-A(\cdot))^\theta f(\cdot) \in E\}$$

endowed with the norm $\|f\|_F := \|(-A(\cdot))^\theta f(\cdot)\|_E$. Observe that F is a Banach space which is continuously embedded in E. Further we consider $\mathcal{B}$ as an operator on $D(\mathcal{B}_F) := \{f \in F : \mathcal{B}f \in E\}$. It can easily be verified that $\mathcal{B} : D(\mathcal{B}_F) \subseteq F \to E$ is a closed operator. Let $f \in E$ and $t > 0$. By (4.1) we obtain

$$\|(-A(s))^\theta\,(T(t)f)(s)\| \le \|(-A(s))^\theta U(s,s-t)\|\,\|\mathcal{R}(t)f\,(s)\| \le C\,t^{-\theta}\,e^{wt}\,\|\mathcal{R}(t)f\,(s)\|\,.$$

Thus, $T(t) \in \mathcal{L}(E,F)$ for $t>0$. For $t,t' \geq t_0 > 0$ and $f \in E$ we have $T(t)f - T(t')f = T(t_0)(T(t-t_0)f - T(t'-t_0)f)$, and hence the function $u:(0,\infty) \to F:\ t \mapsto e^{-\lambda t}T(t)f$ is continuous, where $\lambda > w$ and $f \in E$. Moreover,

$$\|e^{-\lambda t}T(t)f\|_F \leq C\, e^{(w-\lambda)t}\, t^{-\theta} \left(\int_I \|\mathcal{R}(t)f(s)\|^p\,ds\right)^{\frac{1}{p}} \leq C\, e^{(w-\lambda)t}\, t^{-\theta}\, \|f\|_E\,.$$

As a consequence, $u:\mathbb{R}_+ \to F$ is integrable. Since λ dominates the growth bound of $\mathcal{T}$, we infer that $R(\lambda,G):E\to F$ is bounded for $\lambda > w$.

It follows from [1, Thm. 2.3] that $D' \ni (t,s) \mapsto A(t)U(t,s)x$ is continuous for $x \in X$. Hence, by the measurability of $B(\cdot)A(\cdot)^{-1}x$, $x\in X$, the mapping

$$D' \ni (t,s) \mapsto B(t)U(t,s)f(s) = B(t)A(t)^{-1}A(t)U(t,s)f(s)$$

is measurable for $f \in E$. Now let $f \in C_c(I,X)$ with support J. Then

$$\|B(s)(T(t)f)(s)\|^p \leq C^p\, e^{wpt}\, t^{-p\theta}\, \|f\|_\infty^p\, \psi(s)^p\, \chi_J(s-t),$$

for $t>0$. Because of $q>p$ we have $\psi \in L^p_{loc}(I)$, and thus $T(t)f \in D(\mathcal{B}_F) \subseteq D(\mathcal{B})$, $t>0$. Moreover,

$$\begin{aligned}\|\mathcal{B}T(t)f - \mathcal{B}T(t')f\|_E^p \ \leq\ & \int_I \psi(s)^p\, \|(-A(s))^\theta U(s,s-t_0)\|^p \\ & \|U(s-t_0,s-t)\,(\mathcal{R}(t)f)(s) - U(s-t_0,s-t')\,(\mathcal{R}(t')f)(s)\|^p\,ds\end{aligned}$$

for $t',t \geq t_0 > 0$. Clearly, the integrand converges to 0 for a.e. $s \in I$ as $t' \to t$. Also, the integrand is bounded by $C_4\,\psi(\cdot)^p\chi_K(\cdot)$ for a constant C_4 and a compact interval $K \subseteq I$. Therefore the mapping $(0,\infty) \ni t \mapsto \mathcal{B}T(t)f \in E$ is continuous for $f \in C_c(I,X)$ due to the dominated convergence theorem. Further, we derive for $\lambda > \mu > w$ and $u(t) = e^{-\lambda t}T(t)f$, $f \in C_c(I,X)$, that

$$\begin{aligned}\int_0^\infty \|\mathcal{B}u(t)\|_E\,dt \ &=\ \int_0^\infty e^{(\frac{\mu}{p}-\lambda)t}\left(\int_I e^{-\mu t}\,\|B(s)U(s,s-t)\chi_I(s-t)f(s-t)\|^p\,ds\right)^{\frac{1}{p}} dt \\ &\leq\ C_5\left(\int_0^\infty\int_I e^{-\mu t}\,\|B(s)U(s,s-t)\chi_I(s-t)f(s-t)\|^p\,ds\,dt\right)^{\frac{1}{p}} \\ &=\ C_5\left(\int_I\int_0^\infty e^{-\mu t}\,\|B(s+t)U(s+t,s)f(s)\|^p\,dt\,ds\right)^{\frac{1}{p}} \\ &\leq\ C_6\left(\int_I \|f(s)\|^p\,ds\right)^{\frac{1}{p}} = C_6\,\|f\|_E \qquad (4.5)\end{aligned}$$

for constants $C_k > 0$. (Here we have used Hölder's inequality, Fubini's theorem and estimate (4.4).) Thus, $\mathcal{B}u:\mathbb{R}_+ \to E$ is integrable. Now Hille's theorem, [8, Thm. 3.7.12], yields $R(\lambda,G)f = \int_0^\infty u(t)\,dt \in D(\mathcal{B}_F) \subseteq D(\mathcal{B})$ and $\mathcal{B}R(\lambda,G)f = \int_0^\infty \mathcal{B}u(t)\,dt$. In particular, by (4.5) we have $\|\mathcal{B}R(\lambda,G)f\|_E \leq C_6\|f\|_E$ for $f \in C_c(I,X)$.

Finally, we show that $(\mathcal{B},D(\mathcal{B}))$ is G–bounded. Let $g \in D(G)$ and $f = (\lambda - G)g \in E$. Choose a sequence $(f_n) \subseteq C_c(I,X)$, converging to f. Then $g_n := R(\lambda,G)f_n \in D(\mathcal{B}_F)$ tends to g in the norm of F as $n\to\infty$. Moreover, there exists $E - \lim_{n\to\infty}\mathcal{B}R(\lambda,G)f_n =: h$. Since $\mathcal{B}$ is closed in $F\times E$, we infer that $g \in D(\mathcal{B}_F) \subseteq D(\mathcal{B})$ and $\mathcal{B}g = h$. Hence, $\|\mathcal{B}g\|_E \leq C_6\,\|f\|_E = C_6\,\|(\lambda-G)g\|_E$ for $g \in D(G)$. □

So we can apply the results of the preceding section. In fact, we can even improve them considerably and show that the function $t \mapsto U_B(t,s)x$, $s \in I$, $x \in X$, solves (pCP) for a.e. $t \in (s,\infty)$. For this we have to introduce some new concepts.

Let $1 < p, d < \infty$ and let $X = L^d(\Omega)$ for a σ-finite measure space (Ω, μ). Then $E = L^p(I, X)$ is a UMD-space, cf. e.g. [14]. We assume (P1) and that the operators $(A(t), D(A(t)))$, $t \geq 0$, satisfy the following condition, where ϕ is determined by (P1).

(P2) The operators $(A(t), D(A(t)))$ admit *bounded imaginary powers* $(-A(t))^{ir}$ and $\|(-A(t))^{ir}\| \leq c\, e^{(\pi-\phi)|r|}$ for $t \geq 0$, $r \in \mathbb{R}$, and a constant c,

see [14] and the references cited therein. Clearly, the extension of $A(\cdot)$ to $I = (-1, \infty)$ given by $A(t) = A(0)$, $-1 < t < 0$, also satisfies (P2). In Section 5 we will see that (P1) and (P2) can be verified for a large class of elliptic operators $A(t)$. We define a linear operator $\mathcal{A}$ on E by setting $\mathcal{A}f := A(\cdot)f(\cdot)$ for $f \in D(\mathcal{A}) := \{f \in E : f(s) \in D(A(s)) \text{ for a.e. } s > -1,\ A(\cdot)f(\cdot) \in E\}$. Notice that $(\mathcal{A}, D(\mathcal{A}))$ is closed. Consider the first derivative $G_0 f = -f'$ on E with domain $D(G_0) = \{f \in W^{1,p}(I,X) : f(-1) = 0\}$. Then it was shown in [14], Cor. 2, Thm. 2, that the operator $(G_0 + \mathcal{A}, D(G_0) \cap D(\mathcal{A}))$ is closed and $(\omega, \infty) \subseteq \rho(G_0 + \mathcal{A})$ for some $\omega \in \mathbb{R}$. Further, we set

$$\mathcal{D}_1 := \text{lin}\,\{f \in E : f(\cdot) = \varphi(\cdot)U(\cdot, r)y;\ r \in I, y \in D(A(r)), \varphi \in C_c^1(I) \text{ with } \varphi(s) = 0,\ s \leq r\}.$$

Since the Cauchy problem (CP) related to $A(\cdot)$ is well–posed, it follows from [28, Prop. 1.13], cf. [11, Prop. 2.9], that $\mathcal{D}_1 \subseteq D(\mathcal{A}) \cap D(G_0)$ and the generator $(G, D(G))$ of the evolution semigroup is the closure of $(G_0 + \mathcal{A}, \mathcal{D}_1)$. Hence, $G_0 + \mathcal{A}$ extends G. Since there exists $\lambda \in \rho(G) \cap \rho(G_0 + \mathcal{A})$, this implies

$$(G, D(G)) = (G_0 + \mathcal{A}, D(G_0) \cap D(\mathcal{A})). \tag{4.6}$$

This fact plays a central role in the proof of the next theorem.

Theorem 4.2. *Assume that the operators $(A(t), D(A(t)))$ and $(B(t), Y_t)$, $t \geq 0$, on $X = L^d(\Omega)$ satisfy (P1), (P2), and (B) for $1 < p, d < \infty$. Moreover, assume that $B(\cdot)A(\cdot)^{-1}x$ is measurable for $x \in X$. Then there is a perturbed evolution family $\mathcal{U}_B = (U_B(t,s))_{t \geq s > -1}$ satisfying the conclusions of Theorem 3.4(a) and (b). Set $u(t) = U_B(t,s)x$ for $t \geq s \geq 0$ and $x \in X$. Then $u \in C([s,\infty), X)$ and $u(s) = x$. Further, we have $u(t) \in D(A(t))$ for a.e. $t \in (s,\infty)$, $u \in W^{1,1}_{loc}((s,\infty), X)$, $(A(\cdot) + B(\cdot))u(\cdot) \in L^1_{loc}((s,\infty), X)$, and*

$$\frac{d}{dt} u(t) = (A(t) + B(t))u(t) \qquad \text{for a.e. } t \in (s, \infty).$$

Proof. (1) The existence of $\mathcal{U}_B = (U_B(t,s))_{t \geq s > -1}$ follows from Lemma 4.1 and Theorem 3.4. Let $\mathcal{T}_B$ be the associated evolution semigroup on $E = L^p(I,X)$ with generator $G_B = G + \mathcal{B}$, where $D(G_B) = D(G) \subseteq D(\mathcal{B})$ and $\mathcal{B} = B(\cdot)$. Observe that the generator $(\tilde{G}_0, D(\tilde{G}_0))$ of the left translation semigroup $(\mathcal{R}(-t))_{t \geq 0}$ is given by $\tilde{G}_0 g = g'$ for $g \in D(\tilde{G}_0) = W^{1,p}(I,X)$. Let $f \in D(G)$. We have $T_B(t)f \in D(G) = D(\mathcal{A}) \cap D(G_0) \subseteq D(\tilde{G}_0)$ and $G_B = G_0 + \mathcal{A} + \mathcal{B}$ due to (4.6). Hence,

$$\begin{aligned} \frac{d}{d\tau}\mathcal{R}(-\tau)T_B(\tau)f &= \mathcal{R}(-\tau)G_B T_B(\tau)f + \mathcal{R}(-\tau)\tilde{G}_0 T_B(\tau)f \\ &= \mathcal{R}(-\tau)(\mathcal{A} + \mathcal{B})T_B(\tau)f =: v(\tau), \end{aligned}$$

and $v : [0,t] \to E$ is continuous. Integration from 0 to t yields

$$\mathcal{R}(-t)T_B(t)f = f + \int_0^t \mathcal{R}(-\tau)(\mathcal{A}+\mathcal{B})T_B(\tau)f\,d\tau. \tag{4.7}$$

Moreover, for $\tau \in [0,t]$ we have

$$v(\tau)\,(s) = (A(s+\tau)+B(s+\tau))U_B(s+\tau,s)f(s) =: \tilde{v}(\tau,s)$$

for a.e. $s > -1$. Since $(\mathcal{A}, D(\mathcal{A}))$ is closed and $\mathcal{A}T_B(\tau)f = \mathcal{A}R(\lambda, G_B)T_B(\tau)(\lambda - G_B)f$, we derive from (4.6) the continuity of $\tau \mapsto \mathcal{A}T_B(\tau)f$. By Lemma 3.2 there is a measurable function $\Phi : [0,t]\times I \to X$ such that for a.e. $\tau \in [0,t]$ we have $(\mathcal{R}(-\tau)\mathcal{A}T_B(\tau)f)(s) = \Phi(\tau,s)$ for a.e. $s > -1$. That is, for a.e. $\tau \in [0,t]$

$$\begin{aligned} A(s+\tau)U_B(s+\tau,s)f(s) &= \Phi(\tau,s), \qquad \text{and hence} && (4.8)\\ U_B(s+\tau,s)f(s) &= A(s+\tau)^{-1}\Phi(\tau,s) && (4.9)\end{aligned}$$

for a.e. $s > -1$. Observe that both sides of (4.9) are measurable with respect to (τ,s). Thus (4.9) and, hence, (4.8) hold for a.e. (τ,s), and so the left–hand side of (4.8) is measurable on $[0,t]\times I$. Then the function

$$[0,t]\times I \ni (\tau,s) \mapsto B(s+\tau)U_B(s+\tau,s)f(s) = B(s+\tau)A(s+\tau)^{-1}\,A(s+\tau)U_B(s+\tau,s)f(s)$$

is measurable by the assumption. As a consequence, $\tilde{v} : [0,t]\times I \to X$ is measurable. From Lemma 3.2 we deduce that $\tilde{v}(\cdot,s)$ is integrable for a.e. $s > -1$ and (4.7) implies after a change of variables

$$U_B(t,s)f(s) = f(s) + \int_s^t (A(\tau)+B(\tau))U_B(\tau,s)f(s)d\tau \tag{4.10}$$

for $s \in (-1,\infty)\setminus N(f)$, $t \in [s,\infty)$, and a null set $N(f)$. (Notice that both sides of (4.10) are continuous with respect to t.)

(2) Now fix $x \in X$ and $s \geq 0$. Choose $r_n \searrow s$ and $\varphi_n \in C_c^1(I)$ with $\varphi_n \equiv 0$ on $(-1,s]$ and $\varphi_n \equiv 1$ on $[r_n, r_n+1]$. Set $f_n(\cdot) := \varphi_n(\cdot)U_B(\cdot,s)x$. By Lemma 3.3 we have $f_n \in D(G_B) = D(G)$. There exist $s_n \in [r_n, r_n+1]$ such that $s_n \notin N(f_n)$ and $s_n \searrow s$. From the first part of the proof follows that $U_B(t,s_n)f_n(s_n) \in D(A(t))$ for a.e. $t \in [s_n,\infty)$, $(A(\cdot)+B(\cdot))U_B(\cdot,s_n)f_n(s_n) \in L^1_{loc}([s_n,\infty),X)$ and (4.10) holds for $t \in [s_n,\infty)$. As a consequence, $U_B(\cdot,s_n)f_n(s_n) \in W^{1,1}_{loc}([s_n,\infty),X)$ and

$$\frac{d}{dt}U_B(t,s_n)f_n(s_n) = (A(t)+B(t))U_B(t,s_n)f_n(s_n)$$

for a.e. $t \in [s_n,\infty)$. Set $u(t) := U_B(t,s)x$ for $t \geq s$. Since $u(t) = U_B(t,s_n)f_n(s_n)$ for $t \geq s_n$ the theorem is established. □

Other perturbation results in the parabolic case can be found, for instance, in [10, Thm. 6.1] or [7, Thm. 7.1.3]. There $B(\cdot)$ satisfies certain Hölder conditions which lead to stronger regularity properties of $\mathcal{U}_B$.

5. An application

In this section we investigate the following intial value problem with mixed boundary conditions:

$$
\begin{aligned}
&D_t u(t,\xi) = \sum_{k,l=1}^{n} D_k\left(a_{kl}(t,\xi) D_l u(t,\xi)\right) + \sum_{k=1}^{n} b_k(t,\xi)\, D_k u(t,\xi) + c(t,\xi) u(t,\xi),\ t > s,\ \xi \in \Omega,\\
&u(t,\xi) = 0, \quad \xi \in \Gamma_0\,,\ t > s, \qquad\qquad (IVP)\\
&\sum_{k,l=1}^{n} n_k(\xi)\, a_{kl}(t,\xi)\, D_l u(t,\xi) = 0, \quad \xi \in \Gamma_1\,,\ t > s,\\
&u(s,\xi) = x(\xi), \quad \xi \in \Omega.
\end{aligned}
$$

Here $D_t = \frac{\partial}{\partial t}$ and $D_k = \frac{\partial}{\partial \xi_k}$ denote derivatives in the sense of distributions and the boundary conditions are understood in the sense of traces. Moreover, we consider the following hypotheses:

(A1) $\Omega \subseteq \mathbb{R}^n$ is a bounded domain with compact C^2–boundary $\partial\Omega$, Γ_i are open and closed in $\partial\Omega$ such that $\partial\Omega = \Gamma_0 \cup \Gamma_1$, $\Gamma_0 \cap \Gamma_1 = \emptyset$, $\Gamma_0 \neq \emptyset$, and $n(\xi)$ is the outer normal of $\partial\Omega$ at $\xi \in \partial\Omega$.

(A2) $a_{kl} = a_{lk} : \mathbb{R}_+ \times \overline{\Omega} \to \mathbb{R}$ is continuous with bounded partial derivatives with respect to $\xi \in \Omega$, $a_{kl}, D_j a_{kl} \in C^{\mu}(\mathbb{R}_+, C(\overline{\Omega}))$, $1 \le k,l,j \le n$, with $\mu > \frac{1}{2}$, and

$$\frac{1}{\delta}\,|\eta|^2 \ge \sum_{k,l=1}^{n} a_{kl}(t,\xi)\, \eta_k \eta_l \ge \delta\,|\eta|^2$$

for a constant $\delta > 0$, $\eta \in \mathbb{R}^n$, $t \ge 0$, and $\xi \in \Omega$.

(A3) $b_k, c \in L^q_{loc,u}(\mathbb{R}_+, L^r(\Omega))$, where $r > n$ and $q > \left(\frac{1}{2} - \frac{n}{2r}\right)^{-1}$.

(A3') $b_k \equiv 0$ and $c \in L^q_{loc,u}(\mathbb{R}_+, L^r(\Omega))$, where $2r > n$, $r > 1$, and $q > \left(1 - \frac{n}{2r}\right)^{-1}$.

Assumptions (A1) and (A2) were already considered in [14, §6].

On the space $X = L^d(\Omega)$, $1 < d < \infty$, $d \le r$, we define the second order operator

$$
\begin{aligned}
A(t)x &= \sum_{k,l=1}^{n} D_k\left(a_{kl}(t,\cdot) D_l x(\cdot)\right), \quad t \ge 0,\\
D(A(t)) &= \{x \in W^{2,d}(\Omega) : x(\xi) = 0,\, \xi \in \Gamma_0,\ \sum_{k,l=1}^{n} n_k(\xi) a_{kl}(t,\xi) D_l x(\xi) = 0,\, \xi \in \Gamma_1\}.
\end{aligned}
$$

Then it is shown in [14, §6] that the operators $(A(t), D(A(t)))$, $t \ge 0$, satisfy (P1) (with constants μ and $\nu = \frac{1}{2}$) and (P2). (See also [1] and [31] for the verification of (P1) and the references in [14] concerning (P2).) Denote by $\mathcal{U}$ the evolution family solving the Cauchy problem (CP) corresponding to $A(\cdot)$.

Next, we introduce the (formal) first order operator

$$B(t)x = \sum_{k=1}^{n} b_k(t,\cdot)\, D_k x(\cdot) + c(t,\cdot) x(\cdot), \qquad t \ge 0.$$

Assuming (A3), resp. (A3'), one can see that there are numbers $\theta > \frac{1}{2} + \frac{n}{2r}$, resp. $\theta > \frac{n}{2r}$, $p > 1$, and $\rho < 1$ such that $1 > \rho > p\theta$ and $q = \left(\frac{1}{p} - \frac{\theta}{\rho}\right)^{-1}$. As in the preceding section Y_t is defined as the domain of $(-A(t))^\theta$ endowed with the norm $\|\cdot\|_{Y_t}$. Let $\frac{1}{\vartheta} = \frac{1}{d} - \frac{1}{r} \geq 0$. Then $\vartheta \geq d$ and $\frac{j}{2} + \frac{n}{2r} = \frac{j}{2} + \frac{n}{2d} - \frac{n}{2\vartheta}$ for $j \in \{0,1\}$. Since $\theta > \frac{j}{2} + \frac{n}{2r}$, we obtain by [7, Thm. 1.6.1] that $Y_t \subseteq W^{j,\vartheta}(\Omega)$ and $\|x\|_{W^{j,\vartheta}} \leq K\|x\|_{Y_t}$. The proof of [7, Thm. 1.6.1] and the estimate

$$\|A(t)^{-1}x\|_{W^{2,d}} \leq K' \,\|x\|_d\,, \qquad x \in X, \tag{5.1}$$

(see for instance [14, (6.6)] or [3, (IV.2.6.19)]) imply that the constant K does not depend on t. Moreover, we have $\frac{1}{r} + \frac{1}{d'} + \frac{1}{\vartheta} = 1$, where $\frac{1}{d} + \frac{1}{d'} = 1$. Hence, by Hölder's inequality we derive for $x \in Y_t$ and $y \in L^{d'}(\Omega)$

$$\begin{aligned}
|\langle B(t)x, y\rangle| &\leq \int_\Omega \left(|c(t,\xi)x(\xi)y(\xi)| + \sum_{k=1}^n (|b_k(t,\xi)D_k x(\xi)y(\xi)|\right) d\xi \\
&\leq \sum_{k=1}^n (\|D_k x\|_\vartheta \,\|b_k(t)\|_r \,\|y\|_{d'}) + \|x\|_\vartheta \,\|c(t)\|_r \,\|y\|_{d'} \\
&\leq \tilde{\psi}(t)\,\|x\|_{W^{1,\vartheta}}\,\|y\|_{d'} \\
&\leq K\tilde{\psi}(t)\,\|x\|_{Y_t}\,\|y\|_{d'},
\end{aligned}$$

where $\tilde{\psi}(t) := \max\{\|c(t)\|_r, \|b_k(t)\|_r, k = 1, \cdots, n\}$. Thus $B(t) \in \mathcal{L}(Y_t, X)$, $\|B(t)\|_{\mathcal{L}(Y_t,X)} \leq \psi(t) := K\tilde{\psi}(t)$, and $\psi \in L^q_{loc,u}(\mathbb{R}_+)$.

Now we show that $B(\cdot)A(\cdot)^{-1}x$, $x \in X$, is measurable. The Gagliardo–Nirenberg inequality (see e.g. [22, Lemma 8.4.1]), (5.1), and (P1) imply

$$\begin{aligned}
\|D_k\,(A(t)^{-1} - A(s)^{-1})x\|_d &\leq C_1\,\|(A(t)^{-1} - A(s)^{-1})x\|_d^{\frac{1}{2}}\,\|(A(t)^{-1} - A(s)^{-1})x\|_{W^{2,d}}^{\frac{1}{2}} \\
&\leq C_2\,\|x\|_d\,|t-s|^{\frac{\mu}{2}}
\end{aligned}$$

for constants C_l. Hence, $B(\cdot)A(\cdot)^{-1}x$ is measurable if $b_k, c \in L^\infty(\mathbb{R}_+ \times \Omega)$, cf. [26, Lemma 2.4]. The general case follows by truncation of the coefficients b_k and c and an application of the dominated convergence theorem. Finally, from $B(t)U(t,s)x = B(t)A(t)^{-1}\,A(t)U(t,s)x$ we derive the measurability of $B(\cdot)U(\cdot,s)x$ for $x \in X$.

As a consequence, Theorem 4.2 yields the following result which extends [24, Ex. 4.6], where the case $b_k \equiv 0$ and $A(t) \equiv A$ was considered.

Proposition 5.1. *Assume (A1), (A2), and (A3), resp. (A3'). Let $X = L^d(\Omega)$, $1 < d < \infty$, $d \leq r$, $x \in X$, and $s \geq 0$. Then there is a function $u \in W^{1,1}_{loc}((s,\infty), X) \cap C([s,\infty), X)$ with $u(s) = x$ such that $u(t) \in D(A(t))$ for a.e. $t \in (s,\infty)$, $(A(\cdot) + B(\cdot))u(\cdot) \in L^1_{loc}((s,\infty), X)$, and u satisfies (IVP) for a.e. $t \in (s,\infty)$ and $\xi \in \Omega$.*

In [4] D.G. Aronson has studied a more general version of (IVP) (however with Dirichlet boundary conditions) and obtained weak solutions of (IVP) by PDE methods. We point out that in [4] the conditions $r > n$ and $q > \left(\frac{1}{2} - \frac{n}{2r}\right)^{-1}$, cf. (A3), also play a crucial role.

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