An existence result for fully Nonlinear Parabolic Systems of PDE's

Dr. Marco Squassina
e-mail: *squazz@numerica.it*

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Contents

Abstract

By means of some suriectivity abstract results for pseudomonotone operators like in [1], we prove existence (Theorem II.1.6) of weak solutions to fully nonlinear parabolic systems in divergence form of the type

$$\begin{cases} D_t u_h - \sum\limits_{k=1}^{N} \sum\limits_{i,j=1}^{n} D_i \left(a_{ij}^{hk}(x, u, Du) D_j u_k \right) = g_h(x, u, Du) & \text{in }]0, T[\times\Omega \quad (h = 1, .., N) \\ u(t, x) = 0 & \text{on }]0, T[\times\partial\Omega \\ u(0, x) = 0 & \text{in } \Omega \end{cases}$$

We'll study this problem by means of the following operators

$$A_1, A_2 : L^2\left(0, T; H_0^1\left(\Omega; \mathbb{R}^N\right)\right) \to L^2\left(0, T; H^{-1}\left(\Omega; \mathbb{R}^N\right)\right),$$

$$L : \mathcal{D}(L) \to L^2\left(0, T; H^{-1}\left(\Omega; \mathbb{R}^N\right)\right),$$

defined setting

$$\begin{aligned} \langle A_1(u), v\rangle &= \int_0^T \int_\Omega \sum_{h,k=1}^{N} \sum_{i,j=1}^{n} a_{ij}^{hk}(x, u, Du) D_i u_h D_j v_k \, dx \, dt \ ; \\ \langle A_2(u), v\rangle &= -\int_0^T \int_\Omega g(x, u, Du) \cdot v \, dx \, dt \ ; \\ \langle Lu, v\rangle &= -\int_0^T \int_\Omega u D_t v \, dx \, dt \, . \end{aligned}$$

Under suitable assumptions on the data, the operators A_1, A_2 and L are well defined, L's maximal monotone linear and $(A_1 + A_2)$ is pseudo-monotone and coercive. From the abstract results of the first section we'll get the announced existence result.

Chapter I

Abstract preliminary results

1 Variational inequalities for pseudomonotone operators

In the following paper, $M \subseteq X \times X'$ will denote a "multi-valued" operator over X.

(1.1) Definition *We say that M is a* monotone operator *if*

$$\forall u_1, u_2 \in \mathcal{D}(M),\ \ \forall \varphi_1 \in M(u_1),\ \ \forall \varphi_2 \in M(u_2):\ \ \langle \varphi_2 - \varphi_1, u_2 - u_1 \rangle \geq 0.$$

Moreover, if M is "single-valued", this is equivalent to

$$\forall u_1, u_2 \in \mathcal{D}(M):\ \ \langle M(u_2) - M(u_1), u_2 - u_1 \rangle \geq 0.$$

(1.2) Definition *Let M be a monotone operator on X. We say that M is* maximal, *if the unique maximal monotone extension of M is M.*

(1.3) Definition *Let X be a reflexive Banach space and $T \in]0, +\infty[$. We say that M is an* emicontinuous operator *from X to X', if $\mathcal{D}(M)$ is convex and for every $u, v \in \mathcal{D}(M)$ the function*

$$\{\lambda \longmapsto M(\lambda u + (1-\lambda)v)\}$$

*is continuous from $[0, T]$ to X', where X' is endowed with the weak * topology.*

(1.4) Definition *Let X be a reflexive Banach space and T an operator over X to X'. We say that T is a* pseudo monotone operator *if for each sequence (u_h) in $\mathcal{D}(T)$ weakly convergent to u such that*

$$\limsup_h \langle T(u_h), u_h - u \rangle \leq 0,$$

we have

$$\forall v \in \mathcal{D}(T):\ \ \liminf_h \langle T(u_h), u_h - v \rangle \geq \langle T(u), u - v \rangle.$$

(1.5) Definition *Let X a reflexive Banach space and T an operator over X to X'. We say that T is a* bounded operator *if T maps bounded subsets of X into bounded subsets of X'.*

(1.6) Proposition *Let X be a reflexive Banach space, T_1 a pseudomonotone operator over X to X' and T_2 a completely continuous map over X to X'. Let us assume that for each sequence (u_h) weakly convergent to u with*

$$\limsup_h \langle T_1(u_h), u_h - u\rangle \leq 0,$$

(u_h) converges strongly to u. Then $(T_1 + T_2)$ is pseudomonotone.

Proof. If (u_h) is a sequence in X weakly convergent to u with

$$\limsup_h \langle T_1(u_h) + T_2(u_h), u_h - u\rangle \leq 0.$$

up to a further subsequence $(T_2(u_h))$ is strongly convergent in X', so that

$$\lim_h \langle T_2(u_h), u_h - u\rangle = 0.$$

Whence we have

$$\limsup_h \langle T_1(u_h), u_h - u\rangle \leq 0,$$

so that $u_h \to u$ and

$$\forall v \in X: \;\; \liminf_h \langle T_1(u_h), u_h - v\rangle \geq \langle T_1(u), u - v\rangle.$$

Finally, being

$$\forall v \in X: \;\; \liminf_h \langle T_2(u_h), u_h - v\rangle = \langle T_2(u), u - v\rangle,$$

the thesys follows. ∎

We now come to a finite dimentional existence result which is of fundamental importance for what follows.

(1.7) Lemma *let X be a reflexive finite dimentional Banach space, C a closed, convex, bounded of X, $M \subseteq C \times X'$ a monotone operator and $T : C \to X'$ a pseudomonotone bounded operatore. Then there exists $u \in C$ such that*

$$\forall (v, \varphi) \in M: \;\; \langle \varphi + T(u), v - u\rangle \geq 0.$$

Proof. Let us assume by contradiction that for each $u \in C$ there exists $(v, \varphi) \in M$ such that

$$\langle \varphi + T(u), v - u\rangle < 0.$$

For every $(v,\varphi) \in M$ we set

$$\mathcal{N}_{v,\varphi} := \{u \in C : \ \langle \varphi + T(u), v - u\rangle < 0\}.$$

Evidently we have

$$C = \bigcup_{(v,\varphi)\in M} \mathcal{N}_{v,\varphi}.$$

Let now prove that for every $(v,\varphi) \in M$, $\mathcal{N}_{v,\varphi}$ is an open subset of C. Let (u_h) be a sequence in $C \backslash \mathcal{N}_{v,\varphi}$ such that

$$u_h \to u.$$

Being (u_h) bounded in C and T bounded, $(T(u_h))$ is bounded itself in X' so that we get

$$\lim_h \langle T(u_h), u_h - u\rangle = 0.$$

Since T is pseudomonotone, we deduce that for each $v \in C$

$$\lim_h \inf \langle T(u_h), u_h - v\rangle \geq \langle T(u), u - v\rangle.$$

Therefore, we conclude that

$$\begin{aligned}
\langle \varphi + T(u), v - u\rangle &= \\
&= \langle \varphi, v - u\rangle - \langle T(u), u - v\rangle \geq \\
&\geq \lim_h \langle \varphi, v - u_h\rangle - \lim_h \inf \langle T(u_h), u_h - v\rangle \geq \\
&\geq \lim_h \sup \langle \varphi + T(u_h), v - u_h\rangle \geq 0.
\end{aligned}$$

and so $u \in C\backslash \mathcal{N}_{v,\varphi}$. Since C is a compact subset of X, we deduce that there exist

$$v_1, ..., v_k \in C, \qquad \varphi_1, ..., \varphi_k \in X'$$

such that

$$C = \bigcup_{j=1}^{k} \mathcal{N}_{v_j,\varphi_j}.$$

Let now

$$\{\Theta_j : C \to [0,1] : j = 1, .., k\}$$

be a partition of unity associated to the finite covering

$$\{\mathcal{N}_{v_j,\varphi_j} : j = 1, .., k\}.$$

As it's well known we have

$$\forall u \in \bigcup_{j=1}^{k} \mathcal{N}_{v_j,\varphi_j} : \ \sum_{j=1}^{k} \Theta_j(u) = 1.$$

Let define two continuous mappings

$$\vartheta_1 : C \to C,$$

$$\vartheta_2 : C \to X',$$

by setting

$$\forall u \in C : \ \ \vartheta_1(u) = \sum_{j=1}^{k} \Theta_j(u) v_j \ ;$$

$$\forall u \in C : \ \ \vartheta_2(u) = \sum_{j=1}^{k} \Theta_j(u) \varphi_j \ .$$

From Brower fixed point theorem we get the existence of $\tilde{u} \in C$ such that

$$\vartheta_1(\tilde{u}) = \tilde{u}.$$

On the other hand, if we define

$$\Lambda : C \to \mathbb{R},$$

by setting

$$\forall u \in C : \ \ \Lambda(u) = \langle \vartheta_2(u) + T(u), \vartheta_1(u) - u \rangle,$$

we deduce that

$$\begin{aligned}
\Lambda(u) &= \\
&= \left\langle \sum_{i=1}^{k} \Theta_i(u)(\varphi_i + T(u)), \sum_{j=1}^{k} \Theta_j(v_j - u) \right\rangle = \\
&= \sum_{\substack{i,j=1 \\ i=j}}^{k} \Theta_j(u)^2 \langle \varphi_j + T(u), v_j - u \rangle + \\
&+ \sum_{\substack{i,j=1 \\ i<j}}^{n} \Theta_i(u)\Theta_j(u)(\langle \varphi_i + T(u), v_i - u \rangle + \\
&+ \langle \varphi_j + T(u), v_j - u \rangle - \langle \varphi_i - \varphi_j, v_i - v_j \rangle).
\end{aligned}$$

Since for every $i, j = 1, .., k$ we have

$$\langle \varphi_j + T(u), v_j - u \rangle < 0,$$

$$\langle \varphi_i - \varphi_j, v_i - v_j \rangle > 0,$$

we can conclude that

$$\forall u \in C: \ \Lambda(u) < 0,$$

which is a contradiction, since $\Lambda(\tilde{u}) = 0$. ■

(1.8) Lemma *Let X be a reflexive Banach space, C a cloed, convex and bounded subset of X, $M \subseteq X \times X'$ a monotone operator and $T : C \to X'$ a bounded pseudomonotone operator. Then there exists $u \in C$ such that*

$$\forall (v, \varphi) \in M_{|C}: \ \langle \varphi + T(u), v - u \rangle \geq 0.$$

Proof. Let denote by $\tilde{M}$ the maximal monotone extension of $M_{|C}$. It sufficies to prove that for each $(v, \varphi) \in \tilde{M}$ it is

$$\langle \varphi + T(u), v - u \rangle \geq 0.$$

Let (Y_h) an encreasing sequence of finite dimentional vectorial subspaces of X and for each $h \in \mathbb{N}$ let

$$i_{Y_h} : Y_h \to X$$

be the inclusion mapping associated to Y_h. Clearly we have

$$i_{Y_h} \in \mathcal{L}(Y_h, X), \qquad i^*_{Y_h'} \in \mathcal{L}(X', Y_h').$$

Set now

$$M_{Y_h} = \{(v, i^*_{Y_h'}\varphi): \ (v, \varphi) \in \tilde{M}_{|Y_h}\}, \qquad T_{Y_h} := i^*_{Y_h'} \circ T \circ i_{Y_h}.$$

Then, $M_{Y_h} \subseteq (Y_h \cap C) \times Y_h'$ is monotone and T_{Y_h} is pseudo-monotone and bounded over $Y_h \cap C$ to Y_h'. From previous Lemma we deduce that for each $h \in \mathbb{N}$ there exists $u_h \in Y_h \cap C$ such that

$$\forall (v, \varphi) \in M_{|Y_h}: \ \langle i^*_{Y_h'}\varphi + T_{Y_h}(u_h), v - u_h \rangle \geq 0.$$

Since (u_h) is a bounded subsequence of C, we get that $(T(u_h))$ is bounded in Y_h'. For the weak compactness theorem, there exist a subsequence (u_{h_k}) of (u_h), $u \in C$ and $\Lambda \in X'$ such that

$$u_{h_k} \rightharpoonup u, \qquad T(u_{h_k}) \overset{*}{\rightharpoonup} \Lambda.$$

For each $(v, \varphi) \in \tilde{M}$ and $k \in \mathbb{N}$ we get

$$\begin{aligned} \langle T(u_{h_k}), u_{h_k} - u \rangle &= \\ &= \langle T(u_{h_k}), u_{h_k} - v \rangle + \langle T(u_{h_k}), v - u \rangle \leq \\ &\leq \langle \varphi, v - u_{h_k} \rangle + \langle T(u_{h_k}), v - u \rangle \leq \\ &\leq \langle \varphi, v \rangle - \langle \varphi, u_{h_k} \rangle + \langle T(u_{h_k}), v - u \rangle. \end{aligned}$$

so that

$$\limsup_k \langle T(u_{h_k}), u_{h_k} - u\rangle \le \langle \varphi + \Lambda, v - u\rangle,$$

and whence

$$\limsup_k \langle T(u_{h_k}), u_{h_k} - u\rangle \le \inf_{(v,\varphi)\in\tilde{M}} \langle \varphi + \Lambda, v - u\rangle.$$

If it was

$$\inf_{(v,\varphi)\in\tilde{M}} \langle \varphi + \Lambda, v - u\rangle > 0,$$

we'd have $(u, -\varphi) \in \tilde{M}$ being $\tilde{M}$ maximal monotone, a contradiction. Since T is pseudomonotone, we have

$$\forall v \in C: \ \liminf_k \langle T(u_{h_k}), u_{h_k} - v\rangle \ge \langle T(u), u - v\rangle.$$

Therefore, taking into account that eventually for $k \to +\infty$ it is $v \in Y_{h_k} \cap C$, we conclude that

$$\begin{aligned} \langle \varphi, v - u\rangle &= \\ &= \lim_k \langle \varphi, v - u_{h_k}\rangle \ge \\ &\ge \liminf_k \langle T(u_{h_k}), u_{h_k} - v\rangle \ge \\ &\ge \langle T(u), u - v\rangle, \end{aligned}$$

from which the thesis follows. ■

We can now state and prove the main result of this section.

(1.9) Theorem *Let X be a reflexive Banach space, C a close, convex subset of X, $M \subseteq X \times X'$ a monotone operator and $T : C \to X'$ a bounded pseudomonotone operator. Let us assume that there exists $(u_0, \varphi_0) \in M$ such that*

$$\lim_{\substack{\|u\|\to+\infty \\ u\in C}} \langle T(u) + \varphi_0, u - u_0\rangle = +\infty.$$

Then there exists $u \in C$ such that

$$\forall (v, \varphi) \in M: \ \langle \varphi + T(u), v - u\rangle \ge 0.$$

Proof. Denote again by $\tilde{M}$ the maximal monotone extension of $M_{|C}$. For each $h \in \mathbb{N}$ the set

$$C_h := \{u \in C : \| u \| \le h\}$$

is closed, convex and bounded and $T_h = T_{|C_h}$ is a bounded pseudomonotone operator over C_h to X'. By the previous Lemma, for every $h \in \mathbb{N}$ there exists $u_h \in C_h$ such that

$$\forall (v, \varphi) \in \tilde{M}_{|C_h}: \ \langle \varphi + T_h(u_h), v - u_h\rangle \ge 0.$$

Since $(u_0, \varphi_0) \in M_{|_C}$, we have

$$\forall h \in \mathbb{N} : \parallel u_0 \parallel \leq h \Longrightarrow \langle \varphi_0 + T(u_h), u_0 - u_h \rangle \geq 0.$$

Whence, since by assumption it is

$$\lim_{\substack{\|u\| \to +\infty \\ u \in C}} \langle T(u) + \varphi_0, u - u_0 \rangle = +\infty,$$

we deduce that (u_h) is a bounded sequence of C and being T bounded, $(T(u_h))$ is bounded in X' too. By the weak and weak * compactness Theorems we get a subsequence (u_{h_k}) of (u_h), $\hat{u} \in C$ e $\hat{\varphi} \in X'$ such that

$$u_{h_k} \rightharpoonup \hat{u}, \qquad T(u_{h_k}) \overset{*}{\rightharpoonup} \hat{\varphi}.$$

Let now $(v, \varphi) \in \tilde{M}$. For every $k \in \mathbb{N}$ we have

$$\parallel v \parallel \leq h_k \Longrightarrow \langle \varphi + T(u_{h_k}), v - u_{h_k} \rangle \geq 0.$$

Therefore, we obtain

$$\begin{aligned} \limsup_k \langle T(u_{h_k}), u_{h_k} - \hat{u} \rangle &= \\ &= \limsup \langle T(u_{h_k}), u_{h_k} - v \rangle + \lim_k \langle T(u_{h_k}), v - \hat{u} \rangle \leq \\ &\leq \lim_k \{ \langle \varphi, v - u_{h_k} \rangle + \langle T(u_{h_k}), v - \hat{u} \rangle \} = \\ &= \langle \varphi + \hat{\varphi}, v - \hat{u} \rangle . \end{aligned}$$

Then it follows that

$$\limsup_k \langle T(u_{h_k}), u_{h_k} - \hat{u} \rangle \leq \inf_{(v,\varphi) \in \tilde{M}} \langle \varphi + \hat{\varphi}, v - \hat{u} \rangle.$$

Like in the previous Lemma we have that

$$\inf_{(v,\varphi) \in \tilde{M}} \langle \varphi + \hat{\varphi}, v - \hat{u} \rangle \leq 0,$$

so that

$$\limsup_k \langle T(u_{h_k}), u_{h_k} - \hat{u} \rangle \leq 0.$$

Since T is a pseudomonotone operator, we deduce that

$$\langle T(u), u - v \rangle \leq \liminf_k \langle T(u_{h_k}), u_{h_k} - v \rangle \leq \langle \varphi, v - u \rangle,$$

which is the Theorem. ■

(1.10) Corollary *Let X a reflexive Banach space, $M \subseteq X \times X'$ a maximal monotone operator and $T : X \to X'$ a bounded pseudomonotone operator. Let us assume that there exists $(u_0, \varphi_0) \in M$ such that*

$$\lim_{\|u\| \to +\infty} \langle T(u) + \varphi_0, u - u_0 \rangle = +\infty.$$

Then there exists $u \in \mathcal{D}(M)$ such that

$$(M + T)u \ni 0.$$

Proof. If $C = X$, by Theorem (1.9) there's $u \in X$ such that

$$\forall (v, \varphi) \in M : \ \langle \varphi + T(u), v - u \rangle \geq 0.$$

It then follows that

$$(u, -T(u)) \in M,$$

and the proof is over. ■

(1.11) Corollary *Let X be a reflexive Banach space, $M \subseteq X \times X'$ a maximal monotone and $T : X \to X'$ a bounded pseudomonotone operator. Let us assume that there exist $(u_0, \varphi_0) \in M$ such that*

$$\lim_{\|u\| \to +\infty} \frac{\langle T(u) + \varphi_0, u - u_0 \rangle}{\| u \|} = +\infty.$$

Then, for each $\varphi \in X'$ there exists $u \in \mathcal{D}(M)$ such that

$$(M + T)u \ni \varphi.$$

Proof. Set $\hat{T}u = Tu - \varphi$, with $\varphi \in X'$. Then, $\hat{T}$ is a pseudomonotone bounded operator over X to X'. Since

$$\begin{aligned}
\lim_{\|u\| \to +\infty} \langle \hat{T}u + \varphi_0, u - u_0 \rangle \ &= \\
&= \lim_{\|u\| \to +\infty} \langle Tu - \varphi + \varphi_0, u - u_0 \rangle = \\
&= \lim_{\|u\| \to +\infty} \| u \| \left\{ \frac{\langle T(u) + \varphi_0, u - u_0 \rangle}{\| u \|} - \frac{\langle \varphi, u - u_0 \rangle}{\| u \|} \right\} = +\infty\,.
\end{aligned}$$

By the previous Corollary we get $u \in \mathcal{D}(M)$ such that

$$(M + \hat{T})u \ni 0,$$

so that the thesis is proven. ■

The following result will allow us to get a weak solution of our parabolic problem.

(1.12) Corollary *Let X be a reflexive Banach space, $L \subseteq X \times X'$ a linear maximal monotone densily defined operator and $T : X \to X'$ a bounded pseudomonotone operator. Let us assume that there's $u_0 \in \mathcal{D}(L)$ such that*

$$\lim_{\|u\| \to +\infty} \frac{\langle T(u) + Lu_0, u - u_0 \rangle}{\| u \|} = +\infty.$$

Then for every $\varphi \in X'$ there exists $u \in \mathcal{D}(L)$ such that

$$(L + T)u \ni \varphi.$$

Proof. It's just a trivial conseguence of the previous Corollary. ∎

(1.13) Corollary [**Minty**] *Let X be a Hilbert and $A \subseteq X \times X$. Then A is a maximal monotone operator if and only if there exists $\lambda > 0$ such that*

$$\mathcal{R}(\lambda I + A) = X.$$

Proof. If A is maximal monotone, it sufficies to choose $M = A$, $T = I$, $\lambda = 1$ and $(u_0, \eta_0) \in A$ in Corollary (1.11). The viceversa can be proven easily. ∎

2 A maximal monotonocity sufficient condition

(2.1) Definition *We say that a function* $\jmath : [0, +\infty[\to \mathbb{R}$ *is a* gauge function *if* $\jmath$ *è strictly increasing,* $\jmath(0) = 0$ *and*

$$\lim_{\varrho \to +\infty} \jmath(\varrho) = +\infty.$$

(2.2) Definition *Let* X *be a Banach space. We say that* X *is strictly convex if*

$$\forall u, v \in X, \ \ \forall \lambda \in]0, 1[: \ \| u \| = \| v \| \Longrightarrow \| (1 - \lambda) u + \lambda v \| < \| u \| .$$

(2.3) Definition *Let* X *be a Banach and* $T \subseteq X \times X'$. *We say that* T *is a* duality map *from* X *to* X' *with gauge function* $\jmath$, *if*

$$T(u) = \left\{ \varphi \in X' : \ \ \langle \varphi, u \rangle = \| \varphi \| \| u \|, \| \varphi \| = \jmath(\| u \|) \right\}$$

(2.4) Observation *It's easy to verify that* T *is well defined and monotone over* X. *Moreover, if* X *and* X' *are strictly convex, you can prove that* T *is single-valued e iniective.*

(2.5) Lemma *Let* X *a reflexive Banach space such that* X *and* X' *are strictly convex,* $M \subseteq X \times X'$ *a monotone operator and* T *a duality mapping from* X *to* X'. *Assume that*

$$\mathcal{R}(M + T) = X'.$$

Then M *is maximal monotone.*

Proof. For the proof, see [1, Proposition 6.4.1]. ∎

(2.6) Theorem *Let* X *a reflexive Banach space with* X *and* X' *strictly convex and* L *a closed, monotone, linear, from* X *to* X'. *Assume that* L^* *is monotone. Then* L *is maximal monotone.*

Proof. By the previous Lemma, it sufficies to prove that

$$\mathcal{R}(L + T) = X',$$

where T is a duality map with gauge function $\jmath(\rho) = \rho$. Clearly $\mathcal{D}(L)$ is a reflexive Banach space if endowed with the grafic norm. Given $\varphi \in X'$ and $\varepsilon > 0$, for each $u, v \in \mathcal{D}(L)$ set

$$\langle \Phi_\varepsilon u, v \rangle := \varepsilon \langle Lv, T^{-1}(Lu) \rangle + \langle Lu, v \rangle + \langle T(u), v \rangle - \langle \varphi, v \rangle.$$

Then,

$$\Phi_\varepsilon : \mathcal{D}(L) \to \mathcal{D}(L)'$$

is an emicontinuous and monotone map. Moreover, since we have

$$\frac{\langle \Phi_\varepsilon u, u\rangle}{\| u \|_{\mathcal{D}(L)}} \geq \frac{1}{\| u \|_{\mathcal{D}(L)}} \left\{ \varepsilon \| Lu \|^2 + \| u \|^2 - \| \varphi \| \| u \| \right\},$$

we deduce that

$$\lim_{\|u\|_{\mathcal{D}(L)}} \frac{\langle \Phi_\varepsilon u, u\rangle}{\| u \|_{\mathcal{D}(L)}} = +\infty.$$

It then follows by Corollary (1.12) that

$$\mathcal{R}(\Phi_\varepsilon) = \mathcal{D}(L)'.$$

Therefore, for each $\varepsilon > 0$, there is $u_\varepsilon \in \mathcal{D}(L)$ such that $\Phi_\varepsilon u_\varepsilon = 0$. Since for each $v \in \mathcal{D}(L)$ we have

$$\varepsilon\langle Lv, T^{-1}(Lu_\varepsilon)\rangle + \langle Lu_\varepsilon, v\rangle + \langle T(u_\varepsilon), v\rangle = \langle \varphi, v\rangle,$$

for each $\varepsilon > 0$ it is $\| u_\varepsilon \| \leq \| \varphi \|$. Moreover, since

$$\varepsilon L^* T^{-1}(Lu_\varepsilon) + Lu_\varepsilon + T(u_\varepsilon) = \varphi,$$

we deduce that

$$\varepsilon\langle L^*T^{-1}(Lu_\varepsilon), T^{-1}(Lu_\varepsilon)\rangle + \langle Lu_\varepsilon, T^{-1}(Lu_\varepsilon)\rangle + \langle T(u_\varepsilon), T^{-1}(Lu_\varepsilon)\rangle = \langle \varphi, T^{-1}(Lu_\varepsilon)\rangle$$

Since by assumption

$$\varepsilon\langle L^*T^{-1}(Lu_\varepsilon), T^{-1}(Lu_\varepsilon)\rangle \geq 0,$$

we deduce that for each $\varepsilon > 0$ we have

$$\| Lu_\varepsilon \| \leq 2 \| \varphi \| .$$

It follows that (u_ε) and (Lu_ε) both are bounded sequences, and whence by weak compactness Theorems we have

$$u_\varepsilon \rightharpoonup u, \qquad Lu_\varepsilon \overset{*}{\rightharpoonup} Lu.$$

Set now $v = u_\varepsilon - u$, we have

$$\begin{aligned} \langle Tu_\varepsilon, u_\varepsilon - u\rangle &= \langle \varphi, u_\varepsilon - u\rangle - \varepsilon\langle L(u_\varepsilon - u), T^{-1}Lu_\varepsilon\rangle - \langle Lu_\varepsilon, u_\varepsilon - u\rangle \leq \\ &\leq \langle \varphi, u_\varepsilon - u\rangle - \varepsilon\langle L(u_\varepsilon - u), T^{-1}Lu_\varepsilon\rangle - \langle Lu, u_\varepsilon - u\rangle, \end{aligned}$$

and so

$$\limsup_{\varepsilon\to 0}\langle T(u_\varepsilon), u_\varepsilon - u\rangle \le 0.$$

Since T is pseudomonotone operator, we get

$$\forall v \in X: \ \liminf_{\varepsilon\to 0}\langle T(u_\varepsilon), u_\varepsilon - v\rangle \ge \langle T(u), u - v\rangle.$$

Let us assume that $T(u_\varepsilon)$ weakly * converges to Λ. Therefore, for each $v \in X$ we get

$$\begin{aligned}
0 &\ge \limsup_{\varepsilon\to 0}\langle T(u_\varepsilon), u_\varepsilon - u\rangle \ge \\
&\ge \limsup_{\varepsilon\to 0}\{\langle T(u_\varepsilon) - T(v), u_\varepsilon - v\rangle + \langle T(v), u_\varepsilon - v\rangle + \langle T(u_\varepsilon), v - u\rangle\} \ge \\
&\ge \limsup_{\varepsilon\to 0}\{\langle T(v), u_\varepsilon - v\rangle + \langle T(u_\varepsilon), v - u\rangle\} = \\
&= \langle \Lambda - T(v), v - u\rangle.
\end{aligned}$$

Then we conclude that

$$\forall v \in X: \ \langle \Lambda - T(v), v - u\rangle \le 0,$$

and su we have $\Lambda = T(u)$ and

$$\forall v \in X: \ \langle Lu + T(u), v\rangle = \langle \varphi, v\rangle.$$

The Theorem is now proven. ■

Chapter II

Nonlinear Parabolic Systems

1 Existence of weak solutions

Let Ω be an open bounded subset of $\mathbb{R}^n$, $N \geq 1$, $T > 0$, for every $i, j = 1, .., n$ e $h, k = 1, .., N$

$$a_{ij}^{hk} : \Omega \times \mathbb{R}^N \times \mathbb{R}^{nN} \to \mathbb{R},$$

continuous function and let

$$g : \Omega \times \mathbb{R}^N \times \mathbb{R}^{nN} \to \mathbb{R}^N,$$

be a Carathéodory function such that for each $\varepsilon > 0$ there is $a_\varepsilon \in L^{\frac{2n}{n+2}}(\Omega)$ such that

$$|g(x, \xi, \eta)| \leq a_\varepsilon(x) + \varepsilon |\xi|^{\frac{(n+2)}{(n-2)}} + \varepsilon |\eta|^{\frac{(n+2)}{n}}$$

for q.o. $x \in \Omega$ and for each $(\xi, \eta) \in \mathbb{R}^N \times \mathbb{R}^{nN}$. In this section, by using the abstract framework of the previous section, we want to obtain an existence result for parabolic nonlinear systems in divergence form of the type

$$\begin{cases} D_t u_h - \sum\limits_{k=1}^{N} \sum\limits_{i,j=1}^{n} D_i(a_{ij}^{hk}(x, u, Du) D_j u_k) = g_h(x, u, Du) & \text{in }]0, T[\times \Omega \quad (h = 1, .., N) \\ u(t, x) = 0 & \text{on }]0, T[\times \partial\Omega \\ u(0, x) = 0 & \text{in } \Omega \end{cases}$$

Assume that there exist $\nu > 0$ and $C \in [0, \nu]$ such that for every $x \in \Omega$, $\xi \in \mathbb{R}^N$ and $\eta \in \mathbb{R}^{nN}$ we have

$$|a_{ij}^{hk}(x, \xi, \eta)| \leq C, \qquad \sum_{h,k=1}^{N} \sum_{i,j=1}^{n} a_{ij}^{hk}(x, \xi, \eta) \eta_h^i \eta_k^j \geq \nu |\eta|^2,$$

Define the operators

$$A_1, A_2 : L^2\left(0, T; H_0^1\left(\Omega; \mathbb{R}^N\right)\right) \to L^2\left(0, T; H^{-1}\left(\Omega; \mathbb{R}^N\right)\right),$$

by setting for every $u, v \in L^2\left(0,T; H_0^1\left(\Omega;\mathbb{R}^N\right)\right)$

$$\langle A_1(u), v\rangle = \int_0^T \int_\Omega \sum_{h,k=1}^N \sum_{i,j=1}^n a_{ij}^{hk}(x,u,Du) D_i u_h D_j v_k \, dx\, dt,$$

$$\langle A_2(u), v\rangle = -\int_0^T \int_\Omega g(x,u,Du)\cdot v \, dx\, dt.$$

Let moreover L the operator defined setting

$$\mathcal{D}(L) = \{u \in L^2\left(0,T; H_0^1\left(\Omega;\mathbb{R}^N\right)\right): \ D_t u \in L^2\left(0,T; H^{-1}\left(\Omega;\mathbb{R}^N\right)\right), \ u(0,\cdot)=0\},$$

$$\forall u \in \mathcal{D}(L): \ Lu = D_t u.$$

(1.1) Definition *We say that a function u is a* weak solution *of the parabolic problem*

$$\begin{cases} D_t u_h - \sum_{k=1}^N \sum_{i,j=1}^n D_i\left(a_{ij}^{hk}(x,u,Du) D_j u_k\right) + g_h(x,u,Du) = 0 & \text{in }]0,T[\times\Omega \quad (h=1,..,N) \\ u(t,x) = 0 & \text{on }]0,T[\times\partial\Omega \\ u(0,x) = 0 & \text{in } \Omega \end{cases}$$

if $u \in L^2\left(0,T; H_0^1\left(\Omega;\mathbb{R}^N\right)\right)$ and

$$\forall v \in L^2\left(0,T; H_0^1\left(\Omega;\mathbb{R}^N\right)\right): \ \langle (L + A_1 + A_2)(u), v\rangle = 0.$$

(1.2) Definition *We say that the operator A_1 satisfies the $(\mathcal{S})_+$ condition, if the following condition holds*

$$\begin{cases} u^{(m)} \rightharpoonup u \\ \limsup_h \langle A_1(u^{(m)}) - A_1(u), u^{(m)} - u\rangle \le 0 \end{cases} \Longrightarrow u^{(m)} \to u.$$

We now prove a basic compactness result for Nemyckii operator.

(1.3) Theorem *Let us assume that for each $\varepsilon > 0$ there exists $a_\varepsilon \in L^{\frac{2n}{n+2}}(\Omega)$ such that*

$$|g(x,\xi,\eta)| \le a_\varepsilon(x) + \varepsilon|\xi|^{\frac{(n+2)}{(n-2)}} + \varepsilon|\eta|^{\frac{(n+2)}{n}}$$

for q.o. $x \in \Omega$ and for each $(\xi,\eta) \in \mathbb{R}^N \times \mathbb{R}^{nN}$. Then the application

$$\begin{array}{ccc} H_0^1(\Omega;\mathbb{R}^N) & \longrightarrow & H^{-1}(\Omega;\mathbb{R}^N) \\ u & \longmapsto & g(x,u,Du) \end{array}$$

is completely continuous.

Proof. It's known that g is continuous. Assume that there exist $R > 0$ and $a \in L^\infty(\Omega)$ such that

$$|g(x,\xi,\eta)| \le a(x) \quad \text{for q.o. } x \in \Omega \text{ and every } (\xi,\eta) \in \mathbb{R}^N \times \mathbb{R}^{nN},$$

$$a(x) = 0 \quad \text{for q.o. } x \in \Omega \setminus \mathrm{B}_R(0).$$

If (u_h) is a bounded sequence in $H_0^1(\Omega;\mathbb{R}^N)$, we have that $(g(x,u_h,Du_h))$ is bounded in $L^2(\Omega;\mathbb{R}^N)$ and $g(x,u_h,Du_h) = 0$ q.o. in $\Omega \setminus \mathrm{B}_R(0)$. Let $\vartheta \in C_c^\infty(\mathrm{B}_{R+1}(0))$ with $\vartheta(x) = 1$ om $\overline{\mathrm{B}_R(0)}$. By Rellich Theorem, up toa further subsequence, $(g(x,u_h,Du_h))$ is a Cauchy sequence in $H^{-1}(\Omega \cap \mathrm{B}_{R+1}(0);\mathbb{R}^N)$. For each $v \in H_0^1(\Omega;\mathbb{R}^N)$ we have

$$\left| \int_\Omega (g(x,u_h,Du_h) - g(x,u_k,Du_k)) \cdot v \, dx \right| =$$

$$= \left| \int_{\Omega \cap \mathrm{B}_{R+1}(0)} (g(x,u_h,Du_h) - g(x,u_k,Du_k)) \cdot \vartheta \, v \, dx \right| \le$$

$$\le \|g(x,u_h,Du_h) - g(x,u_k,Du_k)\|_{H^{-1}(\Omega \cap \mathrm{B}_{R+1}(0);\mathbb{R}^N)} \, \|\vartheta v\|_{H_0^1(\Omega \cap \mathrm{B}_{R+1}(0);\mathbb{R}^N)} \le$$

$$\le c_\vartheta \, \|g(x,u_h,Du_h) - g(x,u_k,Du_k)\|_{H^{-1}(\Omega \cap \mathrm{B}_{R+1}(0);\mathbb{R}^N)} \, \|v\|_{H_0^1(\Omega;\mathbb{R}^N)} .$$

It follows that

$$\|g(x,u_h,Du_h) - g(x,u_k,Du_k)\|_{H^{-1}(\Omega;\mathbb{R}^N)} \le$$

$$\le c_\vartheta \, \|g(x,u_h,Du_h) - g(x,u_k,Du_k)\|_{H^{-1}(\Omega \cap \mathrm{B}_{R+1}(0);\mathbb{R}^N)} ,$$

so that $(g(x,u_h,Du_h))$ is Cauchy, and then convergent, in $H^{-1}(\Omega;\mathbb{R}^N)$. Assume now that there is $a \in L^{\frac{2n}{n+2}}(\Omega;\mathbb{R}^N)$ such that

$$|g(x,\xi,\eta)| \le a(x) \quad \text{for q.o. } x \in \Omega \text{ and every } (\xi,\eta) \in \mathbb{R}^N \times \mathbb{R}^{nN}.$$

Set

$$g_h(x,\xi,\eta) = \begin{cases} g(x,\xi,\eta) & \text{if } a(x) \le h \text{ and } |x| < h, \\ 0 & \text{elsewhere,} \end{cases}$$

$$a_h(x) = \begin{cases} a(x) & \text{if } a(x) \le h \text{ and } |x| < h, \\ 0 & \text{elsewhere.} \end{cases}$$

Clearly g_h is a Carathéodory function and

$$|g(x,\xi,\eta)| \le a_h(x) \quad \text{for q.o. } x \in \Omega \text{ and every } (\xi,\eta) \in \mathbb{R}^N \times \mathbb{R}^{nN}.$$

Then by the previous step the mappping

$$\{u \longmapsto g_h(x,u,Du)\}$$

is completely continuous over $H_0^1(\Omega;\mathbb{R}^N)$ to $H^{-1}(\Omega;\mathbb{R}^N)$. On the other hand for each $u \in H_0^1(\Omega;\mathbb{R}^N)$ we have

$$\int_\Omega |g(x,u,Du) - g_h(x,u,Du)|^{\frac{2n}{n+2}}\, dx \le \int_\Omega |a|^{\frac{2n}{n+2}} \left(1 - \chi_{\{x:\, a(x)\le h\}}\, \chi_{\mathrm{B}_h(0)}\right) dx\,.$$

By dominated convergence Theorem we deduce that

$$\lim_h \|g_h(x,u,Du) - g(x,u,Du)\|_{\frac{2n}{n+2}} = 0$$

uniformly on $H_0^1(\Omega;\mathbb{R}^N)$. By Sobolev Theorem, it follows

$$\lim_h \|g_h(x,u,Du) - g(x,u,Du)\|_{-1,2} = 0$$

uniformly on $H_0^1(\Omega;\mathbb{R}^N)$. As it is well knowno, this sufficies to conclude that g is completely continuous. Let us come to the general case. Set $\varepsilon = \frac{1}{h}$, for every $h \ge 1$ let $a_h \in L^{\frac{2n}{n+2}}(\Omega)$ such that

$$|g(x,\xi,\eta)| \le a_h(x) + \frac{1}{h}|\xi|^{\frac{(n+2)}{(n-2)}} + \frac{1}{h}|\eta|^{\frac{(n+2)}{2n}}$$

for q.o. $x \in \Omega$ and every $(\xi,\eta) \in \mathbb{R}^N \times \mathbb{R}^{nN}$. Set

$$g_h(x,\xi,\eta) = \min\left\{\max\left\{g(x,\xi,\eta), -a_h(x)\right\}, a_h(x)\right\}\,.$$

Obviously g_h is Carathéodory and

$$|g_h(x,\xi,\eta)| \le a_h(x) \quad \text{for q.o. } x \in \Omega \text{ and every } (\xi,\eta) \in \mathbb{R}^N \times \mathbb{R}^{nN}.$$

By the previous step the map

$$\{u \longmapsto g_h(x,u,Du)\}$$

is completely continuous over $H_0^1(\Omega;\mathbb{R}^N)$ to $H^{-1}(\Omega;\mathbb{R}^N)$. On the other hand it's easily to verify that

$$|g(x,\xi,\eta) - g_h(x,\xi,\eta)| \le \frac{1}{h}|\xi|^{\frac{(n+2)}{(n-2)}} + \frac{1}{h}|\eta|^{\frac{(n+2)}{2n}}$$

for q.o. $x \in \Omega$ and each $(\xi,\eta) \in \mathbb{R}^N \times \mathbb{R}^{nN}$, and so, for every $u \in H_0^1(\Omega;\mathbb{R}^N)$, we have

$$\int_\Omega |g(x,u,Du) - g_h(x,u,Du)|^{\frac{2n}{n+2}}\, dx \le$$

$$\le 2^{\frac{n-2}{n+2}} \left(\frac{1}{h}\right)^{\frac{2n}{n+2}} \left(\int_\Omega |u|^{\frac{2n}{n-2}}\, dx + \int_\Omega |Du|^2\, dx\right)\,.$$

Therefore we get

$$\lim_h \|g_h(x,u,Du) - g(x,u,Du)\|_{\frac{2n}{n+2}} = 0$$

uniformly on the bounded sets of $H_0^1(\Omega;\mathbb{R}^N)$. In particular we have

$$\lim_h \|g_h(x,u,Du) - g(x,u,Du)\|_{-1,2} = 0$$

uniformely on bounded sets of $H_0^1(\Omega;\mathbb{R}^N)$, and the proof's now complete. ■

(1.4) Lemma *If A_1 is a monotone operator, then A_1 is a pseudo-monotone operator.*

Proof. First of all, let us prove that A_1 is emicontinuous. Evidently, since for each i, j, h, k

$$a_{ij}^{hk} \in C(\Omega \times \mathbb{R}^n \times \mathbb{R}^N;\mathbb{R}),$$

for every $u, v \in L^2\left(0,T;H_0^1\left(\Omega;\mathbb{R}^N\right)\right)$ and each $(t,x) \in [0,T]\times\Omega$, the maps

$$\{\lambda \longmapsto a_{ij}^{hk}(x, \lambda u(t,x) + (1-\lambda)v(t,x), \lambda Du(t,x) + (1-\lambda)Dv(t,x))\}$$

are continuous from $[0,T]$ to $\mathbb{R}$. Let now $\lambda_0 \in [0,T]$ and (λ_m) a sequence in $[0,T]$ such that

$$\lambda_m \to \lambda_0.$$

Clearly, for every $(t,x) \in [0,T]\times\Omega$ and $\zeta \in L^2(0,T;H_0^1)$ we have

$$\lim_m a_{ij}^{hk}(x, \lambda_m u(t,x) + (1-\lambda_m)v(t,x), \lambda_m Du(t,x) + (1-\lambda_m)Dv(t,x)) =$$

$$= a_{ij}^{hk}(x, \lambda_0 u(t,x) + (1-\lambda_0)v(t,x), \lambda_0 Du(t,x) + (1-\lambda_0)Dv(t,x)) \ ;$$

$$\lim_m D_i(\lambda_m u_h(t,x) + (1-\lambda_m)v_h(t,x))D_j\zeta_k(t,x) = D_i(\lambda_0 u_h(t,x) + (1-\lambda_0)v_h(t,x))D_j\zeta_k(t,x) \ ;$$

Moreover, for each $m \in \mathbb{N}$ and $\zeta \in L^2(0,T;H_0^1)$, we have

$$\left|\sum_{i,j=1}^n \sum_{h,k=1}^N a_{ij}^{hk}(x, \lambda_m u + (1-\lambda_m)v, \lambda_m Du + (1-\lambda_m)Dv) D_i(\lambda_m u_h + (1-\lambda_m)v_h)D_j\zeta_k\right| \le$$

$$\le C\sum_{i,j=1}^n \sum_{h,k=1}^N \max\{|D_i u_h||D_j\zeta_k|, |D_i v_h||D_j\zeta_k|\} \in L^1([0,T]\times\Omega) \ ;$$

By the dominated convergence Theorem it follows that for each $\zeta \in L^2\left(0,T;H_0^1\left(\Omega;\mathbb{R}^N\right)\right)$

$$\lim_m \int_0^T \int_\Omega \left\{ \sum_{h,k=1}^N \sum_{i,j=1}^n a_{ij}^{hk}(x, \lambda_m u + (1-\lambda_m)v, \lambda_m Du + \right.$$

$$+(1-\lambda_m)Dv)D_i(\lambda_m u_h + (1-\lambda_m)v_h)D_j\zeta_k\Big\}\, dx\, dt =$$

$$= \int_0^T \int_\Omega \sum_{h,k=1}^N \sum_{i,j=1}^n a_{ij}^{hk}(x, \lambda_0 u + (1-\lambda_0)v, \lambda_0 Du + (1-\lambda_0)Dv)D_i(\lambda_0 u_h + (1-\lambda_0)v_h)D_j\zeta_k\, dx\, dt \ \ ;$$

Then, for each $u, v \in L^2\left(0,T; H_0^1\left(\Omega;\mathbb{R}^N\right)\right)$ it is

$$\forall \zeta \in L^2\left(0,T; H_0^1\left(\Omega;\mathbb{R}^N\right)\right):\ \lim_m \langle A_1(\lambda_m u + (1-\lambda_m)v), \zeta\rangle = \langle A_1(\lambda_0 u + (1-\lambda_0)v), \zeta\rangle.$$

Therefore we conclude that for each $u, v \in L^2\left(0,T; H_0^1\left(\Omega;\mathbb{R}^N\right)\right)$ the map

$$\{\lambda \to A_1(\lambda u + (1-\lambda)v)\}$$

is continuous from $[0,T]$ to $L^2\left(0,T; H^{-1}\left(\Omega;\mathbb{R}^N\right)\right)$, with $L^2\left(0,T; H^{-1}\left(\Omega;\mathbb{R}^N\right)\right)$ endowed with the weak * topology. Let us now assume that A_1 is monotone and let (u_h) a sequence weakly convergent to u in $L^2\left(0,T; H_0^1\left(\Omega;\mathbb{R}^N\right)\right)$ such that

$$\limsup_h \langle A_1(u_h), u_h - u\rangle \le 0.$$

Being A_1 monotone, for every $h \in \mathbb{N}$ we have

$$\langle A_1(u_h), u_h - u\rangle \ge \langle A_1(u), u_h - u\rangle.$$

It follows that

$$\liminf_h \langle A_1(u_h), u_h - u\rangle \ge \lim_h \langle A_1(u), u_h - u\rangle = 0.$$

Therefore it is

$$\lim_h \langle A_1(u_h), u_h - u\rangle = 0.$$

If now $v \in L^2\left(0,T; H_0^1\left(\Omega;\mathbb{R}^N\right)\right)$, by monotonocity we have

$$\liminf_h \langle A_1(u_h), u_h - v\rangle \ge \lim_h \langle A_1(v), u_h - v\rangle = \langle A_1(v), u - v\rangle.$$

For every $v \in L^2\left(0,T; H_0^1\left(\Omega;\mathbb{R}^N\right)\right)$ and each $\vartheta \in]0,1[$, set

$$\eta = \vartheta v + (1-\vartheta)u.$$

We have

$$\forall \vartheta \in]0,1[:\ \liminf_h \langle A_1(u_h), u_h - u + \vartheta(u-v)\rangle \ge \langle A_1(\eta), \vartheta(u-v)\rangle,$$

and so we get

$$\liminf_h \langle A_1(u_h), u - v\rangle \ge \langle A_1(\eta), u - v\rangle.$$

Taking the limit $\vartheta \to 0^+$, by the previous step we get that

$$\liminf_h \langle A_1(u_h), u - v\rangle \geq \langle A_1(u), u - v\rangle.$$

Finally, by putting together the inequalities obtained we conclude that

$$\forall v \in L^2\left(0, T; H_0^1\left(\Omega; \mathbb{R}^N\right)\right): \ \liminf_h \langle A_1(u_h), u_h - v\rangle =$$

$$= \liminf_h \{\langle A_1(u_h), u_h - u\rangle + \langle A_1(u_h), u - v\rangle\} \geq \langle A_1(u), u - v\rangle,$$

and the thesis follows. ∎

Let us observe that $L^2(0, T; H_0^1)$ and it's dual are strictly convex Banach spaces.

(1.5) Lemma *L is a maximal monotone operator.*

Proof. From [1, Lemma 5.5.1], we have that

$$\forall u \in L^2\left(0, T; H_0^1\left(\Omega; \mathbb{R}^N\right)\right): \ \ u \in \mathcal{D}(L) \Longrightarrow u \in C([0, T]; L^2(\Omega; \mathbb{R}^N)).$$

Therefore $u(0)$ is well defined as an element of $L^2(\Omega; \mathbb{R}^N)$. It's easy to verify that

$$L : L^2\left(0, T; H_0^1\left(\Omega; \mathbb{R}^N\right)\right) \to L^2\left(0, T; H^{-1}\left(\Omega; \mathbb{R}^N\right)\right),$$

is a linear and closed and $\mathcal{D}(L)$ is dense in $L^2\left(0, T; H_0^1\left(\Omega; \mathbb{R}^N\right)\right)$. Observe moreover that

$$\forall u \in \mathcal{D}(L): \ \langle Lu, u\rangle = \langle D_t u, u\rangle = \int_0^T \int_\Omega D_t u u \, dx \, dt =$$

$$= \frac{1}{2}\int_0^T \int_\Omega D_t |u|^2 \, dx \, dt = \frac{1}{2}\int_\Omega |u(T, x)|^2 \, dx = \frac{1}{2} \parallel u(T, \cdot) \parallel_2^2 \geq 0.$$

Finally, the adjoint of L is given by

$$\mathcal{D}(L^*) = \{u \in L^2\left(0, T; H_0^1\left(\Omega; \mathbb{R}^N\right)\right) : D_t u \in L^2\left(0, T; H^{-1}\left(\Omega; \mathbb{R}^N\right)\right), \ \ u(T, \cdot) = 0\},$$

$$\forall u \in \mathcal{D}(L^*): \ \ L^* u = -D_t u,$$

Since L^* is maximal monotone, by Theorem (2.6) L is maximal monotone. ∎

We finally come to the announced existence result for solutions to nonlinear parabolic systems in divergence form of the type

$$\begin{cases} D_t u_h - \sum\limits_{k=1}^N \sum\limits_{i,j=1}^n D_i \left(a_{ij}^{hk}(x, u, Du) D_j u_k\right) = g_h(x, u, Du) & \text{in }]0, T[\times\Omega \quad (h = 1, .., N) \\ u(t, x) = 0 & \text{on }]0, T[\times\partial\Omega \\ u(0, x) = 0 & \text{in } \Omega \end{cases}$$

(1.6) Theorem *Assume that A_1 satisfies the $(\mathcal{S})_+$ condition and that for each $\varepsilon > 0$ there exists $a_\varepsilon \in L^{\frac{2n}{n+2}}(\Omega)$ such that*

$$|g(x,\xi,\eta)| \leq a_\varepsilon(x) + \varepsilon|\xi|^{\frac{(n+2)}{(n-2)}} + \varepsilon|\eta|^{\frac{(n+2)}{n}}$$

for q.o. $x \in \Omega$ and for $(\xi,\eta) \in \mathbb{R}^N \times \mathbb{R}^{nN}$. Then there exists a weak solution to the problem

$$\begin{cases} D_t u_h - \sum\limits_{k=1}^{N}\sum\limits_{i,j=1}^{n} D_i\left(a_{ij}^{hk}(x,u,Du)D_j u_k\right) = g_h(x,u,Du) & \text{in }]0,T[\times\Omega \quad (h=1,..,N) \\ u(t,x) = 0 & \text{on }]0,T[\times\partial\Omega \\ u(0,x) = 0 & \text{in } \Omega \end{cases}$$

Proof. For every $u, v \in L^2\left(0,T;H_0^1\left(\Omega;\mathbb{R}^N\right)\right)$ we have

$$\begin{aligned}
\langle A_1(v) - A_1(u), v-u\rangle &= \\
&= \int_0^T\int_\Omega \sum_{h,k=1}^{N}\sum_{i,j=1}^{n} a_{ij}^{hk}(x,v,Dv)D_i v_h D_j(v_k-u_k)\,dx\,dt + \\
&- \int_0^T\int_\Omega \sum_{h,k=1}^{N}\sum_{i,j=1}^{n} a_{ij}^{hk}(x,u,Du)D_i u_h D_j(v_k-u_k)\,dx\,dt + \\
&\geq \int_0^T\int_\Omega \sum_{h,k=1}^{N}\sum_{i,j=1}^{n} a_{ij}^{hk}(x,v,Dv)D_i v_h D_j v_k\,dx\,dt + \\
&+ \int_0^T\int_\Omega \sum_{h,k=1}^{N}\sum_{i,j=1}^{n} a_{ij}^{hk}(x,u,Du)D_i u_h D_j u_k\,dx\,dt + \\
&- \int_0^T\int_\Omega \sum_{h,k=1}^{N}\sum_{i,j=1}^{n} a_{ij}^{hk}(x,v,Dv)D_i u_h D_j v_k\,dx\,dt + \\
&- \int_0^T\int_\Omega \sum_{h,k=1}^{N}\sum_{i,j=1}^{n} a_{ij}^{hk}(x,u,Dv)D_i u_h D_j v_k\,dx\,dt \geq \\
&\geq \nu\int_0^T \| Du \|^2_{L^2(\Omega;\mathbb{R}^{nN})}\,dt + \nu\int_0^T \| Dv \|^2_{L^2(\Omega;\mathbb{R}^{nN})}\,dt + \\
&- C\int_0^T\int_\Omega \sum_{h,k=1}^{N}\sum_{i,j=1}^{n} |D_i u_h||D_j v_k|\,dx\,dt + \\
&- C\int_0^T\int_\Omega \sum_{h,k=1}^{N}\sum_{i,j=1}^{n} |D_i u_h||D_j v_k|\,dx\,dt \geq \\
&\geq \nu\int_0^T \| Du \|^2_{L^2(\Omega;\mathbb{R}^{nN})}\,dt + \\
&- 2C\int_0^T \| Du \|_{L^2(\Omega;\mathbb{R}^{nN})}\| Dv \|_{L^2(\Omega;\mathbb{R}^{nN})}\,dt +
\end{aligned}$$

$$\begin{aligned}
&+ \quad \nu \int_0^T \| Dv \|^2_{L^2(\Omega;\mathbb{R}^{nN})} \ dt \geq \\
&\geq \quad \nu \int_0^T \left(\| Du \|^2_{L^2} -2 \| Du \|_{L^2} \| Dv \|_{L^2} \ dt + \| Dv \|^2_{L^2} \right) \ dt = \\
&= \quad \nu \int_0^T \left(\| Dv \|_{L^2(\Omega;\mathbb{R}^{nN})} - \| Du \|_{L^2(\Omega;\mathbb{R}^{nN})} \right)^2 \ dt \geq \\
&\geq \quad 0\,.
\end{aligned}$$

It follows that A_1 is a monotone operator. Let now $(u^{(m)})$ be a sequence in $L^2(0,T;H_0^1)$ weakly convergent to u with

$$\limsup_m \left\langle A_1(u^{(m)}), u^{(m)} - u \right\rangle \leq 0.$$

Since A_1 is pseudomonotone, we ealisy get

$$\limsup_h \left\langle A_1(u^{(m)}) - A_1(u), u^{(m)} - u \right\rangle \leq 0.$$

Since A_1 satisfies condizione $(\mathcal{S})_+$, we have that $(u^{(m)})$ converges stronly to u in $L^2(0,T;H_0^1)$. Whence, by mixing together Proposition (I.1.6), Theorem (1.3) and Lemma (1.4), we can conclude that (A_1+A_2) is pseudomonotone. By taking into account Sobolev Theorem and Poincarè inequality, for every $u \in L^2\left(0,T;H_0^1\left(\Omega;\mathbb{R}^N\right)\right)$ we have

$$\begin{aligned}
\frac{\langle A_1(u) + A_2(u), u\rangle}{\| Du \|_{L^2(0,T;L^2)}} \quad &= \\
&= \quad \frac{1}{\| Du \|_{L^2(0,T;L^2)}} \left\{ \int_0^T \int_\Omega \sum_{h,k=1}^N \sum_{i,j=1}^n a_{ij}^{hk}(x,u,Du) D_i u_h D_j u_k \, dx \, dt + \right. \\
&- \quad \left. \int_0^T \int_\Omega g(x,u,Du)\cdot u \right\} \geq \\
&\geq \quad \frac{1}{\| Du \|_2} \left\{ \nu \| Du \|_2^2 - \| a_\varepsilon \|_{\frac{2n}{n+2}} \| u \|_{\frac{2n}{n-2}} - \varepsilon \| u \|_2^2 - \varepsilon \| Du \|_2^2 \right\} \geq \\
&\geq \quad \frac{1}{\| Du \|_{L^2(0,T;L^2)}} \left\{ \nu \| Du \|_2^2 - c_{n,2} \| a_\varepsilon \|_{\frac{2n}{n+2}} \| Du \|_2 + \right. \\
&- \quad \left. c_{n,2} \mathcal{L}^n(\Omega)^{\frac{1}{n}} \varepsilon \| Du \|_2^2 - \varepsilon \| Du \|_2^2 \right\}.
\end{aligned}$$

Choosen now $\varepsilon > 0$ in such a way that

$$\left(c_{n,2} \mathcal{L}^n(\Omega)^{\frac{1}{n}} + 1 \right) \varepsilon < \nu,$$

with $u_0 = 0$, we get

$$\lim_{\|Du\|_{L^2(0,T;H_0^1)} \to +\infty} \frac{\langle A_1(u) + A_2(u) + Lu_0, u - u_0\rangle}{\| Du \|_{L^2(0,T;H_0^1)}} = +\infty.$$

Let now C a bounded subset of $L^2(0,T;H_0^1)$ and let $R>0$ such that

$$\forall u \in C: \parallel u \parallel_{L^2(0,T;H_0^1)} \leq R.$$

If $\varphi \in A_1(C)$, there exists $u \in C$ such that

$$\begin{aligned}
\parallel \varphi \parallel &= \\
&= \sup_{\substack{v \in L^2(0,T;H_0^1) \\ \|v\| \leq 1}} |\langle A_1(u), v\rangle| \leq \\
&\leq \sup_{\substack{v \in L^2(0,T;H_0^1) \\ \|v\| \leq 1}} C \parallel Du \parallel_{L^2(0,T;L^2)} \parallel v \parallel_{L^2(0,T;H_0^1)} \leq \\
&\leq CR \sup_{\substack{v \in L^2(0,T;H_0^1) \\ \|v\| \leq 1}} \parallel v \parallel_{L^2(0,T;H_0^1)} \leq \\
&\leq CR.
\end{aligned}$$

Taking into account Theorem (1.3), it's easy to prove that also A_2 is bounded. It then follows that $A_1 + A_2$ is a bounded operatore. By Lemma (1.5) L is maximal monotono linear. By Corollary (I.1.12) there exists $u \in L^2\left(0,T;H_0^1\left(\Omega;\mathbb{R}^N\right)\right)$ such that

$$(L + A_1 + A_2)(u) \ni 0,$$

so that, being A_1, A_2 and being L "single-valued", we get

$$\forall v \in L^2\left(0,T;H_0^1\left(\Omega;\mathbb{R}^N\right)\right): \langle (L + A_1 + A_2)(u), v\rangle = 0,$$

Therefore, it follows that u is a weak solution of the parabolic system

$$\begin{cases}
D_t u_h - \sum\limits_{k=1}^{N} \sum\limits_{i,j=1}^{n} D_i\left(a_{ij}^{hk}(x,u,Du)D_j u_k\right) = g_h(x,u,Du) & \text{in }]0,T[\times\Omega \quad (h=1,..,N) \\
u(t,x) = 0 & \text{on }]0,T[\times\partial\Omega \\
u(0,x) = 0 & \text{in } \Omega
\end{cases},$$

and the Theorem is now completely proven. ∎

For any suggestions and improvements about the present paper please e-mail me at the e-mail address: *squazz@numerica.it* . In particular I invite you to solve the following question:

" Is it possible to weaken the assumption $C \in [0,\nu]$? "

List of symbols

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Bibliografy

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