Linearized stability for semilinear non-autonomous evolution equations with applications to retarded differential equations

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Abstract

We consider the semilinear non-autonomous evolution equation $\frac{d}{dt}u(t) = Au(t) + G(t, u(t))$, $t > s \geq 0$, where $(A, D(A))$ is a Hille-Yosida operator on a Banach space X and G is a continuous function on $\mathbb{R}_+ \times \overline{D(A)}$ with values in the extrapolated Favard class corresponding to A. In our main results, we present principles of linearized stability and instability for a solution of such an equation. Our approach is based on the theory of extrapolation spaces and the theory of evolution semigroups. We finally apply the obtained results to non-autonomous semilinear retarded differential equations.

1 Introduction

The purpose of this paper is to present a principle of linearized (in-)stability for the semilinear non-autonomous evolution equation

$$(SNP)\begin{cases} \frac{d}{dt}u(t) &= Au(t) + G(t, u(t)), \quad t \geq s \geq 0, \\ u(s) &= x, \end{cases}$$

where $(A, D(A))$ is a Hille-Yosida operator on a Banach space X and G is a continuous function on $\mathbb{R}_+ \times \overline{D(A)}$ with values in the extrapolated Favard class F_0 corresponding to A satisfying certain differentiability conditions. This kind of equations naturally occurs when dealing with boundary conditions that change in time (see [12]) but also if one linearizes along nonstationary solutions (see [25]).
Principles of linearized stability and instability for the semilinear equation (SNP) with A being the generator of a C_0-semigroup on X and $G : X \to X$ satisfying certain differentiability conditions are intensively studied e.g. in [30]. In a more general situation equation (SNP) has been investigated e.g. by Clément et al. [1], Desch, Schappacher and Zhang [4],

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Greiner [8] and Thieme [27]. In particular, they obtained existence and uniqueness of mild solutions and, in the *autonomous case*, principles of linearized (in-)stability for *stationary solutions*. In Section 3 we recall some of their results in a slightly more general situation and in unified terminology. Note, however, that none of the above authors investigated linearized (in-)stability when G also depends on t. In order to obtain principles of linearized (in-)stability for the non-autonomous equation (SNP), we deal with (linear and nonlinear) non-autonomous perturbations of the operator A. Our approach is based on the theory of extrapolation spaces developed by DaPrato and Grisvard [2] and Nagel and Sinestrari [16] and on the theory of evolution semigroups as presented in [19], [26]. The basic facts on extrapolation spaces and Favard classes are collected in Section 2. The use of extrapolation spaces permits to consider perturbations G in (SNP) belonging to a rather general class of functions, whereas the use of evolution semigroups reduces the non-autonomous situation to an equivalent autonomous one. In particular, this allows to deduce principles of linearized (in)stability for (SNP) from corresponding results for autonomous equations. All this is part of Section 4 and Section 5. Finally, in Section 6, we apply the previous results to non-autonomous semilinear retarded differential equations of the form

$$\begin{cases} \frac{d}{dt}v(t) &= Bv(t) + K(t, v_t), \quad t \geq s \geq 0, \\ \quad v_s &= g \in C([-1,0], Y), \end{cases}$$

where $(B, D(B))$ is the generator of a C_0-semigroup on a Banach space Y and K is a continuous function on $\mathbb{R}_+ \times C([-1,0], Y)$ with values in Y satisfying a certain differentiability condition.

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2 Extrapolation spaces and Favard classes

Throughout the whole paper, X denotes a Banach space and $(A, D(A))$ is a *Hille-Yosida operator* on X, i.e. A is linear, there exists $\omega \in \mathbb{R}$ such that (ω, ∞) is contained in the resolvent set $\rho(A)$ of A and

$$\sup\{\|(\lambda - \omega)^n R(\lambda, A)^n\| : \lambda > \omega; n \in \mathbb{N}\} < \infty, \tag{2.1}$$

where $R(\lambda, A) := (\lambda - A)^{-1}$ is the *resolvent* of A at λ. The following result is well-known.

Lemma 2.1.*([11], Theorem 12.2.4)*
Let $(A, D(A))$ be a Hille-Yosida operator on the Banach space X. Then the part $(A_0, D(A_0))$ of A in $X_0 := \overline{D(A)}$ given by

$$D(A_0) := \{x \in D(A) : Ax \in X_0\}, \quad A_0 x := Ax \quad \textit{for} \quad x \in D(A_0)$$

generates a C_0-semigroup $(T_0(t))_{t\geq 0}$ on X_0. Moreover, $\rho(A) \subseteq \rho(A_0)$ and $R(\lambda, A_0) = R(\lambda, A)_{|X_0}$ for $\lambda \in \rho(A)$.

On X_0 we introduce a new norm by setting $\|x\|_{-1} := \|R(\lambda, A_0)x\|$ for $x \in X_0$ and $\lambda \in \rho(A_0)$. The completion X_{-1} of $(X_0, \|\cdot\|_{-1})$ is called the *extrapolation space* of X_0 associated with A_0. Note that, for different $\lambda \in \rho(A_0)$, we obtain equivalent norms on X_{-1} (see [18], Ch.3, Section 1). The operators $T_0(t)$ have a unique bounded linear extension $T_{-1}(t)$ to the Banach space X_{-1}. Then $(T_{-1}(t))_{t\geq 0}$ is a C_0-semigroup on X_{-1} (see [16] Proposition 1.3), the so-called *extrapolated semigroup* of $(T_0(t))_{t\geq 0}$, and we denote its generator by $(A_{-1}, D(A_{-1}))$. The subsequent lemmas can be found in [16].

Lemma 2.2.*([16] Proposition 1.3 and Theorem 1.4)*
Let $\lambda \in \rho(A_0)$. Then the following properties hold:

(i) $\|T_{-1}(t)\|_{\mathcal{L}(X_{-1})} = \|T_0(t)\|_{\mathcal{L}(X_0)}$,

(ii) $D(A_{-1}) = X_0$,

(iii) $A_{-1} : X_0 \to X_{-1}$ *is the unique continuous extension of* $A_0 : D(A_0) \subseteq (X_0, \|\cdot\|) \to (X_{-1}, \|\cdot\|_{-1})$ *and* $\lambda - A_{-1}$ *is an isometry from* $(X_0, \|\cdot\|)$ *onto* $(X_{-1}, \|\cdot\|_{-1})$,

(iv) $\lambda - A_{-1}$ *is invertible and* $(\lambda - A_{-1})^{-1} \in \mathcal{L}(X_{-1})$. *In particular,* $\lambda \in \rho(A_{-1})$ *and* $R(\lambda, A_{-1})_{|X_0} = R(\lambda, A_0)$.

The following lemma shows that the space X lies between X_0 and X_{-1}. We write $Y \hookrightarrow Z$ if Y and Z are Banach spaces such that $Y \subseteq Z$ and $id : Y \hookrightarrow Z$ is continuous.

Lemma 2.3.*([16], Theorem 1.7)*
Let $(A, D(A))$ be a Hille-Yosida operator on the Banach space X and $\lambda \in \rho(A)$. For the norm $\|x\|_{-1} := \|R(\lambda, A)x\|$, $x \in X$, the space $X_0 := \overline{D(A)}$ is dense in $(X, \|\cdot\|_{-1})$. Hence the extrapolation space X_{-1} is also the completion of $(X, \|\cdot\|_{-1})$ and therefore $X \hookrightarrow X_{-1}$. Moreover, the operator A_{-1} is an extension of A. In particular, $R(\lambda, A_{-1})_{|X} = R(\lambda, A)$ and $R(\lambda, A_{-1})X = D(A)$.

Next we introduce the Favard class corresponding to a C_0-semigroup.

Definition 2.4.
Let $(V(t))_{t\geq 0}$ be a C_0-semigroup with generator $(C, D(C))$ on the Banach space Z such that $\|V(t)\| \leq Me^{\nu t}$ for constants $M \geq 1$ and $\nu \in \mathbb{R}$. The *Favard class* of $(V(t))_{t\geq 0}$ is the Banach space

$$F := \{x \in Z : \sup_{t>0} \frac{1}{t}\|e^{-\nu t}V(t)x - x\| < \infty\}$$

equipped with the norm $\|x\|_F := \|x\| + \sup_{t>0} \frac{1}{t}\|e^{-\nu t}V(t)x - x\|$.

It is easy to see that F is invariant under $V(t)$ for $t \geq 0$ and that $D(C) \subseteq F$. In the following we denote by $F_1 \subseteq X_0$ the Favard class of the C_0-semigroup $(T_0(t))_{t\geq 0}$ and by $F_0 \subseteq X_{-1}$ the Favard class of $(T_{-1}(t))_{t\geq 0}$.

Lemma 2.5.*([16], Proposition 3.2)*
For $\lambda \in \rho(A_0)$ and the Favard classes F_0 and F_1 the following holds:

(i) $(\lambda - A_{-1})F_1 = F_0$,

(ii) $\|(\lambda - A_{-1})x\|_{F_0} = \|x\|_{F_1}$ *for* $x \in F_1$,

(iii) $T_{-1}(t)F_0 \subseteq F_0$ *for* $t \geq 0$,

(iv) $D(A_0) \hookrightarrow D(A) \hookrightarrow F_1 \hookrightarrow X_0 \hookrightarrow X \hookrightarrow F_0 \hookrightarrow X_{-1}$, *where* $D(A)$ *is equipped with the graph norm* $\|x\|_A := \|x\| + \|Ax\|$, $x \in D(A)$.

The following lemma, due to Nagel and Sinestrari [16], is fundamental for our investigations.

Lemma 2.6. *([16], Proposition 3.3)*
Let $M \geq 1$ *and* $\omega \in \mathbb{R}$ *such that* $\|T_0(t)\| \leq Me^{\omega t}$ *for* $t \geq 0$. *For* $f \in L^1_{loc}(\mathbb{R}_+, F_0)$ *define*

$$[T_{-1} \star f](t) := \int_0^t T_{-1}(t-s)f(s)\,ds, \quad t \geq 0.$$

Then

(i) $[T_{-1} \star f](t) \in X_0$,

(ii) $\|[T_{-1} \star f](t)\|_{X_0} \leq M_1 \int_0^t e^{\omega(t-s)}\|f(s)\|_{F_0}\,ds$ *for a constant* $M_1 > 0$, *independent of* f *and* t,

(iii) $\lim_{t \to 0} \|[T_{-1} \star f](t)\|_{X_0} = 0$.

Remark 2.7.
As an easy consequence of Lemma 2.6, we obtain that $t \mapsto [T_{-1} \star f](t)$ *is continuous from* $\mathbb{R}_+$ *into* X_0.

In the following, we use the notations introduced above without further comment.

3 Existence and uniqueness of solutions of (SNP)

The main purpose of this paper is to discuss the semilinear non-autonomous Cauchy problem

$$(SNP)_{s,x} \quad \begin{cases} \frac{d}{dt}u(t) = Au(t) + G(t, u(t)), & t \geq s \geq 0, \\ u(s) = x \in X_0, \end{cases}$$

on the Banach space X, where $(A, D(A))$ is a Hille-Yosida operator on X and $G : \mathbb{R}_+ \times X_0 \to F_0$ is a function satisfying certain conditions to be specified below. We will discuss existence and uniqueness and, first of all, stability properties of solutions of (SNP). For our purposes the concept of a mild solution is most useful.

Definition 3.1.
A function $u = u(\cdot; s; x) \in C([s,T], X_0)$ is called a *mild solution* of $(SNP)_{s,x}$ on the interval $[s,T]$ if

$$u(t) = T_0(t-s)x + \int_s^t T_{-1}(t-\sigma)G(\sigma, u(\sigma))\,d\sigma \quad \text{for } t \in [s,T]. \tag{3.1}$$

Remark 3.2.
Our definition of a mild solution coincides with that in [1], but also with the definition of an F-solution [3], a weak solution [4] and an integral solution [27]. It also includes mild solutions as defined in [8]. This is shown, for example, in [3], Proposition 12.3, [4], Section 4, [18], Proposition 3.2.6, [15], Theorem 4.5 and Section 4.3, in combination with [28], Section 6.

By using Lemma 2.6 and a fixed point argument, one obtains existence and uniqueness of a mild solution of (SNP) as in the case where A is the generator of a C_0-semigroup and $X_0 = X$ (see [22], Theorem 6.1.2). We therefore omit the proof of our next proposition. Related results can be found e.g. in [1],Theorem 3.1, [4], Theorem 1, [8], Theorem 1.3, [10], Theorems 3.3.3 and 3.3.4, [14], Theorem 7.1.2 and Proposition 7.1.8, [15], Corollary 4.13, and [27], Theorem 1.3.

Proposition 3.3.
Suppose $G : \mathbb{R}_+ \times X_0 \to F_0$ is continuous and locally Lipschitz continuous with respect to the second variable uniformly on compact intervals of $\mathbb{R}_+$. Then for every initial value $x \in X_0$ and every initial time $s \in \mathbb{R}_+$ there exists a unique mild solution $u = u(\cdot; s; x)$ of (SNP) defined on a maximal interval $[s, t_{max})$. Moreover, $t_{max} = \infty$ or $\|u(t; s; x)\| \to \infty$ as $t \to t_{max}$.

When we discuss stability and instability of a solution v of (SNP) it suffices to know existence and uniqueness of solutions for initial values only on a neighborhood of the orbit of v. If v is the zero solution this is guaranteed by the following condition on G. Here and in the following we denote by B_Z the open unit ball of a Banach space Z.

(G0) The function $G : \mathbb{R}_+ \times X_0 \to F_0$ is continuous, $G(t, 0) = 0$ for $t \in \mathbb{R}_+$ and for every $t_0 > 0$ there exist constants $q(t_0)$, $r(t_0) > 0$ such that $\|G(t, x) - G(t, y)\|_{F_0} \leq q(t_0)\|x - y\|$ for $x, y \in r(t_0)B_{X_0}$ and $t \in [0, t_0]$.

Theorem 3.4.
Suppose condition (G0) is satisfied. Then for every $s \geq 0$ and every $T > 0$ there exists an $\epsilon := \epsilon(T, s) > 0$ such that $(SNP)_{s,x}$ has a uniquely determined mild solution on $[s, s + T]$ for every initial value $x \in \epsilon B_{X_0}$.

Proof. Step 1. Fix $s \geq 0$ and $T > 0$. At first, we show that there exist $\epsilon_0 = \epsilon_0(s+T) > 0$ and $h = h(s + T) > 0$, such that $(SNP)_{s_0,x}$ has a uniquely determined mild solution on the interval $[s_0, s_0 + h]$ for every $s_0 \in [s, s + T]$ and for every $x \in \epsilon_0 B_{X_0}$. In order to prove this let $M \geq 1$ and $\omega > 0$ such that $\|T_0(t)\| \leq Me^{\omega t}$. We set $\epsilon_0 := \frac{r(s+T)}{4Me^{\omega}}$ and $h := \min\{1, (4M_1 e^{\omega} q(s+T))^{-1}\}$ where M_1 is the constant occuring in Lemma 2.6. For fixed $x \in \epsilon_0 B_{X_0}$ and $s_0 \in [s, s + T]$, we define $F : C([s_0, s_0 + h], X_0) \to C([s_0, s_0 + h], X_0)$ by

$$Fu(t) := T_0(t - s_0)x + \int_{s_0}^{t} T_{-1}(t - \sigma)G(\sigma, u(\sigma))\, d\sigma, \quad s_0 \leq t \leq s_0 + h.$$

Then F maps $r(s+T)B_{C([s_0,s_0+h],X_0)}$ into itself. In fact, if $u \in r(s+T)B_{C([s_0,s_0+h],X_0)}$ we obtain with Lemma 2.6,

$$\begin{aligned} \|Fu(t)\| &\leq Me^{\omega h}\epsilon_0 + M_1 \int_{s_0}^t e^{\omega(t-\sigma)} q(s+T) r(s+T)\, d\sigma \\ &\leq Me^{\omega h}\epsilon_0 + M_1 e^{\omega h} q(s+T) r(s+T) h \leq \frac{r(s+T)}{2} \qquad \text{for } t \in [s_0, s_0+h]. \end{aligned} \tag{3.2}$$

Furthermore, if u_1, $u_2 \in r(s+T)B_{C([s_0,s_0+h],X_0)}$, then

$$\begin{aligned} &\|Fu_1(t) - Fu_2(t)\| = \|\int_{s_0}^t T_{-1}(t-\sigma)\big(G(\sigma,u_1(\sigma)) - G(\sigma,u_2(\sigma))\big)\, d\sigma\| \\ &\leq M_1 \int_{s_0}^t e^{\omega(t-\sigma)} \|G(\sigma,u_1(\sigma)) - G(\sigma,u_2(\sigma))\|_{F_0}\, d\sigma \\ &\leq M_1 \int_{s_0}^t e^{\omega(t-\sigma)} q(s+T)\|u_1(\sigma) - u_2(\sigma)\|\, d\sigma \\ &\leq M_1 e^{\omega h} q(s+T) h \|u_1 - u_2\|_{C([s_0,s_0+h],X_0)} \leq \frac{1}{4}\|u_1 - u_2\|_{C([s_0,s_0+h],X_0)} \end{aligned}$$

for $0 \leq t \leq T$, i.e., F is a contraction on $r(s+T)B_{C([s_0,s_0+h],X_0)}$. By Banach's fixed point theorem there is a unique fixed point u of F in $r(s+T)B_{C([s_0,s_0+h],X_0)}$. It remains to show that the fixed point u is unique in $C([s_0,s_0+h],X_0)$. In fact, let $\tilde{u} \in C([s_0,s_0+h],X_0)$ such that

$$\tilde{u}(t) = T_0(t-s_0)x + \int_{s_0}^t T_{-1}(t-\sigma)G(\sigma,\tilde{u}(\sigma))\, d\sigma \qquad \text{for } t \in [s_0, s_0+h].$$

The uniqueness assertion in Banach's fixed point theorem yields $u(t) = \tilde{u}(t)$ on every interval $[s_0, s_0+\tilde{h}] \subseteq [s_0, s_0+h]$ such that $\|\tilde{u}(t)\| \leq r(s+T)$ on $[s_0, s_0+\tilde{h}]$. On the other hand, by (3.2) we have $\|\tilde{u}(t)\| \leq \frac{r(s+T)}{2}$ on $[s_0, s_0+\tilde{h}]$. Thus $h = \sup\{\tilde{h} \in [0,h] : u(t) = \tilde{u}(t) \text{ on } [s_0, s_0+\tilde{h}]\}$, i.e., $u = \tilde{u}$.

Step 2. For $x \in \epsilon_0 B_{X_0}$ and $s_0 \in [s, s+T]$ denote by $u(\cdot; s_0; x)$ the mild solution of $(SNP)_{s_0,x}$ defined on $[s_0, s_0+h]$. An easy application of Gronwall's inequality yields

$$\lim_{\|x\|\to 0} \sup_{s \leq s_0 \leq s+T} \sup_{s_0 \leq t \leq s_0+h} \|u(t; s_0; x)\| = 0.$$

This allows to apply Step 1 n-times for initial values sufficiently close to 0. Precisely, for every $n \in \mathbb{N}$ we can find $\epsilon > 0$ such that for $x \in \epsilon B_X$ there exist unique mild solutions $u_1 = u_1(\cdot; s; x) \in C([s, s+h], X_0)$, $u_2 = u_2(\cdot; s+h; u_1(s+h)) \in C([s+h, s+2h], X_0)$,...,$u_n = u_n(\cdot; s+(n-1)h; u_{n-1}(s+(n-1)h)) \in C([s+(n-1)h, s+nh], X_0)$. Now choose $n \in \mathbb{N}$ such that $nh \geq T$. Then $u \in C([s, s+nh], X_0)$ defined by $u(t) := u_k(t)$ if $t \in [s+(k-1)h, s+kh]$ is the unique mild solution of $(SNP)_{s,x}$ defined on $[s, s+nh] \supseteq [s, s+T]$. □

Remark 3.5.
By looking at the proof of Theorem 3.4, one can see that the constant ϵ does not depend on s if q and r are t-independent. In particular, this is true for the autonomous equation

$$(SAP)_x \quad \begin{cases} \frac{d}{dt}u(t) &= Au(t) + G(u(t)), \quad t \geq 0, \\ u(0) &= x \in X_0, \end{cases}$$

and $G : X_0 \to F_0$ such that G is Lipschitz continuous in a neighborhood of 0 and $G(0) = 0$.

We now consider the situation where $G : X_0 \to F_0$ is t-independent and of the form $G = B + o^\star$, i.e., we look at the semilinear autonomous equation

$$(SAP')_x \quad \begin{cases} \frac{d}{dt}u(t) = & Au(t) + Bu(t) + o^\star(u(t)), \quad t \geq 0, \\ u(0) = & x \in X_0, \end{cases}$$

where $B : X_0 \to F_0$ is linear and bounded and $o^\star$ satisfies certain continuity conditions to be specified below. It is shown in [18], Theorem 4.2.2, and [20], Theorem 3.6, that the part $(A+B)_0$ of $A+B$ in X_0 is the generator of a C_0-semigroup $(T_B(t))_{t\geq 0}$ on X_0 which is uniquely determined by the variation-of-constants formula

$$T_B(t)x = T_0(t)x + \int_0^t T_{-1}(t-\sigma)BT_B(\sigma)x\,d\sigma \quad \text{for } t \geq 0 \text{ and } x \in X_0. \tag{3.3}$$

Moreover, the extrapolation spaces associated with A_0 and $(A+B)_0$ coincide and have equivalent norms (see [18], Theorem 3.2.6, Lemma 4.1.3, and the discussion on p.75).
The following proposition connects mild solutions of (SAP') with mild solutions of

$$\begin{cases} \frac{d}{dt}u(t) = & (A+B)u(t) + o^\star(u(t)), \quad t \geq 0, \\ u(0) = & x \in X_0. \end{cases}$$

A similiar result is shown in [1], Proposition 2.5.

Proposition 3.6.
Suppose $G = B + o^\star$ where $B : X_0 \to F_0$ is linear and bounded and $o^\star : X_0 \to F_0$. Moreover, let $u \in C([0,T], X_0)$ such that $o^\star \circ u \in L^1([0,T], F_0)$. Then u is a mild solution of (SAP'), i.e.,

$$u(t) = T_0(t)u(0) + \int_0^t T_{-1}(t-\sigma)\Big(Bu(\sigma) + o^\star(u(\sigma))\Big)\,d\sigma \quad \text{for } 0 \leq t \leq T$$

if and only if

$$u(t) = T_B(t)u(0) + \int_0^t (T_B)_{-1}(t-\sigma)o^\star(u(\sigma))\,d\sigma \quad \text{for } 0 \leq t \leq T. \tag{3.4}$$

Proof. We first prove the 'if' part. We claim that

$$\begin{aligned} &\int_0^t (T_B)_{-1}(t-\sigma)o^\star(u(\sigma))\,d\sigma \\ &\quad = \int_0^t T_{-1}(t-\sigma)o^\star(u(\sigma))\,d\sigma + \int_0^t T_{-1}(t-\sigma)B\int_0^\sigma (T_B)_{-1}(\sigma-\tau)o^\star(u(\tau))\,d\tau\,d\sigma \end{aligned} \tag{3.5}$$

for $t \in [0,T]$. In order to show this consider $x \in X_0$ and $f \in L^1([0,T], F_0)$. Note that $\int_0^t B(T_B)_{-1}(\tau)x\,d\tau = B\int_0^t (T_B)_{-1}(\tau)x\,d\tau$ and, using Remark 2.7, $\int_0^s B\int_0^t (T_B)_{-1}(t-\tau)f(\tau)\,d\tau\,dt = B\int_0^s\int_0^t (T_B)_{-1}(t-\tau)f(\tau)\,d\tau\,dt$ (where all integrals are Bochner integrals in X_{-1}). From equation (3.3) we know that

$$(T_B)_{-1}(t)x = T_{-1}(t)x + \int_0^t T_{-1}(t-\tau)B(T_B)_{-1}(\tau)x\,d\tau \quad \text{for } x \in X_0.$$

Integrating this expression from 0 to $\sigma - s$ yields

$$\begin{aligned}
\int_0^{\sigma-s}(T_B)_{-1}(t)x\,dt &= \int_0^{\sigma-s} T_{-1}(t)x\,dt + \int_0^{\sigma-s}\Big(\int_0^t T_{-1}(\tau)B(T_B)_{-1}(t-\tau)x\,d\tau\Big)\,dt \\
&= \int_0^{\sigma-s} T_{-1}(t)x\,dt + \int_0^{\sigma-s} T_{-1}(\tau)B\int_\tau^{\sigma-s}(T_B)_{-1}(t-\tau)x\,dt\,d\tau \\
&= \int_0^{\sigma-s} T_{-1}(t)x\,dt + \int_s^{\sigma} T_{-1}(\tau-s)B\int_{\tau-s}^{\sigma-s}(T_B)_{-1}(t+s-\tau)x\,dt\,d\tau \\
&= \int_0^{\sigma-s} T_{-1}(t)x\,dt + \int_s^{\sigma} T_{-1}(\tau-s)B\int_0^{\sigma-\tau}(T_B)_{-1}(t)x\,dt\,d\tau \\
&= \int_0^{\sigma-s} T_{-1}(t)x\,dt + \int_s^{\sigma} T_{-1}(\sigma-\tau)B\int_0^{\tau-s}(T_B)_{-1}(t)x\,dt\,d\tau \qquad (3.6)
\end{aligned}$$

for $\sigma \geq s \geq 0$. Now let $x \in F_0$. Choose $(x_n)_{n\in\mathbb{N}} \subseteq X_0$ such that $\lim_{n\to\infty}\|x_n - x\|_{-1} = 0$. By Lebesgue's dominated convergence theorem we have

$$\begin{aligned}
\int_0^{\sigma-s}(T_B)_{-1}(t)x\,dt &= \lim_{n\to\infty}\int_0^{\sigma-s}(T_B)_{-1}(t)x_n\,dt \\
&= \lim_{n\to\infty}\int_0^{\sigma-s} T_{-1}(t)x_n\,dt + \lim_{n\to\infty}\int_s^{\sigma} T_{-1}(\sigma-\tau)B\int_0^{\tau-s}(T_B)_{-1}(t)x_n\,dt\,d\tau \\
&= \int_0^{\sigma-s} T_{-1}(t)x\,dt + \int_s^{\sigma} T_{-1}(\sigma-\tau)B\int_0^{\tau-s}(T_B)_{-1}(t)x\,dt\,d\tau
\end{aligned}$$

for $\sigma \geq s \geq 0$ (where the limits exist in the norm of X_{-1}). Thus equation (3.6) also holds for $x \in F_0$. If we replace x by $o^\star(u(s))$ for $s \in [0,\sigma]$ and integrate from 0 to σ we obtain

$$\begin{aligned}
&\int_0^\sigma\int_0^t(T_B)_{-1}(t-s)o^\star(u(s))\,ds\,dt = \int_0^\sigma\int_0^{\sigma-s}(T_B)_{-1}(t)o^\star(u(s))\,dt\,ds \\
&= \int_0^\sigma\int_0^{\sigma-s} T_{-1}(t)o^\star(u(s))\,dt\,ds + \int_0^\sigma\int_s^\sigma T_{-1}(\sigma-\tau)B\Big(\int_0^{\tau-s}(T_B)_{-1}(t)o^\star(u(s))\,dt\Big)\,d\tau\,ds \\
&= \int_0^\sigma\int_0^{\sigma-s} T_{-1}(t)o^\star(u(s))\,dt\,ds + \int_0^\sigma\int_0^\tau T_{-1}(\sigma-\tau)B\Big(\int_0^{\tau-s}(T_B)_{-1}(t)o^\star(u(s))\,dt\Big)\,ds\,d\tau \\
&= \int_0^\sigma\int_0^{\sigma-s} T_{-1}(t)o^\star(u(s))\,dt\,ds + \int_0^\sigma T_{-1}(\sigma-\tau)B\int_0^\tau\Big(\int_0^t(T_B)_{-1}(t-s)o^\star(u(s))\,ds\Big)\,dt\,d\tau \\
&= \int_0^\sigma\int_0^{\sigma-s} T_{-1}(t)o^\star(u(s))\,dt\,ds + \int_0^\sigma\int_0^\tau T_{-1}(\sigma-\tau)B\Big(\int_0^t(T_B)_{-1}(t-s)o^\star(u(s))\,ds\Big)\,dt\,d\tau \\
&= \int_0^\sigma\int_0^{\sigma-s} T_{-1}(t)o^\star(u(s))\,dt\,ds + \int_0^\sigma\int_0^{\sigma-\tau} T_{-1}(\tau)B\Big(\int_0^t(T_B)_{-1}(t-s)o^\star(u(s))\,ds\Big)\,dt\,d\tau \\
&= \int_0^\sigma\int_0^{\sigma-s} T_{-1}(t)o^\star(u(s))\,dt\,ds + \int_0^\sigma\int_\tau^\sigma T_{-1}(\tau)B\Big(\int_0^{t-\tau}(T_B)_{-1}(t-s-\tau)o^\star(u(s))\,ds\Big)\,dt\,d\tau \\
&= \int_0^\sigma\int_0^t T_{-1}(t-s)o^\star(u(s))\,ds\,dt + \int_0^\sigma\int_0^t T_{-1}(t-\tau)B\Big(\int_0^\tau(T_B)_{-1}(\tau-s)o^\star(u(s))\,ds\Big)\,d\tau\,dt.
\end{aligned}$$

If we differentiate both sides with respect to σ we obtain equation (3.5).
In order to show the desired implication we replace in (3.4) $T_B(t)u(0)$ by the expression (3.3) and apply (3.5). This leads to

$$\begin{aligned}
u(t) &= T_B(t)u(0) + \int_0^t (T_B)_{-1}(t-\sigma)o^\star(u(\sigma))\,d\sigma \\
&= T_{-1}(t)u(0) + \int_0^t T_{-1}(t-\sigma)BT_B(\sigma)u(0)\,d\sigma + \int_0^t (T_B)_{-1}(t-\sigma)o^\star(u(\sigma))\,d\sigma \\
&= T_{-1}(t)u(0) + \int_0^t T_{-1}(t-\sigma)BT_B(\sigma)u(0)\,d\sigma + \int_0^t T_{-1}(t-\sigma)o^\star(u(\sigma))\,d\sigma
\end{aligned}$$

$$
\begin{aligned}
&+\int_0^t T_{-1}(t-\sigma)B\int_0^\sigma (T_B)_{-1}(\sigma-\tau)o^\star(u(\tau))\,d\tau\,d\sigma \\
=\;& T_{-1}(t)u(0)+\int_0^t T_{-1}(t-\sigma)\Big(Bu(\sigma)+o^\star(u(\sigma))\Big)\,d\sigma \quad \text{for } t\in[0,T].
\end{aligned}
$$

The 'only if' part follows from the 'if' part if we write $o^\star = (o^\star + B) - B$ and perturb $A+B$ by $-B$. □

Consider now the following condition on G.

(G1) $G = B + o^\star : X_0 \to F_0$ where $B : X_0 \to F_0$ is a bounded linear operator and $o^\star : X_0 \to F_0$ satisfies $o^\star(0) = 0$ and for every $q > 0$ there exists $\rho > 0$ such that

$$
\|o^\star(x) - o^\star(y)\|_{F_0} \le q\|x-y\| \quad \text{for } x, y \in \rho B_{X_0}.
$$

Theorem 3.4 implies that for initial values sufficiently close to zero (SAP') has a uniquely determined mild solution defined on a fixed interval $[0,T]$. If the C_0-semigroup $(T_B(t))_{t\ge 0}$ is *exponentially stable*, i.e., $\|T_B(t)\| \le Me^{-\omega t}$ for $t \ge 0$ and constants $M \ge 1$ and $\omega > 0$, we even obtain mild solutions defined on $[0,\infty)$. Results of this kind can also be found in [1], Theorem 4.2, [8], Theorem 2.4, [10], Theorem 5.1.1, [14], Theorem 9.1.2, and [27], Theorem 4.2.

Theorem 3.7.
Suppose condition (G1) is satisfied and let the perturbed C_0-semigroup $(T_B(t))_{t\ge0}$ be exponentially stable. Then there exists $\epsilon > 0$ such that for $x \in \epsilon B_{X_0}$ equation $(SAP')_x$ has a uniquely determined mild solution $u(\cdot,x)$ defined on $[0,\infty)$. Moreover, $\|u(t,x)\| \le M_0 e^{-\omega_0 t}$ for $t \ge 0$ and constants $M_0 \ge 0$ and $\omega_0 > 0$.

Proof. One easily sees that $B + o^\star : X_0 \to F_0$ satisfies condition (G0). Hence, by Remark 3.5 applied to $G = B + o^\star$ for fixed $T > 0$, there exists $\epsilon > 0$ such that $(SAP')_x$ has a unique mild solution $u(\cdot,x)$ defined on $[0,T]$ for every $x \in \epsilon B_X$. Let $\|T_B(t)\| \le Me^{-\omega t}$ for $t \ge 0$ and constants $M \ge 1$ and $\omega > 0$. We set $q := \frac{\omega}{2M_1}$, where M_1 is the constant appearing in Lemma 2.6. Because of condition $(G1)$ there exists $\rho > 0$ such that $\|o^\star(y)\| \le q\|y\|$ for $y \in \rho B_{X_0}$. Let $x \in \frac{\min\{\rho,\epsilon\}}{M} B_{X_0}$. Take $T_0 \in (0,T]$ such that $\|u(t,x)\| < \min\{\rho,\epsilon\}$ for $t \in [0,T_0)$. Lemma 2.6 yields

$$
\begin{aligned}
\|u(t,x)\| &\le Me^{-\omega t}\|x\| + M_1\int_0^t e^{-\omega(t-\sigma)}\|o^\star(u(\sigma,x))\|_{F_0}\,d\sigma \\
&\le Me^{-\omega t}\|x\| + M_1\int_0^t e^{-\omega(t-\sigma)}q\|u(\sigma,x)\|\,d\sigma \quad \text{for } t\in[0,T_0).
\end{aligned}
$$

An application of Gronwall's inequality gives

$$
e^{\omega t}\|u(t,x)\| \le \min\{\rho,\epsilon\}e^{M_1 q t} \le \min\{\rho,\epsilon\}e^{\frac{\omega}{2}t} \quad \text{for } t\in[0,T_0),
$$

and hence

$$
\|u(t,x)\| \le \min\{\rho,\epsilon\}e^{-\frac{\omega}{2}t} \quad \text{for } t\in[0,T_0). \tag{3.7}
$$

In order to prove that the solution $u(\cdot,x)$ exists on $[0,\infty)$ and satisfies (3.7) for $t\in[0,\infty)$ it suffices to show $\|u(t,x)\| < \min\{\rho,\epsilon\}$ for $t\in[0,\infty)$. If this is not the case we can find

$T_1 > 0$ such that $\|u(T_1, x)\| = \min\{\rho, \epsilon\}$ and $\|u(t, x)\| < \min\{\rho, \epsilon\}$ for $t \in [0, T_1)$, which is a contradiction to (3.7). $\square$

Remark 3.8.
Theorem 3.7 can be interpreted as a principle of linearized stability for equation (SAP'). In fact, exponential stability of the zero solution of the linear equation

$$\begin{cases} \frac{d}{dt}u(t) &= (A+B)u(t), \quad t \geq 0, \\ u(0) &= x \in X_0, \end{cases}$$

implies the exponential stability (see Definition 4.1) of the zero solution of (SAP').

4 A principle of linearized stability

In the following two sections we derive principles of linearized stability and instability for the semilinear non-autonomous equation (SNP). The main difficulty arises from the fact that the nonlinearity G maps X_0 into F_0 and, at the same time, G is $t-$dependent. At first, we introduce the following notions.

Definition 4.1.
Let $u(\cdot; s_0; x)$ be a mild solution of $(SNP)_{s_0,x}$ defined on $[s_0, \infty)$.

a) Then $u(\cdot; s_0; x)$ is called *stable* if for every $\epsilon > 0$ and every $s \geq s_0$ there exists $\delta := \delta(\epsilon, s) > 0$ such that for all $y \in X_0$ with $\|y - u(s; s_0; x)\| < \delta$ the mild solution $u(\cdot; s; y)$ exists on $[s, \infty)$ and $\|u(t; s; y) - u(t; s_0; x)\| < \epsilon$ for all $t \in [s, \infty)$.

b) The mild solution $u(\cdot; s_0; x)$ is called *exponentially stable* if there exist constants $M \geq 1$ and $\omega > 0$ and for every $s \geq s_0$ there exists $\delta := \delta(s) > 0$ such that for $y \in X_0$ with $\|y - u(s; s_0; x)\| < \delta$ the mild solution $u(\cdot; s; y)$ is defined on $[s, \infty)$ and $\|u(t; s_0; x) - u(t; s; y)\| \leq Me^{-\omega(t-s)}$ for $t \in [s, \infty)$.

c) The mild solution $u(\cdot; s_0; x)$ is called *uniformly exponentially stable* if the constant δ in b) is independent of s.

d) The mild solution $u(\cdot; s_0; x)$ is *unstable* if it is not stable.

Throughout the whole section $v = v(\cdot; 0; x_0)$ denotes a fixed mild solution of $(SNP)_{0,x_0}$ defined on the half-line $[0, \infty)$. Moreover, we impose the following conditions on the nonlinearity G.

(G2) The map $G : \mathbb{R}_+ \times X_0 \to F_0$ is continuous, differentiable in the second variable along the solution v and for every $q > 0$ there exists $r > 0$ such that

$$\|G(t, v(t) + h_1) - G(t, v(t) + h_2) - DG(t, v(t))(h_1 - h_2)\| \leq q\|h_1 - h_2\|,$$

for $t \in \mathbb{R}_+$ and $h_1, h_2 \in rB_{X_0}$.

(G3) The map $\mathbb{R}_+ \ni t \mapsto DG(t, v(t)) \in \mathcal{L}(X_0, F_0)$ is uniformly bounded and strongly continuous.

Remark 4.2.
If $v \equiv x_0$ is a stationary solution of (SNP), then conditions (G2) and (G3) are satisfied if G is uniformly continuously differentiable with respect to the second variable in a neighborhood V of x_0 uniformly in $t \in \mathbb{R}_+$. In fact, then for every $q > 0$ there is $r > 0$ such that

$$\|G(t, \nu + h) - G(t, \nu) - DG(t, \nu)h\| \leq q\|h\|$$

for every $t \in \mathbb{R}_+$, $\nu \in V$ and $h \in rB_{X_0}$, and hence

$$\begin{aligned}
&\|G(t, x_0 + h_1) - G(t, x_0 + h_2) - DG(t, x_0)(h_1 - h_2)\| \\
&\quad = \|G(t, x_0 + h_1) - G(t, x_0 + h_2) - DG(t, x_0 + h_2)(h_1 - h_2)\| \\
&\quad + \|DG(t, x_0 + h_2)(h_1 - h_2) - DG(t, x_0)(h_1 - h_2)\| \\
&\quad \leq q\|h_1 - h_2\| + q\|h_1 - h_2\| \quad \text{for } h_1, h_2 \in r_0 B_{X_0} \text{ and appropriate } r_0 \leq r.
\end{aligned}$$

Our way towards a principle of linearized stability for (SNP) is divided into several steps.

Step 1. From v to the zero solution and linearization of (SNP)

At first, we pass from (SNP) to the equation

$$(SNP')_{s,x} \begin{cases} \frac{d}{dt}u(t) &= Au(t) + B(t)u(t) + \rho(t, u(t)), \qquad t \geq s \in \mathbb{R}_+, \\ u(s) &= x \in X_0, \end{cases}$$

where

$$B(t) := DG(t, v(t)) \text{ and } \rho(t, h) := G(t, v(t) + h) - G(t, v(t)) - DG(t, v(t))h$$

for $t \in \mathbb{R}_+$ and $h \in X_0$.
The following proposition connects mild solutions of (SNP) with mild solutions of (SNP').

Proposition 4.3.
Let $u : [s, T] \to X_0$. Then u is a mild solution of $(SNP)_{s,x}$ if and only if $u - v$ is a mild solution of $(SNP')_{s,x-v(s)}$ on $[s, T]$. In particular, the mild solution v of (SNP) corresponds to the zero solution of (SNP') and v is stable (exponentially stable, uniformly exponentially stable, unstable) if and only if the same holds for the zero solution of (SNP').

Proof. Since

$$v(t) = T_0(t - s)v(s) + \int_s^t T_{-1}(t - \sigma)G(\sigma, v(\sigma))\, d\sigma \quad \text{for } t \geq s,$$

we obtain that

$$u(t; s; x) = T_0(t - s)x + \int_s^t T_{-1}(t - \sigma)G(\sigma, u(\sigma; s; x))\, d\sigma \quad \text{for } t \geq s$$

if and only if

$$\begin{aligned}
&u(t; s; x) - v(t) \\
&\quad = T_0(t - s)(x - v(s)) + \int_s^t T_{-1}(t - \sigma)\Big(G(\sigma, u(\sigma; s; x)) - G(\sigma, v(\sigma))\Big)\, d\sigma \\
&\quad = T_0(t - s)(x - v(s)) \\
&\qquad + \int_s^t T_{-1}(t - \sigma)\Big(DG(\sigma, v(\sigma))(u(\sigma; s; x) - v(\sigma)) + \rho(\sigma, u(\sigma; s; x) - v(\sigma))\Big)\, d\sigma
\end{aligned}$$

for $t \geq s$. The remaining assertions are obvious. $\square$

Remark 4.4.
From (G2) and (G3) we obtain that the map $\mathbb{R}_+ \times X_0 \ni (t,x) \to B(t)x + \rho(t,x)$ satisfies condition (G0). Then Proposition 4.3 together with Theorem 3.4 imply that for $T > 0$ and $s \geq 0$ there is $\epsilon > 0$ such that $(SNP)_{s,x}$ has a uniquely determined mild solution on $[s, s+T]$ for $x \in v(s) + \epsilon B_{X_0}$.

Note that condition (G2) implies $\|\rho(t,h)\| = o(\|h\|), h \in X_0$, uniformly for $t \in \mathbb{R}_+$. So if we pass to the linearized non-autonomous equation

$$(LNP)_{s,x} \quad \begin{cases} \frac{d}{dt}u(t) &= Au(t) + B(t)u(t), \quad t >\geq s, \\ u(s) &= x \in X_0, \end{cases}$$

one should expect that (in-)stability of the zero solution of (LNP) leads to (in-)stability of the zero solution of (SNP') and hence to (in-)stability of v. This turns out to be the desired principle of linearized (in-)stability (see Step 3 and Section 5).
We point out that $(LNP)_{s,x}$ always has a unique mild solution defined on $[s,\infty)$. To make this more precise we introduce the following notion. A family $(U(t,s))_{t\geq s\geq 0} \subseteq \mathcal{L}(X_0)$ is called an *evolution family* (EVF) on X_0 if $U(t,t) = Id$, $U(t,r)U(r,s) = U(t,s)$ for $t \geq r \geq s \geq 0$, $(t,s) \mapsto U(t,s)x$ is continuous for $x \in X_0$, and there are constants $M \geq 1$ and $\omega \in \mathbb{R}$ such that $\|U(t,s)\| \leq Me^{\omega(t-s)}$ for $t \geq s \geq 0$. By a perturbation theorem of Rhandi ([24], Theorem 2.3), there is a unique EVF $(U_B(t,s))_{t\geq s\geq 0}$ on X_0 such that

$$U_B(t,s)x = T_0(t-s)x + \int_s^t T_{-1}(t-\sigma)B(\sigma)U_B(\sigma,s)x\, d\sigma \tag{4.1}$$

for $t \geq s$ and $x \in X_0$. In particular, $U_B(\cdot,s)x$, $s \geq 0$, $x \in X_0$, is a mild solution of $(LNP)_{s,x}$. The uniqueness of the mild solution follows from a standard argument based on Gronwall's inequality.

Step 2. From (SNP') to the autonomous equation $(\mathcal{SAP})$

In the next step, we introduce a semilinear autonomous equation $(\mathcal{SAP})$ such that the zero solution has the same stability properties as the zero solution of (SNP'). The way to obtain this equation is based on the construction of evolution semigroups (see [19], [26] and the references therein). For a Banach space Z denote by $C_{00}(\mathbb{R}_+, Z)$ the space of continuous Z-valued functions on $\mathbb{R}_+$ vanishing at 0 and ∞ (equipped with the sup-norm). Let $\mathcal{X}_0 := C_{00}(\mathbb{R}_+, X_0)$, $\mathcal{X} := C_{00}(\mathbb{R}_+, F_0)$ and define operators $\mathcal{T}_0(t)$, $t \geq 0$, on $\mathcal{X}_0$ by

$$(\mathcal{T}_0(t)f)(s) := \begin{cases} T_0(t)f(s-t) & \text{for } s \geq t \\ 0 & \text{for } 0 \leq s < t. \end{cases}$$

The operator family $(\mathcal{T}_0(t))_{t\geq 0}$ is a C_0-semigroup on $\mathcal{X}_0$, the so-called *evolution semigroup* induced by $(T_0(t))_{t\geq 0}$, and we denote its generator by $\mathcal{A}_0$. In the same way the evolution semigroup induced by $(T_{-1}(t))_{t\geq 0}$ leads to a C_0-semigroup $(\mathcal{S}(t))_{t\geq 0}$ on $C_{00}(\mathbb{R}_+, X_{-1})$.
On $\mathcal{X}_0$ we consider the semilinear autonomous equation

$$(\mathcal{SAP})_g \quad \begin{cases} \frac{d}{dt}w(t) &= \mathcal{A}_0 w(t) + \mathcal{B}w(t) + \mathcal{O}^\star(w(t)), \qquad t \geq 0, \\ w(0) &= g \in \mathcal{X}_0, \end{cases}$$

where $\mathcal{B} : \mathcal{X}_0 \to \mathcal{X} : g \mapsto [s \mapsto B(s)g(s)]$ and $\mathcal{O}^\star : \mathcal{X}_0 \to \mathcal{X} : g \mapsto [s \mapsto \rho(s, g(s))]$. Note that condition (G3) on G implies that $\mathcal{B}$ is bounded.
The following result allows to apply the results of Section 2 in the discussion of $(\mathcal{SAP})$.

Proposition 4.5.
There is a Hille-Yosida operator $\mathcal{A}$ on $\mathcal{X}$ such that $\mathcal{A}_0$ is the part of $\mathcal{A}$ in $\mathcal{X}_0$. Moreover, $\|\cdot\|_{\mathcal{X}_0}$ and $\|\cdot\|_{\mathcal{X}}$ are equivalent norms, on $\mathcal{X}_0$ and $\mathcal{X}_0 = \overline{D(\mathcal{A})}^{\|\cdot\|_{\mathcal{X}}}$.

Proof. Choose $\omega \in \mathbb{R}$ and $M \geq 1$ such that $\|T_0(t)\| \leq Me^{\omega t}$, $t \geq 0$. For $Re\lambda > \omega$ and $f \in \mathcal{X}$ set

$$(R(\lambda)f)(s) := \int_0^\infty e^{-\lambda\tau}(\mathcal{S}(\tau)f)(s)\, d\tau \quad \text{for } s \geq 0. \tag{4.2}$$

By Lemma 2.6 and the definition of $(\mathcal{S}(t))_{t\geq 0}$ we have

$$\begin{aligned} (R(\lambda)f)(s) &= \int_0^s e^{-\lambda\tau}T_{-1}(\tau)f(s-\tau)\, d\tau \in X_0 \quad \text{and} \\ \|(R(\lambda)f)(s)\|_{X_0} &\leq M_1 \int_0^s e^{-(\lambda-\omega)\tau}\|f(s-\tau)\|_{F_0}\, d\tau \quad \text{for } f \in \mathcal{X} \text{ and } s \geq 0. \end{aligned}$$

Thus $R(\lambda)$ is a bounded linear operator from $\mathcal{X}$ to $\mathcal{X}_0$.
By definition $R(\lambda)$ is the restriction of the resolvent of the generator of $(\mathcal{S}(t))_{t\geq 0}$ to $\mathcal{X}$. In particular, $(R(\lambda))_{Re\lambda>\omega}$ satisfies the resolvent identity and $\ker R(\lambda) = \{0\}$ for $Re\lambda > \omega$. As a consequence there is a linear operator $\mathcal{A}$ on $\mathcal{X}$ such that $R(\lambda) = R(\lambda, \mathcal{A})$ for $Re\lambda > \omega$ (see [11], Theorem 5.8.3) and $R(\lambda, \mathcal{A})\mathcal{X} = D(\mathcal{A}) \subseteq \mathcal{X}_0$.
In order to show that $\mathcal{A}$ is a Hille-Yosida operator we differentiate $\lambda \mapsto R(\lambda)$ $(n-1)$-times. Then

$$(-1)^{(n-1)}(n-1)!R(\lambda)^n f = \frac{d^{n-1}}{d\lambda^{n-1}}R(\lambda)f = (-1)^{n-1}\int_0^\infty \tau^{n-1}e^{-\lambda\tau}\mathcal{S}(\tau)f\, d\tau \tag{4.3}$$

for $Re\lambda > \omega$ and $f \in \mathcal{X}$. Lemma 2.6 yields

$$\begin{aligned} &\|(R(\lambda)^n f)(s)\|_{F_0} \leq c\|(R(\lambda)^n f)(s)\| \\ &= \frac{c}{(n-1)!}\left\|\int_0^\infty \tau^{n-1}e^{-\lambda\tau}(\mathcal{S}(\tau)f)(s)\, d\tau\right\| = \frac{c}{(n-1)!}\left\|\int_0^s \tau^{n-1}e^{-\lambda\tau}T_{-1}(\tau)f(s-\tau)\, d\tau\right\| \\ &\leq \frac{cM_1}{(n-1)!}\int_0^s \tau^{n-1}e^{-(\lambda-\omega)\tau}\|f(s-\tau)\|_{F_0}\, d\tau \leq \frac{cM_1}{(\lambda-\omega)^n}\|f\|_{\mathcal{X}} \end{aligned}$$

for constants c, M_1 independent of $f \in \mathcal{X}$, $n \in \mathbb{N}$ and $s \in \mathbb{R}_+$. Thus $\mathcal{A}$ is a Hille-Yosida operator on $\mathcal{X}$. If $g \in D(\mathcal{A}_0)$ and $Re\lambda > \omega$, then there exists $f \in \mathcal{X}_0$ such that

$$g = R(\lambda, \mathcal{A}_0)f = \int_0^\infty e^{-\lambda\tau}\mathcal{T}_0(\tau)f\, d\tau = \int_0^\infty e^{-\lambda\tau}\mathcal{S}(\tau)f\, d\tau = R(\lambda, \mathcal{A})f.$$

Hence

$$D(\mathcal{A}_0) \subseteq D(\mathcal{A}) \subseteq \mathcal{X}_0 \text{ and } \mathcal{A}_0 g = \mathcal{A}g \quad \text{for } g \in D(\mathcal{A}_0).$$

On the other hand, if $g \in D(\mathcal{A}) \subseteq \mathcal{X}_0$ satisfies $\mathcal{A}g \in \mathcal{X}_0$, then

$$g = R(\lambda, \mathcal{A})(\lambda - \mathcal{A})g = \lambda R(\lambda, \mathcal{A}_0)g - R(\lambda, \mathcal{A}_0)\mathcal{A}g \in D(\mathcal{A}_0)$$

for $Re\lambda > \omega$. Thus $\mathcal{A}_0$ is the part of $\mathcal{A}$ in $\mathcal{X}_0$.
Finally, it can be easily seen that for the generator of a C_0-semigroup the norm of the Favard class and the graph norm are equivalent on the domain of the generator. Applying this to A_{-1} and the Favard class F_0 we obtain that on X_0 the norms $\|\cdot\|_{X_0}$ and $\|\cdot\|_{F_0}$ coincide. Hence the corresponding sup-norms on $\mathcal{X}_0$ are equivalent. Since $D(\mathcal{A}_0) \subseteq D(\mathcal{A}) \subseteq \mathcal{X}_0$ and $\overline{D(\mathcal{A}_0)}^{\|\cdot\|_{\mathcal{X}_0}} = \mathcal{X}_0$ we obtain $\mathcal{X}_0 = \overline{D(\mathcal{A})}^{\|\cdot\|_{\mathcal{X}}}$ □

Let $\mathcal{X}_{-1}$ be the extrapolation space of $\mathcal{X}_0$ associated with $\mathcal{A}_0$ and denote by $\mathcal{F}_0 \subseteq \mathcal{X}_{-1}$ the corresponding extrapolated Favard class. The previous proposition and Lemma 2.5 (iv) yield

$$\mathcal{X}_0 \subseteq \mathcal{X} \hookrightarrow \mathcal{F}_0 \subseteq \mathcal{X}_{-1}. \tag{4.4}$$

Since $\mathcal{B} : \mathcal{X}_0 \to \mathcal{X}$ is bounded the operator $\mathcal{A}+\mathcal{B}$ with domain $D(\mathcal{A}+\mathcal{B}) = D(\mathcal{A})$ is a Hille-Yosida operator on $\mathcal{X}$ (see [18], Lemma 4.1.1) and its part $(\mathcal{A}+\mathcal{B})_0$ in $\mathcal{X}_0$ generates a C_0-semigroup $(\mathcal{T}_B(t))_{t\geq 0}$ on $\mathcal{X}_0$. Moreover, the extrapolation spaces of $\mathcal{X}_0$ associated with $\mathcal{A}_0$ and $(\mathcal{A}+\mathcal{B})_0$, respectively, are isomorphic and $(\mathcal{A}+\mathcal{B})_{-1} = \mathcal{A}_{-1}+\mathcal{B}$ (see [18], Lemma 4.1.4). If $(\mathcal{T}_{-1}(t))_{t\geq 0}$ is the extrapolated C_0-semigroup of $(\mathcal{T}_0(t))_{t\geq 0}$ on $\mathcal{X}_{-1}$ then $(\mathcal{T}_B(t))_{t\geq 0}$ is uniquely determined by the formula

$$\mathcal{T}_B(t)f = \mathcal{T}_0(t)f + \int_0^t \mathcal{T}_{-1}(t-\sigma)\mathcal{B}\mathcal{T}_B(\sigma)f\,d\sigma \tag{4.5}$$

for $t \geq 0$ and $f \in \mathcal{X}_0$.
We now show that in (4.5) we can replace the semigroup $(\mathcal{T}_{-1}(t))_{t\geq 0}$ by $(\mathcal{S}(t))_{t\geq 0}$.

Lemma 4.6.
Let $\varphi \in L^1_{loc}(\mathbb{R}_+, \mathcal{X})$. Then

$$\int_0^t \mathcal{T}_{-1}(t-\sigma)\varphi(\sigma)\,d\sigma = \int_0^t \mathcal{S}(t-\sigma)\varphi(\sigma)\,d\sigma \qquad \textit{for } t \geq 0. \tag{4.6}$$

Proof. Note that $\mathcal{T}_{-1}(t)f = \mathcal{T}_0(t)f = \mathcal{S}(t)f$ for $t \geq 0$ and $f \in \mathcal{X}_0$. If $\mathcal{D}$ denotes the generator of $(\mathcal{S}(t))_{t\geq 0}$, then by Lemma 2.3 and the definition of $\mathcal{A}$ we have $R(\lambda, \mathcal{A}) = R(\lambda, \mathcal{A}_{-1})_{|\mathcal{X}} = R(\lambda, \mathcal{D})_{|\mathcal{X}}$ for $\lambda > 0$ sufficiently large. Moreover, by Lemma 2.6 (applied to the semigroups $(\mathcal{T}_0(t))_{t\geq 0}$ and $(T_0(t))_{t\geq 0}$) both integrals in (4.6) belong to $\mathcal{X}_0$. Thus for $\lambda > 0$ sufficiently large and $t \geq 0$ we obtain

$$\begin{aligned}
&R(\lambda, \mathcal{A}_0)\int_0^t \mathcal{T}_{-1}(t-\sigma)\varphi(\sigma)\,d\sigma \\
&\quad = \int_0^t R(\lambda, \mathcal{A}_{-1})\mathcal{T}_{-1}(t-\sigma)\varphi(\sigma)\,d\sigma = \int_0^t \mathcal{T}_{-1}(t-\sigma)R(\lambda, \mathcal{A})\varphi(\sigma)\,d\sigma \\
&\quad = \int_0^t \mathcal{T}_0(t-\sigma)R(\lambda, \mathcal{A})\varphi(\sigma)\,d\sigma = \int_0^t \mathcal{S}(t-\sigma)R(\lambda, \mathcal{A})\varphi(\sigma)\,d\sigma \\
&\quad = \int_0^t R(\lambda, \mathcal{D})\mathcal{S}(t-\sigma)\varphi(\sigma)\,d\sigma = R(\lambda, \mathcal{A}_0)\int_0^t \mathcal{S}(t-\sigma)\varphi(\sigma)\,d\sigma.
\end{aligned} \tag{4.7}$$

Since $R(\lambda, \mathcal{A}_0)$ is injective the assertion follows. □

With Proposition 4.5 and Lemma 4.6 we are able to show how the C_0-semigroup $(\mathcal{T}_B(t))_{t\geq 0}$ and the evolution family $(U_B(t,s))_{t\geq s\geq 0}$ defined in (4.1) are connected. Set

$$\mathcal{S}_B(t)f(s) := \begin{cases} U_B(s,s-t)f(s-t), & s\geq t\geq 0,\\ 0, & 0\leq s<t, \end{cases}$$

for $f\in\mathcal{X}_0$. Then $(\mathcal{S}_B(t))_{t\geq 0}$ is a C_0-semigroup on $\mathcal{X}_0$, the so-called *evolution semigroup* induced by $(U_B(t,s))_{t\geq s\geq 0}$ (see [19], [26]).

Proposition 4.7.
The semigroups $(\mathcal{T}_B(t))_{t\geq 0}$ and $(\mathcal{S}_B(t))_{t\geq 0}$ coincide, i.e., $\mathcal{T}_B(t)=\mathcal{S}_B(t)$ for $t\geq 0$.

Proof. Since $(\mathcal{T}_B(t))_{t\geq 0}$ is uniquely determined by formula (4.5) it suffices to show that (4.5) holds with $(\mathcal{T}_B(t))_{t\geq 0}$ replaced by $(\mathcal{S}_B(t))_{t\geq 0}$. Fix $f\in\mathcal{X}_0$ and $t\geq 0$. If $s\geq t$, then (4.1) and Lemma 4.6 yield

$$\begin{aligned}
\mathcal{S}_B(t)f(s) &= U_B(s,s-t)f(s-t)\\
&= T_0(t)f(s-t)+\int_{s-t}^{s} T_{-1}(s-\sigma)B(\sigma)U_B(\sigma,s-t)f(s-t)\,d\sigma\\
&= \mathcal{T}_0(t)f(s)+\int_0^t T_{-1}(t-\sigma)B(s-t+\sigma)U_B(s-t+\sigma,s-t)f(s-t)\,d\sigma\\
&= \mathcal{T}_0(t)f(s)+\Big(\int_0^t \mathcal{S}(t-\sigma)\mathcal{B}\mathcal{S}_B(\sigma)f\,d\sigma\Big)(s) = \mathcal{T}_0(t)f(s)+\Big(\int_0^t \mathcal{T}_{-1}(t-\sigma)\Big)\mathcal{B}\mathcal{S}_B(\sigma)f\,d\sigma\Big)(s).
\end{aligned}$$

Similarly, if $t>s\geq 0$ we obtain

$$\mathcal{S}_B(t)f(s)=0=\mathcal{T}_0(t)f(s)+(\int_0^t \mathcal{T}_{-1}(t-\sigma)\mathcal{B}\mathcal{S}_B(\sigma)f\,d\sigma\Big)(s).$$

□

Step 3. Linearized stability for $(\mathcal{SAP})$

In order to obtain a principle of linearized stability for $(\mathcal{SAP})$, we need additional information on the nonlinearity $\mathcal{O}^\star$. The next lemma follows immediately from (G2) and the definition of $\mathcal{O}^\star$.

Lemma 4.8.
The mapping $\mathcal{O}^\star:\mathcal{X}_0\to\mathcal{X}$ satisfies $\mathcal{O}^\star(0)=0$ and for every $q>0$ there exists $r>0$ such that

$$\|\mathcal{O}^\star(g)-\mathcal{O}^\star(h)\|\leq q\|g-h\| \text{ for } g,h\in rB_{\mathcal{X}_0}.$$

Lemma 4.8 and (4.4) show that the map $\mathcal{B}+\mathcal{O}^\star$ satisfies condition (G1) with X_0 and F_0 replaced by $\mathcal{X}_0$ and $\mathcal{F}_0$, respectively. Remark 3.5 implies that for initial values $g\in\mathcal{X}_0$ sufficiently close to zero $(\mathcal{SAP})_g$ has a uniquely determined mild solution w defined on a fixed interval $[0,T]$, i.e., $w\in C([0,T],\mathcal{X}_0)$ and

$$w(t)=\mathcal{T}_0(t)g+\int_0^t \mathcal{T}_{-1}(t-\sigma)\Big(\mathcal{B}w(\sigma)+\mathcal{O}^\star(w(\sigma))\Big)\,d\sigma \quad \text{for } t\in[0,T].$$

Furthermore from Theorem 3.6 we obtain that $w \in C([0,T],\mathcal{X}_0)$ is a mild solution of $(\mathcal{SAP})_g$ if and only if

$$w(t) = \mathcal{T}_B(t)g + \int_0^t \mathcal{T}_B(t-\sigma)\mathcal{O}^\star(w(\sigma))\,d\sigma \qquad \text{for } t \in [0,T]. \tag{4.8}$$

The following principle of linearized stability for $(\mathcal{SAP})$ is now a direct consequence of Theorem 3.7.

Theorem 4.9.
Let the perturbed C_0-semigroup $(\mathcal{T}_B(t))_{t\geq 0}$ be exponentially stable. Then there exists $\epsilon > 0$ such that for $g \in \epsilon B_{\mathcal{X}_0}$ equation $(\mathcal{SAP})_g$ has a uniquely determined mild sloution $w(t;g)$ defined on $[0,\infty)$, and there are constants $M_0 \geq 0$ and $\omega_0 > 0$ such that $\|w(t;g)\| \leq M_0 e^{-\omega_0 t}$ for $t \geq 0$.

The next corollary connects exponential stability of the zero solution of $(\mathcal{SAP})$ and (LNP). The proof follows immediately from Theorem 4.9 and Proposition 4.7.

Corollary 4.10.
Let $\|U_B(t,s)\| \leq Me^{-\omega(t-s)}$, $t \geq s \geq 0$, for constants $M \geq 1$ and $\omega > 0$. Then the zero solution of $(\mathcal{SAP})$ is exponentially stable.

Step 4. Linearized stability for $(\boldsymbol{SNP})$

In the last step we derive from Corollary 4.10 a principle of linearized stability for (SNP). The following proposition shows how mild solutions of $(\mathcal{SAP})$ and (SNP') are connected.

Proposition 4.11.
Let $w(\cdot;g)$, $g \in \mathcal{X}_0$, be a mild solution of $(\mathcal{SAP})_g$ defined on the interval $[0,T]$. Then $t \mapsto w(t-s;g)(t)$ is a mild solution of $(SNP')_{s,g(s)}$ on $[s,s+T]$.
Conversely, let $\epsilon > 0$ such that $(\mathcal{SAP})_g$ has a unique mild solution $w(\cdot;g)$ defined on $[0,T]$ for every $g \in \epsilon B_{\mathcal{X}_0}$. If $u(\cdot;s;y)$ is a mild solution of $(SNP')_{s,y}$ on $[s,s+T]$ and $y \in \epsilon B_{X_0}$, then $u(t) = w(t-s;g)(t)$ for every $g \in \epsilon B_{X_0}$ such that $g(s) = y$.

Proof. Let $w(\cdot;g)$, $g \in \mathcal{X}_0$, be a mild solution of $(\mathcal{SAP})_g$ on the interval $[0,T]$, i.e.,

$$w(t;g) = \mathcal{T}_0(t)g + \int_0^t \mathcal{T}_{-1}(t-\sigma)(\mathcal{B}w(\sigma;g) + o^\star(w(\sigma;g)))\,d\sigma \text{ for } t \in [0,T].$$

Together with Lemma 4.6 this implies

$$w(t-s;g) = \mathcal{T}_0(t-s)g + \int_0^{t-s} \mathcal{S}(t-s-\sigma)\Big(\mathcal{B}w(\sigma;g) + o^\star(w(\sigma;g))\Big)\,d\sigma$$

for $t \in [s,s+T]$ and fixed $s \in \mathbb{R}_+$. If we evaluate this equation at $t \in [s,s+T]$ we obtain

$$\begin{aligned}
w(t-s;g)(t) &= T_0(t-s)g(s) \\
&\quad + \int_0^{t-s} T_{-1}(t-s-\sigma)\Big(B(s+\sigma)(w(\sigma;g)(s+\sigma)) + \rho(s+\sigma, w(s+\sigma;g)(s+\sigma))\Big)\,d\sigma \\
&= T_0(t-s)g(s) + \int_s^t T_{-1}(t-\sigma)\Big(B(\sigma)(w(\sigma;g)(\sigma)) + \rho(\sigma, w(\sigma;g)(\sigma))\Big)\,d\sigma,
\end{aligned}$$

i.e., $w(\cdot - s; g)(\cdot)$ is a mild solution of $(SNP')_{s,g(s)}$ on $[s, s+T]$. The second statement holds since the existence and uniqueness of a mild solution of $(\mathcal{SAP})_g$ is garuanteed by the discussion following Lemma 4.8. $\square$

The main result of this section is the following principle of linearized stability for (SNP) and the (fixed) mild solution v.

Theorem 4.12.
Let $(U_B(t,s))_{t\geq s\geq 0}$ be exponentially stable. Then the mild solution v of (SNP) is uniformly exponentially stable.

Proof. By Corollary 4.10 the zero solution of $(\mathcal{SAP})$ is exponentially stable. In particular, there exist constants $M \geq 0$, $\omega > 0$ and $\epsilon > 0$ such that for every $g \in \epsilon B_{X_0}$ there is a unique mild solution $w(\cdot; g) : \mathbb{R}_+ \to \mathcal{X}_0$ of $(\mathcal{SAP})_g$ and $\|w(t; g)\| \leq Me^{-\omega t}$, $t \geq 0$. From Proposition 4.11 we obtain that for $y \in \epsilon B_{X_0}$ and $s \geq 0$ the mild solution $u(\cdot; s; y)$ of $(SNP')_{s,y}$ is defined on $[s, \infty)$ and satisfies $\|u(t; s; y)\| \leq Me^{-\omega(t-s)}$, $t \geq s$. Thus, the zero solution of (SNP') is uniformly exponentially stable and hence, by Proposition 4.3, the mild solution v of (SNP) is uniformly exponentially stable. $\square$

5 A principle of linearized instability

As in the previous section, we assume that $v = v(\cdot; 0; x_0)$ is a fixed mild solution of (SNP) defined on $\mathbb{R}_+$ and that the nonlinearity G satisfies conditions (G2) and (G3). Moreover, as in Step 1 of Section 4, we pass to equation

$$(SNP') \quad \frac{d}{dt}u(t) = Au(t) + B(t)u(t) + \rho(t, u(t)), \quad t \geq s \geq 0,$$

and hence it suffices to study the zero solution of (SNP') instead of v. Let $(U_B(t,s))_{t\geq s\geq 0}$ be the evolution family associated with the linearized equation

$$(LNP) \quad \frac{d}{dt}u(t) = Au(t) + B(t)u(t), \quad t \geq s \geq 0,$$

(see Step 1 of Section 4). Our aim is to find conditions on $(U_B(t,s))_{t\geq s}$ such that the zero solution of (SNP') and hence v is unstable. The following concept plays a crucial role.

Definition 5.1.
An evolution family $(U(t,s))_{t\geq s\geq 0}$ on the Banach space Z is said to have an *exponential dichotomy* (with constants $\beta > \alpha$) if there exists a family of projections $(P(t))_{t\geq 0} \subseteq \mathcal{L}(Z)$ and constants M, $L \geq 1$ such that

(i) $P(t)U(t,s) = U(t,s)P(s)$,

(ii) $\|U(t,s)z\| \leq Me^{\alpha(t-s)}\|z\|$ for all $z \in P(s)Z$,

(iii) $\|U(t,s)z\| \geq Le^{\beta(t-s)}\|z\|$ for all $z \in (Id - P(s))Z$,

(iv) the map $U_|(t,s) : (Id - P(s))Z \to (Id - P(t))Z$: $z \mapsto U(t,s)z$ is invertible for every $t \geq s \geq 0$.

Note that in [19], Lemma 4.2, it is shown that the family $(P(t))_{t\geq 0}$ appearing in Definition 5.1 is automatically bounded and strongly continuous.
Next we introduce a Green's operator function associated with $(U(t,s))_{t\geq s\geq 0}$.

Definition 5.2.
Let $(U(t,s))_{t\geq s\geq 0}$ be an exponentially dichotomic evolution family on the Banach space Z with corresponding family of projections $(P(t))_{t\geq 0} \subseteq \mathcal{L}(Z)$. Then $\Gamma_U(\cdot) : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathcal{L}(Z)$ given by

$$(t,s) \mapsto \begin{cases} U(t,s)P(s) & \text{for } t \geq s \geq 0 \\ -[U_|(s,t)]^{-1}(Id - P(s)) & \text{for } 0 \leq t < s \end{cases}$$

is called *the Green's operator function* corresponding to $(U(t,s))_{t\geq s\geq 0}$ and $(P(t))_{t\geq 0}$.

A variant of the following lemma can be found in [13], Proposition 1.2. The proofs are essentially the same and straightforward.

Lemma 5.3.
Let $(U(t,s))_{t\geq s\geq 0}$ be an exponentially dichotomic evolution family on the Banach space Z with constants $\beta > 0 > \alpha$ and corresponding family of projections $(P(t))_{t\geq 0}$. Then for every $z_s \in P(s)Z$ and every bounded continuous function $f : [s,\infty) \to Z$ there is a unique bounded continuous function $z : [s,\infty) \to Z$ such that

$$z(t) = U(t,s)z(s) + \int_s^t U(t,\sigma)f(\sigma)\,d\sigma \text{ for } t \geq s$$

and $P(s)z(s) = z_s$. Moreover, the function $z(\cdot)$ is given by

$$z(t) = U(t,s)P(s)z(s) + \int_s^\infty \Gamma_U(t,\sigma)f(\sigma)\,d\sigma \text{ for } t \geq s.$$

As in Section 4 we first turn from the linearized equation (LNP) to the autonomous nonlinear equation $(\mathcal{SAP})$. Assuming that $(U_B(t,s))_{t\geq s\geq 0}$ (i.e., the family of mild solutions of (LNP)) has an exponential dichotomy with constant $\beta > 0$ we will prove instability of the zero solution of $(\mathcal{SAP})$. In order to exploit the lower exponential estimate of $(U_B(t,s))_{t\geq s\geq 0}$, it is advantageous to rewrite formula (4.8). Recall that $\mathcal{C} := \mathcal{A}+\mathcal{B}$ is a Hille-Yosida operator on $\mathcal{X}$ and the part $\mathcal{C}_0$ of $\mathcal{C}$ in $\mathcal{X}_0$ generates a C_0-semigroup (see Section 3, Step 2). Thus $f = \lim_{\lambda\to\infty} \lambda R(\lambda,\mathcal{C}_0)f = \lim_{\lambda\to\infty} \lambda R(\lambda,\mathcal{C})f$ for $f \in \mathcal{X}_0$. If $w(\cdot;g)$, $g \in \mathcal{X}_0$, is a mild solution of $(\mathcal{SAP})_g$ on $[0,T]$, this together with Proposition 3.6 leads to

$$\begin{aligned} w(t;g) &= \lim_{\lambda\to\infty} \lambda R(\lambda,\mathcal{C})\mathcal{T}_B(t)g + \lim_{\lambda\to\infty} \lambda R(\lambda,\mathcal{C}) \int_0^t (\mathcal{T}_B)_{-1}(t-\sigma)\mathcal{O}^\star(w(\sigma;g))\,d\sigma \\ &= \lim_{\lambda\to\infty} \mathcal{T}_B(t)\lambda R(\lambda,\mathcal{C})g + \lim_{\lambda\to\infty} \int_0^t \lambda R(\lambda,\mathcal{C}_{-1})(\mathcal{T}_B)_{-1}(t-\sigma)\mathcal{O}^\star(w(\sigma;g))\,d\sigma \\ &= \lim_{\lambda\to\infty} \mathcal{T}_B(t)\lambda R(\lambda,\mathcal{C})g + \lim_{\lambda\to\infty} \int_0^t \mathcal{T}_B(t-\sigma)\lambda R(\lambda,\mathcal{C})\mathcal{O}^\star(w(\sigma;g))\,d\sigma \quad \text{for } t \in [0,T]. \end{aligned}$$

Evaluating $w(t-s;g)$ at t we obtain with Proposition 4.7

$$
\begin{aligned}
&w(t-s;g)(t)\\
&\quad= \lim_{\lambda\to\infty} U_B(t,s)\Big(\lambda R(\lambda,\mathcal{C})g\Big)(s) + \lim_{\lambda\to\infty}\int_0^{t-s} \mathcal{T}_B(t-s-\sigma)\Big(\lambda R(\lambda,\mathcal{C})\mathcal{O}^\star(w(\sigma;g))\Big)(t)\,d\sigma\\
&\quad= \lim_{\lambda\to\infty} U_B(t,s)\Big(\lambda R(\lambda,\mathcal{C})g\Big)(s) \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (5.1)\\
&\qquad+ \lim_{\lambda\to\infty}\int_0^{t-s} U_B(t,s+\sigma)\Big(\lambda R(\lambda,\mathcal{C})\mathcal{O}^\star(w(\sigma;g))\Big)(s+\sigma)\,d\sigma\\
&\quad= \lim_{\lambda\to\infty} U_B(t,s)\Big(\lambda R(\lambda,\mathcal{C})g\Big)(s) + \lim_{\lambda\to\infty}\int_s^{t} U_B(t,\sigma)\Big(\lambda R(\lambda,\mathcal{C})\mathcal{O}^\star(w(\sigma-s;g))\Big)(\sigma)\,d\sigma
\end{aligned}
$$

for $0\le s\le t$ in $[0,T]$ and

$$
\begin{aligned}
w(t-s;g)(t) &= \lim_{\lambda\to\infty}\int_{-s}^{t-s} U_B(t,s+\sigma)\Big(\lambda R(\lambda,\mathcal{C})\mathcal{O}^\star(w(\sigma;g))\Big)(s+\sigma)\,d\sigma \qquad (5.2)\\
&= \lim_{\lambda\to\infty}\int_0^{t} U_B(t,\sigma)\Big(\lambda R(\lambda,\mathcal{C})\mathcal{O}^\star(w(\sigma-s;g))\Big)(\sigma)\,d\sigma
\end{aligned}
$$

for $t\in[0,T]$ and $s<0$. Now, with (5.1) and (5.2), we can show instability of the zero solution of $(\mathcal{SAP})$ in the following situation.

Theorem 5.4.
If the EVF $(U_B(t,s))_{t\ge s\ge 0}$ has an exponential dichotomy with constants $\beta>\alpha$, $\beta>0$, and projections $(P(t))_{t\ge 0}$ such that $\ker P(s_0)\neq 0$ for some $s_0\in\mathbb{R}_+$, then the zero solution of $(\mathcal{SAP})$ is unstable.

Proof. There exist constants $L,M\ge 1$ such that $\|U_B(t,s)P(s)x\|\le Me^{\alpha(t-s)}\|P(s)x\|$ and $\|U_B(t,s)(Id-P(s))x\|\ge Le^{\beta(t-s)}\|(Id-P(s))x\|$ for $t\ge s\ge 0$ and $x\in X_0$. Choose $\gamma>0$ such that $\beta>\gamma>\alpha$ and define the rescaled EVF $W(t,s):=e^{-\gamma(t-s)}U_B(t,s)$, $t\ge s\ge 0$. Note that $(W(t,s))_{t\ge s\ge 0}$ has an exponential dichotomy with constants $\beta-\gamma>0>\alpha-\gamma$. Let $C:=\sup_{0\le t<\infty}\{\|P(t)\|,\|Id-P(t)\|\}$ and $M_0:=\overline{\lim_{\lambda\to\infty}}\|\lambda R(\lambda,\mathcal{C})\|$. Now take $0<q<(2CM_0\{\frac{M}{\gamma-\alpha}+\frac{L}{\beta-\gamma}\})^{-1}$. By Lemma 4.8 there exists $r>0$ such that $\|\mathcal{O}^\star(g)\|\le q\|g\|$ for $g\in rB_{\mathcal{X}_0}$. We prove the theorem by contradiction, i.e., we assume that the zero solution of $(\mathcal{SAP})$ is stable. In particular, there is $\delta>0$ such that each mild solution $w(\cdot;g)$ of $(\mathcal{SAP})_g$, $g\in\delta B_{\mathcal{X}_0}$, is defined on $[0,\infty)$ and satisfies $\|w(\cdot;g)\|\le r$. Consider the projection $\mathcal{Q}$ on $\mathcal{X}_0$ defined by $(\mathcal{Q}g)(s)=(Id-P(s))g(s)$, $s\ge 0$. Then from the representation of $(\mathcal{T}_B(t))_{t\ge 0}$ given in Proposition 4.7 and property (i) of Definition 5.1 it follows that $\mathcal{Q}\mathcal{T}_B(t)=\mathcal{T}_B(t)\mathcal{Q}$, $t\ge 0$. As a consequence $\mathcal{Q}R(\lambda,\mathcal{C}_0)=R(\lambda,\mathcal{C}_0)\mathcal{Q}$ for $\lambda\in\rho(\mathcal{C}_0)$. Now choose $0\neq g_0\in\delta B_{\mathcal{X}_0}$ such that $g_0(s)\in(Id-P(s))X_0$ for every $s\ge 0$. Then $R(\lambda,\mathcal{C}_0)g_0\neq 0$ and $R(\lambda,\mathcal{C}_0)g_0(s)\in(Id-P(s))X_0$ for $\lambda\in\rho(\mathcal{C}_0)$ and $s\ge 0$. If $s\in\mathbb{R}$ and $\lambda\in\rho(\mathcal{C}_0)$ then $z=e^{-\gamma(\cdot-s)}\Big(\lambda R(\lambda,\mathcal{C}_0)w(\cdot-s;g_0)\Big)(\cdot)$ and $f=e^{-\gamma(\cdot-s)}\Big(\lambda R(\lambda,\mathcal{C}_0)\mathcal{O}^\star(w(\cdot-s;g_0))\Big)(\cdot)$ are bounded continuous X_0-valued functions on $[\max\{s,0\},\infty)$. Moreover, as in (5.1) and (5.2) we obtain

$$
e^{-\gamma(t-s)}\Big(\lambda R(\lambda,\mathcal{C}_0)w(t-s;g_0)\Big)(t)
$$

$$
= \begin{cases} W(t,s)\Big(\lambda R(\lambda,\mathcal{C}_0)g_0\Big)(s) \\ \quad + \displaystyle\int_s^t W(t,\sigma)e^{-\gamma(t-s)}\Big(\lambda R(\lambda,\mathcal{C})\mathcal{O}^\star(w(\sigma-s;g_0))\Big)(\sigma)\,d\sigma & \text{for } t\geq s\geq 0 \\ \\ \displaystyle\int_0^t W(t,\sigma)e^{-\gamma(t-s)}\Big(\lambda R(\lambda,\mathcal{C})\mathcal{O}^\star(w(\sigma-s;g))\Big)(\sigma)\,d\sigma & \text{for } t\geq 0 \text{ and } s<0. \end{cases}
$$

Thus we can apply Lemma 5.3 to z, f and $(W(t,s))_{t\geq s\geq 0}$ and hence

$$
\begin{aligned}
&e^{-\gamma(t-s)}\Big(\lambda R(\lambda,\mathcal{C}_0)w(t-s;g_0)\Big)(t) \\
&\quad = \begin{cases} \displaystyle\int_s^\infty \Gamma_W(t,\sigma)e^{-\gamma(\sigma-s)}\Big(\lambda R(\lambda,\mathcal{C})\mathcal{O}^\star(w(\sigma-s;g_0))\Big)(\sigma)\,d\sigma & \text{for } t\geq s\geq 0 \\ \displaystyle\int_0^\infty \Gamma_W(t,\sigma)e^{-\gamma(\sigma-s)}\Big(\lambda R(\lambda,\mathcal{C})\mathcal{O}^\star(w(\sigma-s;g_0))\Big)(\sigma)\,d\sigma & \text{for } t\geq 0 \text{ and } s<0. \end{cases}
\end{aligned}
$$

By taking norms and passing to $\lambda\to\infty$ this leads to

$$
\begin{aligned}
\sup_{t\geq s\geq 0}\|e^{-\gamma(t-s)}w(t-s;g_0)(t)\| &= \sup_{t\geq s\geq 0}\lim_{\lambda\to\infty}\|e^{-\gamma(t-s)}\lambda R(\lambda,\mathcal{C}_0)w(t-s;g_0)(t)\| \\
&\leq \sup_{t\geq s\geq 0}\lim_{\lambda\to\infty}\{\int_s^t \|W(t,\sigma)P(\sigma)\|e^{-\gamma(\sigma-s)}\|\lambda R(\lambda,\mathcal{C})\|_{\mathcal{L}(\mathcal{X},\mathcal{X})}q\|w(\sigma-s;g_0)\|\,d\sigma \\
&\quad + \int_t^\infty \|W_|(\sigma,t)^{-1}(Id-P(\sigma))\|e^{-\gamma(\sigma-s)}\|\lambda R(\lambda,\mathcal{C})\|_{\mathcal{L}(\mathcal{X},\mathcal{X})}q\|w(\sigma-s;g_0)\|\,d\sigma\} \\
&\leq CqM_0\sup_{0\leq\tau<\infty}\|e^{-\gamma\tau}w(\tau;g_0)\|\{\frac{M}{\gamma-\alpha}+\frac{L}{\beta-\gamma}\}
\end{aligned}
$$

and

$$
\begin{aligned}
\sup_{s<0,t\geq 0}\|e^{-\gamma(t-s)}w(t-s;g_0)(t)\| &= \sup_{s<0,t\geq 0}\lim_{\lambda\to\infty}\|e^{-\gamma(t-s)}\lambda R(\lambda,\mathcal{C}_0)w(t-s;g_0)(t)\| \\
&\leq \sup_{s<0,t\geq 0}\lim_{\lambda\to\infty}\{\int_0^t\|W(t,\sigma)P(\sigma)\|e^{-\gamma(\sigma-s)}\|\lambda R(\lambda,\mathcal{C})\|_{\mathcal{L}(\mathcal{X},\mathcal{X})}q\|w(\sigma-s;g_0)\|\,d\sigma \\
&\quad + \int_t^\infty \|W_|(\sigma,t)^{-1}(Id-P(\sigma))\|e^{-\gamma(\sigma-s)}\|\lambda R(\lambda,\mathcal{C})\|_{\mathcal{L}(\mathcal{X},\mathcal{X})}q\|w(\sigma-s;g_0)\|\,d\sigma\} \\
&\leq CqM_0\sup_{0\leq\tau<\infty}\|e^{-\gamma\tau}w(\tau;g_0)\|\{\frac{M}{\gamma-\alpha}+\frac{L}{\beta-\gamma}\}.
\end{aligned}
$$

Thus

$$
\sup_{s\in\mathbb{R},t\geq\max\{s,0\}}\|e^{-\gamma(t-s)}w(t-s;g_0)(t)\| \leq \frac{1}{2}\sup_{0\leq\tau<\infty}\|e^{-\gamma\tau}w(\tau;g_0)\|_{\mathcal{X}_0}. \tag{5.3}
$$

Inequality (5.3) can only be true if $w(\cdot;g_0)\equiv 0$ which is a contradiction to $g_0\neq 0$. □

We now state the main result of this section. It is a principle of linearized instability for (SNP) and the fixed mild solution v.

Theorem 5.5.
If the EVF $(U_B(t,s))_{t\geq s\geq 0}$ has an exponential dichotomy with constants $\beta>\alpha$, $\beta>0$, and projections $(P(t))_{t\geq 0}$ such that $\ker P(s_0)\neq 0$ for some $s_0\in\mathbb{R}_+$, then the zero solution of (SNP) is unstable.

Proof. By condition (G2) there exist q, $r > 0$ such that $\|\rho(t,x)\| \le q\|x\|$ for $t \ge 0$ and $x \in rB_{X_0}$ (see Section 4, Step 1). Moreover, by Theorem 5.4 the zero solution of $(\mathcal{SAP})$ is unstable. In particular, there exists $0 < \epsilon \le r$ such that for every $\delta > 0$ one can find $0 \ne g_\delta \in \delta B_{\mathcal{X}_0}$ and $t_\delta > 0$ such that the mild solution $w(\cdot; g_\delta)$ of $(\mathcal{SAP})_{g_\delta}$ satisfies $\|w(t_\delta; g_\delta)\| = \epsilon$ and $\|w(t; g_\delta)\| \le \epsilon$ for $0 \le t \le t_\delta$. Choose $s_\delta \in \mathbb{R}_+$ such that $\|w(t_\delta; g_\delta)\| = \|w(t_\delta; g_\delta)(s_\delta)\|$. We claim that $s_\delta \ge t_\delta$. In fact, let $0 \le s < t_\delta$. Choose $M \ge 1$ and $\omega \in \mathbb{R}$ such that $\|T_0(t)\| \le Me^{\omega t}$, set $C := \sup_{t\ge 0}\|B(t)\|$ and let M_1 be as in Lemma 2.6. Then

$$
\begin{aligned}
&\|w(s + t_\delta - s; g_\delta)(s)\| = \|w(t_\delta; g_\delta)(s)\| \\
&\quad = \|(\mathcal{T}_0(t_\delta)g_\delta)(s) + \int_0^{t_\delta} \mathcal{T}_{-1}(t_\delta - \sigma)(\mathcal{B}w(\sigma; g_\delta) + \mathcal{O}^\star(w(\sigma; g_\delta)))\, d\sigma(s)\| \\
&\quad \le M_1 \int_0^s e^{\omega(s-\sigma)} \|B(\sigma)w(\sigma + t_\delta - s; g_\delta)(\sigma) + \rho(\sigma, w(\sigma + t_\delta - s; g_\delta))\|\, d\sigma \\
&\quad \le M_1 \int_0^s e^{\omega(s-\sigma)} (C + q)\|w(\sigma + t_\delta - s; g_\delta)(\sigma)\|\, d\sigma.
\end{aligned}
$$

An application of Gronwall's inequality yields $w(t_\delta; g_\delta)(s) = 0$, and the claim follows. From Proposition 4.11 we know that $t \mapsto u(t) := w(t - s_\delta + t_\delta; g_\delta)(t)$ is a mild solution of $(SNP')_{s_\delta - t_\delta, g(s_\delta - t_\delta)}$. Since $g(s_\delta - t_\delta) \in \delta B_{X_0}$ and $\|u(s_\delta)\| = \epsilon$, we obtain instability of the zero solution of (SNP'). Hence, by Proposition 4.3, v is an unstable solution of (SNP). $\square$

6 Non-autonomous retarded differential equations

In this section we apply the results obtained for (SNP) to retarded differential equations. Throughout the whole section Y is a fixed Banach space and we set $E := C([-1,0], Y)$. For $s \ge 0$, a function $v \in C([s-1,\infty), Y)$ and $t \ge s$ we define $v_t \in E$ by $v_t(r) := v(t + r)$, $r \in [-1, 0]$. We consider the non-autonomous semilinear retarded differential equation

$$
(RSNP_a)_{s,g} \qquad \left\{ \begin{array}{ll} \frac{d}{dt}v(t) &= Bv(t) + K(t, v_t), \qquad t \ge s \ge 0, \\ \qquad v_s &= g \in C([-1,0], Y), \end{array} \right.
$$

where $(B, D(B))$ is the generator of a C_0-semigroup $(S(t))_{t\ge 0}$ on Y and $K : \mathbb{R}_+ \times E \to Y$ is a continuous nonlinear mapping such that

(K1) $K(t, 0) = 0$ for every $t \in \mathbb{R}_+$,

(K2) K is Fréchet-differentiable at $0 \in E$ with Fréchet-derivative $DK(t, 0)$ and for every $q > 0$ there exists $r > 0$ such that $\|K(t, h_1) - K(t, h_2) - DK(t, 0)(h_1 - h_2)\|_Y \le q\|h_1 - h_2\|_E$ for $t \in \mathbb{R}_+$ and $h_1, h_2 \in rB_E$,

(K3) the map $\mathbb{R}_+ \ni t \mapsto DK(t, 0)$ is strongly continuous and bounded.

A comprehensive discussion of retarded differential equations can be found in [9]. For results connected with existence, uniqueness and stability of solutions of $(RSNP_a)$ we also refer to [21], [29] in the autonomous case and to [6], [7], [23] in the non-autonomous situation. We now give the definition of a mild solution of $(RSNP_a)$ (cf. [21]).

Definition 6.1.
A function $v(\cdot;s;g) \in C([s-1,T],Y)$, $T > s \geq 0$, is called a *mild solution* of $(RSNP_a)_{s,g}$ on the interval $[s-1,T]$ if

$$\begin{cases} v(t;s;g) = & S(t-s)g(0) + \int_s^t S(t-\sigma)K(\sigma, v_\sigma(\cdot;s;g))\,d\sigma, \qquad t \in [s,T], \\ v(t;s;g) = & g(t-s), \qquad t \in [s-1,s]. \end{cases}$$

We now transform $(RSNP_a)$ into another equation such that we can apply the results developed in the previous sections. For that we set $X := Y \times E$ and consider the equation

$$(RSNP_b)_{s,x} \quad \begin{cases} \frac{d}{dt}u(t) &= Au(t) + G(t,u(t)), \qquad t \geq s \geq 0, \\ u(s) &= x \in E, \end{cases}$$

where $A : D(A) \to X$ is the linear operator on X given by

$$A\begin{pmatrix} 0 \\ f \end{pmatrix} = \begin{pmatrix} -f'(0) + Bf(0) \\ f' \end{pmatrix}, D(A) := \{\begin{pmatrix} 0 \\ f \end{pmatrix} \in \{0\} \times E : f \in C^1([-1,0],Y), f(0) \in D(B)\},$$

and $G : \mathbb{R}_+ \times \overline{D(A)} \to X : (t, \binom{0}{f}) \mapsto \binom{K(t,f)}{0}$. Note that the conditions on K imply that G satisfies (G2) and (G3) (see Section 4). Moreover, in [24] it is shown that $(A, D(A))$ is a Hille-Yosida operator on X and for $\lambda \in \rho(B) \cap \rho(A_0)$ the resolvent $R(\lambda; A)$ is given by

$$R(\lambda, A)\begin{pmatrix} x \\ f \end{pmatrix} = \begin{pmatrix} 0 \\ e_\lambda \otimes R(\lambda, B)x + R(\lambda, A_0)f \end{pmatrix}, \tag{6.1}$$

where $e_\lambda \otimes y : [-1,0] \to Y : s \mapsto e^{\lambda s}y$ for $y \in Y$. It is known that on the Banach space E the operator A_0 defined by

$$A_0 f := f', \quad D(A_0) := \{f \in C^1([-1,0],Y) : f(0) \in D(B) \text{ and } f'(0) = Bf(0)\},$$

is the generator of a C_0-semigroup $(T_0(t))_{t\geq 0}$ given by

$$(T_0(t)f)(r) = \begin{cases} f(t+r) & \text{if } t+r \leq 0, \\ S(t+r)f(0) & \text{if } t+r > 0, \end{cases}$$

(see [17] B.IV.3.1). Obviously, the operator A_0 can be identified with the part of A on $\{0\} \times E = \overline{D(A)}$. Therefore we do not distinguish between $(T_0(t))_{t\geq 0}$ and the C_0-semigroup generated by the part of A on $\overline{D(A)}$. From Theorem 3.4 we know that for every $s \geq 0$ and every $T > s$ there is $\epsilon > 0$ such that $(RSNP_b)_{s,g}$ has a unique mild solution on $[s,T]$ for every $g \in \epsilon B_E$. Note that $u \in C([s,T],E)$ is a mild solution of $(RSNP_b)_{s,g}$ if

$$\begin{pmatrix} 0 \\ u(t) \end{pmatrix} = T_0(t-s)\begin{pmatrix} 0 \\ g \end{pmatrix} + \int_s^t T_{-1}(t-\sigma)G(\sigma, u(\sigma))\,d\sigma, \quad t \in [s,T]. \tag{6.2}$$

We show how mild solutions of $(RSNP_b)$ are connected with mild solutions of $(RSNP_a)$.

Theorem 6.2.
If $u \in C([s,T],E)$ satisfies equation (6.2) for all $t \in [s,T]$, then $t \mapsto u(t)(0)$ is a mild solution of $(RSNP_a)_{s,g}$. Conversely, if $t \mapsto v(t)$ is a mild solution of $(RSNP_a)_{s,g}$ on $[s-1,T]$, then $t \mapsto v_t$ satisfies equation (6.2).

Proof. At first we look at the expression

$$\int_0^t T_{-1}(\sigma)\binom{x}{0}\,d\sigma \in \{0\}\times E \text{ for } x\in Y \text{ and } t\geq 0.$$

Since A is a Hille-Yosida operator, we have

$$\begin{aligned}\int_0^t T_{-1}(\sigma)\binom{x}{0}\,d\sigma &= \lim_{\lambda\to\infty}\lambda R(\lambda,A_0)\int_0^t T_{-1}(\sigma)\binom{x}{0}\,d\sigma\\ &= \lim_{\lambda\to\infty}\int_0^t T_0(\sigma)\lambda R(\lambda,A)\binom{x}{0}\,d\sigma\end{aligned}$$

for $x\in Y$ and $t\geq 0$. If $\tau\in[-1,0]$ and pr_E is the coordinate projection from X onto E, we obtain from (6.1)

$$\begin{aligned}&pr_E(T_0(\sigma)\lambda R(\lambda,A)\binom{x}{0})(\tau)\\ &\quad= (T_0(\sigma)\lambda e_\lambda\otimes R(\lambda,B)x)(\tau) = \begin{cases}\lambda e^{\lambda(\tau+\sigma)}R(\lambda,B)x & \text{if } \tau+\sigma\leq 0,\\ S(\tau+\sigma)\lambda R(\lambda,B)x & \text{if } \tau+\sigma>0.\end{cases}\end{aligned}$$

This yields

$$\begin{aligned}&pr_E(\int_0^t T_0(\sigma)\lambda R(\lambda,A)\binom{x}{0}\,d\sigma)(\tau)\\ &\quad=\begin{cases}R(\lambda,B)(e^{\lambda(t+\tau)}-e^{\lambda\tau})x & \text{if } t+\tau\leq 0,\\ R(\lambda,B)(1-e^{\lambda\tau})x+\int_0^{t+\tau}S(\sigma)\lambda R(\lambda,B)x\,d\sigma & \text{if } t+\tau>0.\end{cases}\end{aligned}$$

Passing to the limit $\lambda\to\infty$ we obtain

$$pr_E(\int_0^t T_{-1}(\sigma)\binom{x}{0}\,d\sigma)(\tau)=\begin{cases}0 & \text{if } t+\tau\leq 0,\\ \int_0^{t+\tau}S(\sigma)x\,d\sigma & \text{if } t+\tau>0.\end{cases}$$

Now let $u\in C([s,T],E)$ such that (6.2) holds. Then for $t\in[s,T]$ and $\tau\in[-1,0]$ we have

$$\begin{aligned}(u(t))(\tau) &= (T_0(t-s)g)(\tau)+pr_E(\frac{d}{dt}\int_s^t\int_s^r T_{-1}(r-\sigma)G(\sigma,u(\sigma))\,d\sigma\,dr)(\tau)\\ &= (T_0(t-s)g)(\tau)+pr_E(\frac{d}{dt}\int_s^t\int_\sigma^t T_{-1}(r-\sigma)G(\sigma,u(\sigma))\,dr\,d\sigma)(\tau)\\ &= (T_0(t-s)g)(\tau)+pr_E(\frac{d}{dt}\int_s^t\int_0^{t-\sigma} T_{-1}(r)G(\sigma,u(\sigma))\,dr\,d\sigma)(\tau)\\ &= \begin{cases}(T_0(t-s)g)(\tau) & \text{if } t+\tau\leq s,\\ (T_0(t-s)g)(\tau)+\frac{d}{dt}\int_s^{t+\tau}\int_0^{t-\sigma+\tau}S(r)K(\sigma,u(\sigma))\,dr\,d\sigma & \text{if } t+\tau>s.\end{cases}\end{aligned}\tag{6.3}$$

From (6.3) we obtain for $\tau=0$ and $t\in[s,T]$

$$\begin{aligned}[u(t)](0) &= S(t-s)g(0)+\frac{d}{dt}\int_s^t\int_0^{t-\sigma}S(r)K(\sigma,u(\sigma))\,dr\,d\sigma\\ &= S(t-s)g(0)+\frac{d}{dt}\int_s^t\int_\sigma^t S(r-\sigma)K(\sigma,u(\sigma))\,dr\,d\sigma\\ &= S(t-s)g(0)+\int_s^t S(t-\sigma)K(\sigma,u(\sigma))\,d\sigma.\end{aligned}$$

We define

$$z(t) = \begin{cases} u(t)(0) & \text{for } t \in [s,T], \\ g(t-s) & \text{for } t \in [s-1,s]. \end{cases}$$

With (6.3) we obtain

$$z_\sigma(\tau) = z(\sigma+\tau) = u(\sigma)(\tau) \text{ for } \tau \in [-1,0] \text{ and } \sigma \in [s,T].$$

Thus $z_\sigma(\cdot) = u(\sigma)(\cdot)$ for $\sigma \in [s,T]$ and hence z is a mild solution of $(RSNP_a)$. The second statement is proved in the same way. □

Corollary 6.3.
If $(U(t,s))_{t\geq s\geq 0}$ is the evolution family on E determined by the integral formula

$$\begin{pmatrix} 0 \\ U(t,s)g \end{pmatrix} = T_0(t-s)\begin{pmatrix} 0 \\ g \end{pmatrix} + \int_s^t T_{-1}(t-\sigma)DG(\sigma,0)\begin{pmatrix} 0 \\ U(\sigma,s)g \end{pmatrix} d\sigma \tag{6.4}$$

for $t \geq s \geq 0$, $g \in E$ and $DG(\sigma,0)\binom{0}{h} = \binom{DK(\sigma,0)h}{0}$, $h \in E$, then the map $t \mapsto [U(t,s)g](0)$ is the unique mild solution of the linear problem

$$(RLNP)_{s,g} \quad \begin{cases} \frac{d}{dt}v(t) = Bv(t) + DK(t,0)v_t, & t \geq s \geq 0, \\ v(s) = g \in C([-1,0],Y). \end{cases}$$

On the other hand, if $t \mapsto v(t)$ is a mild solution of $(RLNP)_{s,g}$, then $t \mapsto v_t$ satisfies equation (6.4).

Proof. By (4.1) the operator family $(U(t,s))_{t\geq s}$ defined by (6.4) is an evolution family. The assertion follows by applying Theorem 6.2 to $(RLNP)$ instead of $(RSNP_a)$. □

We are now in a position to prove the following principle of linearized stability. It should be clear how uniform exponential stability of a mild solution of $(RSNP_a)$ is defined.

Theorem 6.4.
If the zero solution of (RLNP) is uniformly exponentially stable, then the zero solution of $(RSNP_a)$ is uniformly exponentially stable.

Proof. Let the zero solution of $(RLNP)$ be uniformly exponentially stable. Then we can find $\delta > 0$, $M \geq 1$ and $\omega > 0$ such that for every $g \in \delta B_{C([-1,0],Y)}$ and every $s \geq 0$ the mild solution $v(\cdot;s;g)$ of $(RLNP)_{s,g}$ is defined on $[s-1,\infty)$ and

$$\|v(t+r;s;g)\|_Y \leq Me^{-\omega(t+r-s)} \text{ for } t \geq s \text{ and } r \in [-1,0].$$

Together with Corollary 6.3 this implies

$$\|U(t,s)g\|_E = \|v_t(\cdot;s;g)\|_E \leq \tilde{M}e^{-\omega(t-s)} \text{ for } t \geq s \text{ and a constant } \tilde{M} \geq 1,$$

where $(U(t,s))_{t\geq s\geq 0}$ is the evolution family given by formula (6.4). Combining Theorem 4.12 and Theorem 6.2 we obtain the desired result. □

As a concrete example we discuss the retarded differential equation

$$(E)\quad \begin{cases} \frac{\partial}{\partial t}v(x,t) = & -\frac{\partial}{\partial x}(g(x)v(x,t)) + k(t)f(v(x,t-1)), \quad 0 \le x \le 1, \quad t \ge s \ge 0, \\ v(x,t) = & \varphi(x,t-s), \quad 0 \le x \le 1, \quad s-1 \le t \le s, \end{cases}$$

with initial value $\varphi \in C([0,1]\times[-1,0])$. We assume that $g \in C^1[0,1]$, $g(0)=0$, $g(x)>0$ for $0 < x \le 1$, $\int_0^1 \frac{1}{g(s)}\,ds = \infty$, $f : \mathbb{R} \to \mathbb{R}$ is continuously differentiable in a neighborhood of 0, $f(0)=0$ and $k : \mathbb{R}_+ \to \mathbb{R}$ is continuous. For $k \equiv c \in \mathbb{R}$, a similar equation is studied in [5]. There it is shown that on the Banach space $Y = C[0,1]$ the operator $(B, D(B))$ defined by

$$(B\psi)(x) = \begin{cases} -(g(x)\psi(x))' & \text{if } 0 < x \le 1, \\ (B\psi)(0) = -g'(0)\psi(0) & \text{if } x = 0, \end{cases}$$

$$D(B) = \{\psi \in C[0,1] : \psi \text{ is differentiable on } (0,1], \psi' \in C[0,1], \lim_{x\to 0} g(x)\psi'(x) = 0\},$$

is the infinitesimal generator of a C_0-semigroup $(S(t))_{t\ge 0}$ given by

$$(S(t)\psi)(x) = exp(-\int_0^t g'(h^{-1}(h(x)e^{-s}))\,ds)\psi(h^{-1}(h(x)e^{-t})) \text{ for } \psi \in C[0,1],$$

where $h(x) = exp(\int_1^x \frac{1}{g(s)}\,ds)$. In particular, if $g'(x) > 0$ on $[0,1]$ and $I := \inf_{0\le x\le 1} g'(x)$ then $\|S(t)\| \le e^{-It}$, $t \ge 0$. We define $E = C([-1,0],Y)$,

$$K : \mathbb{R}_+ \times E \to Y : \quad (t,\varphi) \mapsto [x \mapsto k(t)f(\varphi(-1)(x))]$$

and assume that for every $q > 0$ there is $r > 0$ such that $|f(x_1) - f(x_2) - f'(0)(x_1-x_2)| \le q|x_1-x_2|$ for $x_1, x_2 \in rB_{\mathbb{R}}$. This is for example the case if $f(x) := x(x+1)$ or $f(x) := x(x-1)$, $x \in [0,1]$. In addition we assume that the function k is bounded. Then K satisfies conditions (K1) and (K2) and

$$(DK(t,0)\varphi)(x) = k(t)f'(0)\varphi(-\tau)(x) \text{ for } t \ge 0,\ x \in [0,1] \text{ and } \varphi \in C[-1,0].$$

Therefore with Theorem 3.4 and Theorem 6.2 we obtain that there is a unique function $u \in C([0,1]\times\mathbb{R}_+)$ satisfying the following integrated version of (E)

$$\begin{aligned} v(x,t) = {} & \varphi(h^{-1}(h(x)e^{-(t-s)}),0)exp(-\int_0^{t-s} g'(h^{-1}(h(x)e^{-\sigma}))\,d\sigma) + \\ & \int_s^t k(\sigma)f(v(h^{-1}(h(x)e^{-(t-\sigma)}),\sigma-1))exp(-\int_0^{t-\sigma} g'(h^{-1}(h(x)e^{-\xi}))\,d\xi)\,d\sigma \end{aligned}$$

for $0 \le x \le 1$, $t > s \ge 0$, and

$$v(x,t) = \varphi(x,t-s) \quad \text{for } 0 \le x \le 1,\ -1+s \le t \le s.$$

Now assume $|k(t)f'(0)| < I$ for $t \ge 0$. Then K also satisfies condition (K3). Moreover, with Gronwalls's lemma we obtain that the zero solution of the linearized equation

$$\begin{aligned} v(x,t) = {} & \varphi(h^{-1}(h(x)e^{-(t-s)}),0)exp(-\int_0^{t-s} g'(h^{-1}(h(x)e^{-\sigma}))\,d\sigma) + \\ & \int_s^t k(\sigma)f'(0)v(h^{-1}(h(x)e^{-(t-\sigma)}),\sigma-1)exp(-\int_0^{t-\sigma} g'(h^{-1}(h(x)e^{-\xi}))\,d\xi)\,d\sigma \end{aligned}$$

for $0 \le x \le 1$, $t > s \ge 0$, and

$$v(x,t) = \varphi(x,t-s) \quad \text{for } 0 \le x \le 1,\ -1+s \le t \le s,$$

is uniformly exponentially stable. Thus, by Theorem 6.4, the zero solution of (E) is uniformly exponentially stable.

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