ASYMPTOTIC BEHAVIOUR OF A NON-AUTONOMOUS POPULATION EQUATION WITH DIFFUSION IN L^1

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Abstract. We prove existence and uniqueness of positive solutions of an age-structured population equation of McKendrick type with spatial diffusion in L^1. The coefficients may depend on age and position. Moreover, the mortality rate is allowed to be unbounded and the fertility rate is time dependent. In the time periodic case, we estimate the essential spectral radius of the monodromy operator which gives information on the asymptotic behaviour of solutions. Our work extends previous results in [NR], [Rh1], [Th1], and [Th2] to the non–autonomous situation. We use the theory of evolution semigroups and extrapolation spaces.

1. Introduction

The investigation of an age–structured population of McKendrick type with age and space dependent spatial diffusion leads to the mathematical model

$$(P)\begin{cases} \partial_t u(t,a,x) + \partial_a u(t,a,x) = \displaystyle\sum_{k,l=1}^{n} \partial_k\, a_{kl}(a,x)\, \partial_l\, u(t,a,x) + \sum_{k=1}^{n} b_k(a,x)\, \partial_k u(t,a,x) \\ \qquad\qquad +c(a,x)u(t,a,x) - \mu(a,x)u(t,a,x),\ t \geq s,\ 0 \leq a \leq a_m,\ x \in \Omega, \\ \displaystyle\sum_{k=1}^{n} \alpha_k(a,x)\, \partial_k u(t,a,x) + \gamma(a,x)u(t,a,x) = 0, \quad t \geq s,\ 0 \leq a \leq a_m,\ x \in \Gamma_1, \\ u(t,a,x) = 0, \quad t \geq s,\ 0 \leq a \leq a_m,\ x \in \Gamma_0, \\ u(t,0,x) = \displaystyle\int_0^{a_m} \beta(t,a,x)u(t,a,x)\, da, \quad t \geq s,\ x \in \Omega, \\ u(s,a,x) = f(a,x) \geq 0, \quad 0 \leq a \leq a_m,\ x \in \Omega. \end{cases}$$

Here $u(t,a,x)$ is the population density at time t, age a, and position x, Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega = \Gamma_0 \dot\cup \Gamma_1$, and $a_m \in (0,\infty]$ is the maximal life expectancy. Let $I = [0,a_m]$ for finite a_m and $I = \mathbb{R}_+$ for $a_m = \infty$. The coefficients a_{kl}, b_k, c, α_k, γ are assumed to be real, sufficiently smooth and uniformly elliptic. They describe the movement of the population and the behaviour at the boundary. Further, the mortality rate $\mu \geq 0$ is allowed to be singular with respect to x and a. Nonintegrability of μ at $a = a_m$ ensures that no individual reaches maximal age, see e.g. [CG, (8)], and a singularity of $x \mapsto \mu(a,x)$ may represent a very hostile part of the domain. Finally, the fertility rate

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$\beta \geq 0$ depends on the time to reflect, e.g., seasonal changes and is supposed to be uniformly continuous. (See Section 4 for a precise statement of our hypotheses.)

In the present paper, we show existence and uniqueness of positive (generalized) solutions of (P) and discuss, for time periodic β, spectral and asymptotic properties of the solution operators. In the autonomous case, problem (P) has been solved in $L^2(I\times\Omega)$ by a semigroup approach, see [CG], [DIS], or [Hu]. However, the natural state space is $E = L^1(I \times \Omega)$ because $\|u(t)\|_1$ gives the size of the population at time t. In the L^1–setting, we can treat (P) by means of more elaborated perturbation techniques. Similar methods were used in [NR] and [Rh1], see also [Th1] and [Th2], for time independent fertility rates β and bounded mortality rates μ. More references to related literature can be found in the above mentioned papers and in [We, p.24]. Further, G.F. Webb's monograph [We] extensively treats nonlinear versions of (P) without diffusion.

Let us sketch our approach. Consider the realization $A(a)$ in $X = L^1(\Omega)$ of the diffusion operator $A(a,x,D) = \sum_{kl} \partial_k\, a_{kl}(a,x)\, \partial_l + \sum_k b_k(a,x)\, \partial_k + c(a,x)\, Id$ subject to the mixed boundary conditions given in the second and third equation of (P), see Section 4. The operator $Lf := -f' + A(\cdot)f(\cdot)$ defined on a suitable subspace of $E \cong L^1(I,X)$ has a closure G in E, Proposition 4.5. It is very important for our analysis that the restriction G_0 of G to functions with $f(0) = 0$ generates the *evolution semigroup* $(T_0(t)f)(a) = \chi_I(a-t)\, U(a, a-t)f(a-t)$ on $L^1(I,X)$, where $U(a,r)$ solves the Cauchy problem (4.2) related to $A(\cdot)$ and χ_M is the characteristic function of a set M. This allows to use the perturbation theory of Miyadera type developped in [RRSV]. So we show that $D(G)$ is contained in the domain of the multiplication operator V induced by μ on E and that the operator $\mathcal{G}_V$ on $X \times E$ defined by $(0,f) \mapsto (-f(0), (G-V)f)$ for $f \in D(G)$ is a Hille-Yosida operator, see (2.1). Now the birth law in (P), given by the operators $B(t)f = \int_0^{a_m} \beta(t,a,\cdot)f(a,\cdot)da$, can be expressed by a perturbation of $\mathcal{G}_V$ in $X \times E$ of the form $(0,f) \mapsto (B(t)f, 0)$. From results in [Rh2] and [GK], we then derive the existence of a positive evolution family $(W(t,s))_{t\geq s\geq 0}$ on E solving (P), see Theorem 4.4. If β does not depend on t, we obtain $W(t,s) = S(t-s)$ for a $\mathcal{C}_0$–semigroup $S(\cdot)$ whose generator can be described precisely.

Our main interest, however, is directed to spectral and asymptotic properties of the evolution family $W(\cdot,\cdot)$ in the case of a fertility rate β which is p-periodic in t. First, in Proposition 2.1, we extend a perturbation theorem for the essential spectral radius due to J. Voigt, [Vo1], to our situation. To apply this result, we need the Dyson-Phillips expansion (2.4) of $W(t,s)$ and certain regularity properties of $A(\cdot)$. As a consequence, we can estimate in Theorem 4.8 the essential spectral radius $r_e(W(p,0))$ of the monodromy operator $W(p,0)$. For instance, if $a_m < \infty$, then $r_e(W(p,0)) = 0$ and the spectrum of $W(p,0)$ consists of a sequence of finite eigenvalues accumulating at 0. The implications of this result to the asymptotic behavior of $W(t,s)$ are described in Corollary 2.2 and Theorem 4.8. Finally, in the autonomous case we recover results from [CG], [NR], [Rh1], [Th1], and [Th2]. In particular, if β is strictly positive, then (after rescaling) the solution semigroup $(S(t))_{t\geq 0}$ converges exponentially to the projection on the unique positive stationary solution, see Remark 4.9. We point out that such results are the starting point for the investigation of the asymptotic behaviour of nonlinear versions of (P) by means of the principle of linearized stability (see [Pr], [We], and the references therein for population equations without diffusion).

Our paper is organized in decreasing order of generality. First, in Section 2, we study the perturbation of a (resolvent positive) Hille-Yosida operator by a certain class of time dependent unbounded perturbations and exhibit conditions which allow to estimate the essential spectral radius of the perturbed evolution family. In Section 3, we consider the Hille-Yosida operator $\mathcal{G}_V$ related to an evolution family $U(\cdot,\cdot)$ and a Miyadera perturbation $V(\cdot)$. Using $\mathcal{G}_V$ and extrapolation methods, we solve a Cauchy problem with boundary perturbation, (3.6), which is an abstract version of (P). The results of Section 2 and 3 are applied to (P) in the last section.

2. Perturbation of the essential spectral radius

We first recall some properties of Hille-Yosida operators and extrapolation spaces. For more details and proofs we refer to [NS], see also [Am2, Chap.V] and the references therein. A linear operator $(A, D(A))$ on a Banach space X is called a *Hille-Yosida operator* if there are constants $M \geq 1$ and $w \in \mathbb{R}$ such that

$$(w, \infty) \subset \rho(A) \text{ and } \|(\lambda - w)^n R(\lambda, A)^{-n}\| \leq M \quad \text{for all } \lambda > w \text{ and } n \in \mathbb{N} \tag{2.1}$$

It is well-known that the part A_0 of A in $X_0 := \overline{D(A)}$ generates a $\mathcal{C}_0$–semigroup $(T_0(t))_{t\geq 0}$ on X_0. Also, the resolvent $R(\lambda, A_0)$ is the restriction of $R(\lambda, A)$ to X_0 for $\lambda \in \rho(A) = \rho(A_0)$. We define on X_0 the norm $\|x\|_{-1} := \|R(\lambda, A_0)x\|$ for a fixed $\lambda \in \rho(A)$ (different $\lambda \in \rho(A)$ yield equivalent norms). The completion X_{-1} of X_0 with respect to $\|\cdot\|_{-1}$ is called *extrapolation space*. The *extrapolated semigroup* $(T_{-1}(t))_{t\geq 0}$ is the unique continuous extension of $(T_0(t))_{t\geq 0}$ to X_{-1}. It is strongly continuous and its generator $A_{-1} \in \mathcal{L}(X_0, X_{-1})$ is the unique continuous extension of A_0. Moreover, X is continuously embedded in X_{-1} and $R(\lambda, A_{-1})$ is an extension of $R(\lambda, A)$ for $\lambda \in \rho(A_{-1}) = \rho(A)$. Finally, A_0 and A are the parts of A_{-1} in X_0 and X, respectively.

It follows from [NS, Prop. 3.3] that we have $\int_s^t T_{-1}(t-\tau)f(\tau)\,d\tau \in X_0$ and

$$\left\| \int_s^t T_{-1}(t-\tau)f(\tau)\,d\tau \right\|_{X_0} \leq M \int_s^t e^{w(t-\tau)} \|f(\tau)\|_X \,d\tau \tag{2.2}$$

for all $f \in L^1_{loc}(\mathbb{R}_+, X)$ and some constant M (where we may and shall assume that this constant coincides with the one in (2.1)). This estimate is crucial for our analysis.

A family $(U(t,s))_{(t,s)\in D}$ of bounded linear operators on a Banach space Y is called *evolution family* if

(a) $U(t,r)U(r,s) = U(t,s)$ and $U(s,s) = Id$ for $t, r, s \in I$ with $t \geq r \geq s$ and
(b) $D \ni (t,s) \mapsto U(t,s)$ is strongly contiunous,

where $D = \{(t,s) \in I^2 : t \geq s\}$ for an interval $I \subseteq \mathbb{R}$ The *exponential growth bound* $\omega(U)$ of $U(\cdot,\cdot)$ is defined by

$$\omega(U) := \inf\{w \in \mathbb{R} : \text{ there is } M_w \geq 1 \text{ with } \|U(t,s)\| \leq M_w e^{w(t-s)} \text{ for } (t,s) \in D\}.$$

The evolution family is said to be *exponentially bounded* if $\omega(U) < \infty$ and *positive* if Y is a Banach lattice and $U(t,s)$ is a positive operator for $(t,s) \in D$. In the remainder of this section we let $I = \mathbb{R}_+$.

We now consider a perturbation $B(\cdot) \in C_b(\mathbb{R}_+, \mathcal{L}_s(X_0, X))$, the space of strongly continuous, uniformly bounded operator–valued functions. Then, by [Rh2, Thm. 2.3], there is a unique evolution family $(U(t,s))_{t\ge s\ge 0}$ on X_0 satisfying

$$U(t,s)x = T_0(t-s)x + \int_s^t T_{-1}(t-\tau)B(\tau)U(\tau,s)x\,d\tau, \quad t\ge s\ge 0,\, x\in X_0. \tag{2.3}$$

Further, $U(t,s)$ is given by the Dyson-Phillips expansion

$$U(t,s) = \sum_{n=0}^{\infty} U_n(t,s), \quad t\ge s\ge 0, \tag{2.4}$$

where the series converges in $\mathcal{L}(X_0)$ uniformly for $0\le s\le t\le b$ and

$$U_0(t,s) = T_0(t-s), \quad U_{n+1}(t,s)x = \int_s^t T_{-1}(t-\tau)B(\tau)U_n(\tau,s)x\,d\tau, \quad x\in X_0.$$

Also, $\|U(t,s)\| \le Me^{(w+Mc)(t-s)}$ for $t\ge s\ge 0$ and $c := \sup_{\tau\ge 0}\|B(\tau)\|_{\mathcal{L}(X_0,X)}$.

If $B(t) = B(t+p)$ for some $p>0$ and all $t\ge 0$, the expansion (2.4) implies that $U(t+p,s+p) = U(t,s)$ for $t\ge s\ge 0$, that is, $(U(t,s))_{t\ge s\ge 0}$ is *p–periodic*. Finally, if $B(t)\equiv B$, then $U(t,s) = S(t-s)$ for a $\mathcal{C}_0$–semigroup $(S(t))_{t\ge 0}$ generated by

$$C = A_{-1} + B \qquad \text{with} \quad D(C) = \{x\in X_0 : A_{-1}x + Bx \in X_0\}, \tag{2.5}$$

see [NR, Thm. 3.6]. We point out that a variety of closely related perturbation results can be found in the literature and refer to the bibliography of [NR], [Rh2], [Th1], and [Th2].

Next, we adopt Voigt's perturbation result [Vo1, Thm. 2.2] for the essential spectral radius of a semigroup to our situation. To that purpose, we recall some definitions. Let $R_n(t,s) := \sum_{k=n}^{\infty} U_k(t,s)$ be the nth remainder of the expansion for $U(t,s)$. Notice that

$$R_{n+1}(t,s)x = \int_s^t T_{-1}(t-\tau)B(\tau)R_n(\tau,s)x\,d\tau, \quad t\ge s\ge 0,\, x\in X_0.$$

For a linear operator C on a Banach space Y, a complex number λ is called an *eigenvalue of finite algebraic multiplicity* of C if λ is an isolated point of $\sigma(C)$ and a pole of $R(\cdot,C)$ such that the residue of $R(\cdot,C)$ has finite dimensional range, see e.g. [GGK, §XV.2] or [Na, A-III]. The *essential spectral radius* of $C\in\mathcal{L}(Y)$ is defined by

$$r_e(C) := \sup\{|\lambda| : \lambda\in\sigma(C) \text{ is not an eigenvalue of finite algebraic multiplicity}\}.$$

Due to [GGK, XI.5.3, XI.8.4], $r_e(C)$ coincides with the spectral radius of the canonical image of C in the algebra $\mathcal{L}(Y)$ modulo the ideal of compact operators. Further, the set $\sigma(C)\cap\{|\lambda| > r_e(C)\}$ consists of at most countably many eigenvalues of finite algebraic multiplicity which can only accumulate at the circle $|\lambda| = r_e(C)$.

An operator $C\in\mathcal{L}(Y)$ is called *strictly power compact* if there is $j\in\mathbb{N}$ such that $(CS)^j$ is compact for all $S\in\mathcal{L}(Y)$. Of course, a compact operator is strictly power compact.

Proposition 2.1. *Let A be a Hille-Yosida operator on X and $B(\cdot)\in C_b(\mathbb{R}_+,\mathcal{L}_s(X_0,X))$. Let $R_n(t_m,s)$ be strictly power compact for some $n\in\mathbb{N}$, $s\ge 0$, and all $t_m\ge s$ with $t_m\to\infty$ as $m\to\infty$. Then, for all $\varepsilon>0$, there is $T_\varepsilon>0$ such that*

$$r_e(U(t_m,s)) \le e^{w_\varepsilon(t_m-s)} \qquad \text{for } t_m - s\ge T_\varepsilon, \tag{2.6}$$

where $w_\varepsilon := \omega(T_0)+\varepsilon$ if $\omega(T_0) > -\infty$ and $w_\varepsilon\to-\infty$ as $\varepsilon\searrow 0$ if $\omega(T_0) = -\infty$.

Proof. Let $c = \sup_{\tau \geq 0} \|B(\tau)\|_{\mathcal{L}(X_0,X)}$ and $w'_\varepsilon := w_\varepsilon - \frac{\varepsilon}{2}$. Using (2.2), it is straightforward to show $\|U_k(t,s)\| \leq M_\varepsilon^{k+1} c^k (t-s)^k e^{w'_\varepsilon(t-s)} \frac{1}{k!}$ for $t \geq s \geq 0$ and $k \in \mathbb{N}$, see the proof of [Rh2, Thm. 2.3]. Hence, there is $T_\varepsilon > 0$ such that

$$\left\| \sum_{k=0}^{n-1} U_k(t,s) \right\| \leq e^{w_\varepsilon(t-s)} \qquad \text{for } t-s \geq T_\varepsilon.$$

Now [Vo1, Cor. 1.4] implies $r_e(U(t_m,s)) \leq e^{w_\varepsilon(t_m-s)}$ for $t_m - s \geq T_\varepsilon$. □

In the autonomous or periodic case the above result has important consequences for the asymptotic behaviour of the evolution family $(U(t,s))_{t\geq s\geq 0}$. Recall that an evolution family $(V(t,s))_{t\geq s\geq 0}$ in $\mathcal{L}(Y)$ has an *exponential splitting* with exponents $\alpha < \beta$ if there exists projections $P(\cdot) \in C_b(\mathbb{R}_+; \mathcal{L}_s(Y))$ and a constant $N \geq 1$ such that

(a) $P(t)V(t,s) = V(t,s)P(s)$ for $t \geq s \geq 0$;
(b) the restriction $V_Q(t,s) : Q(s)X \to Q(t)X$ is invertible for $t \geq s \geq 0$, where $Q(t) := Id - P(t)$;
(c) $\|V(t,s)P(s)\| \leq Ne^{\alpha(t-s)}$ and $\|(V_Q(t,s)Q(s))^{-1}\| \leq Ne^{-\beta(t-s)}$ for $t \geq s \geq 0$.

Corollary 2.2. *Let $B(t) = B(t+p)$ for some $p > 0$ and all $t \geq 0$. Assume that the hypotheses of Proposition 2.1 hold for $s = 0$, $t_m = mp$, $m \in \mathbb{N}$ and a natural number n. Then $\omega_e \leq \omega(T_0)$, where ω_e is given by $r_e(U(p,0)) = e^{\omega_e p}$. Moreover, for $\beta > \alpha > \omega_e$ such that $|\sigma(U(p,0))| \cap [e^{\alpha p}, e^{\beta p}] = \emptyset$, the evolution family $(U(t,s))_{t\geq s\geq 0}$ has an exponential splitting with exponents $\alpha < \beta$ and projections $P(s)$ satisfying* $\dim\ker P(s) = k < \infty$ *and $P(s+p) = P(s)$ for $s \geq 0$. Finally, if $B(t) = B$ for $t \geq 0$, then $P(t) = P(0)$ and $r_e(S(t)) = e^{\omega_e t}$ for $t > 0$, where $S(t-s) = U(t,s)$.*

Proof. By Proposition 2.1, for $\varepsilon > 0$ there exists $m_\varepsilon \in \mathbb{N}$ such that

$$r_e(U(p,0))^m = r_e(U(mp,0)) \leq e^{w_\varepsilon mp} \quad \text{ for } m \geq m_\varepsilon.$$

Hence, $r_e(U(p,0)) \leq e^{\omega(T_0)p}$. Let $\beta > \alpha > \omega_e$ such that $|\sigma(U(p,0))| \cap [e^{\alpha p}, e^{\beta p}] = \emptyset$. As in [He, Lemma 7.2.2], we see that $\sigma(U(p,0)) \setminus \{0\} = \sigma(U(s+p,s)) \setminus \{0\}$ for $s \geq 0$. In particular, the circle $\Gamma = \{\lambda \in \mathbb{C} : |\lambda| = e^{\gamma p}\}$ is contained in $\rho(U(s+p,s))$ for $\gamma \in (\alpha,\beta)$ and $s \geq 0$. As in [He, Thm. 7.2.3], it can be shown that

$$P(s)x := \frac{1}{2\pi i} \int_\Gamma R(\lambda, U(s+p,s))x \, d\lambda, \qquad s \geq 0,\ x \in X_0, \tag{2.7}$$

defines projections $P(s)$ on X_0 yielding an exponential splitting for $U(t,s)$ with exponents $\alpha < \beta$. Clearly, $P(\cdot)$ is p-periodic (and constant if $B(\cdot)$ is constant). Finally, $\ker P(0)$ is the span of the eigenspaces corresponding to the spectral set $\sigma(U(p,0)) \cap \{\lambda : |\lambda| \geq e^{\gamma p}\}$. Since $U_Q(t,0) : \ker P(0) \to \ker P(t)$ is an isomorphism, the dimension of $\ker P(t)$ is constant and finite. The last assertion is shown in [Vo1, Lemma 2.1]. □

In order to apply this corollary, we verify compactness of the remainder $R_3(t,s)$ for $t \geq s \geq 0$ assuming the following hypotheses which hold in our application in Section 4. Recall that an operator C on a Banach lattice Y is called *resolvent positive* if $(w,\infty) \subseteq \rho(C)$ and $R(\lambda, C) \geq 0$ for $\lambda > w$. Clearly, if C generates a $\mathcal{C}_0$–semigroup $(S(t))_{t\geq 0}$ on Y, then $S(\cdot)$ is positive if and only if C is resolvent positive. Also, Y has *order continuous norm* if every decreasing net in the positive cone Y_+ with infimum 0 converges to 0 in norm. For instance, L^p–spaces have order continuous norm if $1 \le p < \infty$, [Sch, p.92].

(H1) A is a resolvent positive Hille-Yosida operator on a Banach lattice X with order continuous norm. The perturbations satisfy $0 \le B(\cdot) \in C_b(\mathbb{R}_+, \mathcal{L}(X_0, X))$ and the mapping $t \mapsto B(t_0)T_0(t) \in \mathcal{L}(X_0, X)$ is continuous for $t > 0$ and all $t_0 \ge 0$.

(H2) For all $\varepsilon > 0$, there is a positive, compact operator $K_\varepsilon : X_0 \to X$ such that $0 \le B(t)T_0(\varepsilon) \le K_\varepsilon$ for $t \ge 0$ and the mapping $t \mapsto K_\varepsilon T_0(t) \in \mathcal{L}(X_0, X)$ is continuous for $t > 0$.

Remark 2.3. *Let the first sentence in (H1) hold. Then, by* [AB, Prop. A]*,* $X_0 = \overline{D(A)}$ *is a sublattice (even an ideal) of* X*. Hence,* A_0 *is resolvent positive and* $T_0(\cdot)$ *is positive.*

Proposition 2.4. *Let* $A^{(1)}$ *and* $A^{(2)}$ *be Hille-Yosida operators with the same domain on a Banach lattice* X *with order continuous norm and let* $(T_0^{(k)}(t))_{t\ge 0}$*,* $k = 1, 2$*, be the corresponding* $\mathcal{C}_0$*–semigroups on* X_0*. Assume that* $0 \le R(\lambda, A^{(1)}) \le R(\lambda, A^{(2)})$ *for* $\lambda > w$*. Then the following assertions hold.*

(a) $0 \le \int_0^t T_{-1}^{(1)}(t-\tau)f(\tau)\,d\tau \le \int_0^t T_{-1}^{(2)}(t-\tau)f(\tau)\,d\tau$ *for* $0 \le f \in L^1_{loc}(\mathbb{R}_+, X)$ *and* $t \ge 0$*.*

(b) *Let* $0 \le B(\cdot) \in C_b(\mathbb{R}_+, \mathcal{L}_s(X_0, X))$ *and let* $(U^{(k)}(t, s))_{t\ge s\ge 0}$*,* $k = 1, 2$*, be the evolution families satisfying* (2.3)*. Then* $0 \le U^{(1)}(t, s) \le U^{(2)}(t, s)$ *for* $t \ge s \ge 0$*.*

Proof. (a) Set $f(\lambda) := R(\lambda, A^{(2)})x - R(\lambda, A^{(1)})x$ for $x \in X_+$ and $\lambda > w$. Then $(-1)^n f^{(n)}(\lambda) \ge 0$ for $n \in \mathbb{N}$ and $\lambda > w$. Therefore, [Ar, Thm. 5.6] implies that $f(\lambda) = \int_0^\infty e^{-\lambda t}\, d\alpha_x(t)$ for a positive, increasing function $\alpha_x : \mathbb{R}_+ \to X$ with $\alpha_x(0) = 0$. Further, by [Ar, 5.7,6.1] there exist unique increasing, strongly continuous families $(S^{(k)}(t))_{t\ge 0}$ of positive operators on X such that $S^{(k)}(0) = 0$ and

$$R(\lambda, A^{(k)})x = \int_0^\infty e^{-\lambda t}\, dS^{(k)}(t)x = \int_0^\infty \lambda e^{-\lambda t} S^{(k)}(t)x\, dt$$

for $k = 1, 2$, $x \in X$, and $\lambda > \max\{w, 0\}$. Hence, $\alpha_x(t) = S^{(2)}(t)x - S^{(1)}(t)x \ge 0$ by the uniqueness of the Laplace-Stieltjes transform. On the other hand,

$$R(\lambda, A^{(k)})x = R(\lambda, A_{-1}^{(k)})x = \int_0^\infty e^{-\lambda t} T_{-1}^{(k)}(t)x\, dt = \int_0^\infty \lambda e^{-\lambda t} \int_0^t T_{-1}^{(k)}(s)x\, ds\, dt$$

for $x \in X$ and $\lambda > w$. Due to [NS, Prop. 3.3], the integral with respect to dt on the right hand side converges in X. Thus, $S^{(k)}(t) = \int_0^t T_{-1}^{(k)}(s)x\, ds$ for $x \in X$ by the uniqueness of the Laplace transform. Now, (a) follows easily.

(b) is an immediate consequence of the expansion (2.4), Remark 2.3, and part (a). ☐

Throughout the remainder of this section, we assume (H1) and denote by $(U(t, s))_{t\ge s\ge 0}$ the evolution family solving (2.3). Also, C is a positive constant depending on b where $0 \le t - s \le b$. We use the approximation $R_{2,\varepsilon}(t, s)$ of $R_2(t, s)$ defined by the strong integral

$$R_{2,\varepsilon}(t, s) := \int_{s+\varepsilon}^{t-\varepsilon} T_{-1}(t-\tau)B(\tau) \int_s^{\tau-\varepsilon} T_{-1}(\tau-\sigma)B(\sigma)U(\sigma, s)\, d\sigma\, d\tau$$

for $\varepsilon \ge 0$ and $t \ge s \ge 0$ with $t - s \ge 2\varepsilon$. Observe that $R_2(t, s) = R_{2,0}(t, s)$.

Lemma 2.5. *Assume (H1). Then* $B(t)R_{2,\varepsilon}(t, s) \to B(t)R_2(t, s)$ *as* $\varepsilon \searrow 0$ *in* $\mathcal{L}(X_0, X)$ *uniformly for* (t, s) *in sets* $\{(t, s) : b \ge t - \delta \ge s \ge 0,\ \delta > 0\}$*. Further, the function* $t \mapsto B(t)R_2(t, s) \in \mathcal{L}(X_0, X)$ *is continuous for* $t \ge s$*.*

Proof. First notice that

$$
\begin{aligned}
& R_2(t,s)x - R_{2,\varepsilon}(t,s)x \\
= \;& \int_{t-\varepsilon}^{t} T_{-1}(t-\tau)B(\tau)\int_s^\tau T_{-1}(\tau-\sigma)B(\sigma)U(\sigma,s)x\,d\sigma d\tau \\
& + \; T_0(t-s-\varepsilon)\int_s^{s+\varepsilon} T_{-1}(s+\varepsilon-\tau)B(\tau)\int_s^\tau T_{-1}(\tau-\sigma)B(\sigma)U(\sigma,s)x\,d\sigma d\tau \\
& + \; T_0(\varepsilon)\int_{s+\varepsilon}^{t-\varepsilon} T_{-1}(t-\varepsilon-\tau)B(\tau)\int_{\tau-\varepsilon}^{\tau} T_{-1}(\tau-\sigma)B(\sigma)U(\sigma,s)x\,d\sigma d\tau \\
=: \;& I_1 + I_2 + I_3
\end{aligned}
$$

for $x \in X_0$ and $t-s \geq 2\varepsilon$. Now the estimate (2.2) yields

$$
\begin{aligned}
\|I_1\|_{X_0} &\leq C\int_{t-\varepsilon}^{t}\left\|\int_s^\tau T_{-1}(\tau-\sigma)B(\sigma)U(\sigma,s)x\,d\sigma\right\|_{X_0} d\tau \;\leq\; C\,\|x\|_{X_0}\int_{t-\varepsilon}^{t}(\tau-s)\,d\tau, \\
\|I_2\|_{X_0} &\leq C\int_{s}^{s+\varepsilon}\left\|\int_s^\tau T_{-1}(\tau-\sigma)B(\sigma)U(\sigma,s)x\,d\sigma\right\|_{X_0} d\tau \;\leq\; C\,\|x\|_{X_0}\int_{s}^{s+\varepsilon}(\tau-s)\,d\tau, \\
\|I_3\|_{X_0} &\leq C\int_{s+\varepsilon}^{t-\varepsilon}\left\|\int_{\tau-\varepsilon}^\tau T_{-1}(\tau-\sigma)B(\sigma)U(\sigma,s)x\,d\sigma\right\|_{X_0} d\tau \;\leq\; C\|x\|_{X_0}\,(t-s)\,\varepsilon.
\end{aligned}
$$

So the first claim is shown. By a similar argument follows $\lim_{t\searrow s} B(t)R_2(t,s) = 0$ in $\mathcal{L}(X_0,X)$. Thus it remains to prove that the mapping $[s+\delta,\infty) \ni t \mapsto B(t)R_{2,\varepsilon}(t,s) \in \mathcal{L}(X_0,X)$ is continuous, where $\delta \geq 2\varepsilon > 0$. For $t \geq r \geq s+2\varepsilon > s$ and $x \in X_0$, we have

$$
\begin{aligned}
& B(t)R_{2,\varepsilon}(t,s)x - B(r)R_{2,\varepsilon}(r,s)x \\
= \;& (B(t)-B(r))\,R_{2,\varepsilon}(t,s)x \\
& + \; B(r)T_0(\varepsilon)\int_{r-\varepsilon}^{t-\varepsilon} T_{-1}(t-\varepsilon-\tau)B(\tau)\int_s^{\tau-\varepsilon} T_{-1}(\tau-\sigma)B(\sigma)U(\sigma,s)x\,d\sigma d\tau \\
& + \; B(r)\int_{s+\varepsilon}^{r-\varepsilon}\left(T_{-1}(t-\tau)-T_{-1}(r-\tau)\right)B(\tau)\int_s^{\tau-\varepsilon} T_{-1}(\tau-\sigma)B(\sigma)U(\sigma,s)x\,d\sigma d\tau \\
=: \;& J_1 + J_2 + J_3.
\end{aligned}
$$

Since $R_{2,\varepsilon}(t,s)$ is uniformly bounded for $b \geq t \geq s \geq 0$ and $B(\cdot)$ is norm continuous, we obtain $\|J_1\|_X \to 0$ uniformly in x as $(t-r) \to 0$. Further, by (2.2),

$$
\begin{aligned}
\|J_2\|_X &\leq C\left\|\int_{r-\varepsilon}^{t-\varepsilon} T_{-1}(t-\varepsilon-\tau)B(\tau)\int_s^{\tau-\varepsilon} T_{-1}(\tau-\sigma)B(\sigma)U(\sigma,s)x\,d\sigma d\tau\right\|_{X_0} \\
&\leq C\int_{r-\varepsilon}^{t-\varepsilon}\left\|T_0(\varepsilon)\int_s^{\tau-\varepsilon} T_{-1}(\tau-\varepsilon-\sigma)B(\sigma)U(\sigma,s)x\,d\sigma\right\|_{X_0} d\tau \\
&\leq C\,(t-r)\,\|x\|_{X_0} \qquad \text{and} \\
\|J_3\|_X &\leq \|B(r)(T_0(t-r+\varepsilon)-T_0(\varepsilon))\|_{\mathcal{L}(X_0,X)} \\
&\quad \cdot \left\|\int_{s+\varepsilon}^{r-\varepsilon} T_{-1}(r-\varepsilon-\tau)B(\tau)T_0(\varepsilon)\int_s^{\tau-\varepsilon} T_{-1}(\tau-\varepsilon-\sigma)B(\sigma)U(\sigma,s)x\,d\sigma d\tau\right\|_{X_0} \\
&\leq C\,\|x\|_{X_0}\,\|B(r)(T_0(t-r+\varepsilon)-T_0(\varepsilon))\|_{\mathcal{L}(X_0,X)}.
\end{aligned}
$$

Hence, the second assertion follows from (H1). □

Proposition 2.6. *Assume (H1) and (H2). Then $R_3(t,s)$ is compact for all $t \geq s \geq 0$.*

Proof. We first show that $B(t)R_{2,\varepsilon}(t,s)$ is compact. Fix $\varepsilon > 0$ and $t - s \geq 2\varepsilon$. Then,

$$\begin{aligned} B(t)R_{2,\varepsilon}(t,s)x &= B(t)T_0(\varepsilon)\int_0^{t-s-2\varepsilon} T_{-1}(\tau)B(t-\tau-\varepsilon)T_0(\varepsilon) \\ &\quad \cdot \int_0^{t-s-\tau-2\varepsilon} T_{-1}(\sigma)B(t-\tau-\sigma-2\varepsilon)U(t-\tau-\sigma-2\varepsilon,s)x\,d\sigma d\tau \\ &=: B(t)T_0(\varepsilon)L_1(\varepsilon,t,s)(B(\cdot+\varepsilon)T_0(\varepsilon)L_2(\varepsilon,t,s)x) \end{aligned} \tag{2.8}$$

for $x \in X_0$ and linear operators $L_k(\varepsilon,t,s)$ given by

$$\begin{aligned} L_1(\varepsilon,t,s) &: L^1([s,t],X) \to X_0;\ f \mapsto \int_0^{t-s-2\varepsilon} T_{-1}(\tau)f(t-\tau-2\varepsilon)\,d\tau \\ L_2(\varepsilon,t,s) &: X_0 \to L^1([s,t],X_0);\ (L_2(\varepsilon,t,s)x)(r) = \int_0^{r-s} T_{-1}(\sigma)B(r-\sigma)U(r-\sigma,s)x\,d\sigma. \end{aligned}$$

By means of (2.2) and Proposition 2.4 one sees that $L_1(\varepsilon,t,s)$ and $L_2(\varepsilon,t,s)$ are bounded and positive operators. Also, assumption (H2) implies

$$\begin{array}{lcccl} 0 & \leq & B(t)T_0(\varepsilon)L_1(\varepsilon,t,s)f & \leq & K_\varepsilon L_1(\varepsilon,t,s)f \\ 0 & \leq & B(r+\varepsilon)T_0(\varepsilon)\,(L_2(\varepsilon,t,s)x)(r) & \leq & K_\varepsilon\,(L_2(\varepsilon,t,s)x)(r) \end{array} \tag{2.9}$$

for $0 \leq f \in L^1([s,t],X)$, $0 \leq x \in X_0$, and $r \in [s,t]$. Since K_ε is compact, the operator $K_\varepsilon L_1(\varepsilon,t,s) : L^1([s,t],X) \to X$ is compact. To show that $K_\varepsilon L_2(\varepsilon,t,s) : X_0 \to L^1([s,t],X)$ is compact, consider $x_n \in X_0$ with $\|x_n\| \leq 1$. Set $f_n := L_2(\varepsilon,t,s)x_n$. Then, by (2.2),

$$\|f_n(r)\|_{X_0} = \left\| \int_s^r T_{-1}(r-\sigma)B(\sigma)U(\sigma,s)x_n\,d\sigma \right\|_{X_0} \leq C\,(r-s). \tag{2.10}$$

Since $K_\varepsilon \in \mathcal{L}(X_0,X)$ is compact, we can choose for each rational $r \in [s,t]$ a subsequence $\Phi_r(n)$ so that $K_\varepsilon f_{\Phi_r(n)}$ converges to some $g(r) \in X$. By taking the diagonal sequence $\Phi(n)$, we obtain

$$K_\varepsilon f_{\Phi(n)}(r) = K_\varepsilon \int_s^r T_{-1}(r-\sigma)B(\sigma)U(\sigma,s)x_{\Phi(n)}\,d\sigma \ \to\ g(r) \quad \text{as } n \to \infty \tag{2.11}$$

for $r \in \mathbb{Q} \cap [s,t]$. Further, for $t \geq r \geq r' \geq s+\delta > s$, one computes

$$\begin{aligned} &\left\| K_\varepsilon \int_s^{r-\delta} T_{-1}(r-\sigma)B(\sigma)U(\sigma,s)x_n\,d\sigma - K_\varepsilon \int_s^{r'-\delta} T_{-1}(r'-\sigma)B(\sigma)U(\sigma,s)x_n\,d\sigma \right\|_{X_0} \\ &\leq \|K_\varepsilon\| \left\| \int_{r'-\delta}^{r-\delta} T_{-1}(r-\sigma)B(\sigma)U(\sigma,s)x_n\,d\sigma \right\| \\ &\quad + \|K_\varepsilon(T_0(r-r'+\delta) - T_0(\delta))\| \left\| \int_s^{r'-\delta} T_{-1}(r'-\delta-\sigma)B(\sigma)U(\sigma,s)x_n\,d\sigma \right\|_{X_0} \\ &\leq C\,(\|K_\varepsilon\|\,(r-r') + \|K_\varepsilon(T_0(r-r'+\delta) - T_0(\delta))\|). \end{aligned}$$

Therefore, $[s,t] \ni r \mapsto K_\varepsilon f_n(r)$ is continuous uniformly in n by (H2) and (2.10). Using this fact and (2.11), we find $g(r) \in X$ so that

$$K_\varepsilon f_{\Phi(n)}(r) \to g(r) \quad \text{as } n \to \infty \quad \text{for all } r \in [s,t].$$

So from (2.10) and the dominated convergence theorem follows that $\lim_n K_\varepsilon f_{\Phi(n)} = g$ in $L^1([s,t],X)$; that is, $K_\varepsilon L_2(\varepsilon,t,s)$ is compact. Now, due to (2.8), (2.9), and the order continuity of X, we can apply the Dodds-Fremlin-Aliprantis-Burkinshaw theorem, [Za, Thm. 124.3], to derive compactness of $B(t)R_{2,\varepsilon}(t,s)$.

Hence, Lemma 2.5 implies that $B(t)R_2(t,s)$ is compact for $t \geq s \geq 0$. Further, we have

$$R_3(t,s)x = \int_s^t T_{-1}(t-\tau)B(\tau)R_2(\tau,s)x\,d\tau$$

for $x \in X_0$, $t \geq s \geq 0$. Let $x_n \in X_0$ with $\|x_n\| \leq 1$. Set $g_n(\tau) := B(\tau)R_2(\tau,s)x_n$ for $\tau \in [s,t]$. As above we can find a subsequence so that $g_{n_k}(r)$ converges in X for rational r. Due to Lemma 2.5, the function $[s,t] \ni \tau \mapsto g_n(\tau)$ is continuous uniformly in n and, hence, $g_{n_k}(\tau)$ converges for all τ. Consequently, the sequence (g_{n_k}) converges in $L^1([s,t],X)$ by Lebesgue's theorem. Finally, an application of (2.2) shows that $R_3(t,s)x_{n_k}$ converges in X_0. As a result, $R_3(t,s)$ is compact. □

Remark 2.7. *The above proofs show that the conclusion of Proposition 2.6 holds under weaker regularity assumptions in (H1) and (H2). For instance, it suffices to suppose that $B(\cdot)$ is strongly continuous and that the mappings $t \mapsto B(t)$, $t \mapsto B(\varepsilon)T(t)$, and $t \mapsto K_\varepsilon T(t)$ are continous from the right in $\mathcal{L}(X_0,X)$ for a.e. $t > 0$ and each $\varepsilon > 0$.*

3. Mild solutions for a class of evolution equations with boundary perturbation

As a preparation for our investigation of an age–structured population equation in the next section, we now study mild solutions of a certain class of evolution equations, see (3.6) below. Notice that some of the notation we use in this and the following section differs from the one adopted in Section 2.

Let $(U(a,r))_{(a,r)\in D}$ be an exponentially bounded evolution family on a Banach space X, where $I \in \{[0,a_m],\mathbb{R}_+\}$. In particular, for $w > \omega(U)$ there is $M = M_w \geq 1$ such that $\|U(a,r)\| \leq Me^{w(a-r)}$ for $(a,r) \in D$. Observe that $\omega(U) = -\infty$ if I is compact. To simplify notation, we set $U(a,r) := 0$ for $0 \leq a < r$. Further, we assume that there are operators $(V(a),D(V(a)))$, $a \in I$, satisfying the *Miyadera* condition:

(M) $V(a)$ is closed for a.e. $a \in I$. For $x \in X$ and $r \in I$, we have $U(a,r)x \in D(V(a))$ for a.e. $a \in I \cap [r,\infty)$, $V(\cdot)U(\cdot,r)x$ is measurable, and

$$\int_0^\alpha \chi_I(a+r)\|V(a+r)U(a+r,r)x\|\,da \leq \gamma\|x\|$$

for constants $\alpha \in (0,\infty]$ and $\gamma \in [0,1)$.

(See [RRSV] for a somewhat weaker condition.) On the space $E := L^1(I,X)$, we define the multiplication operator $Vf := V(\cdot)f(\cdot)$ with domain $D(V) := \{f \in E : f(a) \in D(V(a))$ for a.e. $a \in I$, $V(\cdot)f(\cdot) \in E\}$. We also need the *evolution semigroup* $(T_0(t))_{t\geq 0}$ on E defined by

$$(T_0(t)f)(a) = \chi_I(a-t)\,U(a,a-t)f(a-t), \qquad t \geq 0,\, a \in I, \tag{3.1}$$

see [LMR, RRSV, Sc] and the references therein. It is easy to show that the semigroup $T_0(\cdot)$ is strongly continuous and $\omega(T_0) = \omega(U)$. We denote its generator by $(G_0,D(G_0))$.

Due to hypothesis (M) and [RRSV, Thm. 3.4, Cor. 3.5], the operator

$$G_V := G_0 - V \quad \text{with} \quad D(G_V) := D(G_0) \subseteq D(V) \tag{3.2}$$

generates an evolution semigroup $(T_V(t))_{t\geq 0}$ on E with a corresponding exponentially bounded evolution family $(U_V(a,r))_{(a,r)\in D}$ on X. Moreover, for $x \in X$ and $r \in I$ we have $U_V(a,r)x \in D(V(a))$ for a.e. $a \in I \cap [r,\infty)$, $V(\cdot)U_V(\cdot,r)x$ is locally integrable, and

$$U_V(a,r)x = U(a,r) - \int_r^a U(a,\tau)V(\tau)U_V(\tau,r)x\,d\tau \tag{3.3}$$

$$U_V(a,r)x = U(a,r) - \int_r^a U_V(a,\tau)V(\tau)U(\tau,r)x\,d\tau \tag{3.4}$$

for all $x \in X$ and $(a,r) \in D$. The evolution family $U_V(\cdot,\cdot)$ is uniquely determined by (3.4).

It is known that the domain $D(G_0)$ (and hence $D(G_V)$) consists of continuous functions vanishing at $a = 0$, [RRSV, Prop. 2.1]. In order to consider functions with $f(0) \neq 0$, we introduce an extension G of G_0. Let $e_\lambda(a)x := e^{-\lambda a}U(a,0)x$ and $e_\lambda^V(a)x := e^{-\lambda a}U_V(a,0)x$ for $\lambda \in \mathbb{C}$, $x \in X$, and $a \in I$. We define for a fixed $\omega > \max\{\omega(U),\omega(U_V)\} =: \omega_1$ the operator

$$D(G) := \{f = f_0 + e_\omega(\cdot)x : \ f_0 \in D(G_0), x \in X\} \quad \text{and} \quad Gf := G_0 f_0 + \omega e_\omega(\cdot)x \tag{3.5}$$

on E. Clearly, f_0 and x are uniquely determined by $f \in D(G)$. (See Proposition 4.5 for the motivation of this definition.)

Concerning the orbits $e_\lambda(\cdot)x$ and $e_\lambda^V(\cdot)x$, we need the following result, where e_λ denotes the operator in $\mathcal{L}(X,E)$ given by $x \mapsto e_\lambda(\cdot)x$ for $\operatorname{Re}\lambda > \omega(U)$ (and analogously for e_λ^V).

Lemma 3.1. *Assume that $(U(a,r))_{(a,r)\in D}$ is an exponentially bounded evolution family on a Banach space X and that the operators $V(a)$, $a \in I$, satisfy (M). Then we have*

(a) *$e_\lambda X \subseteq D(V)$ and $Ve_\lambda \in \mathcal{L}(X,E)$ for $Re\,\lambda > \omega(U)$ and $x \in X$;*
(b) *$e_\lambda x \in D(G)$ and $Ge_\lambda x = \lambda e_\lambda x$ for $Re\,\lambda > \omega(U)$ and $x \in X$;*
(c) *$e_\lambda^V = e_\lambda - R(\lambda, G_V)Ve_\lambda$ for $Re\,\lambda > \omega_1$;*
(d) *$\ker(\lambda - (G - V)) = \{e_\lambda^V x : x \in X\}$ for $Re\,\lambda > \omega_1$ and $x \in X$.*

Proof. (a) We consider $I = \mathbb{R}_+$ since the proof carries over to finite I. For $\operatorname{Re}\lambda > w > \omega(U)$, condition (M) implies

$$\begin{aligned}\int_0^\infty \|V(a)e_\lambda(a)x\|\,da &= \sum_{n=0}^\infty e^{-Re\lambda n\alpha}\int_0^\alpha e^{-Re\lambda a}\|V(a+n\alpha)U(a+n\alpha,n\alpha)U(n\alpha,0)x\|\,da \\ &\leq \gamma\max\{1,e^{-Re\lambda\alpha}\}\sum_{n=0}^\infty e^{-Re\lambda n\alpha}\|U(n\alpha,0)x\| \\ &\leq M\gamma\max\{1,e^{-Re\lambda\alpha}\}(1-e^{(w-Re\lambda)\alpha})^{-1}\,\|x\|.\end{aligned}$$

(b) Let $f = e_\lambda(\cdot)x - e_\mu(\cdot)x$ and $\varphi(a) = e^{-\lambda a} - e^{-\mu a}$ for $\operatorname{Re}\lambda, \operatorname{Re}\mu > \omega(U)$. Then

$$T_0(t)f(a) - f(a) = (\chi_I(a-t)\,\varphi(a-t) - \varphi(a))\,U(a,0)x$$

for $a \in I$ and $t \geq 0$. This implies $f \in D(G_0)$ and $G_0(e_\lambda(\cdot)x - e_\mu(\cdot)x) = \lambda e_\lambda(\cdot)x - \mu e_\mu(\cdot)x$. Considering $e_\lambda = e_\lambda - e_\omega + e_\omega$ yields (b).

(c) follows from (3.4) and

$$R(\lambda, G_V)f(a) = \int_0^\infty \chi_I(a-t)e^{-\lambda t}U_V(a,a-t)f(a-t)\,dt = \int_0^a e^{-\lambda(a-t)}U_V(a,t)f(t)\,dt.$$

(d) Assertion (b) and (c) and (3.2) imply that $e_\lambda^V x \in \ker(\lambda - (G-V))$ for $x \in X$ and $\operatorname{Re}\lambda > \omega_1$. Conversely, if $f \in \ker(\lambda - (G-V))$ then, by part (b) and (c),

$$f - e_\lambda^V f(0) = (f - e_\omega f(0)) + (e_\omega f(0) - e_\lambda f(0)) + (e_\lambda f(0) - e_\lambda^V f(0)) \in D(G_0).$$

Thus, $0 = (\lambda - G_V)(f - e_\lambda^V f(0))$. This yields $f = e_\lambda^V f(0)$ since $\lambda \in \rho(G_V)$. $\square$

Remark 3.2. *By Lemma 3.1(c) we have* $(G-V)f = G_V f_1 + \omega e_\omega^V f(0)$ *for* $f \in D(G)$ *and* $f_1 = f_0 + R(\omega, G_V)Ve_\omega f(0) \in D(G_0) = D(G_V)$. *Observe that* $G - V$ *considered as an operator from* $D(G_V) \times e_\omega X \subseteq E \times e_\omega X$ *to* E *is closed if* $E \times e_\omega X$ *is endowed with the norm* $\|f_1\| + \|x\|$.

Let $B(\cdot) \in C_b(\mathbb{R}_+; \mathcal{L}_s(E,X))$. On $E = L^1(I,X)$ we now investigate the Cauchy problem with boundary perturbation

$$\begin{cases} u'(t) = (G-V)u(t), \\ u(s) = f \in E, \\ u(t,0) = B(t)u(t) \in X, \qquad t \geq s \geq 0. \end{cases} \tag{3.6}$$

A *classical solution* of (3.6) is a function $u \in C^1([s,\infty), E)$ such that $u(t) \in D(G)$ and (3.6) holds for all $t \geq s$. We are also looking for *mild solutions* of (3.6), that is, functions $u \in C([s,\infty), E)$ satisfying

$$\begin{cases} \int_s^t u(\tau)\,d\tau \in D(G), \\ u(t) - f = (G-V)\int_s^t u(\tau)\,d\tau, \\ (\int_s^t u(\tau)\,d\tau)(0) = \int_s^t B(\tau)u(\tau)\,d\tau, \qquad t \geq s \geq 0, \end{cases} \tag{3.7}$$

see [GK] and the references therein. It is straightforward to verify that a classical solution is also a mild solution.

To find mild solutions, we proceed as in [NR] and [Rh1]. On the product space $\mathcal{E} := X \times E$ endowed with the maximum norm we define the matrix operators

$$\mathcal{B}(t) := \begin{pmatrix} 0 & B(t) \\ 0 & 0 \end{pmatrix} \qquad \text{and} \qquad \mathcal{G}_V := \begin{pmatrix} 0 & -\delta_0 \\ 0 & G-V \end{pmatrix},$$

where $D(\mathcal{G}_V) := \{0\} \times D(G)$. Notice that $\overline{D(\mathcal{G}_V)} = \{0\} \times E =: \mathcal{E}_0$. To show that $\mathcal{G}_V$ is a Hille-Yosida operator, we need the bounded operators on $\mathcal{E}$ given by

$$R(\lambda) := \begin{pmatrix} 0 & 0 \\ e_\lambda^V & R(\lambda, G_V) \end{pmatrix} \qquad \text{for } \operatorname{Re}\lambda > \max\{\omega(U), \omega(U_V)\} = \omega_1.$$

Lemma 3.3. *Assume that* $(U(a,r))_{(a,r)\in D}$ *is an exponentially bounded evolution family on a Banach space* X *and that the operators* $V(a)$, $a \in I$, *satisfy (M). Then* $\mathcal{G}_V$ *is a Hille-Yosida operator and* $R(\lambda, \mathcal{G}_V) = R(\lambda)$ *for* $Re\,\lambda > \omega_1$.

Proof. From $D(G_V) = D(G_0)$ and Lemma 3.1(d) we easily derive $R(\lambda)\mathcal{E} \subseteq D(\mathcal{G}_V)$ and $(\lambda - \mathcal{G}_V)R(\lambda) = Id$; that is, $\lambda - \mathcal{G}_V$ is surjective for $\operatorname{Re}\lambda > \omega_1$. On the other hand, Lemma 3.1(c) implies

$$R(\omega)(\omega - \mathcal{G}_V)\begin{pmatrix} 0 \\ f \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ e_\omega^V & R(\omega, G_V) \end{pmatrix}\begin{pmatrix} f(0) \\ (\omega - G_V)f_0 + Ve_\omega f(0) \end{pmatrix} = \begin{pmatrix} 0 \\ f \end{pmatrix}$$

for $f = f_0 + e_\omega f(0) \in D(G)$. Thus, $R(\omega) = R(\omega, \mathcal{G}_V)$ and $\mathcal{G}_V$ is closed. Moreover, $R(\lambda)(\lambda - \mathcal{G}_V)\binom{0}{f_0} = \binom{0}{f_0}$ for $f_0 \in D(G_0)$. If $(\lambda - \mathcal{G}_V)\binom{0}{f} = 0$ for some $f \in D(G)$, then $f(0) = 0$. Therefore, $\lambda - \mathcal{G}_V$ is injective for $\operatorname{Re}\lambda > \omega_1$. Consequently, $R(\lambda) = R(\lambda, \mathcal{G}_V)$ for $\operatorname{Re}\lambda > \omega_1$. Finally, we have

$$R(\lambda)^n = \begin{pmatrix} 0 & 0 \\ R(\lambda, G_V)^{n-1}e_\lambda^V & R(\lambda, G_V)^n \end{pmatrix}$$

and $\|R(\lambda, G_V)^{n-1}e_\lambda^V(\cdot)x\|_E \leq \frac{M}{(\lambda-w)^{n-1}}\|e_\lambda^V(\cdot)x\|_E \leq \frac{M^2}{(\lambda-w)^n}\|x\|$ for $\lambda > w > \omega_1$. So $\mathcal{G}_V$ is a Hille-Yosida operator on $\mathcal{E}$. □

As a consequence, the part $\mathcal{G}_{V,0}$ in $\mathcal{E}_0$ generates a $\mathcal{C}_0$–semigroup $\mathcal{T}_{V,0}(\cdot)$ in $\mathcal{E}_0$. In particular, we obtain

$$D(\mathcal{G}_{V,0}) = \{\tbinom{0}{f} \in \mathcal{E}_0 : f \in D(G),\ f(0) = 0\} = \{\tbinom{0}{f} \in \mathcal{E}_0 : f \in D(G_0)\} \text{ and } \mathcal{G}_{V,0}\tbinom{0}{f} = \tbinom{0}{G_V f}.$$

Thus, on E we can identify $\mathcal{G}_{V,0}$ and $\mathcal{T}_{V,0}(\cdot)$ with G_V and $T_V(\cdot)$, respectively. Moreover, there exists the extrapolated semigroup $\mathcal{T}_{V,-1}(\cdot)$ on $\mathcal{E}_{-1} \hookleftarrow \mathcal{E}$ with generator $\mathcal{G}_{V,-1}$.

We now come to the main result of this section, cf. [NR], [Rh1], [Th1] and [Th2] for the autonomous case.

Theorem 3.4. *Assume that $(U(a,r))_{(a,r)\in D}$ is an exponentially bounded, evolution family on a Banach space X and that the operators $V(a)$, $a \in I$, satisfy (M). Let the operator G on $E = L^1(I, X)$ be given by (3.5) and the multiplication operator V on E be induced by $V(\cdot)$. Finally, suppose $B(\cdot) \in C_b(\mathbb{R}_+; \mathcal{L}_s(E, X))$. Then there is a unique mild solution u of (3.6) given by $u(t) = W(t,s)f$ for an exponentially bounded evolution family $(W(t,s))_{t\geq s\geq 0}$. If $B(\cdot) \in C^1(\mathbb{R}_+; \mathcal{L}_s(E, X))$ and $f(0) = B(s)f$, then u is a classical solution. In the autonomous case, i.e., $B(t) \equiv B$, we have $W(t,s) = S(t-s)$ for a $\mathcal{C}_0$–semigroup $(S(t))_{t\geq 0}$ generated by*

$$D(G_{VB}) = \{f \in D(G) : f(0) = Bf\} \quad \text{and} \quad G_{VB}f = (G - V)f. \tag{3.8}$$

Proof. First observe that $\mathcal{B}(\cdot) \in C_b(\mathbb{R}_+, \mathcal{L}_s(\mathcal{E}_0, \mathcal{E}))$. Due to Lemma 3.3 and [Rh1, Thm. 2.3], there exists an exponentially bounded evolution family $(\mathcal{W}(t,s))_{t\geq s\geq 0}$ on $\mathcal{E}_0$ satisfying

$$\mathcal{W}(t,s)\begin{pmatrix} 0 \\ f \end{pmatrix} = \mathcal{T}_{V,0}(t)\begin{pmatrix} 0 \\ f \end{pmatrix} + \int_s^t \mathcal{T}_{V,-1}(t-\tau)\mathcal{B}(\tau)\mathcal{W}(\tau,s)\begin{pmatrix} 0 \\ f \end{pmatrix} d\tau$$

for $f \in E$. After identifying $(\mathcal{W}(t,s))_{t\geq s\geq 0}$ with an evolution family $(W(t,s))_{t\geq s\geq 0}$ on E, we derive

$$\begin{pmatrix} 0 \\ W(t,s)f \end{pmatrix} = \begin{pmatrix} 0 \\ T_V(t-s)f \end{pmatrix} + \int_s^t \mathcal{T}_{V,-1}(t-\tau)\begin{pmatrix} B(\tau)W(\tau,s)f \\ 0 \end{pmatrix} d\tau. \tag{3.9}$$

We evaluate the integral in (3.9) and obtain

$$\begin{aligned}
&\int_s^t \mathcal{T}_{V,-1}(t-\tau)\begin{pmatrix} B(\tau)W(\tau,s)f \\ 0 \end{pmatrix} d\tau \\
&= \int_s^t \mathcal{T}_{V,-1}(t-\tau)(\lambda-\mathcal{G}_V)R(\lambda,\mathcal{G}_V)\begin{pmatrix} B(\tau)W(\tau,s)f \\ 0 \end{pmatrix} d\tau \\
&= (\lambda-\mathcal{G}_{V,-1})\int_s^t \mathcal{T}_{V,-1}(t-\tau)\begin{pmatrix} 0 \\ e_\lambda^V B(\tau)W(\tau,s)f \end{pmatrix} d\tau \\
&= (\lambda-\mathcal{G}_{V,-1})\begin{pmatrix} 0 \\ \int_s^t T_V(t-\tau)e_\lambda^V B(\tau)W(\tau,s)f\,d\tau \end{pmatrix} =: (\lambda-\mathcal{G}_{V,-1})\begin{pmatrix} 0 \\ g \end{pmatrix}
\end{aligned}$$

for $\lambda > \omega_1$. On the other hand, from (3.9) follows that $(\lambda-\mathcal{G}_{V,-1})\binom{0}{g} \in \mathcal{E}_0$. Since $\mathcal{G}_{V,0}$ is the part of $\mathcal{G}_{V,-1}$ in $\mathcal{E}_0$, we have $\binom{0}{g} \in D(\mathcal{G}_{V,0})$, that is, $g \in D(G_0)$. Then (3.9) yields

$$W(t,s)f = T_V(t-s)f + (\lambda - G_V)\int_s^t T_V(t-\tau)e_\lambda^V B(\tau)W(\tau,s)f\,d\tau \tag{3.10}$$

for $f \in E$ and $t \geq s \geq 0$. Now [GK, Thm. 3.2] shows that the continuous function $u(\cdot) = W(\cdot,s)f$ satisfies (3.10) if and only if it is a mild solution of (3.6). (Here one has to use the description of $\ker(\lambda-(G-V))$ given in Lemma 3.1(d).)

To show uniqueness, let $v, w \in C([s,\infty),X)$ satisfy (3.7). Reversing the above arguments, we see that (3.9) holds with $W(\cdot,s)f$ replaced by $v(\cdot)$ and $w(\cdot)$, respectively. Then (2.2) and Gronwall's inequality imply $v = w$.

Now let $B(\cdot) \in C^1(\mathbb{R}_+,\mathcal{L}_s(E,X))$ and $f(0) = B(s)f$. On $\mathcal{E}$ we define the operators $\mathcal{A}(t) := \mathcal{G}_V + \mathcal{B}(t)$ with domain $D(\mathcal{G}_V)$. Notice that $\mathcal{A}(s)\binom{0}{f} \in \mathcal{E}_0$. Due to [Tan, Thm. 1.10] (which is a version of Kato's well-posedness result in the case of non-dense domains), there exists a function $v \in C^1([s,\infty),X)$ so that $v(t) \in D(G)$, $v(s) = f$, and $\binom{0}{v'(t)} = \mathcal{A}(t)\binom{0}{v(t)}$ for $t \geq s$. As a result, v is a classical, and hence a mild, solution of (3.6). By uniqueness, $v(t) = W(t,s)f$.

If $B(t) \equiv B$, then $\mathcal{W}(t,s) = \mathcal{S}(t-s)$ for a $\mathcal{C}_0$–semigroup $\mathcal{S}(\cdot)$ on $\mathcal{E}_0$ which is generated by the part $\mathcal{G}_{VB}$ of $\mathcal{G}_{V,-1}+\mathcal{B}$ in $\mathcal{E}_0$, see (2.5). For $\binom{0}{f} \in D(\mathcal{G}_{VB})$, we have $\mathcal{G}_{V,-1}\binom{0}{f} + \binom{Bf}{0} \in \mathcal{E}_0$, and hence $\mathcal{G}_{V,-1}\binom{0}{f} \in \mathcal{E}$. Since $\mathcal{G}_V$ is the part of $\mathcal{G}_{V,-1}$ in $\mathcal{E}$, we derive $f \in D(G)$ and $\mathcal{G}_{VB}\binom{0}{f} = \binom{-f(0)+Bf}{(G-V)f} \in \mathcal{E}_0$. After identifying $\mathcal{G}_{VB}$ with G_{VB} on E, this establishes (3.8). $\square$

4. A population equation

We now apply the previous results to the equations (P) introduced in Section 1. On the domain Ω and the coefficients we impose the following conditions.

(O) $\Omega \subseteq \mathbb{R}^n$ is a bounded domain with compact C^4–boundary $\partial\Omega$, Γ_i are open and closed in $\partial\Omega$ such that $\partial\Omega = \Gamma_0 \cup \Gamma_1$ and $\Gamma_0 \cap \Gamma_1 = \emptyset$.

(A) $a_{kl} = a_{lk}, b_k, c \in C_b^1(I, C^2(\overline{\Omega},\mathbb{R}))$ and $\sum_{k,l=1}^n a_{kl}(a,x)\,\eta_k\eta_l \geq \delta_0\,|\eta|^2$ for $\eta \in \mathbb{R}^n$, $a \in I$, $x \in \overline{\Omega}$, and a constant $\delta_0 > 0$.

(B) $\alpha_k, \gamma \in C_b^1(I, C^2(\Gamma_1,\mathbb{R}))$ and $\sum_{k=1}^n \alpha_k(a,x)n_k(x) \geq \delta_1$ for a constant $\delta_1 > 0$ and $(a,x) \in I \times \Gamma_1$, where $n(x)$ is the exterior normal unit vector at $x \in \Gamma_1$.

(V) $0 \leq \mu \in L^q_{loc,u}(I, L^p(\Omega))$ for $p > \frac{n}{2}$ and $q > (1-\frac{n}{2p})^{-1}$.

(V') $I_n = [0,b_n] \uparrow I$ and $\chi_{I_n}\,\mu \geq 0$ satisfies (V) for $n \in \mathbb{N}$

(b) $0 \leq \beta \in BUC(\mathbb{R}_+ \times I, L^\infty(\Omega))$.

Here either $I = [0, a_m]$ for some $a_m > 0$ or $I = \mathbb{R}_+$, where we set $a_m := \infty$. Further $L^q_{loc,u}(I)$ is the space of uniformly locally q-integrable functions on I endowed with the norm $\|\varphi\|_{L^q_{loc,u}} := \sup_{s,s+1\in I} \|\chi_{[s,s+1]}\varphi\|_q$. Of course, $L^q(I) = L^q_{loc,u}(I)$ if I is compact. We let $\partial_t = \frac{\partial}{\partial t}$, $\partial_a = \frac{\partial}{\partial a}$, and $\partial_k = \frac{\partial}{\partial x_k}$. On $X := L^1(\Omega)$, we set $V(a)\varphi := \mu(a,\cdot)\varphi(\cdot)$ with $D(V(a)) = \{\varphi \in X : \mu(a,\cdot)\varphi(\cdot) \in X\}$. The operators $V(a)$ induce on $E := L^1(I \times \Omega) \cong L^1(I, X)$ the multiplication operator $V = V(\cdot)$ with maximal domain. Further, $B(t)f := \int_0^{a_m} \beta(t,a,\cdot) f(a,\cdot)\, da$, $t \geq 0$, defines a bounded operator from E to X. Clearly, $0 \leq B(\cdot) \in C_b(\mathbb{R}_+, \mathcal{L}(E,X))$. To treat the diffusion part of the equations (P), we define

$$
\begin{aligned}
A(a,x,D) &:= \sum_{k,l=1}^{n} \partial_k\, a_{kl}(a,x)\, \partial_l + \sum_{k=1}^{n} b_k(a,x)\, \partial_k + c(a,x)\, Id, \quad x \in \Omega,\ a \in I, \\
B_1(a,x,D) &:= \sum_{k=1}^{n} \alpha_k(a,x)\, \partial_k + \gamma(a,x)\, Id, \quad x \in \Gamma_1,\ a \in I,
\end{aligned}
$$

in the sense of distributions and of trace, respectively. Then the realization $A(a)$ of $A(a,x,D)$, $a \in I$, on $L^1(\Omega)$ with mixed boundary conditions is given by

$$
\begin{aligned}
D(A(a)) &:= \{\varphi \in L^1(\Omega) : \varphi \in W^{1,r}(\Omega) \text{ for } 1 \leq r < \tfrac{n}{n-1};\ A(a,x,D)\varphi \in L^1(\Omega); \\
&\qquad (A(a,x,D)\varphi, \psi) = (\varphi, A'(a,x,D)\psi) \text{ for } \psi \in D(A_s(a)),\ 0 < \tfrac{n}{2}(1 - \tfrac{1}{s}) < \infty\}, \\
A(a)\varphi &:= A(a,x,D)\varphi,
\end{aligned}
$$

see [Ta, §5.4]. Here $A'(a,x,D)$ is the formal adjoint of $A(a,x,D)$, [Ta, (5.2)], [Am1, §4], and $A_s(a)$ is the realization of $A(a,x,D)$, $a \in I$, on $L^s(\Omega)$, $1 < s < \infty$, with domain

$$
D(A_s(a)) = \{\varphi \in W^{2,s}(\Omega) : \varphi = 0 \text{ on } \Gamma_0,\ B_1(a,x,D)\varphi = 0 \text{ on } \Gamma_1\}.
$$

By [Ta, §5.4], there is a constant $d \geq 0$ so that $A_d(a) := A(a) - d$ is invertible and generates a bounded analytic semigroup on $L^1(\Omega)$, see also [Am1]. An inspection of the proof shows that, due to (O), (A), and (B), the constant d and the type (K, ϕ) of $A_d(a)$ do not depend on $a \in I$. Further, in [Ta, §6.13] the estimate

$$
\|A_d(a) R(\lambda, A_d(a))\, (A_d(a)^{-1} - A_d(b)^{-1})\| \leq L\, |a - b|\, |\lambda|^{-\rho} \tag{4.1}
$$

is verified for $\lambda \in \{\lambda \in \mathbb{C} : |\arg \lambda| < \phi\}$, $a, b \in I$, and constants $L \geq 0$ and $\rho \in (0, \frac{1}{2})$. (To be precise, in [Ta, §6.13] only finite intervals I are considered. But the proofs given there imply that the constants in (4.1) do not depend on $I \subseteq \mathbb{R}_+$ if we assume (O), (A), and (B).) Thus, a result of P. Acquistapace and B. Terreni, see [Ac, Thm. 2.3], shows that there is an evolution family $(U(a,r))_{(a,r)\in D}$ on $L^1(\Omega)$ such that for $x \in D(A(r))$ the function $U(\cdot, r)x \in C^1(I \cap [r,\infty), X)$ is the unique solution of the Cauchy problem

$$
w'(a) = A(a)w(a), \quad a \geq r, \qquad w(r) = x. \tag{4.2}
$$

Notice that $U_d(a+t,a) := e^{-dt}U(a+t,a)$ solves (4.2) with $A(\cdot)$ replaced by $A_d(\cdot)$. Moreover, $U(a,a-t)$ maps X into the domain of the fractional power $(-A_d(a))^\theta$ and

$$\begin{aligned} \|(-A_d(a))^\theta U_d(a,a-t)\| &\le M\,e^{wt}\,t^{-\theta}\,, && (4.3)\\ \|(U_d(a,a-t)-Id)(-A_d(a-t))^{-\theta}\| &\le M\,t^\theta\,e^{wt}, && (4.4)\\ \lim_{t\to 0}\frac{1}{t}\,(U_d(a,a-t)-Id)A_d(a-t)^{-1}\varphi &= \varphi && (4.5)\end{aligned}$$

for $(a,a-t)\in D$, $t>0$, $\varphi\in X$, $0\le\theta\le 1$, and constants $M,w\in\mathbb{R}$; see [Ac, Thm. 2.3], [FY, Thm. 2.3], [Ta, Thm. 6.5,6.6], [Am2], and the references therein. Next, we show that $U(a,r)$ is positive using an argument from [Am2, II.6.4.2,IV.2.4.3].

Lemma 4.1. *Assume (O), (A), and (B). Then $U(a,r)\ge 0$ for $(a,r)\in D$.*

Proof. We have $R(\lambda,A(a))\ge 0$ for $a\in I$ and $\lambda>d$ by [Am1, Thm. 10.3]. Hence, for $n\ge n_0$ the Yosida approximation $A_n(a)=nA(a)R(n,A(a))$ is resolvent positive, cf. [Ta, Lemma 6.14]. Also, $A_n(\cdot)$ is bounded and uniformly Hölder continuous for fixed n. So, by [Am2, Thm. II.6.4.2] the evolution family $(U_n(a,r))_{(a,r)\in D}$ solving the Cauchy problem corresponding to $A_n(\cdot)$ is positive. Since $U_n(a,r)$ converges to $U(a,r)$ (in $\mathcal{L}(X)$), see e.g. [Ta, Lemma 6.21], the lemma is proved. ☐

Remark 4.2. *For the biological interpretation it is essential that the diffusion process does not create individuals, that is, $U(a,r)$ is contractive on $L^1(\Omega)$. This is true if $\|e^{tA(a)}\|\le 1$ for $a\in I$ and $t\ge 0$ (which holds if, e.g., $\alpha_k(a,x)=\sum_{l=1}^n a_{kl}(a,x)n_l(x)$ and $b_k=c=\gamma=0$,* [Am1, Thm. 10.3]*). In fact, then the Yosida approximations $A_n(a)$ generate contraction semigroups. Thus, the corresponding evolution families $U_n(\cdot,\cdot)$ are contractive, see e.g.* [Sc, Cor. 4.5]*, and hence $\|U(a,r)\|\le 1$.*

Let $G(0):=Id$ and $G(t)\varphi:=\chi_\Omega(K_t*\varphi)$, $t>0$, for $\varphi\in L^1(\Omega)$, where $K_t(x)=(4\pi t)^{-n/2}\exp(-\frac{|x|^2}{4t})$ for $x\in\mathbb{R}^n$ and φ is extended by 0 to $\mathbb{R}^n$. In [Pa] it is stated that $U(a,r)$ satisfies a *Gaussian estimate*, that is, for $0\le t\le T$ and $(a+t,a)\in D$ there are constants $N,c>0$ (possibly depending on T) such that

$$0\le U(a+t,a)=|U(a+t,a)|\le NG(ct). \tag{4.6}$$

The proof in [Pa] is sketched very briefly. However, it can easily be provided by combining the estimate (3.34) in [PT] with (2.6)–(2.8) in [PT] and Theorem 5.7 in [Ta]. We use (4.6) to verify the Miyadera conditions for the multiplication operators $V(\cdot)$.

Lemma 4.3. *Assume (O), (A), (B), and (V). Then the operators $U(a,r)$ and $V(a)$ defined above satisfy (M). Using the notations of Section 3, we have*

$$0\le U_V(a,r)\le U(a,r),\quad 0\le T_V(t)\le T_0(t),\quad 0\le R(\lambda,G_V)\le R(\lambda,G_0) \tag{4.7}$$

for $(a,r)\in D$, $t\ge 0$, and $\lambda>\omega(U)\ge\omega(U_V)$.

Proof. The measurability condition in (M) can be checked by approximating the function μ pointwise a.e. by bounded functions, cf. [RRSV, §5]. Using (4.6), Hölder's and Young's inequality, and $\|K_t\|_{L^{p'}(\mathbb{R}^n)}=C\,t^{-\frac{n}{2p}}$ for a constant C and $\frac{1}{p}+\frac{1}{p'}=1$, we compute

$$\begin{aligned}\|V(a+t)U(a+t,a)\varphi\|_1 &\le N\,\|\mu(a+t,\cdot)\|_p\,\|G(ct)\,|\varphi|\|_{p'}\\ &\le NC\,t^{-\frac{n}{2p}}\,\|\mu(a+t,\cdot)\|_p\,\|\varphi\|_1 \end{aligned}\tag{4.8}$$

for $(a+t,a) \in D$ and $\varphi \in L^1(\Omega)$. Now, (M) follows from

$$\int_0^t \|V(a+\tau)U(a+\tau,a)\varphi\|_1 \, d\tau \le C_1 \, t^\kappa \, \|\mu\|_{L^q_{loc,u}(I,L^p(\Omega))} \, \|\varphi\|_1 \tag{4.9}$$

for a constant C_1 and $\kappa = 1 - \frac{1}{q} - \frac{n}{2p} > 0$. By [Vo2, Rem. 2.1] and Lemma 4.1, we see that $0 \le T_V(t) \le T_0(t)$. This implies the remaining assertions. $\square$

In the sequel, we use the concepts from Section 3 to solve (P). In particular, a continuous function $u : [s,\infty) \to E$ is called *generalized solution* of (P) if it satisfies (3.7). In the definition of G, see (3.5), we choose ω greater than the constant w used in (4.3) and (4.4). As an immediate consequence of Lemma 4.3, Theorem 3.4 and Proposition 2.4, we obtain the following existence theorem. It generalizes results in the autonomous case from [NR], [Rh1], [Th1], [Th2] (where in [Th1] and [Th2] more general β are considered). See also [CG], [DIS], and [Hu] for the L^2–setting.

Theorem 4.4. *Assume (O), (A), (B), (V), and (b). Then there is a unique generalized solution of (P) for $f \in E$ and $s \ge 0$. It is given by $u = W(\cdot,s)f$ for a positive evolution family $(W(t,s))_{t\ge s\ge 0}$ on E. Moreover, if $\beta \in C^1(\mathbb{R}_+, L^\infty(I\times\Omega))$ and $f(0) = B(s)f$, then u is a classical solution of* (3.6). *If β does not depend on t, then $W(t,s) = S(t-s)$ for the $\mathcal{C}_0$–semigroup $S(\cdot)$ generated by the operator G_{VB} defined in* (3.8).

Observe that in the above theorem the regularity assumption on β could be weakened since we only need strong continuity of $B(\cdot)$ to obtain mild solutions. Of course, it is crucial to determine G in order to understand our notion of a generalized solution of (P). This could be achieved for the state space $L^p(I\times\Omega)$ with $1<p<\infty$, cf. [RRSV, §4]. To give a partial answer in the case $p=1$, we introduce the spaces

$$\begin{aligned} F &:= \{f \in E : f \in W^{1,1}(I,X),\ f(a) \in D(A(a)) \text{ for a.e. } a \in I,\ A(\cdot)f(\cdot) \in E\}, \\ F_0 &:= \{f \in F : f(0) = 0\}, \quad \text{and} \quad F_1 := \{f \in F : f(0) \in D(A(0))\} \end{aligned}$$

and the operator $Lf := -f' + A(\cdot)f \in E$ with domain $F \subseteq E$. Further, $W^{s,1}(\Omega)$ denotes the usual Sobolev space of (fractional) order $s \ge 0$, cf. [BB, §4.3]. The last assertion in the following result was shown for $A(a) \equiv A$ by G. Di Blasio in [DB, Thm. 4.1].

Proposition 4.5. *Assume (O), (A), and (B). Then G_0 is the closure of (L,F_0) in E. Also, $(L,F_1) \subseteq (G,D(G))$ and F_1 is dense in $D(G)$ endowed with the graph norm. In particular, G is the closure of (L,F_1) as an operator from $E\times e_\omega X$ to E, cf. Remark 3.2. On the other hand, $D(G) \subseteq W^{\beta,1}(I,L^1(\Omega))$ for $0 \le \beta < 1$. Moreover, if $\Gamma_1 = \emptyset$, then $D(G) \subseteq L^1(I, W^{2\theta,1}(\Omega) \cap W_0^{1,1}(\Omega))$ for $\frac{1}{2} < \theta < 1$.*

Proof. Considering $L-d$, G_0-d, and $G-d$, we may assume that $d=0$ in (4.3)–(4.5).

(i) It is known that G_0 is the closure of (L,F_{00}) for a space $F_{00} \subseteq F_0 \cap D(G_0)$, see [Sc, Prop. 1.13] and also [LMR, Prop. 2.9]. Further, for $a-t \ge 0$ and $f \in F$, we have

$$T_0(t)f(a) - f(a) = (U(a,a-t) - Id)A(a-t)^{-1}A(a-t)f(a-t) + f(a-t) - f(a).$$

Using (4.4) and (4.5), it is then easy to see that $\frac{1}{t}(T_0(t)f - f) \to Lf$ in E, and so G_0 is the closure of (L,F_0). Due to estimate (4.3), the operator G extends (L,F_1). Further, let $f = f_0 + e_\omega f(0) \in D(G)$ for $f_0 \in D(G_0)$. There are $f_{0,n} \in F_0$ such that $f_{0,n} \to f_0$ and

$Lf_{0,n} \to G_0 f_0$ in E and $D(A(0)) \ni x_n \to f(0)$ in X. Thus $f_n := f_{0,n} + e_\omega x_n \in F_1$ converges to f in the graph norm of G. The other assertion concerning (L, F_1) is then clear.

(ii) Because (4.3) the operators $(-A(\cdot))^\alpha$, $0 \le \alpha < 1$, satisfy (M). Due to [RRSV, Thm. 3.4] and (4.3), this implies $D(G) \subseteq \{f \in E : f(a) \in D((-A(a))^\alpha)$ for a.e. $a \in I$, $(-A(\cdot))^\alpha f(\cdot) \in E\}$. Thus, for $f \in D(G_0)$, we can write

$$\begin{aligned} t^{-\gamma}(T_0(t)f(a) - f(a)) &= t^{-\gamma}(U(a, a-t) - Id)(-A(a-t))^{-\alpha}(-A(a-t))^\alpha f(a-t) \\ &\quad + t^{-\gamma}(f(a-t) - f(a)) \end{aligned}$$

for $a - t \ge 0$ and $0 < \beta < \gamma < \alpha < 1$. The first summand on the right hand side converges to 0 in E as $t \to 0$ by (4.4). Together with (4.3) and (4.4), this shows $D(G) \subseteq W^{\beta,1}(I, L^1(\Omega))$.

(iii) Now assume $\Gamma_1 = \emptyset$. For $0 \le \theta < 1$, we introduce the (real) interpolation space

$$D_{A(a)}(\theta, 1) := \{\varphi \in L^1(\Omega) : \|\varphi\|_{D_{A(a)}(\theta,1)} := \|\varphi\|_{L^1} + \int_0^\infty t^{-\theta} \|A(a)e^{tA(a)}\varphi\|\, dt < \infty\},$$

see e.g. [BB, §3.5]. Then, $D((-A(a))^\alpha) \subseteq D_{A(a)}(\theta, 1)$ for $\theta < \alpha \le 1$, and the embedding is continuous uniformly in a. We have $D_{A(a)}(\theta, 1) \cong W^{2\theta,1}(\Omega) \cap W_0^{1,1}(\Omega)$ for $\theta \in (\frac{1}{2}, 1)$ by [DB, Thm. 3.2]. An inspection of the proof shows that the norms are uniformly equivalent with respect to a. Hence, $D(G) \subseteq L^1(I, W^{2\theta,1}(\Omega) \cap W_0^{1,1}(\Omega))$ for $\theta \in (\frac{1}{2}, 1)$. ☐

We now study the asymptotic behaviour of generalized solutions of (P). First, we have

Lemma 4.6. *Assume (O), (A), (B), and (V). Then,* $\omega(U_V) = \omega(U) = -\infty$ *for finite* a_m *and* $\omega(U_V) \le \omega(U) < 0$ *for* $a_m = \infty$ *if* $c \le 0$ *and* $\Gamma_1 = \emptyset$.

Proof. We only have to show $\omega(U) < 0$ if $a_m = \infty$, $c \le 0$ and $\Gamma_1 = \emptyset$. Due to [vW, pp.14], there are constants $C, \varepsilon > 0$ such that $\|U(a+t, a)\varphi\|_\infty \le Ce^{-\varepsilon t}\|\varphi\|_\infty$ for $t, a \ge 0$ and $\varphi \in D(A_p(a))$, $1 < p < \infty$. By means of (4.6) we derive

$$\|U(a+t, a)\varphi\|_1 \le |\Omega|\, C\, e^{-\varepsilon(t-1)} \|U(a+1, a)\varphi\|_\infty \le C_1\, e^{-\varepsilon t}\, \|\varphi\|_1$$

for $\varphi \in L^\infty(\Omega)$, $t \ge 1$, and a constant C_1. (Here we have used that, by [Ta, p.284], there is a unique evolution family $U_p(\cdot, \cdot)$ solving (4.2) for $A_p(\cdot)$ on $L^p(\Omega)$, $1 < p < \infty$, and that $U_p(a+t, a)\varphi = U(a+t, a)\varphi$ for $\varphi \in L^p(\Omega) \subseteq L^1(\Omega)$.) Moreover, $\|U(a+t, a)\| \le M_1$ for $a \ge 0$ and $0 \le t \le 1$. Consequently, $\omega(U) \le -\varepsilon$. ☐

To verify (H1) and (H2) from Section 2, we need the following regularity result.

Lemma 4.7. *Assume (O), (A), (B), and (V) for* $a_m = \infty$. *Let* $t_0 > 0$. *Then* $U_V(a+t, a) \to U_V(a+t_0, a)$ *in* $\mathcal{L}(X)$ *as* $t \to t_0$ *uniformly in* $a \in \mathbb{R}_+$.

Proof. By considering $e^{-dt}U_V(a+t, a)$, we may and shall assume $d = 0$ in (4.3) and (4.4). Let $t_0 + 1 \ge t \ge t_0 > 0$, $a \ge 0$, and $\varphi \in L^1(\Omega)$. By C we denote a generic constant depending on t_0. From (3.3) follows

$$\begin{aligned} &U_V(a+t, a)\varphi - U_V(a+t_0, a)\varphi \\ &= U(a+t, a)\varphi - U(a+t_0, a)\varphi - \int_{a+t_0}^{a+t} U(a+t, \tau)V(\tau)U_V(\tau, a)\varphi\, dt \\ &\quad - \int_a^{a+t_0} (U(a+t, \tau) - U(a+t_0, \tau))V(\tau)U_V(\tau, a)\varphi\, d\tau =: S_1 + S_2 + S_3. \end{aligned}$$

Using (4.3) and (4.4), we estimate

$$\|S_1\| \le \|(U(a+t,a+t_0)-Id)A(a+t_0)^{-1}\|\,\|A(a+t_0)U(a+t_0,a)\varphi\| \le C\,(t-t_0)\,\|\varphi\|.$$

Further, by means of (4.7) and (4.9),

$$\|S_2\| \le C\int_0^{t-t_0} \|V(a+t_0+\tau)U_V(a+t_0+\tau,a+t_0)U_V(a+t_0,a)\varphi\|\,d\tau \le C\,(t-t_0)^{\kappa}\,\|\varphi\|$$

for some $\kappa>0$. Finally, (4.7), (4.3), (4.4), (4.8), and Hölder's inequality yield

$$\begin{aligned}\|S_3\| &\le \int_a^{a+t_0} \|(U(a+t,a+t_0)-Id)(-A(a+t_0))^{-\theta}\|\,\|(-A(a+t_0))^{\theta}U(a+t_0,\tau)\| \\ &\qquad \cdot\|V(\tau)U(\tau,a)\varphi\|\,d\tau \\ &\le C\,(t-t_0)^{\theta}\,\|\varphi\|\int_0^{t_0}(t_0-\tau)^{-\theta}\|\mu(\tau+a,\cdot)\|_p\,\tau^{-\frac{n}{2p}}\,d\tau \\ &\le C\,(t-t_0)^{\theta}\,\|\varphi\|\end{aligned}$$

for $0<\theta<\frac{1}{q'}$. The case $t_0\ge t>\delta>0$ can be treated in the same way. $\square$

We can now prove the main theorem of this paper. It extends results in the autonomous case shown in [NR, §5], [Rh1, Thm. 3.5], [Th1, §4], [Th2, §5], see also [CG] for the L^2–setting. We refer to [Na, B-IV.2, C-IV.2] concerning quasi–compact semigroups. The *peripherical spectrum* of a bounded operator S is defined by $\sigma_\pi(S):=\sigma(S)\cap\{|\lambda|=r(S)\}$. Recall that $r(V(p,0))=e^{\omega(V)p}$ and $\sigma(V(s+p,s))=\sigma(V(p,0))$, $s\ge 0$, for a p–periodic evolution family $(V(t,s))_{t\ge s\ge 0}$, see Corollary 2.2 and its proof.

Theorem 4.8. *Assume (O), (A), (B), (V), (b) and that β is p–periodic in t. Let $e^{\omega_e p}=r_e(W(p,0))$ for the p–periodic, positive evolution family $(W(t,s))_{t\ge s\ge 0}$ on E given by Theorem 4.4. Then $\omega_e\le\omega(U_V)$, where $\omega(U_V)=-\infty$ if $a_m<\infty$ and $\omega(U_V)<0$ if $a_m=\infty$ and $c\le 0$, $\Gamma_1=\emptyset$. Further, if $\omega_e<\omega(W)$ and if the constants $\alpha<\beta<\omega(W)$ are chosen sufficiently close to $\omega(W)$, then $(W(t,s))_{t\ge s\ge 0}$ has an exponential splitting with exponents α,β and projections $P(s)$ which have the same kernels as the operators $e^{\omega(W)lp}-W(lp+s,s)$ for some $l\in\mathbb{N}$ and all $s\ge 0$.*
Moreover, if β does not depend on t and $\omega_e<\omega(W)$, then $(e^{-\omega(W)t}S(t))_{t\ge 0}$ is quasi–compact, $\omega(W)$ is an eigenvalue of the generator G_{VB} given by Theorem 4.4 with finite algebraic multiplicity and pole order m, and we have

$$\left\| e^{-\omega(W)t}S(t)-\sum_{k=0}^{m-1}\frac{t^k}{k!}\,(G_{VB}-\omega(W))^k P\right\| \le Ce^{-\varepsilon t} \tag{4.10}$$

for $t\ge 0$, constants $C,\varepsilon>0$, and the projection P on the eigenspace corresponding to the eigenvalue $\omega(W)$.

Proof. (i) As in preceding section, we define $\mathcal{B}(t)$, $\mathcal{G}_V$, and $\mathcal{T}_{V,0}(t)$ on $\mathcal{E}:=X\times E$ and $\mathcal{E}_0:=\{0\}\times E\cong E$. By Lemma 3.3 and 4.3, $\mathcal{G}_V$ is a resolvent positive Hille-Yosida operator, and so we obtain a perturbed positive evolution family $(W(t,s))_{t\ge s\ge 0}$ on E which is p–periodic due to the expansion (2.4).

(ii) We first consider the case $a_m = \infty$. In view of Proposition 2.6, we have to verify (H1) and (H2) for $\mathcal{B}(t)$ and $\mathcal{G}_V$ in order to apply Corollary 2.2 to $W(t,s)$ on E. We estimate

$$\begin{aligned}
&\|\mathcal{B}(t_0)\,(\mathcal{T}_{V,0}(t) - \mathcal{T}_{V,0}(s))\tbinom{0}{f}\|_{X\times L^1(\mathbb{R}_+,X)} \qquad (4.11)\\
&= \left\|\int_t^\infty \beta(t_0,a,\cdot)U_V(a,a-t)f(a-t)\,da - \int_s^\infty \beta(t_0,a,\cdot)U_V(a,a-s)f(a-s)\,da\right\|_X\\
&\le \int_0^\infty \|\beta(t_0,a+t,\cdot)-\beta(t_0,a+s,\cdot)\|_\infty\,\|U_V(a+t,a)f(a)\|\,da\\
&\quad + \int_0^\infty \|\beta(t_0,a+s,\cdot)\|_\infty\,\|U_V(a+t,a)-U_V(a+s,a)\|\,\|f(a)\|\,dt\\
&\le \|f\|\,\sup_{a\ge 0}(Me^{wt}\|\beta(t_0,a+t,\cdot)-\beta(t_0,a+s,\cdot)\|_\infty + \|\beta\|_\infty\,\|U_V(a+t,a)-U_V(a+s,a)\|)
\end{aligned}$$

for $t\ge s\ge\delta>0$, $t_0\ge 0$, and $f\in L^1(\mathbb{R}_+,X)$. So assumption (b) and Lemma 4.7 imply (H1). Further, (4.7) and (4.6) yield

$$0\le \mathcal{B}(t)\mathcal{T}_{V,0}(\varepsilon)\begin{pmatrix}0\\ f\end{pmatrix} \le \begin{pmatrix} N\,\|\beta\|_\infty\,G(c\varepsilon)\int_0^\infty f(a)\,da\\ 0\end{pmatrix} =: \mathcal{K}_\varepsilon\begin{pmatrix}0\\ f\end{pmatrix}$$

for $0\le f\in E$ and $\varepsilon>0$, where $\mathcal{K}_\varepsilon\in\mathcal{L}(\mathcal{E}_0,\mathcal{E})$. It is known that $K_t * f\in W^{1,1}(\mathbb{R}^n)$ for $f\in L^1(\mathbb{R}^n)$, see e.g. [BB, 4.3.7, 4.3.14]. Also, by (O) and Rellich's theorem, [Ta, Cor. 3.2], $W^{1,1}(\Omega)$ is compactly embedded in $L^1(\Omega)$. Consequently, $G(c\varepsilon)\in\mathcal{L}(X)$ is compact, and hence $\mathcal{K}_\varepsilon$ is compact. Continuity of $t\mapsto \mathcal{K}_\varepsilon\mathcal{T}_{V,0}(t)\in\mathcal{L}(\mathcal{E}_0,\mathcal{E})$ for $t>0$ is shown as in (4.11). Therefore (H2) holds, and Corollary 2.2 yields $\omega_e\le\omega(\mathcal{T}_{V,0})=\omega(U_V)$. Also, we have $\omega(U_V)<0$ if $c\le 0$ and $\Gamma_1=\emptyset$ due to Lemma 4.6.

(iii) For finite a_m, we extend $A(\cdot)$, $V(\cdot)$, and β by setting $A(a):=A(a_m)$, $V(a):=-2a$, and $\beta(t,a,\cdot):=\beta(t,a_m,\cdot)$ for $a>a_m$. This gives an extended evolution family $(U_V^\infty(a,r))_{a\ge r\ge 0}$ on $L^1(\Omega)$, where $U_V^\infty(a+t,a)=e^{-t^2-2at}e^{tA(a_m)}$ for $a\ge a_m$ and $t\ge 0$. Hence, $(U_V^\infty(a,r))_{a\ge r\ge 0}$ satisfies the conclusion of Lemma 4.7 and $\omega(U_V^\infty)=-\infty$.

Denote by $T_V^\infty(\cdot)$ the corresponding evolution semigroup on $L^1(\mathbb{R}_+,X)$ with generator G_V^∞. Set $Pf:=\chi_I f\in L^1([0,a_m],X)$ for $f\in L^1(\mathbb{R}_+,X)$ and let $Jf\in L^1(\mathbb{R}_+,X)$ be the extension by 0 of $f\in L^1([0,a_m],X)$. Then, $PT_V^\infty(t)=T_V(t)P$ and $PG_V^\infty=G_VP$. Further, we perturb the corresponding Hille-Yosida operator $\mathcal{G}_V^\infty$ on $L^1(\Omega)\times L^1(\mathbb{R}_+\times\Omega)$ by (the matrix operators corresponding to) the operators in $\mathcal{L}(L^1(\mathbb{R}_+,X),X)$ given by $B^\infty(t)f=\int_0^{a_m}\beta(t,a,\cdot)f(a)da$ and $\tilde{B}(t)f=\int_0^\infty\beta(t,a,\cdot)f(a)da$, respectively. This yields two evolution families $(W^\infty(t,s))_{t\ge s\ge 0}$ and $(\tilde{W}(t,s))_{t\ge s\ge 0}$ on $L^1(\mathbb{R}_+,X)$. It follows from the uniqueness of solutions to (3.10) that $W(t,s)=PW^\infty(t,s)J$, and hence

$$W(t,s)=\sum_{k=0}^\infty PW_n^\infty(t,s)J, \tag{4.12}$$

where $W_n^\infty(t,s)$ is the coefficient of the Dyson-Phillips expansion of $W^\infty(t,s)$. Further, by the first part of the proof, the remainder $\tilde{R}_3(t,s)$ of $\tilde{W}(t,s)$ is compact for $t\ge s\ge 0$. From the Dyson-Phillips expansion of $W(t,s)$ and Proposition 2.4 we derive that $0\le R_3^\infty(t,s)S\le\tilde{R}_3(t,s)S$ for $0\le S\in\mathcal{L}(L^1(\mathbb{R}_+,X))$. Since any bounded operator on $L^1(\mathbb{R}_+\times\Omega)$ can be written as linear combination of positive operators, [Sch, IV.1.5], the Dodds-Fremlin theorem, [Za, Thm. 124.3], shows that $(R_3^\infty(t,s)S)^2$ is compact for $t\ge s\ge 0$ and $S\in\mathcal{L}(L^1(\mathbb{R}_+,X))$. Hence, the remainder $PR_3^\infty(t,s)J$ of the expansion (4.12) is strictly power compact in $L^1([0,a_m],X)$. As in in the proof of Proposition 2.1, we can verify (2.6) for $W(t,s)$ and obtain as in Corollary 2.2 that $\omega_e = -\infty$ for finite a_m.

(iv) Let $\omega_e < \omega(W)$. The claims in the autonomous case follow from [Na, C-IV.2.1,2.2]. The remaining assertions can be deduced from Corollary 2.2 provided there is $l \in \mathbb{N}$ such that the boundary spectrum $\sigma_\pi(W(lp,0))$ only consists of $e^{\omega(W)lp}$. But this is a consequence of the cyclicity and finiteness of $\sigma_\pi(W(p,0))$, see [Sch, Thm. IV.4.9]. □

Remark 4.9. *In the situation of Theorem 4.8, assume that β does not depend on t. By Lemma 3.1(d), the eigenfunctions of G_{VB} for an eigenvalue λ with $Re\,\lambda > \omega(U)$ are given by $f(a) = e^{-\lambda a}U_V(a,0)\varphi$, where $\varphi \in L^1(\Omega)$ satisfies*

$$\varphi = \int_0^{a_m} \beta(a,\cdot)e^{-\lambda a}U_V(a,0)\varphi\, da =: B_\lambda\varphi.$$

Let $\omega(W) > \omega(U)$ (which is true if, for instance, $a_m < \infty$ and $\omega(W) > -\infty$ or $\omega(W) \geq 0$ and $\Gamma_1 = \emptyset$, $c \leq 0$). Assume that B_λ maps positive functions to strictly positive functions, that is, B_λ irreducible. (This is the case, e.g., if $\inf_{x\in\Omega}\beta(a,x) > 0$ for a in set of positive measure and if we have Dirichlet boundary conditions, see [Aro, Thm. 9]*.) Now, from* [Sch, Thm. V.5.2] *follows as in* [CG] *that $\omega(W)$ is a simple eigenvalue. Then* (4.10) *holds with $m = 1 = \dim PE$, that is, $(e^{-\omega(W)t}S(t))_{t\geq 0}$ has balanced exponential growth.*

Concluding we indicate how the above results can be generalized if we replace (V) by the more general assumption (V'). For the truncated potentials $V_n(\cdot) := \chi_{[0,b_n]}V(\cdot)$, we obtain evolution families $(U_n(a,r))_{(a,r)\in D}$ on X being uniquely determined by (3.4) for $V_n(\cdot)$. Thus $U_V(a,r) := U_n(a,r)$ for $0 \leq r \leq a \leq b_n$ defines a strongly continuous evolution family on $L^1(\Omega)$ for the interval $[0,a_m)$. Notice that

$$0 \leq U_V(a,r) \leq U_n(a,r) \leq U(a,r) \tag{4.13}$$

for $0 \leq r \leq a < a_m$ and $n \in \mathbb{N}$ So $U_V(\cdot,\cdot)$ induces by (3.1) a $\mathcal{C}_0$–semigroup $T_V(\cdot)$ on E. Its generator is denoted by G_V. By [Vo2, Cor 2.7], G_V is an extension of $(G_0 - V, D(G_0)\cap D(V))$. On the other hand, if $f \in D(G_V)$ with $f(a) = 0$ on $[a_m - \delta, a_m]$ for some $\delta > 0$, then $f \in D(G_0)\cap D(V)$ and $G_V f = (G_0 - V)f$. This implies that $D(G_V)$ consists of continuous functions vanishing at $a = 0$ (use that $\alpha f \in D(G_V)$ for $f \in D(G_V)$ and $\alpha \in C^1[0,a_m)$ with compact support and being equal to 1 on $[0,b]$ for $b < a_m$, cf. [Sc, Thm. 2.6]).

Similar as in (3.5), we define an extension of G_V by

$$\tilde{G}_V f := G_V f_0 + \omega e_\omega^V x \quad \text{for} \quad f \in D(\tilde{G}_V) := \{f = f_0 + e_\omega^V\varphi :\ f_0 \in D(G_V), \varphi \in X\}$$

for a fixed $\omega > \omega(U_V)$. Further, on $\mathcal{E}$ we introduce the operator $\mathcal{G}_V\binom{0}{f} = \binom{-f(0)}{\tilde{G}_V f}$ for $f \in D(\tilde{G}_V)$. By the arguments used in the proof of Lemma 3.1 and 3.3, one can show that $\ker(\lambda - \tilde{G}_V) = \{e_\lambda^V\varphi : \varphi \in X\}$ and that $\mathcal{G}_V$ is a Hille-Yosida operator with resolvent

$$R(\lambda,\mathcal{G}_V) = \begin{pmatrix} 0 & 0 \\ e_\lambda^V & R(\lambda, G_V) \end{pmatrix} \tag{4.14}$$

for $\mathrm{Re}\,\lambda > \omega(U_V)$. Proceeding as in the proof of Theorem 3.4, we then obtain mild solutions of (3.6) for the operator $(\tilde{G}_V, D(\tilde{G}_V))$ given by a positive evolution family $(W(t,s))_{t\geq s\geq 0}$.

Let now β be p–periodic in t. We denote by $(W_n(t,s))_{t\geq s\geq 0}$ the evolution families on E solving (3.7) for $G - V_n$. Due to (4.13), (4.14), and Proposition 2.4, the remainder $R_3^n(t,s)$ of the Dyson-Phillips expansion of $W_n(t,s)$ dominates the remainder $R_3(t,s)$ of $W(t,s)$.

Thus, $0 \leq R_3(t,s)S \leq R_3^n(t,s)S$ for each $0 \leq S \in \mathcal{L}(E)$ and so $(R_3(t,s)S)^2$ is compact for $t \geq s \geq 0$ due to Proposition 2.6 and the Dodds-Fremlin theorem, [Za, Thm. 124.3]. By [Sch, IV.1.5], this implies that $R_3(t,s)$ is strictly power compact for $t \geq s \geq 0$, and thus the conclusions of Corollary 2.2 hold.

Finally, in the autonomous case we obtain the same expression for the eigenfunctions of G_{VB} as in Remark 4.9. Also, $e^{-\omega(W)t}S(t)$ converges exponentially to a one dimensional projection if the additional conditions of Remark 4.9 hold.

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