Heat kernel estimates and L^p-spectral independence of elliptic operators

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Abstract

Let Ω be an open subset of $\mathbb{R}^d$ and let T_p for $p \in [1,\infty)$ be consistent C_0-semigroups given by kernels that satisfy an upper heat kernel estimate. Denoting by A_p their generators, we show that the spectrum $\sigma(A_p)$ is independent of $p \in [1,\infty)$. We also treat the case of weighted L^p-spaces for weights that satisfy a subexponential growth condition. An example shows that independence of the spectrum may fail for an exponential weight.

We apply our result to Schrödinger operators, Petrovskij correct systems with Hölder continuous coefficients, and elliptic operators in divergence form with real, but not necessarily symmetric coefficients and with complex coefficients.

1 Introduction and Main Results

Let Ω be an open subset of $\mathbb{R}^d$ and A be a closed linear operator in $L^2(\Omega) := L^2(\Omega, dx)$ where dx denotes Lebesgue measure.

Assume that A generates a C_0-semigroup T in $L^2(\Omega)$ which induces consistent C_0-semigroups T_p with generators A_p in each $L^p(\Omega)$. This is the case for many realizations of differential operators. It is natural to ask if the spectrum $\sigma(A_p)$ is independent of p. This is false in general (see [1]). Thus in order to obtain L^p spectral independence one needs additional assumptions.

We shall consider the situation where the C_0-semigroup T generated by A in $L^2(\Omega)$ satisfies an upper heat kernel estimate which covers many interesting cases, see [5], [10] and the applications in section 5. Thus we assume that the operators $T(t)$ are integral operators with kernels $k(t,x,y)$ which for some integer $m > 1$ and some $C > 0$ satisfy

$$|k(t,x,y)| \leq Ct^{-d/m} \exp(\omega t) \exp(-b\frac{|x-y|^{m/(m-1)}}{t^{1/(m-1)}}) \tag{1}$$

for all $t > 0$, and a.a. $x, y \in \Omega$ (see [10]). Then for $p \in [1,\infty)$ the family

$$T_p(t)f := T(t)f, \quad f \in L^2(\Omega) \cap L^p(\Omega),\ t \geq 0,$$

defines a C_0-semigroup in $L^p(\Omega)$ with generator A_p (for the convenience of the reader we shall prove this known fact in Lemma 3.1).

Under the assumption (1), L^p spectral independence has been proved for selfadjoint A by E.B. Davies ([6]), who also considered other spaces than open subsets of $\mathbb{R}^d$. W. Arendt ([1]) has given a different proof for the self adjoint case, in proving that, in the general case, the component $\rho_\infty(A_p)$ of the resolvent set $\rho(A_p)$ that contains a right half plane is independent of p. Hence L^p spectral independence holds if the resolvent set $\rho(A)$ is connected.

M. Hieber and E. Schrohe used a different approach via commutator estimates to prove L^p spectral independence ([10]). In the case where a heat kernel estimate like (1) holds, they proved L^p spectral independence under an additional commutator estimate on the resolvents.

In our main result we give a positive answer to the question of L^p spectral independence without other assumptions than (1).

Theorem 1.1 *Assume that $(T(t))$ is a C_0-semigroup in $L^2(\Omega)$ of integral operators whose kernels satisfy (1). Let A denote the generator of T. Then T defines for $p \in [1,\infty)$ consistent C_0-semigroups T_p in $L^p(\Omega)$ with generators A_p, and we have $\sigma(A_p) = \sigma(A)$ for all $p \in [1,\infty)$.*

Thus our result extends those of [1] and [6], and – in the presence of (1) – weakens the assumptions of [10]. The proof we give is based on a refinement of the arguments used in [1] involving kernel estimates for certain operators derived from resolvents, a method that has also been used in [9] and [10].

Of course, our result can be applied to all the applications mentioned in [6] (in euclidian space) and [1]. In particular, we obtain L^p spectral independence for Schrödinger operators, a result due to R. Hempel and J. Voigt (see [8] and [9]). In the section 5 we also reobtain some results from [10] and deal with non-selfadjoint elliptic operators under less restrictive assumptions.

Moreover, we treat the case of weighted L^p-spaces $L^p(\Omega, w(x)\,dx)$ where w is a weight function, i. e. a positive measurable function on Ω which is locally bounded and locally bounded away from 0. We assume that w satisfies a subexponential growth condition. A theorem of this type has been proved in [6] for more general spaces, but for selfadjoint A and polynomial weight functions.

Theorem 1.2 *Let the assumptions of Theorem 1.1 hold and suppose that w is a weight function on Ω such that for all $\epsilon > 0$ there is a $C_\epsilon > 0$ such that*

$$\frac{w(x)}{w(y)} \le C_\epsilon\, e^{\epsilon|x-y|}, \quad x, y \in \Omega. \tag{2}$$

Then T defines for $p \in [1,\infty)$ consistent C_0-semigroups $T_{p,w}$ in $L^p(\Omega, w(x)\,dx)$ with generators $A_{p,w}$, and for each $p \in [1,\infty)$ we have $\sigma(A_{p,w}) = \sigma(A)$.

The proof of this result uses estimates obtained in the proof of Theorem 1.1. We show by a counterexample that the assertion may fail for an exponential weight function which seems to be consistent with results obtained by K.-Th. Sturm on Riemannian manifolds ([12]).

The paper is organized as follows. In section 2 we collect some well-known facts about consistency of operators and resolvents in different Banach spaces. We prove Theorem 1.1 in section 3 and Theorem 1.2 in section 4. Section 5 lists examples and applications.

We use the following notation. If X is a Banach space and A is a closed linear operator in X, then $\mathcal{L}(X)$ denotes the space of all bounded linear operators on X, $\sigma(A)$ denotes the spectrum of A, $\rho(A) := \mathbb{C} \setminus \sigma(A)$ denotes the resolvent set, and $R(\lambda, A) := (\lambda - A)^{-1}$ denotes the resolvent for $\lambda \in \rho(A)$.

2 Consistency of linear operators

Let E and F be Banach spaces, and let G be a Hausdorff topological vector space with $E \hookrightarrow G$ and $F \hookrightarrow G$ where "$\hookrightarrow$" denotes continuous embedding. Assume that $E \cap F$ is dense in both E and F.

Definition 2.1 Two operators $S_E \in \mathcal{L}(E)$ and $S_F \in \mathcal{L}(F)$ are called consistent if $S_E x = S_F x$ holds for all $x \in E \cap F$.

Let T_E and T_F be C_0-semigroups in E and F, respectively, with generators A_E and A_F, respectively. We assume that T_E and T_F are consistent, i. e. $T_E(t)$ and $T_F(t)$ are consistent for all $t \geq 0$.

One of the technical difficulties in the proof of Theorems 1.1 and 1.2 is the consistency of the resolvents $R(\lambda, A_E)$ and $R(\lambda, A_F)$ for $\lambda \in \rho(A_E) \cap \rho(A_F)$.

Of course, consistency is clear for all $Re\,\lambda$ greater than the growth bounds of T_E and T_F since the semigroups are consistent. By a connectedness argument (see [1], Proposition 2.2, and [9], Theorem 1.2) the resolvents are in fact consistent for all λ in the component of $\rho(A_E) \cap \rho(A_F)$ that contains a right half plane, but consistency fails to hold in general (see e.g. [1], example 3.1, or [9]).

Thus in order to assure consistency of *all* resolvents one needs additional assumptions, and we shall use the following (see [1], Proposition 2.5.(a)).

Proposition 2.2 *If $T_E(t)E \subset F$ for some $t > 0$ then $R(\lambda, A_E)$ and $R(\lambda, A_F)$ are consistent for all $\lambda \in \rho(A_E) \cap \rho(A_F)$.*

We will also use the following (see [1], Proposition 2.3, and [10], Lemma 1.4).

Proposition 2.3 *Let $\lambda \in \rho(A_E)$ and assume that there is $Q \in \mathcal{L}(F)$ which is consistent with $R(\lambda, A_E)$. Then $\lambda \in \rho(A_F)$ and $R(\lambda, A_F) = Q$.*

Remark 2.4 If (1) holds, we have $T_p(t)L^p(\Omega) \subset L^q(\Omega)$ for all $1 \leq p < q \leq \infty$, and from Proposition 2.2 we see that in this case the sufficient condition of Proposition 2.3 is also necessary for the conclusion.

3 Proof of Theorem 1.1

We start with the following lemma which proves the first assertion in Theorem 1.1.

Lemma 3.1 *Let the assumptions of Theorem 1.1 hold. Then for* $p \in [1, \infty)$ *the family*

$$T_p(t)f := T(t)f, \quad f \in L^2(\Omega) \cap L^p(\Omega), t \geq 0,$$

defines a C_0*-semigroup in* $L^p(\Omega)$ *with type* $\leq \omega$.

Proof. Multiplying with $\exp(-\omega t)$ we may without loss of generality assume that $\omega = 0$. Define for $b, t > 0$ the function $g_{b,t}$ on $\mathbb{R}^d$ by

$$g_{b,t}(x) := c_{b,d} t^{-d/m} \exp(-b \frac{|x|^{m/(m-1)}}{t^{1/(m-1)}}), \quad x \in \mathbb{R}^d, \tag{3}$$

where $c_{b,d} > 0$ is such that $\int g_{b,1} \, dx = 1$. Then we get from (1) that the norm of $T_p(t)$ in $\mathcal{L}(L^p(\Omega))$ is $\leq \tilde{C} \, \|g_{b,t}\|_1$ where $\tilde{C}$ only depends on b and d. Since

$$g_{b,t} = t^{-d/m} g_{b,1}(\cdot / t^{1/m}) \tag{4}$$

we have by the choice of $c_{b,d}$ that $\|T_p(t)\| \leq \tilde{C}$ for all $t \geq 0$. It rests to show strong continuity on a dense subset, and we take bounded functions with compact support.

Let $p \in [1, 2)$, and let $f \in L^p(\Omega)$ be bounded with support contained in $\{|x| \leq r\}$ for some $r > 0$. Then $f \in L^2(\Omega)$. Set $\Omega_1 := \Omega \cap \{|x| \leq r\}$ and $\Omega_2 := \Omega \setminus \Omega_1$. Then we have by Hölder's inequality and (1)

$$\begin{aligned} \|T_p(t)f - f\|_{L^p(\Omega)} &= \|T_p(t)f - f\|_{L^p(\Omega_1)} + \|T_p(t)f - f\|_{L^p(\Omega_2)} \\ &\leq |\Omega_1|^{\frac{1}{p} - \frac{1}{2}} \|T(t)f - f\|_{L^2(\Omega_1)} + \tilde{C} \|g_{b,t} * |f|\|_{L^p(\Omega_2)} \\ &\leq |\Omega_1|^{\frac{1}{p} - \frac{1}{2}} \|T(t)f - f\|_{L^2(\Omega)} + \tilde{C} \|g_{b,t} * |f| - |f|\|_{L^p(\Omega)}. \end{aligned}$$

For $t \to 0$ the first term tends to 0 by the assumption on T whereas the second term tends to 0 by the choice of $c_{b,d}$ and (4) (mollifier argument).

If $p \in (2, \infty)$ then the dual operators $T_p(t)'$ (not the "Hilbert space adjoints"!) of the operators $T_p(t)$ may be represented by kernels $\tilde{k}(t, x, y) = k(t, y, x)$ which by assumption also satisfy (1). Thus, by the argument above, $(T_p(t)')$ is a C_0-semigroup in $L^q(\Omega)$ where $p^{-1} + q^{-1} = 1$. Since $L^q(\Omega)$ is reflexive, the dual semigroup T_p of $(T_p(t)')$ is a C_0-semigroup in $L^p(\Omega)$. □

As mentioned in the introduction, the proof of Theorem 1.1 is a refinement of certain arguments used in [1]. We will consider perturbed semigroups S in $L^p(\Omega)$ that have kernels

$$\frac{w(x)}{w(y)} k(t, x, y) \tag{5}$$

where w is a weight function on Ω (for the definition of "weight function" see the introduction) satisfying

$$\frac{w(x)}{w(y)} \leq C' \exp(\alpha|x-y|), \quad x, y \in \Omega \tag{6}$$

for some $C', \alpha > 0$. The following lemma shows that this meaningful.

Lemma 3.2 *Asume that w is a weight function satisfying (6). Then the kernels (5) define consistent C_0-semigroups $(S_p(t))$ in all $L^p(\Omega)$ for pin$[1,\infty)$ with type not greater than*

$$\omega(\alpha, b) := \left(\frac{\alpha}{m}\right)^m \left(\frac{m-1}{b}\right)^{m-1} + \omega,$$

and for all $c \in (0,b)$ there is an $M_c > 0$ such that

$$\|S_p(t)\| \leq M_c \, e^{\omega(\alpha,c)t}, \quad t \geq 0, p \in [1,\infty). \tag{7}$$

Proof. We combine (1) and (6) and remark that for any $c \in (0,b)$ we have

$$\omega(c,t) := \sup_{s \geq 0} \alpha s - ct^{-1/(m-1)} s^{m/(m-1)} = t \cdot \left(\frac{\alpha}{m}\right)^m \left(\frac{m-1}{c}\right)^{m-1}.$$

Thus we have established (1) for the kernels (5) with $b-c$ instead of b and $\omega(\alpha, c)$ instead of ω and C only depending on c but not on α. This proves (7). The semigroup property of $(S_p(t))$ is clear from (5) and a density argument. The assertion now follows from Lemma 3.1 if we have shown strong continuity on a dense subset in $L^2(\Omega)$.

Let $f \in L^2(\Omega)$ with support contained in $\{|x| \leq r\}$ for some $r > 0$. Once again we may assume that the growth bound of S_p is negative. Defining Ω_1 and Ω_2 as in the proof of Lemma 3.1 and choosing $c \in (0,b)$ we have

$$\begin{aligned}
\|S_2(t)f - f\|_{L^2(\Omega)} &= \|S_2(t)f - f\|_{L^2(\Omega_1)} + \|S_p(t)f - f\|_{L^p(\Omega_2)} \\
&\leq \|w^2\|_{L^\infty(\Omega_1)} \|T_2(t)(w^{-1}f) - w^{-1}f\|_{L^2(\Omega_1)} + \tilde{C}\|g_{c,t} * |f|\|_{L^2(\Omega_2)} \\
&\leq \|w^2\|_{L^\infty(\Omega_1)} \|T(t)(w^{-1}f) - w^{-1}f\|_{L^2(\Omega)} + \tilde{C}\|g_{c,t} * |f| - |f|\|_{L^2(\Omega)}.
\end{aligned}$$

The second term tends to 0 by the argument in the proof of Lemma 3.1. Since w is locally bounded, $\|w^2\|_{L^\infty(\Omega_1)}$ is finite, and since w is locally bounded away from 0 we have $w^{-1}f \in L^2(\Omega)$. Thus the first term tends to 0 as $t \to 0$ which proves strong continuity. □

We assume now that for each $\epsilon \in \mathbb{R}^d$ we have a weight function w_ϵ on Ω which satisfies (6) for $C' = 1$ and $\alpha(\epsilon)$ instead of α with $\lim_{|\epsilon| \to 0} \alpha(\epsilon) = 0$. We denote by $T_{\epsilon,p}$ the C_0-semigroup induced in $L^p(\Omega)$ by the kernels (5) with w_ϵ instead of w, and we denote its generator by $A_{\epsilon,p}$. The next lemma is an analogue of Proposition 6.4. in [1].

Lemma 3.3 *There are $\epsilon_0 > 0$ and $M_1 > 0$ such that for all $|\epsilon| \leq \epsilon_0$ and all $p \in [1,\infty)$ we have*

$$\|T_{\epsilon,p}(t)\| \leq M_1 \, e^{(\omega+1)t}, \quad t \geq 0. \tag{8}$$

If $\lambda \in \rho(A_p)$ then there is an $\epsilon_1 \in (0, \epsilon_0)$ such that $\lambda \in \rho(A_{\epsilon,p})$ for all $|\epsilon| \leq \epsilon_1$ and

$$\lim_{|\epsilon| \to 0} \|R(\lambda, A_{\epsilon,p}) - R(\lambda, A_p)\| = 0. \tag{9}$$

Proof. The first assertion is clear from a glance at Lemma 3.2. Since

$$R(\lambda, B) = (\mu - \lambda)^{-1}((\mu - \lambda)^{-1} - R(\mu, B))^{-1} R(\mu, B)$$

for any closed operator B in $L^p(\Omega)$ and all $\lambda, \mu \in \rho(B)$, and inversion is continuous on the open subset of invertible elements in $\mathcal{L}(L^p(\Omega))$, it suffices to consider a single $\mu > \omega + 1$. Now for $t > 0$ and $\epsilon \in I\!R^d$ the operator $T_p(t) - T_{\epsilon,p}(t)$ is an integral operator with kernel $k(t,x,y)(1 - w(x)w(y)^{-1})$. From (6) (remember $C' = 1$) we get

$$e^{-\alpha(\epsilon)|x-y|} \le \frac{w(x)}{w(y)} \le e^{\alpha(\epsilon)|x-y|}, \quad x, y \in \Omega,$$

hence

$$\left|1 - \frac{w(x)}{w(y)}\right| \le |1 - e^{\alpha(\epsilon)|x-y|}|, \quad x, y \in \Omega.$$

Using (1), we get for any $\epsilon \in I\!R^d$, $\delta \in (0,1)$, $t \in (\delta, \delta^{-1})$ the estimate

$$\|T_p(t) - T_{\epsilon,p}(t)\| \le C'' \|q_{\epsilon,\delta}\|_1,$$

where

$$q_{\epsilon,\delta}(x) := \delta^{-d/m} \exp(-b\delta^{1/(m-1)}|x|^{m/(m-1)})|1 - e^{\alpha(\epsilon)|x|}|, \quad x \in I\!R^d.$$

By dominated convergence we have $\|q_{\epsilon,\delta}\| \to 0$ as $|\epsilon| \to 0$. We have shown that $T_{\epsilon,p}(t) \to T_p(t)$ for $|\epsilon| \to 0$ in operator norm uniformly for $t \in (\delta, \delta^{-1})$.

Now we have for $\mu > \omega + 1$

$$\begin{aligned}
&\limsup_{|\epsilon|\to 0} \|R(\mu, A_p) - R(\mu, A_{\epsilon,p})\| \\
\le\ &\limsup_{|\epsilon|\to 0} \left(\int_0^\delta + \int_\delta^{1/\delta} + \int_{1/\delta}^\infty \right) e^{-\mu t} \|T_p(t) - T_{\epsilon,p}(t)\|\, dt \\
\le\ &2M(\delta + \frac{1}{\mu - (\omega+1)} e^{-(\mu-(\omega+1))/\delta}).
\end{aligned}$$

Since $\delta > 0$ was arbitrary the claim follows. $\square$

For $\epsilon, x \in I\!R^d$ we denote by ϵx the scalar product $\epsilon x = \sum_{j=1}^d \epsilon_j x_j$ and put

$$w_\epsilon(x) := \exp(\epsilon x), \quad v_\epsilon(x) := \exp(\epsilon x - |\epsilon||x|)$$

for all $x \in \Omega$. Then w_ϵ and v_ϵ are weight functions on Ω, w_ϵ satisfies (6) with $C' = 1$ and $\alpha = |\epsilon|$, and v_ϵ satisfies (6) with $C' = 1$ and $\alpha = 2|\epsilon|$. Thus the Lemmata 3.2 and 3.3 are applicable.

We fix $p, q \in [1, \infty)$ and denote by $T_{\epsilon,p}$ and $S_{\epsilon,p}$, respectively, the C_0-semigroups in $L^p(\Omega)$ that are induced by (5) with w_ϵ and v_ϵ, respectively, instead of w, and their generators by $A_{\epsilon,p}$ and $B_{\epsilon,p}$, respectively.

Define for $\epsilon \in I\!R^d$ the spaces $\tilde{L}^p_\epsilon := L^p(\Omega, e^{-p\epsilon x}\, dx)$ and $\hat{L}^p_\epsilon := L^p(\Omega, e^{-p|\epsilon||x|}\, dx)$. Then $W_{\epsilon,p}f(x) := e^{-\epsilon x}f(x)$ defines an isometric isomorphism $\tilde{L}^p_\epsilon \to L^p(\Omega)$. Thus we have by similarity that $\tilde{T}_{\epsilon,p}(t) := W^{-1}_{\epsilon,p}T_p(t)W_{\epsilon,p}$ defines a C_0-semigroup in $\tilde{L}^p_\epsilon$ with generator $\tilde{A}_{\epsilon,p}$ satisfying $\sigma(\tilde{A}_{\epsilon,p}) = \sigma(A_p)$.

In the same way, $V_{\epsilon,p}f(x) := \exp(-|\epsilon||x|)f(x)$ defines an isometric isomorphism $\hat{L}^p_\epsilon \to L^p(\Omega)$. By similarity, $\hat{T}_{\epsilon,p}(t) := V^{-1}_{\epsilon,p}S_{\epsilon,p}(t)V_{\epsilon,p}$ defines a C_0-semigroup in $\hat{L}^p_\epsilon$ with generator $\hat{A}_{\epsilon,p}$ satisfying $\sigma(\hat{A}_{\epsilon,p}) = \sigma(B_{\epsilon,p})$.

The operators $\tilde{T}_{\epsilon,p}(t)$ and $\hat{T}_{\epsilon,p}(t)$ may be represented by kernels $e^{\epsilon(x-y)}k(t,x,y)$ and are therefore consistent with $T_{\epsilon,p}(t)$.

Let now $\lambda \in \rho(A_p)$. Then it is a simple fact of semigroup theory that

$$R(\lambda, A_p) = \int_0^1 e^{-\lambda t}T_p(t)\, dt + e^{-\lambda}T_p(1)R(\lambda, A_p). \tag{10}$$

We want to use Proposition 2.3. By consistency the first term on the right hand side of (10) defines an operator in $\mathcal{L}(L^q(\Omega))$. According to Lemma 3.3 we choose an ϵ_1 such that $\lambda \in \rho(A_{\epsilon,p}) \cap \rho(B_{\epsilon,p})$ for $|\epsilon| \le \epsilon_1$ and

$$\sup_{|\epsilon|\le\epsilon_1} \|T_{\epsilon,p}(\frac{1}{2})\|_{\mathcal{L}(L^1,L^p)} + \|T_{\epsilon,p}(\frac{1}{2})\|_{\mathcal{L}(L^p,L^\infty)} + \|R(\lambda, A_{\epsilon,p})\|_{\mathcal{L}(L^p,L^p)} < \infty,$$

where we get the estimates on the norms of $T_{\epsilon,p}(1/2)$ from (1) and the definition of w_ϵ. Since the semigroups commute with resolvents of their generators, we thus have

$$c_1 := \sup_{|\epsilon|\le\epsilon_1} \Big(\|T_{\epsilon,p}(1)R(\lambda, A_{\epsilon,p})\|_{\mathcal{L}(L^1,L^\infty)}\Big) < \infty. \tag{11}$$

By the well-known lemma below, the operators $T_{\epsilon,p}(1)R(\lambda, A_{\epsilon,p})$ are for $|\epsilon| \le \epsilon_1$ given by kernels k_ϵ satisfying

$$|k_\epsilon(x,y)| \le c_1, \quad x, y \in \Omega. \tag{12}$$

In particular, we have the kernel k_0 for $T_p(1)R(\lambda, A_p)$. We now will show

$$k_\epsilon(x,y) = e^{\epsilon(x-y)}k_0(x,y), \quad \text{a.a. } x, y \in \Omega, \tag{13}$$

for $|\epsilon| \le \epsilon_1$ which follows if we have established consistency of the resolvents $R(\lambda, A_{\epsilon,p})$ and $R(\lambda, \tilde{A}_{\epsilon,p})$ for $|\epsilon| \le \epsilon_1$ (recall that $\lambda \in \rho(\tilde{A}_{\epsilon,p}) = \rho(A_p)$, and hence $R(\lambda, \tilde{A}_{\epsilon,p})$ makes sense).

It was at this point where connectedness entered in the proof of Theorem 4.2 in [1] (see p. 1167 and the Remark on p. 1166). To circumvent this difficulty we use the semigroup $\hat{T}_{\epsilon,p}$ which is consistent with $T_{\epsilon,p}$ and $\tilde{T}_{\epsilon,p}$, and apply Proposition 2.2 taking $L^p(\Omega)$ and $\tilde{L}^p_\epsilon$ for E, respectively, and $\hat{L}^p_\epsilon$ for F. Remark that then by $E \subset F$ the assumption in Proposition 2.2 is trivially satisfied.

Since $\lambda \in \rho(B_{\epsilon,p}) = \rho(\hat{A}_{\epsilon,p})$ we have that $R(\lambda, A_{\epsilon,p})$ and $R(\lambda, \hat{A}_{\epsilon,p})$ are consistent, and that $R(\lambda, \tilde{A}_{\epsilon,p})$ and $R(\lambda, \hat{A}_{\epsilon,p})$ are consistent. Hence $R(\lambda, A_{\epsilon,p})$ and $R(\lambda, \tilde{A}_{\epsilon,p})$ are consistent which implies (13). From (12) and (13) we get

$$|k_0(x,y)| \le c_1\, e^{-\epsilon_1|x-y|}, \quad x, y \in \Omega. \tag{14}$$

Thus also the second term on the right hand side of (10) defines a bounded operator in $L^q(\Omega)$. By Proposition 2.3 the proof of Theorem 1.1 is finished.

Lemma 3.4 *The mapping $k \mapsto T_k$, where*

$$T_k f(x) := \int k(x,y) f(y)\, dy,$$

defines an isometric isomorphism $L^\infty(\Omega \times \Omega) \to \mathcal{L}(L^1(\Omega), L^\infty(\Omega))$.

Proof. The other statements being rather obvious, we prove surjectivity. So let $T \in \mathcal{L}(L^1, L^\infty)$. Then $\phi(g,f) := < g, Tf >$ defines a continuous bilinear form on $L^1 \times L^1$. Thus its linearization $\hat{\phi}$ belongs to the dual space of $L^1 \hat{\otimes}_\pi L^1 = L^1(L^1) = L^1(\Omega \times \Omega)$. Hence we find $k \in L^\infty(\Omega \times \Omega)$ with

$$< k, g \otimes f > = < \hat{\phi}, g \otimes f > = \phi(g,f) = < g, Tf >$$

for all $f, g \in L^1(\Omega$. Since by Fubini $< k, g \otimes f > = < g, T_k f >$ the lemma is proved. □

4 Proof of Theorem 1.2

Let w, v be a weight functions on Ω satisfying (2). Assume that T is a C_0-semigroup of integral operators on $L^2(\Omega, dx)$ with kernels satisfying (1). By Lemma 3.2 there are for all $p \in [1, \infty)$ consistent C_0-semigroups $T_{p,w}$ and $T_{p,v}$, respectively, on $L^p_w := L^p(\Omega, w(x)\, dx)$ and $L^p_v := L^p_v(\Omega, v(x)\, dx)$, respectively, with generators $A_{p,w}$ and $A_{p,v}$, respectively.

Let $p, q \in [1, \infty)$ and assume $\lambda \in \rho(A_{p,w})$. We use (10) for $A_{p,w}$ and $T_{p,w}$ instead of A_p and T_p and remark that by consistency the first term on the right hand side defines a bounded operator on L^q_v. Now $T_{p,w}$ in L^p_w is similar to the C_0-semigroup $\tilde{T}_{p,w}$ in $L^p(\Omega)$ with kernels

$$\left(\frac{w(x)}{w(y)}\right)^{1/p} k(t,x,y), \quad x, y \in \Omega,$$

and generator $\tilde{A}_{p,w}$. Using (10) for $\tilde{A}_{p,w}$ and $\tilde{T}_{p,w}$ intead of A_p and T_p and the proof of Theorem 1.1 we find that $\tilde{T}_{p,w}(1) R(\lambda, \tilde{A}_{p,w})$ is given by a kernel k_0 satisfying (14) for some $\epsilon_1 > 0$. But then $T_{p,w}(1) R(\lambda, A_{p,w})$ is given by the kernel

$$\left(\frac{w(x)}{w(y)}\right)^{-1/p} k_0(x,y), \quad x, y \in \Omega.$$

We have to show that this defines a bounded operator on L^q_v, i. e. that the kernel

$$k_1(x,y) := \left(\frac{v(x)}{v(y)}\right)^{1/q} \left(\frac{w(x)}{w(y)}\right)^{-1/p} k_0(x,y), \quad x, y \in \Omega,$$

defines a bounded operator on $L^q(\Omega)$. We take $\epsilon := \epsilon_1/3$ in (2) for both v, w and obtain by (14) that $|k_1(x,y)| \le D \exp(-\epsilon|x-y|)$ for all $x, y \in \Omega$ and some $D > 0$. Hence k_1 defines a bounded operator in $L^q(\Omega)$. By Proposition 2.3 this implies $\lambda \in \rho(A_{q,v})$.

5 Applications and Examples

From the applications in the selfadjoint case of [1] and [6] we want to mention

5.1 Schrödinger operators

Let $\Omega = \mathbb{R}^d$, and let V be a real potential satisfying

$$V = V_+ - V_-, V_+, V_- \geq 0, V_- \in K_d, V_+ \in L^1_{\mathrm{loc}}(G),$$

where $G \subset \mathbb{R}^\nu$ is open with complement having measure zero, and K_d is the standard class defined in [11], A.2, or [9], section 2. Then the Schrödinger operator $A = \frac{1}{2}\Delta - V$ generates a C_0-semigroup T on $L^2(\mathbb{R}^d)$ whose integral kernels satisfy (1) for $m = 2$ ([11], Theorem B.6.7). If, for $p \in [1, \infty)$, we denote the generator of the consistent semigroup T_p on $L^p(\mathbb{R}^d)$ by A_p, we reobtain from Theorem 1.1 the following result of R. Hempel and J. Voigt ([8]).

Theorem 5.1 *The spectrum of A_p is independent of $p \in [1, \infty)$.*

In the following we list some of the applications in [10] part of which we can now treat under less restrictive assumptions and additionally in weighted spaces.

5.2 Petrovskij Correct Elliptic Systems

Let $\Omega = \mathbb{R}^d$, and let $\mathcal{A} = \sum_{|\alpha| \leq m} a_\alpha(x) D^\alpha$ be a system of differential operators with

$a_\alpha \in BUC^\rho(\mathbb{R}^d, \mathcal{L}(\mathbb{C}^N))$ for some $\rho \in (0, 1]$ and all $|\alpha| \leq m$,
$\sup_{x \in \mathbb{R}^d, |\xi|=1} Re\, \sigma(\sum_{|\alpha|=m} a_\alpha(x)(i\xi)^\alpha) \leq -\delta$, for some $\delta > 0$,

where we denote by BUC^ρ the space of uniformly Hölder continuous functions and by σ the spectrum in $\mathcal{L}(\mathbb{C}^N)$. Define for $p \in (1, \infty)$ the operator A_p in L^p by

$$A_p f = \mathcal{A} f \qquad \text{for} \quad f \in D(A_p) = W_p^m(\mathbb{R}^d)^N.$$

It is known that A_p generates consistent semigroups T_p in $L^p(\mathbb{R}^d)^N$ which may be represented by kernels k satisfying (1) (see [2], Corollary 9.5, and [7], Theorem 9.4.2). From Theorem 1.1 we thus reobtain (with a different proof) Theorem 2.2 of [10].

Theorem 5.2 *For $p \in [1, \infty)$ the spectrum $\sigma(A_p)$ is independent of p.*

Moreover, we obtain from Theorem 1.2 the following generalization where $A_{p,w}$ denotes the generator of the C_0-semigroup $T_{p,w}$ in $L^p(\mathbb{R}^d, w(x)\, dx)$ which is consistent with T_2.

Theorem 5.3 *For all $p \in [1, \infty)$ and all weight functions w on $\mathbb{R}^d$ satisfying (2) we have $\sigma(A_{p,w}) = \sigma(A_2)$.*

5.3 Elliptic Operators in Divergence Form: Real Coefficients

Let $\Omega = \mathbb{R}^d$, and let $a_{jk} \in L^\infty(\mathbb{R}^d, \mathbb{R})$ for all $j, k \in \{1, \ldots, d\}$. Consider the form $a : W_2^1(\mathbb{R}^d) \times W_2^1(\mathbb{R}^d) \to \mathbb{C}$ defined by

$$a(u, v) = \sum_{j,k=1}^{d} \int a_{jk} \partial_j u \overline{\partial_k v}\, dx. \tag{15}$$

We assume that there is an $\alpha > 0$ such that

$$\sum_{j,k=1}^{d} a_{jk}(x) \xi_j \xi_k \geq \alpha |\xi|^2 \tag{16}$$

for all $\xi \in \mathbb{R}^d$ and a.a. $x \in \mathbb{R}^d$. If A denotes the operator associated with the form a, then A generates a C_0-semigroup of integral operators on $L^2(\Omega)$ whose kernels satisfy (1) (see [3] and [5]). If, for $p \in [1, \infty)$ and weight functions w satisfying (2), we denote the generators of the consistent semigroups $T_{p,w}$ on $L^p(\mathbb{R}^d, w(x)\, dx)$ by $A_{p,w}$, we get from Theorem 1.2 the following.

Theorem 5.4 *$\sigma(A_{p,w}) = \sigma(A)$ for all $p \in [1, \infty)$ and all weight functions w satisfying* (2).

This has been shown in [10] (Theorem 2.14) for $w = 1$ under additional assumptions involving $D(A_p)$ and $D_{p,c} := \{f \in D(A_p) : \operatorname{supp} f \text{ compact}\}$.

5.4 Elliptic Operators in Divergence Form: Complex Coefficients

Let $\Omega = \mathbb{R}^d$, where $d = 1, 2$, and let $a_{jk} \in L^\infty(\mathbb{R}^d, \mathbb{C})$ for all $j, k \in \{1, \ldots, d\}$. Consider the form a given as above, and assume that there is an $\alpha > 0$ such that

$$Re \sum_{j,k=1}^{d} a_{jk}(x) \xi_j \xi_k \geq \alpha |\xi|^2 \tag{17}$$

for all $\xi \in \mathbb{C}^d$ and a.a. $x \in \mathbb{R}^d$. If A denotes the operator associated with the form a, then according to the results in [4], A generates a C_0-semigroup T in $L^2(\mathbb{R}^d)$ which may be represented by kernels satisfying (1). If, for $p \in [1, \infty)$ and weight functions w satisfying (2), we denote the generators of the consistent semigroups $T_{p,w}$ on $L^p(\mathbb{R}^d, w(x)\, dx)$ by $A_{p,w}$, we get from Theorem 1.2 the following.

Theorem 5.5 *$\sigma(A_{p,w}) = \sigma(A)$ for all $p \in [1, \infty)$ and all weight functions w satisfying* (2).

This has been shown in [10] (Theorem 2.16) for $w = 1$ under additional assumptions involving $D(A_p)$ and $D_{p,c} := \{f \in D(A_p) : \operatorname{supp} f \text{ compact}\}$.

5.5 A Counterexample

According to Theorem 1.2 we have independence of the spectrum in weighted L^p-spaces for generators of C_0-semigroups satisfying heat kernel estimates as long as the weight function w satisfies (2). The following example shows that the spectrum may depend on p for exponential weights. This observation seems to be consistent with results obtained by K.-Th. Sturm ([12]) on Riemannian manifolds.

Take the heat semigroup on $L^2(I\!R)$ generated by the Laplacian and $w(x) := e^x$. According to Lemma 3.2 there is a consistent C_0-semigroup $T_{p,w}$ in $L^p(\Omega, e^x\,dx)$ with generator $A_{p,w}$. According to Example 3.4 in [1] we have

$$\sigma(A_{p,w}) = \{(\frac{1}{p} + is)^2 : s \in I\!R\}$$

for $p \in [1, \infty)$ and the assertion of Theorem 1.2 fails.

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