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Convolution Kernels and Generalized Functions

1. Introduction

The operation of convolution $k \star f : t \mapsto \int_{-\infty}^{\infty} k(t-s)f(s)\, ds$ is basic in classical analysis. It generalizes derivatives $\delta^{(n)} \star f$, antiderivatives $\frac{t^n}{n!}\chi_{[0,\infty)} \star f$ and arises when two Laplace or Fourier integrals are multiplied together. The linearity and time invariance of the convolution operator (i.e., convolution commutes with translation) puts it in the center of signal processing applications and the theory of Markov processes. Moreover, convolution defines a zero divisor free multiplication of functions with appropriate support (see below). Thus, the totality of complex valued functions with support in some interval $[a, \infty)$ extends to a field of generalized functions. This field provides a mathematically clear and technically simple basis for an "operational calculus" applicable to linear ordinary differential equations, difference equations, and integral equations (see for example J. Mikusiński [Mi1], K. Yosida [Yo], or B. Bäumer [Ba]).

Concerning the convolution transform, two results are of particular importance. One is E. C. Titchmarsh's "Injectivity Theorem" ([Ti]), the other is C. Foiaş' "Dense Range Theorem" ([Fo]). The aim of this paper is to extend these results to (a) Banach space valued functions f, (b) defined on intervals or point inflection invariant bounded open sets $\Omega \subset \mathbb{R}^n$, and (c) scalar or semigroup valued kernels K. For Banach space valued functions f on real intervals we give outlines of two proofs of Titchmarsh's and Foiaş' results: an elegant proof by contradiction following the original proofs in the numerical case and a new, constructive proof yielding a sequence f_n of smooth functions "converging" to the generalized solution f of the convolution equation $k \star f = g$ (for given k, g). We extend Foiaş' result to functions defined on regions $\Omega \subset \mathbb{R}^n$ that are bounded and symmetric to a point; i.e., we show that the injectivity of the convolution transform as a mapping from $C(\Omega; X) \to C(\Omega; X)$ implies that the closure of the image contains $C_0(\Omega; X)$. As example of an operator valued kernel we consider strongly continuous semigroups K and show that the convolution operator $Tf := \int_0^t K(t-s)f(s)\,ds$ is injective and has dense range in $C_0([0,a]; X)$.

In the following, Banach spaces are denoted by X, their dual by $X^\star$. We denote by $C_0([0,a]; X)$ the space of continuous functions $f : [0,a] \to X$, $f(0) = 0$, endowed with the supremums norm; if $X = \mathbb{C}$, then we write $C_0[0,a]$ instead. We loosely identify any function defined on some interval with its zero extension to the whole real line. The convolution between two functions $f \in C([a,b]; X)$ and $k \in L^1[c,d]$ is therefore defined to be the convolution of their zero extensions[(1)].

2. Extensions to Banach space valued functions on intervals

This section contains outlines of two proofs of Banach space valued versions of statements obtained for the scalar case by E. C. Titchmarsh in 1926 [Ti] and C. Foiaş in 1961 [Fo] (see also J. Mikusiński [Mi2]).

Theorem 1 (Titchmarsh-Foiaş). *Let $k \in L^1_{loc}[0,\infty)$ and $a > 0$. Consider the convolution $T : f \mapsto k \star f$ as an operator from $C([0,a]; X)$ into $C_0([0,a]; X)$. Then the following are equivalent.*

(i) T has dense range,

(ii) T is injective,

(iii) $0 \in supp(k)$.

Moreover, if $0 \in supp(k)$, then $\|f\|_T := \|Tf\|$ defines a norm on $C([0,a]; X)$ and T extends to an isometric isomorphism between the completion of $C([0,a]; X)$ under that norm and $C_0([0,a]; X)$.

First proof (using the scalar versions of Titchmarsh's and Foiaş' results). *(i)* => *(ii)*: Let $\tilde{T} : h \mapsto k \star h$ for $h \in C[0,a]$. Let $g \in C_0[0,a]$, $x \in X$ with $\|x\| = 1$

(1) In particular, if $k \in L^1_{loc}[0,\infty)$ and $f \in C([0,a]; X)$, then $(k \star f)(t) = \int_0^t k(t-s)f(s)\,ds$ for $t \in [0,a]$.

and let $x^\star \in X^\star$ be such that $\langle x, x^\star\rangle = 1$. Since the range of T is dense in $C_0([0,a];X)$, there exist $f_n \in C([0,a];X)$ such that $k\star f_n \to g\cdot x$ and thus $k\star\langle f_n, x^\star\rangle = \langle k\star f_n, x^\star\rangle \to \langle g\cdot x, x^\star\rangle = g$. Hence, the range of $\tilde{T}$ is dense in $C_0[0,a]$.

Now let $f \in C([0,a];X)$. Suppose $Tf = 0$. Choose $g_n \in C[0,a]$ such that $Tg_n \to Id$. Then $0 = k\star f = k\star f\star g_n = \lim k\star g_n \star f = Id\star f$. Hence $\int_0^t (t-s)f(s)\,ds = 0$ for all $t \geq 0$ and thus $f = 0$.

(ii) => *(iii)*: Suppose 0 is not in the support of k. Then $k = 0$ a.e. on $[0,\epsilon]$ for some $\epsilon > 0$. Then $(Tf)(t) = \int_0^{t-\epsilon} k(t-s)f(s)\,ds$ for all $t \in [0,a]$ and therefore $Tf = 0$ for all $f \in C([0,a];X)$ with $f = 0$ on $[0, a-\epsilon]$, contradicting *(ii)*.

(iii) => *(ii)*: Suppose $k\star f = 0$ for some $f \in C([0,a];X)$. Then $0 = \langle k\star f, x^\star\rangle = k\star\langle f, x^\star\rangle$ for all $x^\star \in X^\star$. By the full[(2)] theorem of Titchmarsh, this implies that $\langle f, x^\star\rangle = 0$ for all $x^\star \in X^\star$ and thus $f = 0$.

(ii) => *(i)*: Let $g \in C_0([0,a];X)$. Let $(h_n)_{n\in\mathbb{N}} \subset C_0[0,a]$ be an approximate identity; i.e., $h_n \geq 0$, $\int_0^a h_n(t)\,dt = 1$ and $h_n = 0$ on $[1/n, a]$. By Foiaş' theorem, which is proved by contradiction using the Hahn-Banach theorem on the dual of $C[0,a]$ (see, for example, [Fo], [Ki-Sk], or [Mi2]) there exist $f_n \in C[0,a]$ such that $\|k\star f_n - h_n\| \to 0$. This implies that

$$k\star(f_n\star g) = (k\star f_n)\star g \to g.$$

It remains to show that if $0 \in supp(k)$, then the convolution operator $T: f \mapsto k\star f$ defines an *isometric isomorphism* between the completion of the vector space $C([0,a];X)$ with norm $\|f\|_T := \|Tf\|$ and $C_0([0,a];X)$ (with the usual supremums norm). This follows from the following proposition, whose elementary proof is omitted (see [Ba], Theorem 2.1 or [Na]).

Proposition 1. *Let Z, Y be Banach spaces and let $T \in \mathcal{L}(Z,Y)$ be injective with dense range in Y. Let $\overline{Z}^T$ be the completion of the normed vector space $Z^T :=$*

(2) E. C. Titchmarsh proved that $f, g \in C[0,\infty)$ and $f\star g = 0$ on $[0,a]$ implies that $f = 0$ on $[0,a_1]$ and $g = 0$ on $[0,a_2]$ for some $a_1, a_2 \geq 0$ with $a_1 + a_2 \geq a$. The proof (see, for example, [Ti] or [Mi1]) is slightly more complicated than the well known ones for the half-line.

$(Z, \|\cdot\|_T)$, *where* $\|z\|_T := \|Tz\|$. *Then* T *extends to an isometric isomorphism between* $\overline{Z}^T$ *and* Y. $\diamondsuit$

Second proof (by approximations): The implications (i) => (ii) and (ii) => (iii) are proved as above. The proof of (iii) => (i) is based on the following new inversion formula for Laplace transforms

Let (β_n) be a Müntz sequence; i.e., there exists $\delta > 0$, such that for all $n \in \mathbb{N}$,

$$\beta_n > 0, \quad \beta_{n+1} - \beta_n > \delta > 0, \text{ and } \sum_{n=1}^{\infty} \frac{1}{\beta_n} = \infty. \tag{M}$$

For all $n \in \mathbb{N}$, choose $N_n \in \mathbb{N}$ such that $\sum_{j=1}^{N_n} \frac{1}{\beta_{nj}} \geq a$ and define

$$\alpha_{n,i} := \frac{1}{e} \prod_{\substack{j=1 \\ j \neq i}}^{N_n} \frac{\beta_{nj}}{\beta_{nj} - \beta_{ni}} e^{-\frac{\beta_{ni}}{\beta_{nj}}}.$$

Then $|\alpha_{n,i}| \leq e^{2\beta_{ni}/n\delta}$ and for all $f \in L^1([0,a];X)$ and $q(\lambda) = \int_0^a e^{-\lambda t} f(t)\, dt$ one has

$$\int_0^t f(s)\, ds = \lim_{n\to\infty} \sum_{i=1}^{N_n} \alpha_{n,i} e^{\beta_{ni} t} q(\beta_{ni}), \tag{1}$$

where the limit is uniform for $t \in [0,a]$. For a proof of the "Phragmén-Mikusiński" inversion formula (1) see [Ba].

Now let $g \in C_0([0,a];X)$ and $0 < \epsilon < a$. Then there exists $\epsilon' > 0$ and $g_\epsilon \in C_0^1([0,a];X)$ with $\|g_\epsilon - g\| \leq \epsilon$ and $supp(g_\epsilon) \subset [\epsilon', a]$. Thus, it suffices to show that there exists a sequence f_n such that $k \star f_n \to g_\epsilon$. Define $\hat{g_\epsilon'}(\lambda) := \int_0^a e^{-\lambda t} g_\epsilon'(t)\, dt$ and $\hat{k}(\lambda) := \int_0^a e^{-\lambda t} k(t)\, dt$. Choose an integer Müntz sequence $\left(\tilde{\beta}_n\right)_{n\in\mathbb{N}} \subset \mathbb{N}$. Drop all elements for which $|\hat{k}(\tilde{\beta}_n)| \leq e^{-\tilde{\beta}_n \epsilon'/2}$ and call the obtained sequence $(\beta_n)_{n\in\mathbb{N}}$. The dropped elements cannot form a Müntz sequence $(\gamma_n)_{n\in\mathbb{N}}$, because otherwise the Phragmén-Mikusiński inversion (1) would yield

$$\begin{aligned} |\int_0^t k(s)\, ds| &= |\lim_{n\to\infty} \sum_{i=1}^{N_n} \alpha_{n,i} e^{\gamma_{ni} t} \hat{k}(\gamma_{ni})| \leq \lim_{n\to\infty} \sum_{i=1}^{N_n} |\alpha_{n,i}| e^{\gamma_{ni} t} |\hat{k}(\gamma_{ni})| \\ &\leq \lim_{n\to\infty} \sum_{i=1}^{N_n} e^{2\gamma_{ni}/n} e^{\gamma_{ni} t} e^{-\gamma_{ni} 2\epsilon/3}) \leq \lim_{n\to\infty} \sum_{i=1}^{N_n} e^{2ni(1/n + t - \epsilon'/2)} = 0 \end{aligned}$$

for all $t \in [0, \epsilon'/2)$ contradicting $0 \in supp(k)$. Thus $\sum_{i=1}^{\infty} \frac{1}{\gamma_i} < \infty$, and therefore $(\beta_n)_{n\in\mathbb{N}}$ is still a Müntz sequence satisfying $|\frac{1}{\hat{k}(\beta_n)}| < e^{\beta_n \epsilon'/2}$.

Now consider the sum of exponential functions

$$f_n(t) := \sum_{i=1}^{N_n} \alpha_{n,i} \frac{\hat{g}'_\epsilon(\beta_{ni})}{\hat{k}(\beta_{ni})} e^{\beta_{ni} t}, \tag{2}$$

where $\alpha_{n,i}$ and N_n are as above. Then

$$\begin{aligned}
(k \star f_n)(t) &= \sum_{i=1}^{N_n} \alpha_{n,i} \frac{\hat{g}'_\epsilon(\beta_{ni})}{\hat{k}(\beta_{ni})} \int_0^t k(s) e^{\beta_{ni}(t-s)}\, ds \\
&= \sum_{i=1}^{N_n} \alpha_{n,i} \frac{\hat{g}'_\epsilon(\beta_{ni})}{\hat{k}(\beta_{ni})} e^{\beta_{ni} t} \int_0^t e^{-\beta_{ni} s} k(s)\, ds \\
&= \sum_{i=1}^{N_n} \alpha_{n,i} \frac{\hat{g}'_\epsilon(\beta_{ni})}{\hat{k}(\beta_{ni})} e^{\beta_{ni} t} \int_0^a e^{-\beta_{ni} s} k(s)\, ds \\
&\quad - \sum_{i=1}^{N_n} \alpha_{n,i} \frac{\hat{g}'_\epsilon(\beta_{ni})}{\hat{k}(\beta_{ni})} e^{\beta_{ni} t} \int_t^a e^{-\beta_{ni} s} k(s)\, ds \\
&= \sum_{i=1}^{N_n} \alpha_{n,i} e^{\beta_{ni} t} \hat{g}'_\epsilon(\beta_{ni}) - \sum_{i=1}^{N_n} \alpha_{n,i} \frac{\hat{g}'_\epsilon(\beta_{ni})}{\hat{k}(\beta_{ni})} e^{\beta_{ni} t} \int_t^a e^{-\beta_{ni} s} k(s)\, ds.
\end{aligned}$$

By the Phragmén-Mikusiński inversion (1), the first term converges uniformly for $t \in [0, a]$ to g_ϵ. We conclude the proof by showing that the second term converges to zero. This follows from the fact that there exists a constant C such that, as $n \to \infty$,

$$\begin{aligned}
\| \sum_{i=1}^{N_n} \alpha_{n,i} \frac{\hat{g}'_\epsilon(\beta_{ni})}{\hat{k}(\beta_{ni})} e^{\beta_{ni} t} \int_t^a e^{-\beta_{ni} s} k(s)\, ds \| &\leq \sum_{i=1}^{\infty} e^{2\beta_{ni}/n} \frac{C e^{-\epsilon' \beta_{ni}}}{e^{-\beta_{ni}\epsilon'/2}} e^{\beta_{ni} t} \int_t^a e^{-\beta_{ni} s} \|k(s)\|\, ds \\
&\leq \sum_{i=1}^{\infty} C e^{2\beta_{ni}/n - \beta_{ni}\epsilon'/2 + \beta_{ni} t - \beta_{ni} t} \\
&\leq C \sum_{i=1}^{\infty} e^{\beta_{ni}(2/n - \epsilon'/2)} \leq C \sum_{i=1}^{\infty} e^{ni(2/n - \epsilon'/2)} \to 0.
\end{aligned}$$

$\diamondsuit$

3. Remarks

Usually generalized functions are defined as functionals on a space of test functions. Starting from this definition, one sheds light on their local structure, for example, by characterizing them (locally) as (pseudo)-derivatives of continuous functions. In other words, the local structure theorems for generalized functions θ give a regularizing function k such that $k\star\theta$ is a continuous function g; or, at least formally, that $\theta = \frac{1}{\hat{k}(D)} g$, where $\hat{k}(\lambda) := \int_0^\infty e^{-\lambda t} k(t)\, dt$ and D denotes the first derivative operator.

The Titchmarsh-Foiaş result leads to an alternative approach to generalized functions suggested earlier by S. Bochner ([Bo]) and J. Mikusiński and R. Sikorski [Mi-Si]. There, one takes the (local) structure theorems as (local) definitions and defines generalized functions $\theta : [0, a] \to X$ as elements of the (Banach space) completion $C^{[k]}([0, a]; X)$ of the vector space(3) $C([0, a]; X)$ with norm $\|f\|_T := \|k \star f\|$; i.e., generalized functions θ are regarded as equivalence classes $[f_n]$ of continuous(4) functions f_n such that $k \star f_n \to g = k \star \theta \in C_0([0, a]; X)$ for some $k \in L^1_{loc}[0, \infty)$. Note that (2) allows to construct for each $\theta \in C^{[k]}([0, a]; X)$ a concrete approximating sequence $[f_n] = \theta$, where $f_n(t) = \sum_{i=1}^{N_n} \gamma_{i,n} e^{\beta_{in} t}$ for some Müntz sequence $(\beta_n)_{n\in\mathbb{N}} \subset \mathbb{N}$ and coefficients $\gamma_{i,n} \in X$.

Example 1. Let $k(t) = 1$ for $t \geq 0$. By the Titchmarsh-Foiaş theorem, the antiderivative operator $(Tf)(t) := \int_0^t f(s)\, ds$ extends to an isometric isomorphism between the completion $R([0, 1]; X)$ of the space $C([0, 1]; X)$ with 'mean'-norm $\|f\|_M := \|Tf\| = \sup_{t\in[0,1]} \|\int_0^t f(s)\, ds\|$ and the space $C_0([0, 1]; X)$ with the supremums norm. Thus, the Banach space $R([0, 1]; X)$ consists of (generalized) derivatives of functions in $C_0([0, 1]; X)$. Let $\theta \in R([0, 1]; X)$. Then there exists $f_n \in C([0, 1]; X)$ and $g, g_n \in C_0([0, 1]; X)$ such that $T\theta = g = \lim g_n = \lim T f_n$. Thus, one may define for $0 \leq a < b \leq 1$

$$\int_a^b \theta(s)\, ds := g(b) - g(a) = \lim g_n(b) - g_n(a) = \lim \int_a^b f_n(s)\, ds.$$

If $C([0, 1]; X) \ni f_n \to \theta \in R([0, 1]; X)$, then f_n might not converge pointwise, but f_n

(3) Since the convolution is shift invariant; i.e., $k\star f = S_{-a}(k\star S_a(f))$ for any shift S, the Titchmarsh-Foiaş theorem also holds if the convolution operator is considered as an operator on $C([a, b]; X)$ for $a < b \in \mathbb{R}$. Furthermore, using the Frechét topology, the same holds for the spaces $C([a, \infty); X)$, $a \in \mathbb{R}$.

(4) One could also take C^∞-functions, polynomials or any dense set of continuous functions as core elements.

converges towards f in the mean; i.e., for all $0 \leq a < b \leq 1$,

$$\frac{1}{b-a}\int_a^b f_n(s)\,ds \to \frac{1}{b-a}\int_a^b f(s)\,ds.$$

In fact, a sequence $f_n \in C([0,1];X)$ is Cauchy in $R([0,1];X)$ if and only if f_n is Cauchy in the mean. The Banach space $R([0,1];X)$ strictly contains the spaces $L^p([0,1];X)$ for $1 \leq p \leq \infty$. In general, $R([0,1];X)$ also contains non-measurable functions. To see this, take $X = L^\infty[0,1]$ and $f(t) := \chi_{[0,t]}$ (the characteristic function of the interval $[0,t]$. Then $f : [0,1] \to X$ is nowhere continuous. Since $\|f(s) - f(t)\| = 1$ for $t \neq s$, f is not almost separably valued; by Pettis' theorem, f is not measurable. However, $f \in R([0,1];X)$. Moreover, f is Riemann integrable and the antiderivative $(Tf)(t) = F(t) := \int_0^t f(s)\,ds = (t-\cdot)\chi_{[0,t]}$ is a Lipschitz continuous function in $C_0([0,1];X)$ which is nowhere differentiable (in the classical sense). However, F is differentiable in the extended sense and $F' = f$. In fact, any function $H \in C_0([0,1];X)$ is differentiable in the extended sense; i.e., let $H_n \in C_0^1([0,1];X) \to H$ in $C_0([0,1];X)$. Then H_n' is Cauchy in the Banach space $R([0,1];X)$ and $\theta := [H_n']$ is called the generalized derivative of H. This yields the following version of the fundamental theorem of calculus.

Corollary 1 (Fundamental Theorem of Calculus). *Let X be a Banach space. Any $H \in C_0([0,1];X)$ is differentiable in the extended sense, $H' \in R([0,1];X)$ is integrable and $\int_a^b H'(s)\,ds = H(b) - H(a)$ for all $0 \leq a < b \leq 1$. Moreover, each $\theta \in R([0,1];X)$ is integrable, $H(t) := \int_0^t \theta(s)\,ds$ is differentiable (in the extended sense) and $H' = \theta$.* $\diamondsuit$

J. Mikusiński developed his 'Operational Calculus' on the basis of a field extension of the zero-divisor free convolution ring of continuous functions on $[0,\infty)$. We obtain the same field by considering generalized functions $f \in C^{[k]}[a,\infty)$ for some $a \in \mathbb{R}$ and $k \in C[0,\infty)$ with $0 \in supp(k)$. The convolution between two generalized functions $f_1 \in C^{[k_1]}[a_1,\infty)$ and $f_2 \in C^{[k_2]}[a_2,\infty)$ is defined via

$$f_1 \star f_2 := T_{k_1 \star k_2}^{-1}(k_1 \star f_1 \star k_2 \star f_2) \in C^{[k_1 \star k_2]}[a_1 + a_2, \infty)$$

and two generalized functions f_1, f_2 are identified if $k_2 \star k_1 \star f_1 = k_1 \star k_2 \star f_2$ (see [Ba] for more details).

In the vector-valued case one has to face the problem that the convolution between two vector-valued functions is in general not defined. However, vector-valued functions form a torsion-free convolution module over the ring of scalar-valued functions, and therefore can be extended to a vector space over the field of generalized scalar-valued functions. This vector space can be identified with the space of generalized functions

$$V_X := \{f \in C^{[k]}([a,\infty); X) : a \in \mathbb{R} \text{ and } k \in C[0,\infty) \text{ with } 0 \in supp(k)\}.$$

If $(K(t))_{t\geq 0} \subset \mathcal{L}(X,Y)$ is a strongly continuous operator family, then $T_K f := K \star f$ defines a bounded operator from $C([0,a];X)$ into $C_0([0,a];Y)$. In general, it does not seem to be known to date how to characterize those K for which T_K is injective and under which additional conditions on K the operator has the dense range property (for some preliminary results employing estimates of the growth of $\|\hat{K}(\lambda)\|$ as $\lambda \to \infty$, see [Ba]). However, for an important class of strongly continuous operator families with $0 \in supp(K)$ one obtains the following result.

Proposition 2. *Let K be a strongly continuous semigroup on a Banach space X and $a > 0$. Then the convolution operator $T_K : C([0,a];X) \to C_0([0,a];X)$ defined by $T_K f := \int_0^t K(t-s)f(s)\,ds$ is one-to-one and has dense range.*[5]

Proof. Let A denote the generator of K and $\mathcal{D}(A)$ its domain. To see that $T_K f$ is one-to-one, observe that $\widehat{T_K f} = \int_0^\infty e^{-\lambda t} \int_0^t K(t-s)f(s)\,ds\,dt = R(\lambda,A)\hat{f}(\lambda)$, where we identify f with its zero continuation onto $[0,\infty)$. Thus, if $T_K f = 0$, then $\hat{f} = 0$ for all sufficiently large λ, and thus $f = 0$.

For the proof of the density of the range of $T_K f$ in $C_0([0,a];X)$ we show first that $C_0^1([0,a];[\mathcal{D}(A)])$ is dense in $C_0([0,a];X)$, where $[\mathcal{D}(A)]$ denotes the Banach space $\mathcal{D}(A)$ endowed with the graph norm $\|x\|_A := \|x\| + \|Ax\|$. It follows from the Hille-Yosida theorem that the operators $\lambda R(\lambda,A)$ are uniformly bounded $(\lambda > \lambda_0)$ and

[5] Since for all $g \in C([0,\infty);X)$, and for all $a > 0, \epsilon > 0$ there exist $f_a \in C([0,\infty);X)$ such that $\sup_{t\in[0,a]} \|\int_0^t K(t-s)f_a(s)\,ds - g(t)\| < \epsilon$, the image is also dense in the Frechét space $C([0,\infty);X)$.

that $\lambda R(\lambda, A)x = R(\lambda, A)Ax + x \to x$ for all $x \in \mathcal{D}(A)$ as $\lambda \to \infty$. Thus, by the density of $\mathcal{D}(A)$, it follows from the Banach-Steinhaus theorem that $\lambda R(\lambda, A)x \to x$ for all $x \in X$, where the convergence is uniform for x in compact subsets of X. Let $g \in C_0[0, a]; X)$. Then

$$g_h : t \mapsto \frac{1}{h}\int_t^{t+h} g(s)\, ds - \frac{1}{h}\int_0^h g(s)\, ds \in C_0^1([0, a]; X)$$

and $g_h \to g$ as $h \to 0$ uniformly on $[0, a]$. Let

$$g_{h,\lambda} : t \mapsto \lambda R(\lambda, A)g_h(t) \in C_0^1([0, a]; [\mathcal{D}(A)]).$$

Since $C_h := \{g_h(t) : t \in [0, a]\}$ is a compact subset of X, it follows that $g_{h,\lambda} \to g_h$ as $\lambda \to \infty$. Now for $f_{h,\lambda} \in C([0, a]; X)$, we have that $T_K f_{h,\lambda} = g_{h,\lambda}$ if and only if $\widehat{T_K f_{h,\lambda}} = R(\lambda, A)\hat{f}_{h,\lambda} = \hat{g}_{h,\lambda}$ or $\hat{f}_{h,\lambda}(\mu) = (\mu - A)\hat{g}_{h,\lambda}(\mu) = \mu\hat{g}_{h,\lambda}(\mu) - \widehat{Ag_{h,\lambda}}(\mu)$. Therefore, $f_{h,\lambda} : t \mapsto g'_{h,\lambda}(t) - Ag_{h,\lambda}(t) \in C([0, a]; X)$ satisfies $T_K f_{h,\lambda} \to g_h$ as $\lambda \to \infty$. Thus, the range of T_K is dense in $C_0([0, a]; X)$. ♢

Results of this type have important applications. For example, consider the inhomogeneous abstract Cauchy problem

$$u'(t) = Au(t) + f(t), \quad u(0) = 0, \qquad (ICP)$$

where A is the generator of a strongly continuous semigroup K on a Banach space X, and where $f \in C([0, \infty); X)$ is a forcing term. Then the unique mild solution of (ICP) is given by

$$u(t) := \int_0^t K(t - s)f(s)\, ds = (T_K f)(t).$$

By the previous proposition, the range of $T_K f$ is dense in $C_0([0, a]; X)$. Thus, for any given target function $h \in C_0([0, a]; X)$ and any $\epsilon > 0$, there exists a forcing term $f_\epsilon \in C([0, a]; X)$ such that the solution $u_\epsilon = T_K f_\epsilon$ of the evolutionary system described by (ICP) satisfies

$$\|T_K f_\epsilon - h\| = \sup_{t\in[0,a]} \|u_\epsilon(t) - h(t)\| < \epsilon.$$

Therefore any evolutionary system governed by a strongly continuous semigroup can be steered by appropriate forcing terms $f_\epsilon \in C([0,a];X)$ to any preassigned target orbit $h \in C_0([0,a];X)$.

Furthermore, since $\|T_K f_\epsilon - h\| = \|u_\epsilon - h\| < \epsilon$, it follows that there exists a generalized forcing term $f_0 \in \overline{C([0,a];X)}^{T_K}$ such that $f_\epsilon \to f_0 \in \overline{C([0,a];X)}^{T_K}$ and

$$h(t) = (\tilde{T}_K f_0)(t) = \lim_{\epsilon \to 0} (T_K f_\epsilon)(t) = \lim_{\epsilon \to 0} u_\epsilon,$$

uniformly in $t \in [0,a]$, where u_ϵ is the solution of

$$u'_\epsilon(t) = Au_\epsilon(t) + f_\epsilon, \quad u_\epsilon(0) = 0.$$

In this sense, any target $z \in C_0([0,a];X)$ is a "limit solution" or generalized solution of the inhomogeneous problem

$$u'(t) = Au(t) + f_0(t), \quad u(0) = 0,$$

for a unique forcing term $f_0 \in \overline{C([0,T];X)}^{T_K}$.

4. Extensions to Banach space valued functions defined on $\Omega \subset \mathbb{R}^n$

It seems to be unknown whether the Titchmarsh-Foiaş theorem holds for domains in more than one dimension. J. Lions [Li] showed in 1951 that for scalar valued continuous functions f, g with compact[(6)] support in $\mathbb{R}^n$ the convex hulls of their supports add up to the convex hull of the support of the convolution product $f \star g(t) = \int_{\mathbb{R}^n} f(t-s)g(s)\,ds$; i.e.,

$$co[supp(f \star g)] = co[supp(f)] + co[supp(g)].$$

This implies that $0 \in supp(k)$ implies injectivity on cones or certain 'triangles' (see J. Mikusiński and C. Ryll-Nardzewski [Mi-Ry], J. Mikusiński [Mi1], or T. K. Boehme [Boe]). However, for cones or triangles it seems not to be known whether injectivity

[(6)] The theorem fails, if we do not require compact support. For example, take $h \in C^\infty[0,\infty)$ with $supp(h) \subset (0,1)$. Then $e^t \chi_{[0,\infty)} \star (h - h') = -h$.

of the convolution operator implies the crucial dense range property. We present a theorem that gives this implication on domains which are symmetric to a point and on some subsets thereof. We do not know whether $0 \in supp(k)$ implies on these domains the injectivity of the convolution product. For many k however, the injectivity is often known by other means; for example, if k comes from a fundamental solution with respect to a differential, ultradifferential, or hyperdifferential, linear local operator $L, Lk = \delta$, then $k \star \mu = 0$ on a region Ω gives $L(k \star \mu) = Lk \star \mu = \delta \star \mu = \mu = 0$ on Ω for any distribution μ that is supported by Ω.

Theorem 2. *Let $V \subset \mathbb{R}^n$ be a bounded open set. Let $b \in \mathbb{R}^n$. Let $\Omega = b + W$, where $W = V \cup (-V)$ or $W = V \cap (-V)$. Let k be a distribution with compact support in $\mathbb{R}^n_+$, and $T_k : C_0^\infty(\Omega) \to C(\overline{\Omega})$ be defined via $T_k\phi := \int_{\mathbb{R}^n} k(t-s)\phi(s)\, ds|_\Omega$. Suppose $k \star \mu = 0$ implies that $\mu = 0$ for all measures μ that are supported by Ω; i.e., $\mu(\mathbb{R}^n \backslash \Omega) = 0$ and $k \star \mu = 0$ implies that $\mu = 0$. Then the closure of the range of T_k contains $C_0(\Omega)$.*

Proof. Suppose the closure of the range of T_k does not contain $C_0(\Omega)$. Then there exists a measure $\mu \neq 0$ supported by Ω such that $0 = \langle k \star \phi, \mu \rangle$ for all $\phi \in C_0^\infty(\Omega)$. However,

$$0 = \langle k \star \phi, \mu \rangle = \left((k^- \star \phi^-) \star \mu \right)(0) = \left((k^- \star \mu) \star \phi^- \right)(0) = \langle k^- \star \mu, \phi \rangle,$$

where $k^-(t) := k(-t)$. Thus $k^- \star \mu = 0$ on Ω. By a change of variables, $k^- \star \mu = k \star \nu$ on $2b - \Omega = \Omega$, where $\nu(m) := \mu(2b - m)$ for any measurable set m. Thus $\nu = 0$ on Ω and hence $\mu = 0$ on $2b - \Omega = \Omega$. $\diamondsuit$

Of course functions in the image of T_k are in general not zero on the boundary. Restricting the region Ω to $\tilde{\Omega} \subset \Omega$, where $(\Omega \backslash \tilde{\Omega} + supp(k)) \cap \tilde{\Omega} = \emptyset$ leads to a better understanding of the size of $\overline{Im(T_k)}$, at least on $\tilde{\Omega}$.

Corollary 2. *Let $\tilde{\Omega} \subset \Omega$ with $(\Omega \backslash \tilde{\Omega} + supp(k)) \cap \tilde{\Omega} = \emptyset$. Then*

$$\overline{Im(T_k)} \supset Z := \{ f \in C(\overline{\tilde{\Omega}}) : f(t) = 0 \text{ for } t \in \partial(\Omega) \}.$$

Proof. Clearly, any function $\tilde{\phi} \in Z$ can be extended to a function $\phi \in C_0(\Omega)$. It remains to be shown that $k \star \tilde{\phi}\big|_{\tilde{\Omega}} = k \star \phi\big|_{\tilde{\Omega}}$. This holds since

$$supp(k \star (\phi - \tilde{\phi})) \cap \tilde{\Omega} \subset (supp(k) + \Omega \backslash \tilde{\Omega}) \cap \tilde{\Omega} = \emptyset.$$

$\diamondsuit$

This result can now be lifted to the vector-valued case by a partition of unity argument. Any $f \in C_0(\Omega; X)$ can be approximated by a finite sum $f(t) \approx \sum_{i=1}^{N} f_i(t)x_i$, where $f_i \in C_0(\Omega)$ and $x_i \in X$. Since each f_i can be approximated by $k \star g_i \approx f_i$, $k \star \sum_{i=1}^{N} g_i(t)x_i \to f$. Thus we have proved the following theorem.

Theorem 3. *Let $V \subset \mathbb{R}^n$ be a bounded open set. Let $b \in \mathbb{R}^n$. Let $\Omega = b + W$, where $W = V \cup (-V)$ or $W = V \cap (-V)$. Let k be a distribution with compact support in $\mathbb{R}^n_+$, and $T_k : C_0^\infty(\Omega; X) \to C(\overline{\Omega}; X)$ be defined via $T_k\phi := \int_{\mathbb{R}^n} k(t-s)\phi(s)\, ds|_\Omega$. Suppose $k \star \mu = 0$ implies that $\mu = 0$ for all measures μ supported by Ω. Then the closure of the range of T_k contains $C_0(\Omega; X)$. Furthermore, for $\tilde{\Omega} \subset \Omega$ with $(\Omega\backslash\tilde{\Omega} + supp(k)) \cap \tilde{\Omega} = \emptyset$ we obtain that*

$$\overline{Im(T_k)} \supset Z := \{f \in C(\overline{\tilde{\Omega}}; X) : f(t) = 0 \text{ for } t \in \partial(\Omega)\}.$$

$\diamondsuit$

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