ASYMPTOTIC STABILITY AND PERRON CONDITION FOR PERIODIC EVOLUTION FAMILIES ON THE HALF LINE

Constantin BUŞE *
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Abstract

Extending earlier results of Datko, Pazy and Neerven on C_0-semigroups, we prove the following. Let $\mathcal{U}$ be a q-periodic evolution family on a Banach space X and E be a normed function space over $\mathbf{R}_+$ with the property that $\lim\limits_{t\to\infty} |\chi_{[0,t]}|_E = \infty$. If for any $x \in X$, the map $t \mapsto ||U(t,0)x|| : \mathbf{R}_+ \to \mathbf{R}_+$ belongs to E, then $\mathcal{U}$ is exponentially stable. On the other hand we prove that if

$$\sup_{t>0} \left\| \int_0^t U(t,\xi) e^{i\mu\xi} f(\xi) d\xi \right\| = M(\mu, f) < \infty$$

for all $\mu \in \mathbf{R}$ and $f \in P_q(\mathbf{R}_+, X)$, (that is f is a q-periodic and continuous function on $\mathbf{R}_+$), then $\mathcal{U}$ is exponentially stable.

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1.INTRODUCTION

Let $\mathbf{T} = \{T(t)\}_{t\geq 0}$ be a C_0-semigroup on a Banach space X, and $\omega(\mathbf{T})$ be its growth bound. It is well known theorem of Datko [Da1], that if

$$\int_0^\infty ||T(t)x||^2 dt < \infty \quad \forall x \in X,$$

then $\omega(\mathbf{T}) < 0$. This result was generalized by Pazy [Pa], who showed that the exponent $p = 2$, may be replaced by $1 \leq p < \infty$, and by Datko [Da2], who showed the following result:

Let $\mathbf{U}$ *be an evolutionary process of class* $C(0,e)$ *on a Banach space* X. *If there exists a real number* $p \geq 1$ *such that*

$$\sup_{s\geq 0} \int_s^\infty ||U(t,s)x||^p ds < \infty \quad \forall x \in X,$$

then $\mathbf{U}$ *is uniformly exponentially stable.*

Let $s \geq 0$ and $x \in X$. We consider the function

$$t \mapsto U_s^x(t) : \mathbf{R}_+ \to \mathbf{R}_+,$$

given by

$$U_s^x(t) = \begin{cases} ||U(t,s)x||, & t \geq s \\ 0, & 0 \leq t < s. \end{cases}$$

Recently in [Bus1, Theorem 0.1] it is shown that the evolutionary process $\mathbf{U}$ of class $C(0,e)$ on X is uniformly exponentially stable if for any $x \in X$ and every $s \geq 0$, the map U_s^x belongs to E and

$$\sup_{s\geq 0} \|U_s^x\|_E = M(x) < \infty.$$

Here E is a normed function space over $\mathbf{R}_+$ with $\Psi_E(\infty) = \infty$, (see [Ne1], [Bus1], [Bus3] or section 2 for the exact definitions in the present context), which satisfies the condition

$$(\mathbf{C1}) \quad \left\|\chi_{[0,t]}\right\|_E \leq \left\|\chi_{[\tau,t+\tau]}\right\|_E \quad \forall t \geq 0, \forall \tau \geq 0.$$

χ_A is the characteristic function of the set A. For example in $L^p(\mathbf{R}_+, \mathbf{C})$ the condition (**C1**) it is verified with equality, so Datko result is an immediate corollary of Theorem 0.1 in [Bus1]. The hypothesis (**C1**) is essential, see [Bus1, Example 3.2], but he may be except in the autonomous case, see [Ne1, Theorem 4.2] or [Bus3, Theorem 0.1]. In this paper we prove that the condition (**C1**) may be except also in the case when $\mathcal{U}$ is a q-periodic evolution family of linear and bounded operators on X.

The concept of exponential dichotomy of linear differential equations was introduced by O. Perron is concerned with the problem of conditional stability of system $\dot{x} = A(t)x + f(t)$. After seminal researches of O. Perron, relevant results concerning the extension of Perron's problem in the general framework of functional Banach space were obtained by M.G.Krein, R.Bellman, J.L.Massera and J.J. Schäffer in the period 1948-1966. In the last three decades a great number of paper about stability and calitative behaviour of C_0-semigroups was published.

Let X be a complex Banach space. We denote by $L(X)$ the Banach space of all linear operators acting on X. We also denote by $||\cdot||$, the norms of vectors and operators. Let $A \in L(X)$ and $\mathbf{R}_+$, the set of the non-negative real numbers. The following results is due to M.G.Krein, see also [Ba].

If the Cauchy problem

$$\dot{x}(t) = Ax(t) + e^{i\mu t}x_0, \quad x(0) = 0, x_0 \in X, \mu \in \mathbf{R}$$

has a bounded solution on $\mathbf{R}_+$ for every $\mu \in \mathbf{R}$, then the homogenous system $\dot{x} = Ax$, is uniformly exponentially stable.

The hypothesis of the above results can be write in the form:

$$\sup_{t>0} \left\| \int_0^t e^{i\mu\xi} e^{\xi A} x_0 d\xi \right\| < \infty, \quad \forall \mu \in \mathbf{R}, \forall x_0 \in X.$$

This result cannot be extended for C_0-semigroups, cf. [RB, Example 3.1]. However, Neerven in [Ne3, Corollary 5] has been shown that if

$$\sup_{\mu\in\mathbf{R}} \sup_{t>0} \left\| \int_0^t e^{i\mu\xi} T(\xi) x_0 d\xi \right\| < \infty, \quad \forall x_0 \in X, \tag{1}$$

then $\omega_1(\mathbf{T}) < 0$. For details concerning $\omega_1(\mathbf{T})$, we refer to [Neu] or [Na, Theorem A IV.1.4]. Moreover under the hypothesis (1) it results that the resolvent $R(z, A_{\mathbf{T}}) = (z - A_{\mathbf{T}})^{-1}$ of the infinitesimal generator of $\mathbf{T}$, exists and is uniformly bounded on $\mathbf{C}_+ := \{\lambda \in \mathbf{C} : \text{ Re } (\lambda) > 0\}$. Combining this with a result of Gearhart in [Ge], it results that if X is a complex Hilbert space and (1) holds, then $\omega(\mathbf{T}) < 0$. In the general case, when X is a Banach space the last results is not true, see e.g. [Bus2, Example 2] or [GVW]. On the other hand in [Ne2, Corollary 2.3] it is been shown that if

$$\sup_{t>0} \left\| \int_0^t T(\xi) f(\xi) d\xi \right\| < \infty, \quad \forall f \in AP(\mathbf{R}_+, X)$$

then $\omega(\mathbf{T}) < 0$. Here $AP(\mathbf{R}_+, X)$ is the linear closed hull in $BUC(\mathbf{R}_+, X)$ of the set of all functions

$$\{e^{i\mu(\cdot)}x, \quad \mu \in \mathbf{R}, x \in X\},$$

and $BUC(\mathbf{R}_+, X)$ is the space of all X-valued bounded uniformly continuous functions on $\mathbf{R}_+$, with sup-norm. For the variants of this result in the non-autonomous case, see [RB, Theorem 3.3], [Bus2, Theorem 2] or [MRS, Theorem 2.2]. A version of this result is proved in the present paper.

2.DEFINITIONS

Let $q > 0$ and $\Delta = \{(t, s) \in \mathbf{R}^2 : t \geq s \geq 0\}$. A mapping $\mathcal{U} : \Delta \to L(X)$ would be called q-periodic evolution family of bounded linear operators on X, iff:

- **(i)** $U(t, s) = U(t, r)U(r, s)$ for all $t \geq s \geq r \geq 0$;
- **(ii)** $U(t, t) = Id$, (Id is the identity on X), for all $t \geq 0$;
- **(iii)** for all $x \in X$, the map $(t, s) \mapsto U(t, s)x : \Delta \to X$, is continuous;
- **(iv)** $U(t + q, s + q) = U(t, s)$ for all $t \geq s \geq 0$.

The operator $\mathcal{U}(t, s)$ we denoted by $U(t, s)$.

Let $(\mathbf{R}_+, \mathcal{L}, m)$ be the Lebesgue measure space and let $\mathcal{M}$ be the linear space of all measurable functions $f : \mathbf{R}_+ \to \mathbf{C}$, identifiyng functions which are equal a.e.. Consider a function $\rho : \mathcal{M} \to [0, \infty]$ with the following properties:

- (**n1**) $\rho(f) = 0$ if and only if $f = 0$;
- (**n2**) $\rho(af) = |a|\rho(f)$ for any scalar $a \in \mathbf{C}$ and any $f \in \mathcal{M}$, with $\rho(f) < \infty$;
- (**n3**) $\rho(f+g) \leq \rho(f) + \rho(g)$ for all $f, g \in \mathcal{M}$.

Let $F = F_\rho$ be the set of all $f \in \mathcal{M}$ such that $|f|_F := \rho(f) < \infty$. It is clear that $(F, |\cdot|)$ is a normed linear space. The normed linear subspace E of F is called *normed function space* if the following two conditions hold:

- (**n4**) if $f \in \mathcal{M}, g \in E$ and $|f| \leq |g|$ a.e., then $f \in E$ and $|f|_E \leq |g|_E$;
- **(n5)** $\chi_{[0,t]} \in E$ for all $t > 0$.

Let E be a normed function space. For all $t > 0$, we define $\Psi_E(t) := \left\|\chi_{[0,t]}\right\|_E$ and $\Psi_E(\infty) = \lim_{t\to\infty} \Psi_E(t)$. With $\sigma(T)$ we denote the *spectrum* of an operator $T \in L(X)$ and with $r(T)$ we denote the *spectral radius* of T. We recall that

$$r(T) = \lim_{n\to\infty} ||T^n||^{\frac{1}{n}} = \sup\{|z| : z \in \sigma(T)\},$$

see e.g. [V, Ex. 1.7.23].

3.THE MAIN RESULTS

We shall prove several lemas which would be used later.

Lemma 1 *The q-periodic evolution family $\mathcal{U}$ on X has exponential growth, (i.e. there are $\omega \in \mathbf{R}$ and $M > 1$ such that*

$$||U(t,s)|| \leq Me^{\omega(t-s)} \quad \forall t \geq s \geq 0. \tag{2)}$$

Proof. The function $t \mapsto U(t,0)x$ is continuous on $[0,q]$ for any $x \in X$, so there exists $M(x) > 1$ such that $||U(t,0)x|| \leq M(x)$. Hence, using uniform boundedness principle it results that there exists $M_q > 1$ such that $||U(t,0)|| \leq M_q$ for any $t \in [0,q]$. Let $\xi = mq + s$ with $m \in \{1, 2, \cdots\}$ and $s \in [0,q)$. Then

$$\begin{aligned} ||U(\xi,0)|| &\leq ||U(\xi,mq)||||U(mq,(m-1)q)||\cdots||U(q,0)|| \\ &\leq M_q^{m+1} = M_q e^{\frac{1}{q}(mq)\ln M_q} \leq M_q e^{\frac{1}{q}(\ln M_q)\xi}. \end{aligned}$$

Now, it is not hard to see, that there are $\omega = \omega(q) > 0$ and $M_q > 1$ such that

$$||U(\xi, 0)|| \leq M_q e^{\omega\xi}, \quad \forall \xi \geq 0. \tag{3}$$

On the other hand, the function $(t, s) \mapsto U(t, s)x$ is continuous on the set $\{(t, s) \in \mathbf{R}^2 : 0 \leq s \leq t \leq q\}$ for any $x \in X$, so there exists $N_q > 1$ such that

$$||U(t, s)|| \leq N_q, \quad \forall (t, s) \in [0, q] \times [0, q], t \geq s. \tag{4}$$

Let $v \geq \xi + q$ and $\xi = nq + u$, with $n \in \mathbf{N}$ and $u \in [0, q)$. Then from (3) and (4), it follows:

$$\begin{aligned} ||U(v, \xi)|| &\leq ||U(v, (n+1)q)U((n+1)q, \xi)|| \\ &\leq ||U(v - nq - q, 0)||||U(q, u)|| \leq M_q N_q e^{\omega(v-\xi)}. \end{aligned}$$

We recall that the evolution family $\mathcal{U}$ is called *exponentially stable* if there are $\omega < 0$ and $M > 1$ such that (2) holds. Let $V = U(q, 0) \in L(X)$.

Lemma 2 *The q-periodic evolution family $\mathcal{U}$ is exponentially stable if and only if $r(V) < 1$.*

Proof. Let $\nu > 0$ such that $r(V) < e^{-\nu} < 1$. Then $r(e^{\nu}V) < 1$, so there is $K > 0$ such that $||e^{\nu n}V^n|| \leq K$, for any $n = 1, 2, \cdots$. It follows $||U(nq, 0)|| \leq Ke^{-\nu n}$. Let $t \geq q, m = 1, 2, \cdots$ and $\rho \in [0, q)$ such that $t = mq + \rho$. Then

$$\begin{aligned} ||U(t, 0)|| &\leq ||U(t, mq)||||U(mq, 0)|| \\ &\leq Me^{\omega q}||U(mq, 0)|| \leq K_1 e^{-\nu_1 t}, \end{aligned}$$

where $\nu_1 = \frac{\nu}{q}$, and $K_1 = MKe^{\omega q}e^{\frac{\nu\rho}{q}}$.

Let $t \in [0, q)$. Then:

$$||U(t, 0)|| \leq K_2 e^{-\nu_1 t}, \text{ where } K_2 = Me^{(\omega+\nu_1)q}.$$

Let $\xi \geq 0$ and $t \geq \xi + q$. It is clear that there are $p \in \mathbf{N}$ and $s \in [0, q)$, such that $\xi = pq + s$. Then:

$$\begin{aligned} ||U(t, \xi)|| &\leq ||U(t, (p+1)q)||||U((p+1)q, s)|| \\ &\leq Me^{\omega q}(K_1 + K_2)e^{-\nu_1(t-\xi)}. \end{aligned}$$

We present without proof a result in the theory of linear and bounded operators. We will denote by $\mathbf{c_0}$ the Banach space of all sequences (a_n) of complex numbers which are convergent to 0, endowed with the norm $||(a_n)|| = \sup\limits_n |a_n|$.

Lemma 3 *Let $T \in L(X)$ with $r(T) \geq 1$ and $\varepsilon \in (0,1)$. For all sequences $(a_n) \in \mathbf{c_0}$ with $||(a_n)|| = 1$, there exists $x_0 \in X$ with $||x_0|| = 1$, such that:*

$$||T^n x_0|| \geq \varepsilon |a_n|, \quad n \in \mathbf{N}.$$

The result contained in Lemma 3, was been proved by V. Müller in [M] and by J. van Neerven in [Ne4, Theorem 1.2].

Lemma 4 *Let $\mathcal{U}$ be a q-periodic evolution family on the Banach space X and $V = U(q,0)$. If $r(V) \geq 1$ then there is $C > 0$ such that for all $\gamma \in C_0(\mathbf{R}_+, \mathbf{C})$, (the space of all continuous functions with $\lim\limits_{t\to\infty} \gamma(t) = 0$), of norm one, there exists $x_0 \in X$ with $||x_0|| = 1$, such that*

$$||U(t,0)x_0|| \geq C|\gamma(t)|, \quad \forall t \geq 0.$$

Proof. Let $\alpha \in C_0(\mathbf{R}_+)$ be a decreasing function of norm one such that $\alpha(s) \geq |\gamma(s)|$ for all $s \geq 0$ and let $\beta \in C_0(\mathbf{R}_+)$, the function defined by

$$\beta(t) = \begin{cases} \alpha(0), & 0 \leq t < q \\ \alpha(t-q), & t \geq q. \end{cases}$$

Let $t > q, k \in \mathbf{N}$ and $r \in [0,q)$ such that $t = nq + r$. From Lemma 3 it follows that for some $x_0 \in X$, $||x_0|| = 1$ we have:

$$\begin{aligned} \tfrac{1}{2}|\gamma(t)| &\leq \tfrac{1}{2}\alpha(t) \leq \tfrac{1}{2}\alpha(kq) = \tfrac{1}{2}\beta((k+1)q) \\ &\leq ||V^{k+1}x_0|| = ||U((k+1)q,0)x_0|| \\ &\leq Me^{\omega_1 q}||U(t,0)x_0||. \end{aligned}$$

Remark. The proof of Lemma 4 follows some ideas which was used in [Ne1, Lemma 4.6].

The following result it is a strongly generalization of Datko-Pazy's theorem.

Theorem 1 *Let $\mathcal{U}$ be a q-periodic evolutionary process on the Banach space X and $V = U(q,0)$. Let also E be a normed function space over $\mathbf{R}_+$, with $\Psi_E(\infty) = \infty$. If for every $x \in X$, the function $t \mapsto ||U(t,0)x||$ belongs to E, then $r(V) < 1$, i.e. $\mathcal{U}$ is exponentially stable.*

Proof. The proof of Theorem 1 in the autonomous case is given in [Bus3, Theorem 0.1]. For reasons of self-containedness we give the proof in some detail. Firstly it is proves by induction that there exists a strictly increasing sequence (t_n) such that

$$\left\|\chi_{[t_n,t_{n+1})}\right\|_E \geq (n+1)^2, \quad n = 0, 1, \cdots .$$

This follows because $\Psi_E(\infty) = \infty$. Suppose that $r(V) \geq 1$. Let $\gamma \in C_0(\mathbf{R}_+)$ such that $\gamma(t) \geq \frac{1}{n}$ for any $t \in [t_{n-1}, t_n)$ and any $n = 1, 2, \cdots$. Let C and x_0 as in Lema 4. Then we have:

$$\begin{aligned} |||U(\cdot,0)x_0|||_E &\geq \left\|\chi_{[t_{n-1},t_n)}||U(\cdot,0)x_0||\right\|_E \\ &\geq C\left\|\chi_{[t_{n-1},t_n)}\gamma\right\|_E \\ &\geq \frac{C}{n}\left\|\chi_{[t_{n-1},t_n)}\right\|_E \geq Cn \end{aligned}$$

for all $n = 1, 2, \cdots$. It follows $|||U(\cdot,0)x_0|||_E = \infty$. This contradiction concludes the proof.

Theorem 1 can be applied to various "weighted" normed function spaces.

Corollary 1 *Let $\alpha \in L^1_{loc}(\mathbf{R}_+)$ such that $\int\limits_0^\infty \alpha(t)dt = \infty, 1 \leq p < \infty$ and $\mathcal{U}$ be a q-peridic evolution family on X. If*

$$\int\limits_0^\infty \alpha(t)||U(t,0)x||^p dt < \infty, \quad \forall x \in X$$

then $\mathcal{U}$ is exponentially stable.

Corollary 2 *Let $\phi \in L^1_{loc}(\mathbf{R}_+)$ such that $\sup\limits_{t\geq 0} \int\limits_t^{t+1} \phi(s)ds = \infty$ and $\mathcal{U}$ be a q-periodic evolution family on X. If*

$$\sup_{t>0} \int_t^{t+1} \phi(s)||U(s,0)x||ds < \infty, \quad \forall x \in X$$

then $\mathcal{U}$ is exponentially stable.

The following result is a generalization of Littman-Neerven's theorem, see [Li] and [Ne1]. For the general non-autonomous case, see [Rol] or [Bus5].

Theorem 2 *Let $\phi : \mathbf{R}_+ \to \mathbf{R}_+$ be a increasing function with $\phi(0) = 0$ and $\phi(u) > 0$ for all $u > 0$. If $\mathcal{U}$ is a q-periodic evolution family on X such that*

$$\int_0^\infty \phi(||U(t,0)x||)dt < \infty, \quad \forall x \in X,$$

then $\mathcal{U}$ is exponentially stable.

Proof. It follows as in [Ne1, Theorem 4.4, p.307] using Lemma 4 in this paper.

Theorem 3 *Let $\mathcal{U}$ be a q-periodic evolution family on the Banach space X and E_1, E_2 be two normed function spaces over $\mathbf{R}_+$, with $\Psi_{E_2}(\infty) = \infty$. If $\mathcal{U}$ is (E_1, E_2)-stable (i.e. the function*

$$t \mapsto \left\| \int_0^t U(t,\xi)f(\xi)d\xi \right\| \text{ belongs to } E_2 \text{ for all } f \in L^1_{loc}(\mathbf{R}_+, X) \text{ with}$$

$$||f(\cdot)|| \in E_1),$$

then $\mathcal{U}$ is exponentially stable.

Proof. Let $h > 0, x \in X$ and $f = \chi_{[0,h]}(\cdot)U(\cdot,0)x$. It is clear that

$$||f(t)|| \leq Me^{\omega h}\chi_{[0,h]}(t)||x|| \quad \forall x \in X,$$

so $||f(\cdot)|| \in E_1$. Moreover,

$$\int_0^t U(t,\xi)f(\xi)d\xi = hU(t,0)x, \quad \forall t \geq h.$$

Then:

$$||U(t,0)x|| \leq h^{-1}\left\|\int_0^t U(t,\xi)f(\xi)d\xi\right\| + Me^{\omega h}\chi_{[0,h]}(t)||x||,$$

so $||U(\cdot,0)x|| \in E_2$. We apply Theorem 1 and it results that $\mathcal{U}$ is exponentially stable.

For the slightly variants of Theorem 3, see [Bus4, Theorems 2.5 and 2.6].

Remark. Let $1 \leq p < \infty$ and $\mathbf{T}_p = \{T_p(t)\}_{t\geq 0}$ the evolution semigroup asociated with the q-periodic evolution family $\mathcal{U}$, on the usual Lebesgue-Bochner space $L^p(\mathbf{R}_+, X)$. Precisely, $T_p(t)$ is given by

$$(T_p(t)f(s) = \begin{cases} U(s,s-t)f(s-t), & s \geq t \\ U(s,0)f(0), & 0 \leq s < t. \end{cases}$$

For more details about evolution semigroups and its connection with asymptotic properties of the evolution families, we refer to [AM], [LRS], [MRS], [Mi], [RS1], [RS2], and the literatur cited therein. Using Theorem 3 it can be proved, as in [MRS, Corollary 2.4], the spectral mapping theorem for $\mathbf{T_p}$, that is:

$$e^{t\sigma(G_p)} = \sigma(T_p(t)) \setminus \{0\}, \quad \forall t \geq 0$$

and

$$\sigma(G_p) = \{\lambda \in \mathbf{C} : \text{ Re } (\lambda) \leq s(G_p)\}$$

where $s(G_p) = \sup\{ \text{ Re } (z) : z \in \sigma(G_p)\}$.

Let X be a complex Banach space, $\mathbf{T} = \{T(t)\}_{t\geq 0}$ be a C_0-semigroup on X and $A_{\mathbf{T}}$ the infinitesimal generator of $\mathbf{T}$. In [RB, Proposition 3.3] it is been shown that if

$$\sup_{t>0}\left\|\int_0^t e^{i\mu\xi}T(\xi)d\xi\right\| < \infty, \quad \forall x \in X, \forall \mu \in \mathbf{R}$$

then

$$\sigma(A_{\mathbf{T}}) \subset \mathbf{C}_- := \{z \in \mathbf{C} : \text{ Re } (z) < 0\}.$$

The discret version of this result is the following:

Lemma 5. *Let $T \in L(X)$. If*

$$\sup_{n \in \mathbf{N}} \left\| \sum_{k=0}^{n} e^{i\mu k} T^k \right\| = M_\mu < \infty \quad \forall \mu \in \mathbf{R},$$

then $r(T) < 1$.

Proof. We will use the identity:

$$\sum_{k=0}^{n} e^{i\mu k} T^k (e^{i\mu} T - Id) = e^{i\mu(n+1)} T^{n+1} - Id. \tag{5}$$

From (5) it follows:

$$||e^{i\mu(n+1)} T^{n+1}|| \leq 1 + M_\mu (1 + ||T||) \quad \forall n \in \mathbf{N}, \tag{6}$$

that is $r(T) \leq 1$. Suppose that $1 \in \sigma(T)$. Then for all $m = 1, 2, \cdots$, there exists $x_m \in X$ with $||x_m|| = 1$ and $(Id - T)x_m \to 0$ as $m \to \infty$, (see [Na, Proposition 2.2, p. 64]). From (6) it results that $T^k (Id - T) x_m \to 0$ as $m \to \infty$, uniformly for $k \in \mathbf{N}$. Let $N \in \mathbf{N}$, $N > 2M_0$ and $m \in \mathbf{N}$ such that

$$||T^k (Id - T) x_m|| \leq \frac{1}{2N}, \quad k = 0, 1, \cdots N.$$

Then

$$\begin{aligned} M_0 \; &\geq \|x_m + \textstyle\sum_{k=1}^{N} (x_m + \sum_{j=0}^{k-1} T^j (T - Id) x_m)\| \\ &= \|(N+1) x_m + \textstyle\sum_{k=1}^{N} \sum_{j=0}^{k-1} T^j (T - Id) x_m\| \\ &\geq (N+1) - \tfrac{N(N+1)}{4N} > \tfrac{N}{2} > M_0. \end{aligned}$$

This contradiction concludes that $1 \notin \sigma(T)$. Now, it is easy to show that $e^{i\mu} \notin \sigma(T)$ for $\mu \in \mathbf{R}$, that is $r(T) < 1$.

Let $P_q(\mathbf{R}_+, X)$ the set of all continuous functions $f : \mathbf{R}_+ \to X$ such that $f(t + q) = f(t)$, for any $t \geq 0$. In the following we prove a result which is an improvement of a results of Neerven, see [Ne2, Corollary 2.3].

Theorem 4. *Let* $\mathcal{U} = \{U(t,s)\}_{t\geq s\geq 0}$ *be a* q*-periodic evolution family on the Banach space* X. *If*

$$\sup_{t>0}\left\|\int_0^t e^{i\mu\xi}U(t,\xi)f(\xi)d\xi\right\| < \infty, \quad \forall \mu\in\mathbf{R}, \forall f\in P_q(\mathbf{R}_+, X), \tag{7}$$

then $\mathcal{U}$ *is exponentially stable.*

Proof. Let $V = U(q,0)$, $x\in X$, $n = 0, 1, \cdots$ and $g\in P_q(\mathbf{R}_+, X)$, such that

$$g(\xi) = \xi(q-\xi)U(\xi,0)x, \quad \forall \xi\in[0,q].$$

From (7), for $t = (n+1)q$, we obtain:

$$\sup_{n\in\mathbf{N}}\left\|\sum_{k=0}^{n}\int_{kq}^{(k+1)q} U((n+1)q,\xi)e^{-i\mu\xi}g(\xi)d\xi\right\| < \infty, \quad \forall\mu\in\mathbf{R}. \tag{8}$$

For every $k = 0, 1, \cdots$, we have:

$$\begin{aligned}
\int_{kq}^{(k+1)q} U((n+1)q,\xi)e^{-i\mu\xi}g(\xi)d\xi &= V^{n-k}\int_{kq}^{(k+1)q} U((k+1)q,\xi)e^{-i\mu\xi}g(\xi)d\xi \\
&= V^{n-k}\int_0^q U((k+1)q, u+kq)e^{-i\mu(u+kq)}g(u)du \\
&= M(\mu,q)e^{-i\mu(n+1)}e^{i\mu(n-k+1)}V^{n-k+1}x,
\end{aligned}$$

where

$$M(\mu,q) = \left(\int_0^q u(q-u)e^{-i\mu u}du\right) \neq 0.$$

We return in (8) and obtain

$$\sup_{n\in\mathbf{N}}\left\|\sum_{j=0}^{n+1} e^{i\mu j}V^j\right\| < \infty,$$

that is $r(V) < 1$ and $\mathcal{U}$ is exponentially stable.

In the end of this paper we formulate two open problems.

Problem 1 *Let $\mathcal{U}$ be a q-periodic evolution family on the Hilbert space H. Is it true that if $1 \leq p < \infty$, and*

$$\int_0^\infty | < U(t,0)x, y > |^p < \infty, \quad \forall x, y \in H$$

then $\mathcal{U}$ is exponentially stable?

A similar results for C_0-semigroups holds and was been shown by Weiss in [W] and by Huang in [Hu].

Problem 2. *Let $\mathcal{U}$ be a q-periodic evolution family on the Hilbert space H. Is it true that if*

$$\sup_{\mu \in \mathbf{R}} \sup_{t>0} \left\| \int_0^t U(t,\xi) e^{i\mu\xi} x d\xi \right\| < \infty, \quad \forall x \in H$$

then $\mathcal{U}$ is exponentially stable?

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DEPARTMENT OF MATHEMATICS, WEST UNIVERSITY OF TIMIŞOARA, Bd. V. PARVAN 4, 1900, TIMIŞOARA, ROMÂNIA
E-mail address: **buse@tim1.uvt.ro**