SOME REMARKS ABOUT THE PERRON CONDITION FOR C_0-SEMIGROUPS

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†

Abstract

We prove the following. Let $\mathbf{T} = \{T(t)\}_{t\geq 0}$ be a bounded C_0-semigroup on a Banach space X and A its infinitesimal generator. Then $\Re\sigma(A) < 0$ if and only if

$$\sup_{t>0} || \int_0^t e^{i\mu\xi} T(\xi) d\xi || < \infty, \quad \forall \mu \in \mathbf{R}, \forall x \in X.$$

In particular we obtain that a strongly continuous and exponentially bounded family of bounded linear operators $\mathcal{U}$ on X is uniformly exponentially stable if and only if the spectrum of the infinitesimal generator of the evolution semigroup associated to $\mathcal{U}$, is lying in $\mathbf{C}_- := \{z \in \mathbf{C} : \Re(z) < 0\}$.

1.INTRODUCTION

In 1930 Oscar Perron found an interesting connection between the property of uniform exponential stability of differential systems and a certain condition (called the *Perron condition*). For the reader's convenience we will

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briefly present Perron's condition for linear nonautonomous finite dimensional differential systems. We consider the system

$$\dot{x}(t) = A(t)x(t) \tag{A}$$

where $x(t) \in \mathbf{R}^n$ and $A(t)$ is a real n-dimensional matrix. We suppose that the mapping $t \mapsto A(t)$ is measurable. We would say that the system (A) satisfies the Perron condition if for every bounded and continuous function $f : \mathbf{R}_+ = [0, \infty) \to \mathbf{R}^n$, the solution of the Cauchy problem

$$\dot{x}(t) = A(t)x(t) + f(t), \quad x(0) = 0$$

is bounded on $\mathbf{R}_+$,, that is

$$\sup_{t>0} || \int_0^t P(t)P^{-1}(\xi)f(\xi)d\xi|| < \infty.$$

Here $P(t)$ is the solution of the operatorial Cauchy problem

$$\dot{U}(t) = A(t)U(t), \quad U(0) = I$$

and I is the identity operator on $\mathbf{R}^n$.

It is well known that the system (A) is uniformly exponentially stable if and only if it satisfies the Perron's condition. This is more or less folklore; a proof can be found in [DK].

Some remarks about this result in the general case when $A(t) = A$ is a unbouned linear operator, will be made in the present paper.

2.DEFINITIONS. NOTATIONS. MAIN RESULT

Let $\mathbf{T} = \{T(t)\}_{t \geq 0}$ be a C_0-semigroup on a Banach space X, $\omega(\mathbf{T})$ be its growth bound (i.e.

$$\omega(\mathbf{T}) = \lim_{t\to\infty} \frac{\ln ||T(t)||}{t} = \inf_{t>0} \frac{\ln ||T(t)||}{t})$$

and A its infinitesimal generator. We denote by $L(X)$ the Banach space of all linear operators acting on X. We also denote by $||\cdot||$, the norms of vectors and operators. Let $\mathbf{C}_+ = \{z \in \mathbf{C} : \Re(z) > 0\}$ and $i\mathbf{R} = \{i\mu : \mu \in \mathbf{R}\}$. With

$\rho(A)$ we denote the *resolvent set* of A and with $\sigma(A) := \mathbf{C} \setminus \rho(A)$ we denote the *spectrum* of A. It is known that $\rho(A)$ is an open set in $\mathbf{C}$. If $A \in L(X)$ then $\sigma(A)$ is a compact subset of $\mathbf{C}$. The following result has been proved in [RB].

Proposition 1 *If*

$$\sup_{t>0} || \int_0^t e^{i\mu\xi} T(\xi) x d\xi || < \infty, \quad \forall x \in X, \forall \mu \in \mathbf{R} \tag{1}$$

then $\sigma(A) \subset \mathbf{C}_-$.

If $\mathbf{T}$ is a uniformly continuous semigroup, i.e. the function $t \mapsto T(t) : \mathbf{R}_+ \to L(X)$ is normic continuous, then $A \in L(X)$ and from (1) it follows $\omega(\mathbf{T}) < 0$. This is not true in the general case of a strongly continuous semigroup, even if X is a complex Hilbert space (cf. [RB, Example 3.1]). However if $\mathbf{T}$ is bounded, i.e. there is $M > 0$, such that

$$\sup_{t \geq 0} ||T(t)|| \leq M \tag{2}$$

then the converse statement of Proposition 1, holds. More precisely, we have:

Proposition 2 *Let* $\mathbf{T} = \{T(t)\}_{t\geq 0}$ *be a bounded* C_0*- semigroup on the Banach space* X*. If* $\sigma(A) \subset \mathbf{C}_-$ *then (1) holds.*

Proof. With previous notations we have $(i\mathbf{R} \cup \mathbf{C}_+) \subset \rho(A)$. Let $\lambda \in \rho(A)$ and $R(\lambda, A) := (\lambda I - A)^{-1}$ the resolvent operator of A. From (2) it follows $\omega(\mathbf{T}) \leq 0$. Then it is well known, see e.g. [Pa]) that

$$R(\lambda, A)x = \int_0^\infty e^{-\lambda t} T(t) x dt, \quad \forall x \in X, \forall \lambda \in \mathbf{C}_+. \tag{3}$$

Let $\mu \in \mathbf{R}$ and $z_n \in \mathbf{C}_+$ with $z_n \to (-i\mu)$ as $n \to \infty$. From (3), it follows

$$\begin{aligned} \int_0^s e^{-tz_n} T(t) x dt &= \int_0^\infty e^{-tz_n} T(t) x dt - \int_0^\infty e^{-(s+u)z_n} T(s+u) x du \\ &= R(z_n, A)x - e^{-sz_n} T(s) R(z_n, A)x. \end{aligned}$$

Using the dominated convergence theorem, we obtain

$$\int_0^s e^{i\mu t} T(t) x dt = R(-i\mu, A)x - e^{i\mu s} T(s) R(-i\mu, A)x. \tag{4}$$

But $||e^{is\mu} T(s) R(-i\mu, A)x|| = ||T(s) R(-i\mu, A)x|| \to 0$, as $s \to \infty$. For the proof of last result, see [AB]. From (4) and (5), it results:

$$\begin{aligned} \sup_{s>0} ||\int_0^s e^{i\mu t} T(t) x dt|| &\leq ||R(-i\mu, A)x|| + \sup_{s>0} ||T(s) R(-i\mu, A)x|| \\ &= M(\mu, x) < \infty. \end{aligned}$$

The proof of Proposition 2 is finished.

Corollary 1 *Let* $\mathbf{T} = \{T(t)\}_{t\geq 0}$ *be a bounded* C_0*-semigroup on the Banach space* X *and* A *its infinitesimal generator. Then* $\sigma(A) \subset \mathbf{C}_+$ *if and only if (1) holds.*

Let us consider the Banach function spaces

$$E_p := \begin{cases} L^p(\mathbf{R}_+, X), & \text{if} \quad 1 \leq p < \infty \\ C_{00}(\mathbf{R}_+, X), & \text{if } p = \infty. \end{cases}$$

Here $L^p(\mathbf{R}_+, X)$, with $1 \leq p < \infty$ are the usual Lebesgue-Bochner spaces and $C_{00}(\mathbf{R}_+, X)$ is the space of all continuous functions $f : \mathbf{R}_+ \to X$ such that $f(0) = \lim_{t\to\infty} f(t) = 0$. Let $\mathcal{U} = \{U(t,s)\}_{t\geq s\geq 0}$ be a strongly continuous and exponentially bounded family of bounded linear operators on the Banach space X, that is $\mathcal{U}$ satisfies the following conditions:

- **(i)** $U(t,s) = U(t,r)U(r,s)$ for all $t \geq s \geq r \geq 0$;
- **(ii)** $U(t,t)x = x$, for all $t \geq 0$ and any $x \in X$;
- **(iii)** for all $x \in X$, the map $(t,s) \mapsto U(t,s)x$ is continuous;
- **(iv)** there exist $\omega > 0$ and $K \geq 1$ such that

$$||U(t,s)|| \leq K e^{\omega(t-s)}, \quad \forall t \geq s \geq 0.$$

It is clear that if $U(t,s) = U(t-s, 0)$ for all $t \geq s \geq 0$ then the family $\mathbf{T} := \{U(t,0)\}_{t\geq 0}$ is a C_0-semigroup of bounded linear operators on X. The *evolution semigroup* $\mathcal{T}_p = \{T_p(t)\}_{t\geq 0}$ associated with the family $\mathcal{U}$ on E_p, is defined by

$$(T_p(t)f(s) = \begin{cases} U(s, s-t)f(s-t), & s \geq t \\ U(s,0)f(0), & 0 \leq s < t. \end{cases}$$

For more details about evolution semigroups and their connection with asymptotic properties of the evolution families, we refer to [AM], [LRS], [MRS], [Mi], [RS1], [RS2] and the literature cited therein. Let G_p the infinitesimal generator of $\mathcal{T}_p$. It is known the spectral mapping theorem for $\mathcal{T}_p$, that is:

$$e^{t\sigma(G_p)} = \sigma(T_p(t)) \setminus \{0\}, \quad \forall t \geq 0$$

and

$$\sigma(G_p) = \{\lambda \in \mathbf{C} : \Re(\lambda) \leq s(G_p)\}$$

where $s(G_p) = \sup\{\Re(z) : z \in \sigma(G_p)\}$.

For the proof of this result we refer to [MRS, Theorem 2.2, Corollary 2.4], for the case $p = \infty$, and to [LMR, Theorem 2.2, Corollary 2.3], for the cases $1 \leq p < \infty$.

Corollary 2 *Let* $\mathcal{U} = \{U(t,s)\}_{t\geq s\geq 0}$ *be a strongly and exponentially bounded family on* X *and* $\mathcal{T}_p = \{T_p(t)\}_{t\geq 0}$*, the evolution semigroup associated with* $\mathcal{U}$*. The following statements are equivalent:*

- **(i)** *there exist* $\omega < 0$ *and* $N > 0$ *such that*

$$||U(t,s)|| \leq Ne^{\omega(t-s)} \quad \forall t \geq s \geq 0;$$

- **(ii)** $\omega(\mathcal{T}_p) < 0$;
- **(iii)** $\sigma(G_p) \subset \mathbf{C}_-$;
- **(iv)** $\sup\limits_{t>0} ||\int\limits_0^t e^{i\mu\xi} T_p(\xi)f(\xi)d\xi|| < \infty$, *for all* $\mu \in \mathbf{R}$ *and all* $f \in E_p$.

Proof. The implications (**i**) $\Rightarrow$ (**ii**) $\Rightarrow$ (**iii**) are obvious. The implication (**iii**) $\Rightarrow$ (**ii**) follows from the spectral mapping theorem. The equivalence (**iii**) $\Leftrightarrow$ (**iv**) follows from Proposition 1 and Proposition 2. For the proof of (**ii**) $\Rightarrow$ (**i**), see e.g. [RS2].

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