MAXIMAL OPERATORS ASSOCIATED WITH DIRICHLET FORMS PERTURBED BY MEASURES

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Abstract. It is shown that a selfadjoint operator defined by a Dirichlet form perturbed by a measure can be described as a suitable maximal operator, for a wide class of measures. The generalized 'distributional notion' employed in the description reduces to the usual one for the case of Schrödinger operators.

1. Introduction

A well-established and very effective way to associate an L_2-operator with a certain differential expression is to define an appropriate form and use the form representation theorem. The operator is then described in the abstract Hilbert space setting. It is a further task to describe the operator analytically. One of these descriptions is by showing that on a suitable core the operator acts as a classical differential operator. This is restricted to cases where the coefficients are sufficiently regular. If this is not the case the operator can sometimes be described distributionally by a maximal operator.

It is the latter access which will be presented in this paper, for operators associated with Dirichlet forms perturbed by measures. In this general context there is, a priori, no well-established distributional notion, and therefore this notion has to be defined suitably. The appropriate distributional notion turns out to be the validity of equation $(*)$ below for a suitable space of test functions.

The main result is the description of an operator as described above in the general context, with a corollary for the case of Schrödinger operators (Sect. 2). The key point in the proof is a version of a result of Brézis and Browder in the context of Dirichlet forms (Sect. 3).

2. preliminary remarks and statement of the results

In order to state our result we shortly have to recall the notions of Dirichlet forms and measure perturbations.

Let X be a locally compact space, m a Radon measure on X with $\mathrm{supp}(m) = X$. Note that in [FOT], [St] it is assumed additionally that X is second countable

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or σ-compact. The properties mentioned below do not need these assumptions. A form $\mathfrak{h}$ in $L_2(m)$ is a *Dirichlet form* if the selfadjoint operator H associated with $\mathfrak{h}$ is such that e^{-tH} is positivity preserving and L_1-contractive for all $t \geq 0$. $\mathfrak{h}$ is further assumed to be *regular*, i. e., $C_c(X) \cap D(\mathfrak{h})$ is dense in $D(\mathfrak{h})$ and $C_0(X)$. We refer to [BH], [FOT], [MR], [SV] for these notions.

From [SV] we recall

$$\begin{aligned} M_0 := \{\mu : \mathcal{B} \to [0,\infty];\ &\mu \text{ is } \sigma\text{-additive, and } \mu(N) = 0 \\ &\text{for every set } N \in \mathcal{B} \text{ of zero capacity}\}, \end{aligned}$$

where $\mathcal{B}$ denotes the σ-algebra of Borel subsets of X. For $\mu \in M_0$ one obtains a closed form $\mathfrak{h} + \mu$ defined by (cf. [St])

$$\begin{aligned} D(\mathfrak{h}+\mu) : &= \{u \in D(\mathfrak{h});\ \int |\tilde{u}|^2\, d\mu < \infty\} \\ &\qquad \text{(where } \tilde{u} \text{ denotes a quasi-continuous version of } u\text{)}, \\ (\mathfrak{h}+\mu)[u,v] : &= \mathfrak{h}[u,v] + \int \tilde{u} \cdot \bar{\tilde{v}}\, d\mu. \end{aligned}$$

$\mu \in M_0$ is called *smooth*, if $D(\mathfrak{h}+\mu)$ is dense in $D(\mathfrak{h})$ (see [FOT], [St] for the original definition and [St] for the above characterization).

In the sequel, we will use the standard assumptions

(SA) $\quad \mu_+,\ \mu_- \in M_0,\ \mu_+$ smooth, μ_- $\mathfrak{h}$-bounded with $\mathfrak{h}$-bound < 1.

Then $\mathfrak{h}+\mu := \mathfrak{h}-\mu_-+\mu_+$ is a closed semibounded form; denote by H_μ the operator associated with this form. For properties of forms and associated operators we refer to [K1], [RSI], [RSIV]. For $u \in D(H_\mu),\ v := H_\mu u$ one has

$$(*) \qquad \mathfrak{h}[u,\varphi] - \int \tilde{u} \cdot \bar{\tilde{\varphi}}\, d\mu_- + \int \tilde{u} \cdot \bar{\tilde{\varphi}}\, d\mu_+ = (v|\varphi)$$

for all $\varphi \in D(\mathfrak{h} - \mu_- + \mu_+) (= D(\mathfrak{h} + \mu_+))$, and H_μ is characterized by this condition.

We note further that the strongly continuous semigroup $(e^{-tH_\mu})_{t\geq 0}$ is positivity preserving (cf. [SV, Theorem 4.1]).

1. Theorem. *Assume (SA). Then $H_\mu = H_{\mu,max}$ where,*

$$\begin{aligned} D(H_{\mu,max}) := \{u \in D(\mathfrak{h});\ &|\tilde{u}|\mu_+ \text{ is a Radon measure and there exists} \\ &v \in L_2(\mu) \text{ such that } (*) \text{ is valid for all } \varphi \in D(\mathfrak{h}) \cap C_c(X)\}, \\ H_{\mu,max} u := v \quad & \end{aligned}$$

for u, v as in the definition of $D(H_{\mu,max})$.

2. Remarks. (a) The condition in the definition of $D(H_{\mu,max})$ should be thought of in a kind of abstract distributional sense. In fact, for the classical

Dirichlet form on an open subset Ω of $\mathbb{R}^d$ one can restrict the test functions $\varphi \in D(\mathfrak{h}) \cap C_c(\Omega)$ further, such that the interpretation becomes the usual distributional one. This is the content of the following Corrollary 3.

(b) If $u \in D(\mathfrak{h})$ then $|\tilde{u}|\mu_-$ is automatically a Radon measure. Indeed, the assumption on μ_- implies $\varphi \in L_1(|\tilde{u}|\mu_-)$ for all $\varphi \in D(\mathfrak{h}) \cap C_c(X)$. Since by the regularity of $\mathfrak{h}$ for every compact $K \subseteq X$ there exists $\varphi \in D(\mathfrak{h}) \cap C_c(X)$, $\varphi \geq 0$, $\varphi(x) \geq 1$ for $x \in K$, this implies $\int_K |\tilde{u}|\, d\mu_- \leq \int_X \varphi|\tilde{u}|\, d\mu_- < \infty$, i. e., $|\tilde{u}|\mu_-$ is a Radon measure.

Note that by an analogous argument one can see that μ_- is also a Radon measure.

(c) If μ_+ itself is a Radon measure then there is an elementary way to see $H_\mu \subseteq H_{\mu,max}$. Note that in this case $D(\mathfrak{h}) \cap C_c(X) \subseteq D(\mathfrak{h} + \mu)$. Let $u \in D(H_\mu)(\subseteq D(\mathfrak{h} + \mu))$. Then $\varphi \in L_1(|\tilde{u}|\mu_+)$ for all $\varphi \in D(\mathfrak{h}) \cap C_c(X)$, therefore, as in (b), one obtains that $|\tilde{u}|\mu_+$ is a Radon measure. Moreover, equation $(*)$, with $v = H_\mu u$, holds in particular for all $\varphi \in D(\mathfrak{h}) \cap C_c(X)$, hence $u \in D(H_{\mu,max})$ and $H_{\mu,max} u = H_\mu u$.

3. Corollary. *Let $\Omega = \mathring{\Omega} \subseteq \mathbb{R}^d$, $\mathfrak{h}$ the classical Dirichlet form on $W^1_{2,0}(\Omega)$, and assume (SA). Then*

$$H_\mu = H_{\mu,max,bc},$$

where

$$\begin{aligned} D(H_{\mu,max,bc}) := \{u \in W^1_{2,0}(\Omega);\ &|\tilde{u}|\mu_+ \text{ is a Radon measure}, \\ &v := -\Delta u + \tilde{u}\mu_+ - \tilde{u}\mu_- \in L_2(\Omega)\}, \\ H_{\mu,max,bc} u := v \quad &(u \in D(H_{\mu,max,bc})). \end{aligned}$$

4. Remarks. (a) The condition $-\Delta u + \tilde{u}\mu_+ - \tilde{u}\mu_- \in L_2(\Omega)$ is meant in the distributional sense, which just means that, for some $v \in L_2(\Omega)$, $(*)$ should hold for all $\varphi \in C_c^\infty(\Omega)$. The condition $u \in W^1_{2,0}(\Omega)$ serves the double purpose of permitting an interpretation of $\tilde{u}\mu_+$ and of imposing a zero boundary condition, the latter being the reason for the index 'bc'.

The notion of suitable maximal operators to describe Schrödinger operators was introduced by Kato in [K2].

(b) Corollary 3 generalizes [BK, Theorem 2.1], where μ is of the form $V dx$ with $V^+ \in L_{1,loc}(\Omega)$, i. e., $\mu_+ = V^+ dx$ is a Radon measure.

(c) In a slightly different context the corresponding extension to smooth (=regular) potentials appears already in [V, Theorem 7.4]. There it is shown that if μ is of the form $V dx$ and $\Omega = \mathbb{R}^d$ (and therefore there is no need imposing a boundary condition), then H_V can be described by the maximal operator $H_{V,max}$,

$$\begin{aligned} D(H_{V,max}) &:= \{u \in L_2(\mathbb{R}^d);\ Vu \in L_{1,loc}(\mathbb{R}^d),\ v := -\Delta u + Vu \in L_2(\mathbb{R}^d)\}, \\ H_{V,max} u &:= v \quad (u \in D(H_{V,max})), \end{aligned}$$

provided the negative part V^- of V is in the Kato class.

(d) Note that [V, Example C.1] also shows that smoothness (=regularity) cannot be dropped from the hypotheses.

5. Example. Let $\Omega := (-1,1)$, $\mathfrak{h}[u,v] := \int u'v'\,dx$, $\mu = \mu_+ = \delta_0$ the Dirac measure at 0. Then

$$\begin{aligned} D(H_{\mu,max,bc}) = \{u \in W^1_{2,0}(-1,1) \cap W^2_2((-1,0) \cup (0,1));\\ u'(0+) - u'(0-) = u(0)\}. \end{aligned}$$

3. PROOFS

Theorem 1 and Corollary 3 are valid in the real and complex case, i. e., $L_2(m) = L_2(m;\mathbb{R})$ and $L_2(m) = L_2(m;\mathbb{C})$. Since by complexification the complex case follows immediately from the real case, it will be sufficient to treat the real case.

The crucial technical tool for the proof of Theorem 1 is the following Lemma, which is a version of [BB] in a generalized context.

6. Lemma. *Let $\mathfrak{h}$ be a regular Dirichlet form, $\varrho_1, \varrho_2 \in M_0$, $\varrho_1 \perp \varrho_2$, $D_0 \subseteq D(\mathfrak{h})$ a dense ideal, $D_0 \subseteq L_1(\varrho_1 + \varrho_2)$. Let the mapping*

$$D_0 \ni \varphi \longmapsto \int \tilde{\varphi}\,d\varrho_1 - \int \tilde{\varphi}\,d\varrho_2 =: T\varphi$$

be $\mathfrak{h}$-continuous. Let $\varphi \in D(\mathfrak{h})$, $\tilde{\varphi}\varrho_1 \geq 0$, $-\tilde{\varphi}\varrho_2 \geq 0$. Then $\tilde{\varphi} \in L_1(\varrho_1 + \varrho_2)$, $\int \tilde{\varphi}\,d\varrho_1 - \int \tilde{\varphi}\,d\varrho_2 = \hat{T}\varphi$, where $\hat{T}$ is the continuous extension of T to $D(\mathfrak{h})$.

(Here by '$\varrho_1 \perp \varrho_2$' we mean that there are disjoint sets X_1, X_2 supporting ϱ_1, ϱ_2, respectively, and 'ideal' is used in the sense that $u \in D_0$, $v \in D(\mathfrak{h})$ with $|v| \leq |u|$ implies $v \in D_0$.)

Proof. Let $(\psi_n)_{n\in\mathbb{N}} \subseteq D_0$, $\psi_n \to \varphi$ in $D(\mathfrak{h})$, without restriction $\tilde{\psi}_n \to \tilde{\varphi}$ quasi-everywhere (cf. [FOT, Theorem 2.1.4]). For $n \in \mathbb{N}$ we define

$$\varphi_n := (\operatorname{sgn}\varphi)(|\psi_n| \wedge |\varphi|)$$

We now use that $\mathbb{R}^2 \ni (x,y) \longmapsto \operatorname{sgn}x(|y| \wedge |x|) \in \mathbb{R}$ is a multiple of a normal contraction because of the inequalities, for $x, y, \tilde{x}, \tilde{y} \in \mathbb{R}$,

$$\begin{aligned} &\Big|\operatorname{sgn}x(|y| \wedge |x|) - \operatorname{sgn}\tilde{x}(|\tilde{y}| \wedge |\tilde{x}|)\Big| \\ &\leq |\operatorname{sgn}x|\Big||y| \wedge |x| - |\tilde{y}| \wedge |\tilde{x}|\Big| + |\operatorname{sgn}x - \operatorname{sgn}\tilde{x}|(|\tilde{y}| \wedge |\tilde{x}|) \\ &\leq \Big||y| - |\tilde{y}|\Big| + \Big||x| - |\tilde{x}|\Big| + \Big|\operatorname{sgn}x|\tilde{x}| - x\Big| + \Big|x - \operatorname{sgn}\tilde{x}|\tilde{x}|\Big| \\ &\leq |y - \tilde{y}| + 3|x - \tilde{x}|. \end{aligned}$$

Therefore (see, e. g. [BH, Chapter I, Prop. 3.3.1] or [MR, Chapter I, Theorem 4.12])

$$\varphi_n \in D(\mathfrak{h}), \ \sqrt{\mathfrak{h}[\varphi_n]} \leq 3\sqrt{\mathfrak{h}[\psi_n]} + \sqrt{\mathfrak{h}[\varphi]}.$$

Moreover, since $|\varphi_n| \le |\psi_n|$ and D_0 is an ideal of $D(\mathfrak{h})$ we obtain further $\varphi_n \in D_0$ and

$$\int |\tilde{\varphi_n}|\, d(\varrho_1 + \varrho_2) = \int \tilde{\varphi_n}\, d\varrho_1 - \int \tilde{\varphi_n}\, d\varrho_2 = T(\varphi_n).$$

Since $\varphi_n \to \varphi$ in L_2 and $(\varphi_n)_{n\in\mathbb{N}}$ is bounded in $D(\mathfrak{h})$ one has $\varphi_n \to \varphi$ weakly in $D(\mathfrak{h})$, hence $T(\varphi_n) \to \hat{T}(\varphi)$. Using the quasi-continuous versions $\tilde{\varphi_n} := (\mathrm{sgn}\tilde{\varphi})(|\tilde{\psi_n}| \wedge |\tilde{\varphi}|)$ of φ_n Fatou's lemma implies $\int |\tilde{\varphi}|\, d(\varrho_1 + \varrho_2) < \infty$, i.e., $\tilde{\varphi} \in L_1(\varrho_1 + \varrho_2)$. Finally, the dominated convergence theorem yields

$$\int \tilde{\varphi_n}\, d\varrho_1 - \int \tilde{\varphi_n}\, d\varrho_2 \to \int \tilde{\varphi}\, d\varrho_1 - \int \tilde{\varphi}\, d\varrho_2,$$
$$\int \tilde{\varphi}\, d\varrho_1 - \int \tilde{\varphi}\, d\varrho_2 = \hat{T}(\varphi).$$

□

Proof of Theorem 1. $H_\mu \subseteq H_{\mu,max}$: Let $u \in D(H_\mu)$. Without restriction $u \ge 0$ since $D(H_\mu) = D(H_\mu)_+ - D(H_\mu)_-$, $-H_\mu$ being the generator of a positive semigroup. From the definition of H_μ we know that $(*)$, with $v = H_\mu u$, holds for all $\varphi \in D_0 := D(\mathfrak{h}) \cap D(\mu_+)$. Since μ_+ is smooth D_0 is a dense ideal of $D(\mathfrak{h})$. Using Lemma 6 with $\varrho_1 = \tilde{u}\mu_+$, $\varrho_2 = 0$, $\hat{T}(\varphi) = (H_\mu u|\varphi) - \mathfrak{h}[u,\varphi] + \int \tilde{u}\tilde{\varphi}\, d\mu_-$ $(\varphi \in D(\mathfrak{h}))$, we conclude that, for all $\varphi \in D(\mathfrak{h})_+$, one has $\tilde{\varphi} \in L_1(\tilde{u}\mu_+)$, and $(*)$ holds for all $\varphi \in D(\mathfrak{h})_+$. From $D(\mathfrak{h}) = D(\mathfrak{h})_+ - D(\mathfrak{h})_-$ we obtain $\tilde{\varphi} \in L_1(\tilde{u}\mu_+)$ and $(*)$ for all $\varphi \in D(\mathfrak{h})$. Hence $\tilde{u}\,\mu_+$ is a Radon measure (cf. Remark 2(b)) and in particular $(*)$ holds for all $\varphi \in D(\mathfrak{h}) \cap C_c(X)$. This concludes the first part of the proof.

$H_{\mu,max} \subseteq H_\mu$: Let $u \in D(H_{\mu,max})$, i.e., $|\tilde{u}|\mu_+$ is a Radon measure, and $(*)$, with $v = H_{\mu,max}u$, holds for all $\varphi \in D(\mathfrak{h}) \cap C_c(X)$.

We show first that then $(*)$ holds for all $\varphi \in D_0 := D(\mathfrak{h}) \cap L_{\infty,c}(X)$, where $L_{\infty,c}(X) := \{\varphi \in L_\infty(X);\ \mathrm{supp}\varphi \text{ compact}\}$. There exists a sequence $(\psi_n)_{n\in\mathbb{N}} \subseteq D(\mathfrak{h}) \cap C_c(X)$ such that $\psi_n \to \varphi$ in $D(\mathfrak{h})$, and we may assume $\tilde{\psi_n} \to \tilde{\varphi}$ quasi-everywhere. Due to the regularity of $\mathfrak{h}$ there exists $\psi \in D(\mathfrak{h}) \cap C_c(X)$, $\psi \ge 0$, $\psi(x) \ge ||\varphi||_\infty$ for $x \in \mathrm{supp}\varphi$. For $n \in \mathbb{N}$ we define $\varphi_n := (\psi_n \vee (-\psi)) \wedge \psi \in D(\mathfrak{h}) \cap C_c(X)$. Then $\varphi_n \to \varphi$ weakly in $D(\mathfrak{h})$, and $\tilde{\varphi_n} \to \tilde{\varphi}$ quasi-everywhere. This implies that $(*)$ holds also for φ.

With D_0, $\varrho_1 := (\tilde{u})^+\mu_+$, $\varrho_2 := (\tilde{u})^-\mu_+$, $\hat{T}(\varphi) = (H_{\mu,max}u|\varphi) - \mathfrak{h}[u,\varphi] + \int \tilde{u}\tilde{\varphi}\, d\mu_-$ $(\varphi \in D(\mathfrak{h}))$, $\varphi := u$ we apply Lemma 6, with the result that $\tilde{u} \in L_1(|\tilde{u}|\mu_+)$, therefore $u \in D(\mathfrak{h} + \mu_+) = D(\mathfrak{h} + \mu)$.

Now we can show that $(*)$ holds for all $\varphi \in D(\mathfrak{h} + \mu)$: Without restriction $\varphi \ge 0$. There exists a sequence $(\psi_n)_{n\in\mathbb{N}} \subseteq D(\mathfrak{h}) \cap L_{\infty,c}(X)$ with $\psi_n \to \varphi$ in $D(\mathfrak{h})$, $\tilde{\psi_n} \to \tilde{\varphi}$ q.e. Replacing ψ_n by

$$\varphi_n := (\psi_n \vee (-|\varphi|)) \wedge |\varphi| \in D(\mathfrak{h}) \cap L_{\infty,c}(X) \quad (n \in \mathbb{N})$$

we obtain $\varphi_n \to \varphi$ weakly in $D(\mathfrak{h})$, $\tilde{\varphi_n} \to \tilde{\varphi}$ q. e., and $|\varphi_n| \le |\varphi|$ $(n \in \mathbb{N})$. This implies that in the equation

$$\int \tilde{u}\tilde{\varphi_n}\, d\mu_+ = -\mathfrak{h}[u, \varphi_n] + \int \tilde{u}\tilde{\varphi_n}\, d\mu_- + (H_{\mu,max}u|\varphi_n)$$

both sides converge, and $(*)$ is valid with $v = H_{\mu,max}u$. $\square$

Proof of Corollary 3. $H_\mu \subseteq H_{\mu,max,bc}$: From $C_c^\infty(\Omega) \subseteq D(\mathfrak{h}) \cap C_c(\Omega)$ one obtains $H_{\mu,max} \subseteq H_{\mu,max,bc}$, and therefore the assertion follows from Theorem 1.

$H_{\mu,max,bc} \subseteq H_\mu$: For $u \in D(H_{\mu,max,bc})$, $v := H_{\mu,max,bc}u$, equation $(*)$ holds for all $\varphi \in C_c^\infty(\Omega)$. For $\varphi \in D(\mathfrak{h}) \cap C_c(\Omega) = W_{2,0}^1(\Omega) \cap C_c(\Omega)$ there exists a compact $K \subseteq \Omega$ and a sequence $(\varphi_n)_{n\in\mathbb{N}} \subseteq C_c^\infty(\Omega)$, $\mathrm{supp}\varphi_n \subseteq K$ $(n \in \mathbb{N})$, $\varphi_n \to \varphi$ in $W_{2,0}^1(\Omega)$ as well as uniformly. This implies that $(*)$ carries over from φ_n to φ. $\square$

7. Remark. In the proof of [BK, Theorem 2.1], the result of [BB] was used only for the inclusion $H_{\mu,max} \subseteq H_\mu$. Recall from Remark 2(c) and Remark 4(b) that in the context of [BK, Theorem 2.1] the direction $H_\mu \subseteq H_{\mu,max}$ is elementary. It turned out that the technique of the proof of [V, Theorem7.4(a)], where $H_\mu \subseteq H_{\mu,max}$ has been shown for a special situation (cf. Remark 4(c)) but where μ_+ is no longer required to be a Radon measure, is closely connected with the techniques used in [BB].

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