Asymptotic Laplace transforms and evolution equations

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1 Introduction

The application of classical Laplace transform techniques to differential or integral equations requires one to impose restrictive growth assumptions on the functions involved. To avoid this flaw, the Argentinian mathematician J.C. Vignaux [22] extended the concept of asymptotic power series to Laplace transforms. In the first part of this paper we will give an introduction to Vignaux's method of asymptotic Laplace transforms and prove the following representation results, among others. Consider analytic functions $r : \Omega \to X$, where $\Omega \subset \mathbb{C}$ and X is a Banach space. Then r is the Laplace transform

$$r(\lambda) = \int_0^\infty e^{-\lambda t} u(t)\, dt$$

of a generalized function $u : [0, \infty) \to X$ if its domain Ω contains some right half-plane $\{\lambda \in \mathbb{C} : Re\lambda > \omega\}$ on which it grows less than exponentially.[1] Moreover, the faster a function r grows at infinity, the less regular the generalized function u will be. If the domain Ω does not contain any right half-plane, then r cannot be represented as a Laplace transform. However, if Ω still contains a post-sectorial region[2] on which r has less than exponential growth, then, for all $0 < T \leq \infty$, there exists a generalized function $u : [0, T) \to X$ such that r is an *asymptotic* Laplace

[1] Precise definitions of generalized functions and their Laplace transforms will be given below.

[2] A region S is said to be post-sectorial if for all $0 < \alpha < \pi/2$ there exists a constant $r_\alpha > 0$ such that the rays $re^{i\beta}$ are subsets of S for all $r > r_\alpha$ and all $-\alpha < \beta < \alpha$. The vector space of all analytic functions defined in some post-sectorial region S with values in X is denoted by $\mathcal{A}(\mathcal{P}; \mathcal{X})$.

transform of u. Moreover, more growth of an analytic function r at infinity, a larger interval $[0,T)$, or a smaller post-sectorial region S lead to a decay in regularity of u.

In the second part of the paper we show how asymptotic Laplace transform methods can be used to characterize the existence, uniqueness and regularity of classical or generalized solutions $u:[0,T)\to X$ $(0<T\le\infty)$ of evolution equations

$$u'(t)=Au(t),\ 0\le t<T,\ u(0)=x, \tag{ACP}$$

where A is a closed linear operator with domain $D(A)$ and range in X. We consider the general case of a closed linear operator; i.e., no a priori spectral assumptions will be made and we will allow nondense domains. The aim of the second section is twofold. First, for a given closed linear operator A we characterize the set of all initial values x for which (ACP) has a classical or generalized solution. Second, we give sufficient conditions on A which imply the existence of unique generalized solutions of (ACP) on $[0,T)$ for all $x\in X$.

2 Asymptotic Laplace Transforms

The Laplace transform $\hat f(\lambda)=\int_0^\infty e^{-\lambda t}f(t)\,dt$ of a function $f\in L^1_{loc}([0,\infty);X)$ exists as an improper Bochner integral for some $\lambda\in\mathbb{C}$ if and only if $F(t):=\int_0^t f(r)\,ds$ is of exponential type.[3] In this case $\hat f$ exists and is analytic in some right half-plane. The purpose of this section is to present an asymptotic version of the Laplace transform which has the additional benefit that any $f\in L^1_{loc}$ becomes transformable while the basic operational properties of the Laplace transform remain valid.[4]

For $0<T<\infty$, the *T-asymptotic Laplace transform* of $f\in L^1_{loc}([0,\infty);X)$ is given by the set of analytic functions r defined on some post-sectorial region S with values in X which are asymptotically equal to the finite Laplace transform of f; i.e.,

$$\{f\}_T:=\{r\in\mathcal{A}(\mathcal{P};\mathcal{X}):r(\lambda)-\int_0^T e^{-\lambda t}f(t)\,dt\approx_T 0\},$$

[3]See, for example, G. Doetsch [15], Kap. 2.2. A function F is of exponential type if there exist positive constants M and ω such that $\|F(t)\|\le Me^{\omega t}$ for all $t\ge 0$.

[4]The concept of asymptotic Laplace transforms has its roots in the theory of asymptotic power series of analytic functions; see, e.g., R. Remmert [21], 9.6. With a slightly different definition, asymptotic Laplace transforms were considered first by J.C. Vignaux [22] and further investigated by J.C Vignaux and M. Cotlar [23], W.A. Ditkin [14], L. Berg [7], [8], Yu. I. Lyubich [20], and M. Deakin [13]. However, the method and the scope of its applicability remained largely unnoticed.

where a function $a : (\omega, \infty) \to X$ satisfies $a \approx_T 0$ if [5]

$$\limsup_{\lambda\to\infty} \frac{1}{\lambda} \ln \|a(\lambda)\| \le -T.$$

The *asymptotic Laplace transform* $\{f\}$ of a function $f \in L^1_{loc}([0,\infty); X)$ is defined as the intersection of the sets $\{f\}_T$; i.e.,

$$\{f\} := \cap_{T>0}\{f\}_T = \{r \in \mathcal{A}(\mathcal{P};\mathcal{X}) : r(\lambda) - \int_0^T e^{-\lambda t} f(t)\, dt \approx_T 0 \text{ for all } T > 0\}.$$

To simplify notation let $\{f\}_\infty := \{f\}$. It is easy to see that $f \to \{f\}_T$ is linear; i.e., $\{f\}_T + \{g\}_T = \{f+g\}_T$ and $\{cf\}_T = c\{f\}_T$ for $0 < T \le \infty$, $f, g \in L^1_{loc}$ and $c \in \mathbb{C}$.

Theorem 2.1. (Existence and Uniqueness)[6] *Let $f, g \in L^1_{loc}([0,\infty); X)$ and $0 < T \le \infty$. Then*

(a) $\emptyset \ne \{f\} \subset \{f\}_T$;

(b) if $\hat{f}$ exists, then $\hat{f} \in \{f\}$;

(c) $\{f\}_T \cap \{g\}_T \ne \emptyset$ if and only if $f = g$ a.e. on $[0,T]$.

Proof. (a) For $f \in L^1_{loc}$ and $0 < T < \infty$ define $f_T := f \cdot \chi_{[0,T]}$. Then the finite Laplace transform $\hat{f}_T(\lambda) = \int_0^T e^{-\lambda t} f(t)\, dt = \int_0^\infty f_T(t)\, dt$ is an entire function and

$$\{f\}_T = \hat{f}_T + \{0\}_T \ne \emptyset, \tag{2.1}$$

where $\{0\}_T := \{a \in \mathcal{A}(P;X) : a \approx_T 0\}$. Let $0 < T' < T < \infty$, $F(t) := \int_0^t f(s)\, ds$ $(t \ge 0)$ and $h(\lambda) := \int_{T'}^T e^{-\lambda t} f(t)\, dt = e^{-\lambda T} F(T) - e^{-\lambda T'} F(T') + \lambda \int_{T'}^T e^{-\lambda t} F(t)\, dt$. Since F is bounded on $[T', T]$, it follows that $h \approx_{T'} 0$. Thus $\{f\}_T = \hat{f}_{T'} + h + \{0\}_T \subset \hat{f}_{T'} + h + \{0\}_{T'} = \hat{f}_{T'} + \{0\}_{T'} = \{f\}_{T'}$. This shows that

$$\{f\} = \cap_{T>0}\{f\}_T \subset \{f\}_T \subset \{f\}_{T'} \;\text{ if }\; 0 < T' < T < \infty.$$

[5]If one requires $a \approx_T 0$ if and only if $a \in C^\infty((\omega,\infty); X)$ and $\limsup_{\lambda\to\infty} \frac{1}{\lambda} \ln \|a^{(n)}(\lambda)\| \le -T$ for all $n \in \mathbb{N}_0$, then the asymtotic Laplace transform has the additional operational property $\{(-t)^n f\}_T = \{f\}^{(n)}$, while all other results presented here remain valid; see B. Bäumer [2]. However, the definition above suffices for the purposes of this paper and the proofs extend easily to the more general case. One should notice that $\approx_T$ defines an equivalence relation on $\mathcal{A}(\mathcal{P};\mathcal{X})$. Furthermore, if $r \in \mathcal{A}(\mathcal{P};\mathcal{X})$ and $q \in \mathcal{A}(\mathcal{P};\mathbb{C})$ with $r \approx_T 0$ and $q \approx_S 0$, then $qr \approx_{T+S} 0$.

[6]This theorem is essentially due to Vignaux [22] who extended the Ritt's Theorem about asymtotic series to asymptotic integrals; see, e.g., R. Remmert [21], 9.6.4 and also J.C. Vignaux and M. Cotlar [23], L. Berg [7], and Yu. I. Lyubich [20].

To see that the asymptotic Laplace transform $f \to \{f\}$ is well defined (i.e., $\{f\} \neq \emptyset$ for all $f \in L^1_{loc}((0,\infty);X)$), define

$$r(\lambda) := \lambda \int_0^\infty e^{-\lambda t}(1 - e^{-\frac{d(\lambda)}{G(t)}})\, F(t)\, dt, \tag{2.2}$$

where $G(t) := \max\{\|F(t)\|, 1\}$, and $\lambda \to d(\lambda)$ is an analytic function to be chosen below such that $r \in \{f\}_T$ for all $T > 0$. Since $|1 - e^{-z}| = |z \int_0^1 e^{-zt}\, dt| \leq |z|$ for all $z \in \mathbb{C}$ with $Re(z) \geq 0$, it follows that $\|e^{-\lambda t}(1 - e^{-\frac{d(\lambda)}{G(t)}})\, F(t)\| \leq e^{-Re(\lambda)t}|d(\lambda)|$ for all $t \geq 0$ and $\lambda \in \Omega := \{\lambda \in \mathbb{C} : Re(\lambda) > 0 \text{ and } Re(d(\lambda)) \geq 0\}$. If we assume that Ω contains a post-sectorial region, then $r \in \mathcal{A}(P;X)$ and satisfies the growth estimate

$$\|r(\lambda)\| \leq \frac{1}{Re(\lambda)}|\lambda d(\lambda)|.$$

Consider $a(\lambda) = r(\lambda) - \int_0^T e^{-\lambda t} f(t)\, dt = r(\lambda) + \lambda e^{-\lambda T} F(T) - \lambda \int_0^T e^{-\lambda t} F(t) dt = \lambda \int_T^\infty e^{-\lambda t}(1 - e^{-\frac{d(\lambda)}{G(t)}}) F(t)\, dt - \lambda \int_0^T e^{-\lambda t} e^{-\frac{d(\lambda)}{G(t)}} F(t)\, dt + \lambda e^{-\lambda T} F(T) = a_1(\lambda) - a_2(\lambda) + a_3(\lambda)$. Then $r \in \{f\}$ if and only if $a \approx_T 0$ for all $T > 0$. Clearly, $a_3(\lambda) = \lambda e^{-\lambda T} F(T)$ satisfies $a_3 \approx_T 0$ for all $T > 0$. Consider $a_2(\lambda) = \lambda \int_0^T e^{-\lambda t} e^{-\frac{d(\lambda)}{G(t)}} F(t)\, dt$. If $\lambda \in \Omega \cap \mathbb{R}_+$, then $\|a_2(\lambda)\| \leq MT\lambda e^{-Re(d(\lambda))/M}$, where $M = \sup_{0 \leq t \leq T} \|G(t)\|$, and therefore $\limsup_{\lambda\to\infty} \frac{1}{\lambda} \ln \|a_2(\lambda)\| \leq -\liminf_{\lambda\to\infty} \frac{1}{\lambda} Re(d(\lambda))$. Thus, to ensure that $a_2 \approx_T 0$ for all $T > 0$ we assume that $\frac{1}{\lambda} Re(d(\lambda)) \to \infty$ as $\lambda \to \infty$. Finally, consider $a_3(\lambda) = \lambda \int_T^\infty e^{-\lambda t}(1 - e^{-\frac{d(\lambda)}{G(t)}}) F(t)\, dt$. Then $\|a_3(\lambda)\| \leq |d(\lambda)| e^{-\lambda T}$ for all $\lambda \in \Omega \cap \mathbb{R}_+$. This shows that $a_3 \approx_T 0$ for all $T > 0$ if $\limsup_{\lambda\to\infty} \frac{1}{\lambda} \ln(|d(\lambda)|) \leq 0$.

The considerations above show that r as defined in (2.2) is an element of $\{f\}$ if the damping function d is analytic on a post-sectorial subregion S of the open right half-plane, $Re(d(\lambda)) \geq 0$ for all $\lambda \in S$, $\frac{1}{\lambda} Re(d(\lambda)) \to \infty$ as $\lambda \to \infty$, and $\limsup_{\lambda\to\infty} \frac{1}{\lambda} \ln(|d(\lambda)|) \leq 0$.

An example of a damping function with these properties is given by

$$d(\lambda) = \lambda \ln(\lambda).$$

There, $S = \{\lambda \in \mathbb{C} : Re(\lambda) > 1\} \cap \{\lambda = re^{i\alpha} : r > e^{\alpha \tan(\alpha)}, -\pi/2 < \alpha < \pi/2\}$.

(b) Since $\hat{f}$ exists if and only if there are constants $M, \omega > 0$ such that $\|F(t)\| \leq Me^{\omega t}$ for all $t \geq 0$, there exists a constant $C_T > 0$ such that $\| \int_T^\infty e^{-\lambda t} f(t)\, dt\| = \| -e^{-\lambda T} F(T) + \lambda \int_T^\infty e^{-\lambda t} F(t)\, dt\| \leq e^{-\lambda T}\|F(T)\| + M\frac{\lambda}{\lambda - \omega} e^{-(\lambda-\omega)T} \leq C_T e^{-\lambda T}$ for all $\lambda > 2\omega$. This shows that $\hat{f}(\lambda) - \int_0^T e^{-\lambda t} f(t)\, dt = \int_T^\infty e^{-\lambda t} f(t)\, dt \approx_T 0$ for all $T > 0$. Thus,

$$\{f\}_T = \hat{f} + \{0\}_T \text{ for all } T > 0, \text{ and } \{f\} = \hat{f} + \cap_{T>0}\{0\}_T. \tag{2.3}$$

(c) Clearly, the approximate Laplace transform $\{f\}_T$ of any $f \in L^1_{loc}$ comprises a "large" set of analytic functions. Nevertheless, the sets are disjoint for different f. To see this let $0 < T \leq \infty$, and $r \in \{f\}_T \cap \{g\}_T$. Then there exist $a, b \approx_T 0$ such that

$$r(\lambda) = \int_0^T e^{-\lambda t} f(t)\, dt + a(\lambda) = \int_0^T e^{-\lambda t} g(t)\, dt + b(\lambda)$$

for all $\lambda > \omega$. Thus, $\hat{h}_T(\lambda) = \int_0^T e^{-\lambda t} h(t)\, dt = c(\lambda)$, where $h := f - g$, $c := b - a$, and $c \approx_T 0$. Now statement (c) follows from the following proposition; see also B. Bäumer and F. Neubrander [5]. $\square$

Proposition 2.2. *Let* $T > 0$, $f \in L^1_{loc}([0,\infty); X)$ *and assume* $\hat{f}$ *exists. The following are equivalent.*

(*i*) $\limsup_{\lambda\to\infty} \frac{1}{\lambda} \ln \|\hat{f}(\lambda)\| \leq -T$.

(*ii*) $f = 0$ *a.e. on* $[0, T]$.

Moreover, f *does not identically vanish a.e. on any interval containing* 0 *if and only if* $\limsup_{\lambda\to\infty} \frac{1}{\lambda} \ln \|\hat{f}(\lambda)\| = 0$.

Proof. The existence of $\hat{f}$ implies the exponential boundedness of $F(t) = \int_0^t f(s)\, ds$. Thus, $G(t) := \int_0^t F(s)\, ds$ is Lipschitz-continuous ($\|G(t) - G(s)\| \leq M|\int_s^t e^{\omega r}\, dr|$ for some $M, \omega > 0$ and all $t, s \geq 0$) and $\hat{f}(\lambda) = \lambda^2 \hat{G}(\lambda) = \lambda \widehat{dG}(\lambda)$ for $Re(\lambda) > \omega$. Define $r(\lambda) := \frac{1}{\lambda}\hat{f}(\lambda) = \int_0^\infty e^{-\lambda t}\, dG(t) = \lambda \int_0^\infty e^{-\lambda t} G(t)\, dt$ for $Re(\lambda) > \omega$. The Phragmén-Doetsch inversion formula, whose proof can be found in B. Bäumer and F. Neubrander [5], yields

$$G(t) = \lim_{k\to\infty} \sum_{j=1}^\infty (-1)^{j+1} \frac{1}{j!} e^{tkj} r(kj), \tag{2.4}$$

where the limit exists uniformly for $t \in [0, \infty)$.

Suppose (i) holds. Then $\limsup_{\lambda\to\infty} \frac{1}{\lambda} \ln \|r(\lambda)\| \leq -T$. Thus, for $0 < \xi < T$, there exists $M, \lambda_0 > 0$ such that $\|r(\lambda)\| \leq Me^{-\lambda\xi}$ for all $\lambda > \lambda_0$. Let $t \in [0, \xi)$. Then

$$\|\sum_{j=1}^\infty (-1)^{j+1} \frac{1}{j!} e^{tkj} r(kj)\| \leq M \sum_{j=1}^\infty \frac{1}{j!} e^{(t-\xi)kj} = M\left(e^{e^{(t-\xi)k}} - 1\right) \to 0 \tag{2.5}$$

as $k \to \infty$. Because $r(\lambda) = \int_0^\infty e^{-\lambda t}\, dG(t)$, it follows from (2.4) that $G = 0$ on $[0, \xi)$ for all $0 < \xi < T$. Thus, $f = 0$ on $[0, T]$. This proves that (i) implies (ii).

Suppose (ii) holds. Then $r(\lambda) = \lambda \int_T^\infty e^{-\lambda t} G(t)\, dt = e^{-\lambda T} \lambda \int_0^\infty e^{-\lambda t} G(t + T)\, dt$. Since $t \to G(t+T)$ is exponentially bounded, it follows that $\|\lambda \int_0^\infty e^{-\lambda t} G(t+T)\, dt\| \leq C$ for some $C > 0$ and therefore $\|r(\lambda)\| \leq Ce^{-\lambda T}$ for all sufficiently large $\lambda > 0$. This proves that (ii) implies (i).

If $\hat{f}$ exists, then it follows from the exponential boundedness of the antiderivative F and $\hat{f}(\lambda) = \lambda \int_0^\infty e^{-\lambda t} F(t)\, dt$ that there exist $M, \lambda_0 > 0$ such that $\|\hat{f}(\lambda)\| \leq M$

for all $\lambda > \lambda_0$. Thus, if $\hat{f}$ exists, then $\limsup_{\lambda\to\infty} \frac{1}{\lambda} \ln \|\hat{f}(\lambda)\| \le 0$. This finishes the proof of the proposition. $\square$

Example 2.3. The function $f : t \to e^{t^2}$ is clearly not Laplace transformable. As shown in the proof of Theorem 2.1, the function

$$r_0(\lambda) := \lambda \int_0^\infty e^{-\lambda t}(1 - e^{-\frac{\lambda \ln(\lambda)}{G(t)}})\, F(t)\, dt$$

is an asymptotic Laplace transform of f (i.e., $r_0 \in \{f\}$), where $F(t) = \int_0^t e^{s^2}\, ds$ and $G(t) := \max\{\|F(t)\|, 1\}$. Since r_0 is not a classical function of analysis one might want to find more familar functions in $\{f\}$. We present two further methods of constructing elements of $\{f\}$ which depend more on the special nature of the function $f(t) = e^{t^2}$. The first one is due to L. Berg [7]. Consider the entire function

$$r_1(\lambda) := \int_0^{\lambda/2} e^{-\lambda t + t^2}\, dt = \frac{\lambda}{2} \int_0^1 e^{-\frac{\lambda^2}{2}s + \frac{\lambda^2}{4}s^2}\, ds.$$

Since the convex function $t \to e^{-\lambda t + t^2}$ attains its minimum at $t = \frac{\lambda}{2}$ it follows that $r_1(\lambda) - \int_0^T e^{-\lambda t} e^{t^2}\, dt = \int_T^{\lambda/2} e^{-\lambda t + t^2}\, dt \le (\frac{\lambda}{2} - T) e^{-\lambda T + T^2}$ for all sufficiently large $\lambda > 0$. Therefore, $r_1 \in \{f\}_T$ for all $T > 0$ and thus $r_1 \in \{f\}$.

Since f is analytic, Cauchy's theorem provides another possibility of finding an representative in $\{f\}$. Let Γ be the directed rectangle consisting of the paths $\Gamma_1 := \{t \in [0,T]\}, \Gamma_2(N) := \{t = T + ir; r \in [0,N]\}, \Gamma_3(N) := \{t = r + iN; r \in [0,T]\}$, and $\Gamma_4(N) := \{t = ir; r \in [0,N]\}$. Since $\int_{\Gamma_3(N)} e^{-\lambda t} e^{t^2}\, dt = \int_0^T e^{-\lambda(r+iN)} e^{r^2 - N^2 + 2irN}\, dr$ $\to 0$ as $N \to \infty$, it follows from Cauchy's integral theorem that

$$\int_0^T e^{-\lambda t} e^{t^2}\, dt = \lim_{N\to\infty} \left(- \int_{\Gamma_2(N)} + \int_{\Gamma_4(N)} \right) e^{-\lambda t} e^{t^2}\, dt.$$

Define [7]

$$r_2(\lambda) := \lim_{N\to\infty} \int_{\Gamma_4(N)} e^{-\lambda t} e^{t^2}\, dt = i \int_0^\infty e^{-i\lambda r} e^{-r^2}\, dr = \frac{i}{2}\sqrt{\pi} e^{-\frac{1}{4}\lambda^2} \mathrm{Erfc}(\frac{i\lambda}{2}).$$

Then $r_2(\lambda) - \int_0^T e^{-\lambda t} e^{t^2}\, dt = a(\lambda)$, where the remainder $\lambda \to a(\lambda)$ is given by $a(\lambda) := \lim_{N\to\infty} \int_{\Gamma_2(N)} e^{-\lambda t} e^{t^2}\, dt = i \int_0^\infty e^{-\lambda(T+ir)} e^{T^2 - r^2 + 2iTr}\, dr$. For $\lambda > 0$, the estimate $\|a(\lambda)\| \le e^{-\lambda T} e^{T^2} \int_0^\infty e^{-r^2}\, dr$ implies that $r_2 \in \{f\}$. $\diamondsuit$

[7] The Laplace transform of the function $t \to e^{-t^2}$ is $\lambda \to \frac{1}{2}\sqrt{\pi} e^{-\frac{1}{4}\lambda^2} \mathrm{Erfc}(\frac{\lambda}{2})$.

If $f, g \in L^1_{loc}$ and one of them is scalar-valued, then

$$f * g : t \to \int_0^t f(t-s)g(s)\,ds \quad (t \geq 0)$$

denotes the *convolution* of f and g. For all scalar, locally Lebesgue integrable functions functions g, h and locally Bochner integrable functions f the convolution $f * g$ is in L^1_{loc}, $f * g = g * f$, and $(g * h) * f = g * (h * f)$.[8] The usefulness of the Laplace transform in applications to differential and integral equations is due to the fact that it maps differentiation, integration, and more general, convolution onto multiplication. Next it will be shown that these crucial operational properties extend to the asymptotic Laplace transform as well.

Proposition 2.4. *Let $0 < T \leq \infty$ and $f \in L^1_{loc}([0,\infty); X)$. If f is absolutely continuous and f' exists a.e., then*

$$\{f'\}_T = \lambda\{f\}_T - f(0).$$

If $g \in L^1_{loc}([0,\infty); \mathbb{C})$ and $T < \infty$, then

$$\{f * g\}_T = \{f\}_T \cdot \{g\}_T + \{0\}_T.$$

In particular, $\{f\}_T \cdot \{g\}_T \subset \{f * g\}_T$ for all $0 < T \leq \infty$.

Proof. Let $0 < T < \infty$. If $q \in \{f'\}_T$, then there exist $\omega > 0$ and $a \approx_T 0$ such that $q(\lambda) = \int_0^T e^{-\lambda t} f'(t)\,dt + a(\lambda) = \lambda\left[\int_0^T e^{-\lambda t} f(t)\,dt + \frac{1}{\lambda}e^{-\lambda T} f(T) + \frac{1}{\lambda}a(\lambda)\right] - f(0)$ for all $\lambda > \omega$. Thus $q \in \lambda\{f\}_T - f(0)$. Reversely, if $r \in \lambda\{f\}_T - f(0)$ then there exist $\omega > 0$ and $a \approx_T 0$ such that $r(\lambda) = \lambda\left[\int_0^T e^{-\lambda t} f(t)\,dt + a(\lambda)\right] - f(0) = \int_0^T e^{-\lambda t} f'(t)\,dt - e^{-\lambda T} f(T) + \lambda a(\lambda)$ for all $\lambda > \omega$. Thus, $r \in \{f'\}_T$. This shows that $\{f'\}_T = \lambda\{f\}_T - f(0)$ for all $0 < T < \infty$ and thus also for $T = \infty$. To prove the other statement, let $0 < T < \infty$, $r(\lambda) = \int_0^T e^{-\lambda t} f(t)\,dt + a(\lambda)$, and $q(\lambda) = \int_0^T e^{-\lambda s} g(s)\,ds + b(\lambda)$, where $a, b \approx_T 0$. Define $w(\lambda) := a(\lambda)\int_0^T e^{-\lambda s} g(s)\,ds + b(\lambda)\int_0^T e^{-\lambda t} f(t)\,dt + a(\lambda)b(\lambda)$. Then $w \approx_T 0$ and

$$\begin{aligned}
r(\lambda)q(\lambda) &= \int_0^T \int_0^T e^{-\lambda(t+s)} f(t)g(s)\,dtds + w(\lambda) \\
&= \int_0^T \int_s^{T+s} e^{-\lambda t} f(t-s)g(s)\,dtds + w(\lambda) \\
&= \int_0^T \int_0^t e^{-\lambda t} f(t-s)g(s)\,dsdt + \int_T^{2T} \int_{t-T}^T e^{-\lambda t} f(t-s)g(s)\,dsdt + w(\lambda) \\
&= \int_0^T e^{-\lambda t}(f * g)(t)\,dt + e^{-\lambda T}\int_0^T e^{-\lambda t}\int_t^T f(t+T-s)g(s)\,dsdt + w(\lambda).
\end{aligned}$$

[8]For a proof of the scalar case, see N. Dunford and J. Schwartz [16], VIII.1.24, 633-636.

Thus, $\{f\}_T \cdot \{g\}_T \subset \{f * g\}_T$. If $r \in \{f*g\}_T$, then $r(\lambda) = \int_0^T e^{-\lambda t}(f*g)(t)\,dt + a(\lambda)$ for some $a \approx_T 0$. Since $\int_0^T e^{-\lambda t}(f*g)(t)\,dt = \left(\int_0^T e^{-\lambda t} f(t)\,dt\right)\left(\int_0^T e^{-\lambda s} g(s)\,ds\right) - e^{-\lambda T}\int_0^T e^{-\lambda t}\int_t^T f(t+T-s)g(s)\,dsdt$ it follows that $r \in \{f\}_T\{g\}_T + \{0\}_T$. This shows that $\{f*g\}_T \subset \{f\}_T\{g\}_T + \{0\}_T \subset \{f*g\}_T + \{0\}_T = \{f*g\}_T$. Thus, $\{f*g\}_T = \{f\}_T \cdot \{g\}_T + \{0\}_T$ and $\{f\}_T \cdot \{g\}_T \subset \{f*g\}_T$ for all $0 < T < \infty$. Therefore, $\{f\}\{g\} \subset \cap_{T>0}\{f\}_T\{g\}_T \subset \cap_{T>0}\{f*g\}_T = \{f*g\}$. □

Example 2.5. Consider the ordinary differential equation $v' + v = (2t+1)e^{t^2}$ $(t \geq 0)$, $v(0) = 1$. Suppose there exists an absolutely continuous solution on some interval $[0,T]$ (let us pretend for the moment that we do not know this already). To determine v with the asymptotic Laplace transform, observe that $\{(2t+1)e^{t^2}\}_T = \{v'+v\}_T = \lambda\{v\}_T - 1 + \{v\}_T$. Therefore,

$$\begin{aligned}\{v\}_T &= \frac{1}{\lambda+1} + \frac{1}{\lambda+1}\{(2t+1)e^{t^2}\}_T \\ &\subset \{e^t\}_T + \{e^t\}_T\{(2t+1)e^{t^2}\}_T \subset \{e^t + e^t * (2t+1)e^{t^2}\}_T = \{e^{t^2}\}_T.\end{aligned}$$

It follows from Theorem 2.1 (c) that $\{v\}_T = \{e^{t^2}\}_T$. Thus, the unique absolutely continuous solution is given by $v(t) = e^{t^2}$. ♢

Remark 2.6. (Inversion) If $f \in L^1_{loc}$, then $F := 1 * f$ is bounded on $[0,T]$, and for every $y \in \{f\}_T$ one has that $\frac{1}{\lambda}y \in \{1\}_T\{f\}_T \subset \{F\}_T$. Thus, there exists $a \approx_T 0$ such that

$$\frac{1}{\lambda}y(\lambda) = \int_0^T e^{-\lambda t}F(t)\,dt + a(\lambda) = \int_0^\infty dG(t) + a(\lambda)$$

for all $\lambda > 0$, where G is piecewise defined by $G(t) := 1 * F(t) = \int_0^t (t-s)f(s)\,ds$ for $0 \leq t \leq T$ and $G(t) := 1 * F(T)$ for all $t \geq T$. By (2.5), the Phragmén-Doetsch inversion (2.4) is not affected by exponentially decaying perturbations. It follows that

$$G(t) = \int_0^t (t-s)f(s)\,ds = \lim_{k\to\infty}\sum_{j=1}^{\infty}(-1)^{j+1}\frac{1}{j!}e^{tkj}\frac{y(kj)}{kj} \tag{2.6}$$

for all $f \in L^1_{loc}$, $y \in \{f\}_T$, and $0 \leq t < T \leq \infty$. [9] ♢

[9] A completely different inversion formula is due to Yu. I. Lyubich [20]. Let $F : [0,\infty) \to X$ be of bounded semivariation, $F(0) = 0$ and $r(\lambda) = \int_0^\infty dF(t) + a(\lambda)$ $(\lambda > \omega_0 > 0)$ for some $a \in L^1_{loc}([\omega_0,\infty), X)$ with $a \approx_T 0$. Let $\omega > \omega_0$. Then $H(\mu) := \frac{1}{2\pi i}\int_\omega^\infty e^{\mu t}\frac{r(t)}{t}\,dt$ is welldefined for $Re(\mu) < 0$, has an analytic continuation to the sliced half-plane $\mathbb{C} \setminus [0,T) \setminus \{\mu : Re(\mu) \geq T\}$, and $F(t) = \lim_{\epsilon\to 0} H(t+i\epsilon) - H(t-i\epsilon)$ for all $t \in [0,T)$.

Remark 2.7. (Multiplied Laplace Transforms) Let Ω be an open subset of $\mathbb{C}$ and let $m : \Omega \to \mathbb{C}$ be analytic. Then $q : \Omega \to X$ is said to have an *asymptotic, m-multipled Laplace representation*, if there exist $0 < T \le \infty$, and $f \in C_0([0,\infty;X)$ such that

$$mq \in \{f\}_T. \tag{2.7}$$

The function q is said to have an m-multiplied Laplace representation, if there exists a Laplace transformable $f \in C_0([0,\infty);X)$ (i.e., f is continuous and $f(0) = 0$) such that $mq = \hat{f}$. To see how multiplied Laplace transforms of normalized continuous functions relate to Laplace transforms of generalized functions, let $m = \hat{k}$ for some *regularizing function* k; i.e.,

$$k \in L^1_{loc}([0,\infty);\mathbb{C}) \text{ and } \limsup_{\lambda\to\infty} \frac{1}{\lambda} \ln|\hat{k}(\lambda)| = 0.^{10} \tag{2.8}$$

Then, by Proposition 2.2 and the Titchmarsh-Foiaş Theorem in B. Bäumer, G. Lumer and F. Neubrander [3], the convolution operator $\mathcal{K} : f \to k * f$ is an injective operator on $C([0,\infty);X)$ with dense range in the Frechét space $C_0([0,\infty);X)$. Moreover, $\|f\|_{\mathcal{K},n} := \sup_{t\in[0,n]} \|\mathcal{K}f(t)\|$ defines a family of seminorms on $C([0,\infty);X)$ and $\mathcal{K}$ extends to an isomorphism between the Frechét completion $C^{[k]}([0,\infty);X)$ of $C([0,\infty);X)$ with respect to that family of seminorms and the Frechét space $C_0([0,\infty);X)$.[11]

A *k-generalized function* $u \in C^{[k]}$ is Laplace transformable if the continuous function $f := k * u$ is Laplace transformable and the Laplace transform of u is defined as

$$\hat{u}(\lambda) := \frac{\hat{f}(\lambda)}{\hat{k}(\lambda)}.$$

Similarily, the asymptotic Laplace transform is defined as $\{u\}_T := \frac{1}{\hat{k}(\lambda)}\{f\}_T$ ($0 < T \le \infty$). Thus, if q has a m-multiplied Laplace representation $mq \in \{f\}_T$ or $mq = \hat{f}$ for some $f \in C_0([0,\infty);X)$, where $m = \hat{k}$ for some regularizing function k, then q has a Laplace representation $q \in \{u\}_T$ or $q = \hat{u}$ for $u = \mathcal{K}^{-1}f \in C^{[k]}([0,\infty);X)$.

Next we give examples of classes of analytic functions which can be represented as Laplace transforms of generalized functions. Consider first analytic functions $q : \{Re(\lambda) > \omega \ge 0\} \to X$ with $\sup_{Re(\lambda)>\omega} \|\lambda q(\lambda)\| < \infty$. Let $b > 0$ and

$$f(t) := \lim_{n\to\infty} \frac{1}{2\pi i} \int_{\epsilon-in}^{\epsilon+in} e^{\lambda t} \frac{q(\lambda)}{\lambda^b}\, d\lambda.$$

[10] Typical examples of regularizing functions are $k_n(t) = t^n/n!$ with $\hat{k}_n(\lambda) = 1/\lambda^{n+1}$ and $k_\delta(t) = \frac{1}{2\pi i}\int_{\omega+i\mathbb{R}} e^{t\lambda-\lambda^\delta}\, d\lambda$ ($0 < \delta < 1$) with $\hat{k}_\delta(\lambda) = e^{-\lambda^\delta}$. For the special case $\delta = 1/2$, the regularizer k_δ is given by $k_{1/2}(t) = \frac{1}{2\sqrt{\pi}} t^{-3/2} e^{-\frac{1}{4t}}$. Also, all functions $m(\lambda) = e^{-\lambda^\gamma}$ with $\gamma \ge 1$ can not be represented as Laplace transforms of functions in $k \in L^1_{loc}$; see G. Doetsch [15], p.163, 229, 263.

[11] See alse B. Bäumer [2] and B. Bäumer, G. Lumer and F. Neubrander [3].

Then $f \in C_0([0,\infty);X)$, $\sup_{t>0} \|e^{-\omega t}\frac{1}{t^b}f(t)\| < \infty$ and $q(\lambda) = \lambda^b \hat{f}(\lambda)$ for $Re(\lambda) > \omega$; see, e.g., B. Bäumer and F. Neubrander [5]. Thus $q(\lambda) = \hat{u}(\lambda) = \frac{\hat{f}(\lambda)}{\hat{k}(\lambda)}$, where $k(t) = \frac{1}{\Gamma(b)}t^{b-1}$ and $u = \mathcal{K}^{-1}f \in C^{[k]}$ coincides with the b-th (distributional) derivative of f. More general, if q is an analytic function on some half-plane $\{Re(\lambda) > \omega \geq 0\}$ with values in X which grows less than exponentially, i.e.,

$$\sup_{Re(\lambda)>\omega} \|\lambda \hat{k}_0(\lambda) q(\lambda)\| < \infty$$

for some regularizing function k_0, then it follows from the above mentioned representation theorem that there exists $f \in C_0([0,\infty);X)$ such that $\frac{1}{\lambda}\hat{k}_0(\lambda)q(\lambda) = \hat{k}(\lambda)q(\lambda) = \hat{f}(\lambda)$ or $q(\lambda) = \hat{u}(\lambda)$, where $k := 1 * k_0$ and $u \in C^{[k]}([0,\infty);X)$ is a generalized function such that $f = k * u$.[12] Thus, a faster growing q will have a less regular u such that $q = \hat{u}$. ♢

Next we consider analytic functions $q : \Omega \to X$ with $\sup_{\lambda\in\Omega} \|\lambda q(\lambda)\| < \infty$, where the domain Ω contains a post-sectorial region

$$\Omega_\Psi := \{\lambda : \; Re(\lambda) \geq \beta > 0, \; |Im\lambda| \leq \Psi(Re(\lambda))\},$$

where Ψ is a positive, strictly increasing C^1-function with $\Psi(r) \to \infty$ as $r \to \infty$ and $\sup_{r\geq\beta} \Psi'(r)/r^\alpha\Psi(r) < \infty$ for some $\alpha \geq 0$; i.e., Ψ is dominated by some exponential function $t \to e^{cr^\gamma}$.[13]

Theorem 2.8. *Let $\Omega_\Psi \subset \Omega$, $q : \Omega \to X$ analytic, and $\sup_{\lambda\in\Omega} \|\lambda q(\lambda)\| < \infty$.*

(*a*) *If there exists $c > 0$ and an analytic function $m : \Omega_\Psi \to \mathbb{C}$ such that*

$$|m(\lambda)| \leq e^{-c\Psi^{-1}(|\lambda|)} \quad \textit{for all} \quad \lambda \in \Omega_\Psi,$$

then there exists $f \in C_0([0,c),X)$ such that $mq \in \{f\}_T$ for all $0 < T < c$.

(*b*) *If there exists $c > 0$, $d > 1$, and an analytic function $m : \Omega_\Psi \to \mathbb{C}$ such that*

$$|m(\lambda)| \leq e^{-c(\Psi^{-1}(|\lambda|))^d} \quad \textit{for all} \quad \lambda \in \Omega_\Psi,$$

then there exists $f \in C_0([0,\infty),X)$ such that $mq \in \{f\}$.

Proof. (a) Let Γ be the oriented boundary of the region Ω_Ψ; i.e., $\Gamma = \Gamma_- \cup \Gamma_\beta \cup \Gamma_+$, where $\Gamma_\pm := \{r \pm i\Psi(r); \; \beta \leq r < \infty\}$, and $\Gamma_\beta := \{\beta + ir; \; -\Psi(\beta) \leq r \leq \Psi(r)\}$. Let

[12] If k_i are regularizing functions and $k_1 * k_2 = k_3$, then $C^{[k_1]}$ is continuously embedded in $C^{[k_3]}$.
[13] This growth condition on Ψ does not restrict generality; see Corollary 2.9 (a).

$M, C_0, C_1 > 0$ be such that $\|\lambda q(\lambda)\| \le M$ and $\left|\frac{1 \pm i\Psi'(r)}{r \pm i\Psi(r)}\right| \le C_0 + C_1 r^\alpha$ for all $\lambda \in \Omega_\Psi$ and all $r \ge \beta$. Since Ψ^{-1} is increasing it follows that

$$\Psi^{-1}(|r \pm i\Psi(r)|) \ge \Psi^{-1}(|\Psi(r)|) = r$$

for all $r \ge \beta$. Thus, for $\lambda = r \pm i\Psi(r) \in \Gamma_\pm$, we have the estimate $\|e^{\lambda t} m(\lambda) q(\lambda)\, d\lambda\| \le M e^{rt} e^{-c\Psi^{-1}(|r \pm i\Psi(r)|)} \left|\frac{1 \pm i\Psi'(r)}{r \pm i\Psi(r)}\right| dr \le M(C_0 + C_1 r^\alpha) e^{-r(c-t)} dr$. It follows that

$$f(t) := \frac{1}{2\pi i} \int_\Gamma e^{\lambda t} m(\lambda) q(\lambda)\, d\lambda$$

is welldefined and continuous for $0 \le t < c$. Next it will be shown that

$$\int_0^T e^{-\lambda t} f(t)\, dt = m(\lambda) q(\lambda) - a_T(\lambda)$$

for all $0 < T < c$ and λ in the interior Ω_Ψ^0 of Ω_Ψ, where

$$a_T(\lambda) := \frac{e^{-\lambda T}}{2\pi i} \int_\Gamma \frac{1}{\mu - \lambda} e^{\mu T} m(\mu) q(\mu)\, d\mu.$$

To see this, let $\Gamma(n) := \Gamma \cap \{Re(\lambda) \le n\}$ and $\Pi_n := \{n + ir; -\Psi(n) \le r \le \Psi(r)\}$. Then, for all $\lambda \in \Omega_\Psi^0$,

$$\begin{aligned}
\int_0^T e^{-\lambda t} f(t)\, dt &= \int_0^T e^{-\lambda t} \frac{1}{2\pi i} \int_\Gamma e^{\mu t} m(\mu) q(\mu)\, d\mu\, dt \\
&= \frac{1}{2\pi i} \int_\Gamma \frac{1}{\mu - \lambda} \left[e^{(\mu - \lambda)T} - 1 \right] m(\mu) q(\mu)\, d\mu \\
&= \lim_{n\to\infty} \frac{1}{2\pi i} \left[\int_{\Gamma(n)} \frac{1}{\mu - \lambda} e^{(\mu-\lambda)T} m(\mu) q(\mu)\, d\mu - \int_{\Gamma(n)} \frac{1}{\mu - \lambda} m(\mu) q(\mu)\, d\mu \right] \\
&= \lim_{n\to\infty} \frac{1}{2\pi i} \Big[\int_{\Gamma(n)} \frac{1}{\mu - \lambda} e^{(\mu-\lambda)T} m(\mu) q(\mu)\, d\mu \\
&\qquad + \left(-\int_{\Gamma(n)} + \int_{\Pi_n} - \int_{\Pi_n} \right) \frac{1}{\mu - \lambda} m(\mu) q(\mu)\, d\mu \Big] \\
&= m(\lambda) q(\lambda) + \lim_{n\to\infty} \frac{1}{2\pi i} \Big[\int_{\Gamma(n)} \frac{1}{\mu - \lambda} e^{(\mu-\lambda)T} m(\mu) q(\mu)\, d\mu \\
&\qquad - \int_{\Pi_n} \frac{1}{\mu - \lambda} m(\mu) q(\mu)\, d\mu \Big] \\
&= m(\lambda) q(\lambda) + \frac{1}{2\pi i} \int_\Gamma \frac{1}{\mu - \lambda} e^{(\mu-\lambda)T} m(\mu) q(\mu)\, d\mu,
\end{aligned}$$

since, by the residue theorem,

$$\frac{1}{2\pi i}\left(-\int_{\Gamma(n)}+\int_{\Pi_n}\right)\frac{1}{\mu-\lambda}m(\mu)q(\mu)\,d\mu = m(\lambda)q(\lambda)$$

for $n > Re(\lambda)$, and $\lim_{n\to\infty}\frac{1}{2\pi i}\int_{\Pi_n}\frac{1}{\mu-\lambda}m(\mu)q(\mu)\,d\mu = 0$. The last statement follows from the estimate

$$\begin{aligned}\|\int_{\Pi_n}\frac{1}{\mu-\lambda}m(\mu)q(\mu)\,d\mu\| &\le \int_{-\Psi(n)}^{\Psi(n)}\frac{1}{|n+ir-\lambda|}e^{-c\Psi^{-1}(|n+ir|)}\frac{M}{|n+ir|}\,dr\\ &\le Me^{-c\Psi^{-1}(n)}\int_{-\infty}^{\infty}\frac{1}{|n+ir-\lambda|^2}\,dr.\end{aligned}$$

To show that $a_T \approx_T 0$, it suffices to prove the boundedness of $\int_\Gamma \frac{1}{\mu-\lambda}e^{\mu T}m(\mu)q(\mu)\,d\mu$ for $\lambda\to\infty$. Clearly, $\int_{\Gamma_\beta}\frac{1}{\mu-\lambda}e^{\mu T}m(\mu)q(\mu)\,d\mu\to 0$ as $\lambda\to\infty$. Moreover,

$$\begin{aligned}\|\int_{\Gamma_\pm}\frac{1}{\mu-\lambda}e^{\mu T}m(\mu)q(\mu)d\mu\| &\le \int_\beta^\infty \frac{M}{dist(\lambda,\Gamma_\pm)}e^{rT}e^{-c\Psi^{-1}(|r\pm i\Psi(r)|)}\left|\frac{1\pm i\Psi'(r)}{r\pm\Psi(r)}\right|dr\\ &\le \frac{M}{dist(\lambda,\Gamma_\pm)}\int_\beta^\infty (C_0+C_1r^\alpha)e^{-r(c-T)}\,dr.\end{aligned}$$

This shows that $a_T\approx_T 0$. Thus, $mq\in\{f\}_T$ for all $0<T<c$. Before proving the statement (b), an estimate will be given for $\|f(t)\|$ for $0\le t<c$. Let t be fixed and $\gamma\ge\beta$ be arbitrary. Define $\Gamma^* = \Gamma^*_-\cup\Pi_\gamma\cup\Gamma^*_+$, where $\Gamma^*_\pm := \{r\pm i\Psi(r);\gamma\le r<\infty\}$, and $\Pi_\gamma := \{\gamma+\Psi(\gamma)e^{ir};\frac{\pi}{2}\le r\le\frac{\pi}{2}\}$. Then, by Cauchy's theorem, $f(t) = \frac{1}{2\pi i}\int_{\Gamma^*}e^{\lambda t}m(\lambda)q(\lambda)\,d\lambda$. Moreover,

$$\begin{aligned}\|\frac{1}{2\pi i}\int_{\Pi_\gamma}e^{\lambda t}m(\lambda)q(\lambda)d\lambda\| &\le \frac{M}{2\pi}\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}}|e^{\gamma t+t\Psi(\gamma)e^{ir}}|e^{-c\Psi^{-1}(|\gamma+\Psi(\gamma)e^{ir}|)}\frac{\Psi(\gamma)}{|\gamma+\Psi(\gamma)e^{ir}|}dr\\ &\le \frac{M}{2\pi}\int_{-\pi/2}^{\pi/2}e^{\gamma t+t\Psi(\gamma)\cos(r)}e^{-c\Psi^{-1}(\Psi(\gamma))}\,dr\\ &= \frac{M}{\pi}e^{-\gamma(c-t)}\int_0^{\pi/2}e^{t\Psi(\gamma)\cos(r)}\,dr,\end{aligned}$$

and, since $\|e^{\lambda t}m(\lambda)q(\lambda)\,d\lambda\|\le M(C_0+C_1r^\alpha)e^{-r(c-t)}dr$ along $\Gamma_\pm$,

$$\begin{aligned}\|\frac{1}{2\pi i}\int_{\Gamma^*_\pm}e^{\lambda t}m(\lambda)q(\lambda)\,d\lambda\| &\le \frac{M}{2\pi}\int_\gamma^\infty (C_0+C_1r^\alpha)e^{-r(c-t)}\,dr\\ &= \frac{MC_0}{2\pi}e^{-\gamma(c-t)}\frac{1}{c-t}+\frac{MC_1}{2\pi}\int_\gamma^\infty r^\alpha e^{-r(c-t)}\,dr.\end{aligned}$$

Thus

$$\|f(t)\| \leq \frac{M}{\pi} e^{-\gamma(c-t)} \left(\int_0^{\pi/2} e^{t\Psi(\gamma)\cos(r)}\, dr + \frac{C_0}{c-t} \right) + \frac{MC_1}{\pi} \int_\gamma^\infty r^\alpha e^{-r(c-t)}\, dr.$$

Now choose γ such that $\Psi(\gamma) = \frac{\Psi(\beta)c}{t}$; i.e., $\gamma := \Psi^{-1}(\frac{\Psi(\beta)c}{t})$. Then

$$\begin{aligned}\|f(t)\| \leq{}& \frac{M}{\pi} e^{-\Psi^{-1}(\frac{\Psi(\beta)c}{t})(c-t)} \left(\int_0^{\pi/2} e^{c\Psi(\beta)\cos(r)}\, dr + \frac{C_0}{c-t} \right) \\ &+ \frac{MC_1}{\pi} \int_\gamma^\infty r^\alpha e^{-r(c-t)}\, dr.\end{aligned}$$

for $0 < t < c$. In particular, $f(0) = 0$.

(b) Let the path Γ and the constants M, C_0, C_1 be as above. Then, for $\lambda = r \pm i\Psi(r) \in \Gamma_\pm$,

$$\begin{aligned}\|e^{\lambda t} m(\lambda) q(\lambda)\, d\lambda\| &\leq M e^{rt} e^{-c(\Psi^{-1}(|r \pm i\Psi(r)|))^d} \left| \frac{1 \pm i\Psi'(r)}{r \pm i\Psi(r)} \right| dr \\ &\leq M(C_0 + C_1 r^\alpha) e^{rt - cr^d}\, dr.\end{aligned}$$

Thus, $f(t) := \frac{1}{2\pi i} \int_\Gamma e^{\lambda t} m(\lambda) q(\lambda)\, d\lambda$ is welldefined and continuous for $t \geq 0$. As in part (a) one shows that

$$\int_0^T e^{-\lambda t} f(t)\, dt = m(\lambda) q(\lambda) + \frac{e^{-\lambda T}}{2\pi i} \int_\Gamma \frac{1}{\mu - \lambda} e^{\mu T} m(\mu) q(\mu)\, d\mu.$$

Since $\int_{\Gamma_\beta} \frac{1}{\mu-\lambda} e^{\mu T} m(\mu) q(\mu)\, d\mu \to 0$ as $\lambda \to \infty$ and

$$\begin{aligned}\| \int_{\Gamma_\pm} \frac{1}{\mu - \lambda} e^{\mu T} m(\mu) q(\mu) d\mu \| &\leq \int_\beta^\infty \frac{M}{dist(\lambda, \Gamma_\pm)} e^{rT - c(\Psi^{-1}(|r \pm i\Psi(r)|))^d} \left| \frac{1 \pm i\Psi'(r)}{r \pm \Psi(r)} \right| dr \\ &\leq \frac{M}{dist(\lambda, \Gamma_\pm)} \int_\beta^\infty (C_0 + C_1 r^\alpha) e^{rT - cr^d}\, dr,\end{aligned}$$

it follows that $mq \in \{f\}_T$ for all $T > 0$. Thus, $mq \in \{f\}$. To estimate $\|f(t)\|$ for $0 < t \leq 1$, let t be fixed, $\gamma \geq \beta$ be arbitrary, and define the path Γ^* as above.. Then, $f(t) = \frac{1}{2\pi i} \int_{\Gamma^*} e^{\lambda t} m(\lambda) q(\lambda)\, d\lambda$,

$$\begin{aligned}\| \int_{\Pi_\gamma} e^{\lambda t} m(\lambda) q(\lambda) d\lambda \| &\leq M \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} |e^{\gamma t + t\Psi(\gamma) e^{ir}}| e^{-c(\Psi^{-1}(|\gamma + \Psi(\gamma) e^{ir}|))^d} \frac{\Psi(\gamma)}{|\gamma + \Psi(\gamma) e^{ir}|}\, dr \\ &\leq M \int_{-\pi/2}^{\pi/2} e^{\gamma t + t\Psi(\gamma)\cos(r)} e^{-c(\Psi^{-1}(\Psi(\gamma)))^d}\, dr \\ &= 2M e^{\gamma t - c\gamma^d} \int_0^{\pi/2} e^{t\Psi(\gamma)\cos(r)}\, dr,\end{aligned}$$

and, since $\|e^{\lambda t}m(\lambda)q(\lambda)\,d\lambda\| \le M(C_0 + C_1 r^\alpha)e^{rt-cr^d}dr$ along $\Gamma_\pm$,

$$\|\frac{1}{2\pi i}\int_{\Gamma^*_\pm} e^{\lambda t}m(\lambda)q(\lambda)\,d\lambda\| \le \frac{M}{2\pi}\int_\gamma^\infty (C_0 + C_1 r^\alpha)e^{rt-cr^d}\,dr.$$

Thus, for $t \ge 0$, $\|f(t)\| \le \frac{M}{\pi}e^{\gamma t - c\gamma^d}\int_0^{\pi/2} e^{t\Psi(\gamma)\cos(r)}\,dr + \frac{M}{\pi}\int_\gamma^\infty (C_0 + C_1 r^\alpha)e^{rt-cr^d}\,dr$. If $0 < t \le 1$, one might choose γ such that $\Psi(\gamma) = \frac{\Psi(\beta)}{t}$; i.e., $\gamma := \Psi^{-1}(\frac{\Psi(\beta)}{t})$. Then, for this γ,

$$\|f(t)\| \le \frac{M}{\pi}e^{\gamma t - c\gamma^d}\int_0^{\pi/2} e^{\Psi(\beta)\cos(r)}\,dr + \frac{M}{\pi}\int_\gamma^\infty (C_0 + C_1 r^\alpha)e^{rt-cr^d}\,dr$$

for $0 < t < 1$. In particular, $f(0) = 0$. $\square$

Corollary 2.9. *Let $\Omega_\Psi \subset \Omega$, $q : \Omega \to X$ analytic, and $\sup_{\lambda\in\Omega}\|\lambda q(\lambda)\| < \infty$.*

(a) *Let $\Psi(x) = e^{ax^\gamma}$ $(a > 0, \gamma > 1)$. Then, for all $b > 0$, there exists $f \in C_0([0,\infty);X)$ such that $\frac{1}{\lambda^b}q(\lambda) \in \{f\}$.*

(b) *Let $\Psi(x) = e^{ax}$ $(a > 0)$. Then*

1. *for all $b > 0$ there exists $f \in C_0([0,ab);X)$ such that $\frac{1}{\lambda^b}q(\lambda) \in \{f\}_T$ $(0 < T < ab)$,*
2. *for all $b > 0$, $0 < \delta < 1$ there exists $f \in C_0([0,\infty);X)$ such that $e^{-b\lambda^\delta}q(\lambda) \in \{f\}$.*

(c) *Let $\Psi(x) = x^{1/a}$ $(0 < a < 1)$. Then*

1. *for all $b > 0$ there exists $f \in C_0([0, b\cos(a\pi/2));X)$ such that $e^{-b\lambda^a}q(\lambda) \in \{f\}_T$ $(0 < T < b\cos(a\pi/2))$,*
2. *for all $b > 0$, $a < \gamma < 1$ there exists $f \in C_0([0,\infty);X)$ such that $e^{-b\lambda^\gamma}q(\lambda) \in \{f\}$.*

Proof. (a) Since $\Psi^{-1}(x) = (\frac{1}{a}\ln(x))^{1/\gamma}$, it follows that $e^{-ab(\Psi^{-1}(|\lambda|))^\gamma} = \frac{1}{|\lambda|^b}$. Thus, if one takes $m(\lambda) = \frac{1}{\lambda^b}$, then (a) follows from Theorem 2.8 (b).
(b) Since $\Psi^{-1}(x) = \frac{1}{a}\ln(x)$, it follows that $e^{-ab\Psi^{-1}(|\lambda|)} = \frac{1}{|\lambda|^b}$. Define $m(\lambda) := \frac{1}{\lambda^b}$. Then 1. follows from Theorem 2.8 (a). To prove the second statement note that $|e^{-z^\delta}| \le e^{-|z|^\delta}$ if $Re(z) > 0$ and $0 < \delta < 1$. Let $m(\lambda) = e^{-b\lambda^\delta}$. Then there exists a constant $c > 0$ such that $|m(\lambda)| \le e^{-b|\lambda|^\delta} \le e^{-c(\frac{1}{a}\ln(|\lambda|))^2} = e^{-c(\Psi^{-1}(|\lambda|))^2}$ for all $\lambda \in \Omega_\Psi$. Now statement 2 follows from Theorem 2.8 (b).
(c) Let $m(\lambda) = e^{-b\lambda^a}$. The statement 1 follows from Theorem 2.8 (a) since $|m(\lambda)| \le e^{-b\cos(a\pi/2)|\lambda|^a} = e^{-b\cos(a\pi/2)\Psi^{-1}(|\lambda|)}$ $(Re(\lambda) > 0)$. If $m(\lambda) = e^{-b\lambda^\gamma}$, then statement 2 follows from Theorem 2.8 (b) since $|m(\lambda)| \le e^{-b\cos(\gamma\pi/2)|\lambda|^\gamma} = e^{-b\cos(\gamma\pi/2)\Psi^{-1}(|\lambda|)^{\gamma/a}}$ $(Re(\lambda) > 0)$. $\square$

Clearly, if q is an analytic function on some post-sector Ω_Ψ on which it grows less than exponentially, i.e., $\sup_{\lambda\in\Omega_\Psi} \|\lambda\hat{k}_0(\lambda)q(\lambda)\| < \infty$ for some regularizing function k_0, then the corollary above can be applied that to $\tilde{q}(\lambda) := k_0(\lambda)q(\lambda)$. Thus Corollary 1.9 shows that more growth of an analytic function $q \in \mathcal{A}(P;X)$ at infinity, a larger interval $[0,T)$, or a smaller post-sectorial region Ω_Ψ lead to a decay in regularity of the generalized function μ for which $q \in \{\mu\}_T$.

3 The Abstract Cauchy Problem

This section is concerned with applications of asymptotic Laplace transform methods to abstract Cauchy problems

$$u'(t) = Au(t),\ t \in I,\ u(0) = x, \tag{ACP}$$

where A is a closed[14] linear operator with domain $D(A)$ and range in a Banach space X, and either $I = [0,T]$ or $I = [0,\infty)$. In this section, k will always be a regularizing function whose Laplace transform has no real zeros[15]; i.e.,

$$k \in L^1_{loc}([0,\infty);\mathbb{C}),\ \limsup_{\lambda\to\infty} \frac{1}{\lambda}\ln|\hat{k}(\lambda)| = 0,\ \text{ and } \hat{k}(\lambda) \neq 0 \text{ for all } \lambda > \omega.$$

Since A is closed and $\frac{d}{dt}(k*u)(t) = (k*u')(t) + k(t)u(0)$, the convolution of k with (ACP) yields $(k*u)' = k*u' + k(t)x = A(k*u) + k(t)x$ or $k*u = A(1*k*u) + (1*k)(t)x$. Thus, a function $u \in C^{[k]}(I;X)$ is called a k-generalized (mild) solution of (ACP) if $v = k*u \in C_0([0,\infty);X)$[16]solves

$$v(t) = A\int_0^t v(s)\,ds + (1*k)(t)(x),\ t \in I. \tag{3.1}$$

Conversely, any $v \in C_0(I;X)$ satisfying (3.1) is called a k-regularized solution of (ACP).[17] One of the main tools used in this section are the following operational

[14]The closedness of A in $X\times X$ is assumed for convenience only; the transform methods extend naturally to linear operators whose graphs are closed in $Z\times X$, where Z is a Banach space which contains $D(A)$ and is continuously embedded in X, see B. Bäumer and F. Neubrander [4].

[15]Recall that if k is regularizing, then the convolution operator $u \to k*u$ is an isomorphism between the generalized function space $C^{[k]}(I;X)$ and $C_0(I;X)$; see Proposition 2.2 and Remark 2.7. As seen below, the assumption $\hat{k}(\lambda) \neq 0$ leads to generation results for operators whose resolvent is analytic on some post-sectorial region Ω. Without this assumption one can obtain generation results for operators whose resolvents are meromorphic on Ω. However, the proofs rely on an extension of the Phragmén-Doetsch inversion (2.6) due to B. Bäumer [2] which is beyond the scope of this paper.

[16]If necessary, we continue $v \in C_0([0,T];X)$ onto $[0,\infty)$ by defining $v(t) := v(T)$ for $t > T$.

[17]The notion of k-generalized solutions was introduced by I. Cioranescu and G. Lumer [10][11]. It was shown by B. Bäumer [2] that if $u \in C^{[k]}(I;X)$ is a k-generalized solution of (ACP), then there exists a sequence of continuous functions $u_n \in C(I;[D(A)])$ (where $[D(A)]$ denotes the Banach space $D(A)$ endowed with the graph norm) such that $u_n \to u$ and $u_n - 1*Au_n - x \to 0$ in $C^{[k]}(I;X)$. Because u coincides with the Cauchy sequence $\{u_n\}$ (modulo the 0-sequences) in $C^{[k]}$, any k-generalized solution u of (ACP) can be regarded as a k-limit solution $\{u_n\}$ of (ACP). In fact, the two notions are equivalent.

properties of the asymptotic Laplace transform. Combining them with those of the first section, it is easy to characterize the existence, uniqueness, and regularity of (generalized) solutions of (ACP) in terms of (local or global) spectral properties of the operator A.

Proposition 3.1. *Let A be closed, $0 < T \leq \infty$, and $v, w \in L^1_{loc}([0,\infty; X)$.*

(*a*) *If $r \in \{v\}_T$, and $Ar \in \{w\}_T$, then $Av = w$ on $[0,T]$.*

(*b*) $\{A(1*v)\}_T = \frac{A}{\lambda}\hat{v}_T(\lambda) + \{0\}_T$.

Proof. (a) If $r \in \{v\}_T$, $r(\lambda) \in D(A)$ for all λ in a post-sectorial region S and $Ar \in \{w\}_T$, then the Phragmén-Doetsch inversion (2.6) yields that $\int_0^t (t-s)v(s)\,ds = \lim_{k\to\infty} \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j!} e^{tkj} \frac{r(kj)}{kj}$ and $\int_0^t (t-s)w(s)\,ds = \lim_{k\to\infty}\sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j!} e^{tkj} A \frac{r(kj)}{kj}$ for all $t \in [0,T)$. Because A is closed one obtains that $\int_0^t (t-s)v(s)\,ds \in D(A)$ and $A\int_0^t (t-s)v(s)\,ds = \int_0^t (t-s)w(s)\,ds$ for all $t \in [0,T)$. Again using the closedness of A, $v(t) \in D(A)$ and $Av(t) = w(t)$ for almost all $t \in [0,T)$. (b) By the closedness of A and (3.1) one obtains $\{A(1*v)\}_T = [A\widehat{(1*v)_T}] + \{0\}_T = \int_0^T e^{-\lambda t} A \int_0^t v(s)\,ds + \{0\}_T = A\int_0^T e^{-\lambda t}\int_0^t v(s)\,ds + \{0\}_T = e^{-\lambda T} A \int_0^T v(s)\,ds + \frac{A}{\lambda}\int_0^T e^{-\lambda t} v(t)\,dt + \{0\}_T = \frac{A}{\lambda}\hat{v}_T + \{0\}_T$ □

Proposition 3.2. *Let A be closed, $k \in L^1_{loc}([0,\infty);\mathbb{C})$ be a regularizing function,*[18] *$x \in X$, $0 < T < \infty$, and $v \in L^1_{loc}([0,\infty);X)$. Then the following are equivalent.*

(*a*) *$v(t) = A\int_0^t v(s)\,ds + (1*k)(t)x$ for almost all $t \in [0,T]$.*

(*b*) *There exists a function r such that $(\lambda I - A)r(\lambda) - x \approx_T 0$ and $\hat{k}r \in \{v\}_T$.*

Proof. (a) $\Rightarrow$ (b): It follows from (2.1), (2.3), and the Propositions 2.4 and 3.1 that $\hat{v}_T(\lambda) + \{0\}_T = \{v\}_T = \{A(1*v) + (1*k)(t)x\}_T = \frac{A}{\lambda}\hat{v}_T + \{0\}_T + \{x\}_T\{k\}_T = \frac{A}{\lambda}\hat{v}_T + \{0\}_T + (\hat{x} + \{0\}_T)(\hat{k} + \{0\}_T) = \frac{A}{\lambda}\hat{v}_T + \frac{x}{\lambda}\hat{k}(\lambda) + \{0\}_T$. Thus

$$(\lambda I - A)\hat{v}_T - \hat{k}(\lambda)x \approx_T 0. \tag{3.2}$$

(b) $\Rightarrow$ (a): By Proposition 2.4, $\frac{1}{\lambda}\hat{k}(\lambda)r(\lambda) \in \frac{1}{\lambda}\{v\}_T \subset \{1*v\}_T$. Because k is a regularizer it follows that $A(\frac{1}{\lambda}\hat{k}(\lambda)r(\lambda)) - \hat{k}(\lambda)r(\lambda) + \frac{1}{\lambda}\hat{k}(\lambda)x \in \{0\}_T$ and thus $A(\frac{1}{\lambda}\hat{k}(\lambda)r(\lambda)) \in \hat{k}(\lambda)r(\lambda) - \frac{1}{\lambda}\hat{k}(\lambda)x + \{0\}_T \subset \{v\}_T + \{1*k\}_T x + \{0\}_T \subset \{v + (1*k)x\}_T$. It follows from Proposition 3.1 (a) that $(1*v)(t) \in D(A)$ and $A(1*v)(t) = v(t) + (1*k)(t)x$ for almost all $t \in [0,T]$. □

[18]Here and in the following propostions one can also take $k(t) = \delta(t)$, $\hat{k}(\lambda) = 1$.

The statement of Proposition 3.2 extends easily to the case $T = \infty$. The proof of the equivalence of the statements (a) and (c) below requires an obvious reformulation of Proposition 3.1 for Laplace transforms and is omitted; see also B. Bäumer and F. Neubrander [5].

Proposition 3.3. *Let A be closed, $k \in L^1_{loc}([0,\infty);\mathbb{C})$ be a regularizing function, $x \in X$, and $v \in L^1_{loc}([0,\infty);X)$. Then the following are equivalent.*

(a) *$v(t) = A\int_0^t v(s)\,ds + (1*k)(t)x$ for almost all $t \geq 0$.*

(b) *There exists a function r such that $(\lambda I - A)r(\lambda) - x \approx_T 0$ for all $T > 0$ and $\hat{k}r \in \{v\}$.*

If, in addition, v is Laplace transformable (i.e., $1*v$ is exponentially bounded), then (a) is equivalent to

(c) *There exists a function r such that $(\lambda I - A)r(\lambda) - x = 0$ and $\hat{k}r = \hat{v}$.*

The Cauchy problem (ACP) has the uniqueness property on $[0,T]$ if $v \equiv 0$ is the only continuous function $[0,T] \to X$ for which $v(t) = A\int_0^t v(s)$ for all $t \in [0,T]$. We say that (ACP) has the uniqueness property if it has the uniqueness property on $[0,T]$ for all $T > 0$.

Theorem 3.4. *Let A be closed and $0 < T < \infty$. Then the following are equivalent.*

(a) *(ACP) has the uniqueness property on $[0,T]$.*

(b) *If r is analytic and bounded on some half-plane $\{\lambda \in \mathbb{C} : Re\lambda > \omega\}$ then $(\lambda I - A)r(\lambda) \approx_T 0$ implies that $r \approx_T 0$.*

Proof. Suppose (a) holds, let r be analytic and bounded on some right half-plane and $(\lambda I - A)r(\lambda) \approx_T 0$. Then, as seen in Remark 2.7, $q(\lambda) = \frac{1}{\lambda}r(\lambda)$ is the Laplace transform of a continuous function v, $(\lambda I - A)q(\lambda) \approx_T 0$ and $q \in \{v\}$. It follows from Proposition 3.3 that $v(t) = A\int_0^t v(s)\,ds$ and therefore $v(t) = 0$ for all $t \in [0,T]$. By (2.3), $q \approx_T 0$. Thus $r \approx_T 0$.
Suppose (b) holds. Let v be a continuous function such that $v(t) = A\int_0^t v(s)\,ds$ for all $t \in [0,T]$. By (3.2), $(\lambda I - A)\hat{v}_T(\lambda) \approx_T 0$. Thus, $\hat{v}_T \approx_T 0$. It follows from Proposition 2.2 that $v(t) = 0$ for all $t \in [0,T]$. ☐

Corollary 3.5. [Lyubich's Uniqueness Theorem] *If the resolvent $R(\lambda, A)$ exists for $\lambda > \omega$ and $\limsup_{\lambda\to\infty} \frac{1}{\lambda}\ln\|R(\lambda,A)\| \leq 0$, then (ACP) has the uniqeness property.*

Proof. If $a(\lambda) := (\lambda I - A)r(\lambda) \approx_T 0$ $(\lambda > \omega)$, then $r(\lambda) = R(\lambda, A)a(\lambda) \approx_T 0$. Now the statement follows from Theorem 3.4. □

The Cauchy problem (ACP) is said to be wellposed on $[0,T]$ (or $[0,\infty)$) if there exists a regularizing function k (whose Laplace transform has at most finitely many real zeros) such that, for all $x \in X$, (ACP) has a unique, k-generalized (mild) solution $u \in C^{[k]}([0,T];X)$ (or $u \in C^{[k]}([0,T];X)$).

Proposition 3.6. *If (ACP) is wellposed, then the resolvent $R(\lambda, A)$ exists for $\lambda > \omega$ and $\limsup_{\lambda\to\infty} \frac{1}{\lambda} \ln \|R(\lambda, A)\| \le 0$.*

Proof. Suppose there exists a regularizing function k such that, for all $x \in X$, there exists a unique, continuous function $v : [0,\infty) \to X$ such that $v(t) = A\int_0^t v(s)\,ds + (1 * k)(t)x$ for all $t \in [0,T]$. For $t \ge 0$ define linear operators $S(t)$ by $S(t)x = v(t)$ and $S : X \to C([0,T];X)$ by $x \to S(\cdot)x$. Then S is closed and everywhere defined. Thus S is bounded; i.e., there exists $M > 0$ such that $\|Sx\| = \sup_{0\le t\le T}\|S(t)x\| \le M\|x\|$ for all $x \in X$.[19] It follows from Proposition 3.1 that $\hat{S}_T(\lambda)x \in D(A)$ for all $x \in X$ and $\lambda \in \mathbb{C}$ and $\|\hat{S}_T(\lambda)x\| \le MT\|x\|$ for $x \in X$ and $Re\lambda > 0$. Moreover, $(\lambda I - A)\hat{S}_T(\lambda)x = (\lambda I - A)\int_0^T e^{-\lambda t}S(t)x\,dt = \lambda\hat{S}_T(\lambda)x - A[e^{-\lambda T}\int_0^T S(s)x\,ds + \lambda\int_0^T e^{-\lambda t}\int_0^t S(s)x\,ds] = \lambda\hat{S}_T(\lambda)x - e^{-\lambda T}S(T)x + e^{-\lambda T}(1*k)(T)x - \lambda\int_0^T e^{-\lambda t}(S(t)x - (1*k)(t)x)\,dt = -e^{-\lambda T}S(T)x + e^{-\lambda T}(1*k)(T)x + \lambda\int_0^T e^{-\lambda t}(1*k)(t)x\,dt = -e^{-\lambda T}S(T)x + \int_0^T e^{-\lambda t}k(t)\,dtx = -e^{-\lambda T}S(T)x + \hat{k}(\lambda)x - \int_T^\infty e^{-\lambda t}k(t)\,dtx$. Thus,

$$(\lambda I - A)\frac{1}{\hat{k}(\lambda)}\hat{S}_T(\lambda)x = (I - B(\lambda))x, \tag{3.3}$$

where $B(\lambda) = \frac{1}{\hat{k}(\lambda)}[e^{-\lambda T}S(T) + \int_T^\infty e^{-\lambda t}k(t)\,dt]$. Let $\lambda_0 \in \mathbb{C}$ such that $\hat{k}(\lambda_0) \ne 0$ and $(\lambda_0 I - B(\lambda_0))^{-1}$ exists. Suppose there exists $x_0 \in X$ such that $(\lambda_0 I - A)x_0 = 0$. Then $u(t) = e^{\lambda_0 t}x_0$ solves (ACP) and $S(t)x_0 = v(t) = (k * u)(t) = (k(t) * e^{\lambda_0 t})x_0$. By (3.3), $(I - B(\lambda_0))x_0 = (\lambda_0 I - A)\frac{1}{\hat{k}(\lambda_0)}\hat{S}_T(\lambda_0)x_0 = \frac{1}{\hat{k}(\lambda_0)}\hat{S}_T(\lambda_0)(\lambda_0 I - A)x_0 = 0$. Therefore, $x_0 = 0$. This shows that the set $\Omega := \{\lambda \in \mathbb{C} : \hat{k}(\lambda) \ne 0 \text{ and } (I - B(\lambda))^{-1} \text{ exists}\}$ is contained in the resolvent set $\rho(A)$ and, for all $\lambda \in \Omega$,

$$R(\lambda, A) = \frac{1}{\hat{k}(\lambda)}\hat{S}_T(\lambda)(I - B(\lambda))^{-1}.$$

By (2.3), $\limsup_{\lambda\to\infty}\frac{1}{\lambda}\ln\|B(\lambda)\| \le -T$. Thus, there exists $\omega > 0$ such that $\|B(\lambda)\| \le 1/2$ and $\hat{k}(\lambda) \ne 0$ for all $\lambda > \omega$. Therefore, $(\omega, \infty) \subset \rho(A)$, $\|R(\lambda, A)\| \le \frac{2MT}{|\hat{k}(\lambda)|}$, and thus $\limsup_{\lambda\to\infty}\frac{1}{\lambda}\ln\|R(\lambda, A)\| \le 0$.[20] □

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[19] This shows that existence and uniqueness of k-generalized mild solutions of $u'(t) = Au(t)$ $(t \in [0,T])$, $u(0) = x$, for all $x \in X$ implies their continuous dependence on the initial data; i.e., $\|u\|_{[k]} = \sup_{0\le t\le T}\|(k * u)(t)\| = \sup_{0\le t\le T}\|v(t)\| \le M\|x\|$ for all $x \in X$.

[20] If $1/\hat{k}(\lambda)$ is sufficiently well-behaved, then Ω contains a post-sectorial region Ω_Ψ because

Many authors have considered the abstract Cauchy problem (ACP) for operators whose resolvents exist on a post-sectorial region $\Omega_\Psi := \{\lambda : \ Re\lambda \geq \beta > 0, \ |Im\lambda| \leq \Psi(Re\lambda)\}$ and have less than exponential growth; i.e., there exists a regularizing function k_0 as defined in (2.8))[21] such that $\sup_{\lambda\in\Omega_\Psi} \|\lambda\hat{k}_0(\lambda)R(\lambda, A)\| < \infty$.[22] The asymptotic Laplace transform methods developed in the first section, especially Theorem 2.8, provide a useful, additional tool for a fine, detailed analysis of the degree of wellposedness for such Cauchy problems. We demonstrate this by considering post-sectorial regions Ω_Ψ, where either $\Psi(x) = e^{ax^\gamma}$ $(a > 0, \gamma > 1)$, $\Psi(x) = e^{ax}$ $(a > 0)$, or $\Psi(x) = x^{1/a}$ $(0 < a < 1)$. With the asymptotic representation results from Section 2 at hand (Theorem 2.8, Corollary 2.9) one obtains immediately the (well known) fact that in these cases the abstract Cauchy problem (ACP) is k-wellposed on $[0, T]$ for all $0 < T \leq \infty$. Furthermore, we show that $k = k_0 * k_{\Psi,T}$, where the regularizing function $k_{\Psi,T}$ is explicitely given in terms of Ψ and T. By specifying k in this manner, one gains a better understanding of how the regularity (or irregularity) of the generalized solutions is affected by the boundary of the resolvent set and the growth of the resolvent, and, how a larger time interval $[0, T]$ leads to a decay of regularity of the solutions.

Theorem 3.7. *Let A be a linear operator on a Banach space X such that $\Omega_\Psi \subset \rho(A)$ and $\sup_{\lambda\in\Omega_\Psi} \|\lambda\hat{k}_0(\lambda)R(\lambda, A)\| < \infty$ for some regularizing function k_0.*

- (a) *Let $\Psi(x) = e^{ax^\gamma}$ $(a > 0, \gamma > 1)$. Then, for all $b > 0$, the abstract Cauchy problem (ACP) is $(k_b * k_0)$-wellposed on $[0, \infty)$, where $k_b(t) = t^{b-1}/\Gamma(b)$.*
- (b) *Let $\Psi(x) = e^{ax}$ $(a > 0)$. Then,*
  1. *for all $b > 0$ and $0 < T < ab$, the abstract Cauchy problem (ACP) is $(k_b * k_0)$-wellposed on $[0, T]$, where $k_b(t) = t^{b-1}/\Gamma(b)$, and,*
  2. *for all $b > 0$, $0 < \delta < 1$, the abstract Cauchy problem (ACP) is $(k_{b,\delta} * k_0)$-wellposed on $[0, \infty)$, where $\hat{k}_{b,\delta}(\lambda) = e^{-b\lambda^\delta}$.*
- (c) *Let $\Psi(x) = x^{1/a}$ $(0 < a < 1)$. Then,*
  1. *for all $b > 0$ and $0 < T < b\cos(a\pi/2)$, the abstract Cauchy problem (ACP) is $(k_b * k_0)$-wellposed on $[0, T]$, where $\hat{k}_b(\lambda) = e^{-b\lambda^a}$, and,*
  2. *for all $b > 0$, $a < \gamma < 1$, the abstract Cauchy problem (ACP) is $(k_{b,\gamma} * k_0)$-wellposed on $[0, \infty)$, where $\hat{k}_{b,\gamma}(\lambda) = e^{-b\lambda^\gamma}$.*

$\|B(\lambda)\| \leq 1/2$ for all $\lambda \in \Omega_\Psi$. Moreover, the estimate $\|\hat{k}(\lambda)R(\lambda, A)\| \leq 2MT$ holds for all $\lambda \in \Omega_\Psi$. See, for example, I. Cioranescu and G. Lumer [10], [11], [12].

[21] As the proof below reveals one can also take $k_0(t) = \delta(t)$; i.e., $\hat{k}_0(\lambda) = 1$.

[22] See, for example, W. Arendt, O. El-Mennaoui and V. Keyantuo [1], R. Beals [6], J. Chazarain [9], I. Cioranescu and G. Lumer [10], [11], [12], H. Emamirad [17], H. Komatsu [18], R. deLaubenfels [19], and the references therein.

Proof. It follows from $\limsup_{\lambda\to\infty} \frac{1}{\lambda} \ln |\hat{k}(\lambda)| = 0$ that $\limsup_{\lambda\to\infty} \frac{1}{\lambda} \ln \|R(\lambda, A)\| \leq 0$. Thus, by Corollary 3.5, (ACP) has the uniqueness property. Because $(\lambda I - A)R(\lambda, A)x - x = 0$ for all $x \in X$ and $\lambda \in \Omega_\Psi$, the statements follow immediately from Corollary 2.9 (setting $q(\lambda) = \hat{k}_0(\lambda)R(\lambda, A)x$) and the Propositions 3.2 and 3.3. □

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