Norm Continuity of c_0–Semigroups

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Abstract:

We show that a positive semigroup T_t on $L_p(\Omega, \nu)$ with generator A and $\|R(\alpha+i\beta)\| \to 0$ for $|\beta| \to \infty$ and some $\alpha \in \mathbb{R}$ is continuous in the operator norm for $t > 0$. The proof is based on a criterion for norm continuity in terms of "smoothing properties" of certain convolution operators on general Banach spaces and an extrapolation result for the L_p–scale, which may be of independent interest.

1 Introduction

C_0–semigroups T_t which are continuous in the operator norm of a Banach space for $t > 0$ have a number of interesting properties which do not hold for general c_0–semigroups, e. g.

- the spectral mapping theorem holds, i. e.

$$\sigma(T_t) \setminus \{0\} = e^{t\sigma(A)}$$

  where A is the generator of T_t ([3] Theorem 2.19)

- as a consequence of this, the Liapunov stability theorem holds, i. e. if the right halfplane belongs to the resolvent set of A, then every mild solution $y(t) = T_t x, x \in X$, of the Cauchy problem

$$y'(t) = Ay(t), \quad y(0) = x$$

  is exponentially stable

- the semigroup operators T_t are compact if and only if the Resolvent operators $R(\lambda, A)$ are compact (see [7] Chapter 2 Theorem 3.3.).

It is therefore important to find natural conditions on the resolvent of A that guarantee that T_t is norm continuous. Since

$$R(\lambda, A) = \int_0^\infty e^{-\lambda t} T_t dt, \quad \lambda > \omega(T_t)$$

it follows from the Riemann–Lebesgue lemma that $\|R(\alpha + i\beta, A)\| \to 0$ for $|\beta| \to \infty$ and $\alpha > \omega(T_t)$ is a necessary condition for norm continuity (see [7] Chapter 2 Theorem 3.6.). It was shown by P. You [10] and with a new proof in [5] that in a Hilbert space this necessary condition is also sufficient, but it seems to be an open question whether this is also true in general Banach spaces.

In this paper we give a partial positive answer: If T_t is a positive semigroup on $L_p(\Omega, \nu), 1 < p < \infty$, then the condition $\|R(\alpha + i\beta, A)\| \to 0$ for $|\beta| \to \infty$ for some $\alpha > \omega(T_t)$ implies indeed the norm continuity of T_t (Theorem 3.3. in section 3). The proof uses a characterization of norm continuity in terms of a smoothing property of the convolution operator

$$Kf(t) = \int_0^t T_{t-s}(f(s))ds \tag{1}$$

(see Def. 2.1) on $L_p([0, \tau], X)$, which we also use to give a simple proof of the Hilbert space result quoted above (Theorem 2.4 and Corollary 2.5 in section 2). As a further preparation for the proof of our main result, we prove an extrapolation result for the convolution operator (1), (Proposition 3.1) which allows us to reduce the L_p–case to the L_2–case and which may be of independent interest.

Some further characterizations of norm continuity of a semigroup may be found in [4] and [6].

We use standard notation: X denotes a Banach space and $B(X)$ the bounded linear operators on X with the operator norm. U_X stands for the unit ball of X.

2 Norm continuity and convolution operators

For every c_0–semigroup T_t on a Banach space X we can define a convolution operator $Kf(t) = \int_0^t T_{t-s}(f(s))ds, t \leq \tau$, on $L_p([0, \tau], X)$. We will characterize

the norm continuity of T_t by the following "smoothness" property of K:
2.1. Definition: A bounded operator K on $L_p([0,\tau],X)$ satisfies the Riesz criterion $(R_p), 1 \le p < \infty$, if

$$(R_p) \qquad \int_0^\tau \|(Kf)(t+h) - (Kf)(t)\|^p dt \to 0 \qquad \text{for } h \to 0$$

uniformly for $f \in L_p([0,\tau],X)$ with $\|f\| \le 1$ (and Kf extended periodically from $[0,\tau]$ to $\mathbb{R}$).

2.2. Remarks: On the Riesz condition (R_p):

a) If $\dim X = \infty$ condition (R_p) does not imply that K is a compact operator on $L_p([0,\tau],X)$. But the proof of the Fréchet–Kolmogorov theorem on compactness in $L_p([0,\tau])$ (see e. g. [1] 2.26) shows that K has (R_p) if and only if K is "almost smoothing" in the following sense: For all $\varepsilon > 0$ there is a $M < \infty$ such that

$$(S_p) \qquad K(U_{L_p(X)}) \subset M \cdot U_{W_p^1(X)} + \varepsilon U_{L_p(X)}$$

b) The set of operators on $L_p([0,\tau],X)$ with (R_p) is a closed right sided ideal in $B(L_p([0,\tau],X))$.

c) If $S_h f(s) = f(s+h)$ denotes the shift by h for $f \in L_p([0,\tau],X)$ (again f is extended periodically), then (R_p) can be reformulated as

$$\|(I - S_h)K\|_{L_p([0,\tau],X)} \to 0 \qquad \text{for } h \to 0.$$

Hence condition (R_p) "interpolates".

Sketch of proof for a): It is clear that (S_p) implies (R_p) since

$$|g(t+h) - g(t)|^p \le h^{p/p'} \int_t^{t+h} |g'(s)|^p dt.$$

On the other hand, given (R_p) we choose a $\varphi \in C^\infty(0,\tau)$ with $\varphi \ge 0$, $\int \varphi(t)dt = 1$ and put $\varphi_\varepsilon(t) = 1/\varepsilon\varphi(t/\varepsilon)$. The operators $K_\varepsilon f = \varphi_\varepsilon * (Kf)$ map $L_p([0,\tau],X)$ into $W_p^1([0,\tau],X)$ and since

$$K_\varepsilon f(t) - Kf(t) = \int_{|u|\le\varepsilon} \varphi_\varepsilon(u)[Kf(t-u) - Kf(t)]du$$

we get from (R_p) that for $\varepsilon \to 0$

$$\|K_\varepsilon f - Kf\|_{L_p} \le \sup_{|h|\le\varepsilon} \|Kf(\cdot + h) - Kf\|_{L_p} \to 0.$$

<u>2.3. Example:</u> Let $(t,s) \in [0,\tau]^2 \to k(t,s) \in B(X)$ be a Bochner measurable, operator valued kernel with

$$\int \left(\int \|k(t,s)\|_{B(X)}^{p'} ds \right)^{p/p'} dt < \infty, \quad 1 < p < \infty, \frac{1}{p} + \frac{1}{p'} = 1.$$

Then the operator K on $L_p([0,\tau], X)$ given by

$$Kf(t) = \int_0^\tau k(t,s)(f(s))ds$$

always satisfies the Riesz condition (R_p).
<u>Proof:</u> For $\|f\|_{L_p(\mathbb{R},X)} \le 1$

$$\int_0^{\tau-h} \|Kf(t+h) - Kf(t)\|^p dt \le \int_0^{\tau-h} \left(\int_0^\tau \|k(t+h,s) - k(t,s)\|^{p'} ds \right)^{p/p'} dt \to 0$$

for $h \to 0$, and also

$$\int_A \|Kf(t)\|^p dt \le \int_A \left(\int \|k(t,s)\|^{p'} ds \right)^{p/p'} dt \cdot \|f\|^p_{L_p([0,\tau],X)} \to 0$$

for $A \subset [0,\tau]$ with $m(A) \to 0$. $\square$

<u>2.4. Theorem:</u> A c_0–semigroup T_t on a Banach space X is continuous in the operator norm for $t > 0$ if and only if the operator $K : L_p([0,\tau], X) \to L_p([0,\tau], X)$, $Kf(t) = \int_0^t T_{t-s}(f(s))ds$ satisfies the Riesz condition (R_p) for some (all) $p \in (1,\infty)$.

<u>Proof:</u> If T_t is continuous in the operator norm, then we can write K as $Kf(t) = \int_0^\tau k(t,s)f(s)ds$, where the kernel $k(t,s) = T_{t-s}$ for $0 \le s \le t \le \tau$ and

$k(t,s) = 0$ otherwise satisfies the assumption of Example 2.3. Hence K has (R_p).
To deduce norm continuity from (R_p) put

$$\Delta_h(t)x = (t+h)T_{t+h}x - tT_tx. \tag{1}$$

Observe that for $t+h \le \tau$

$$\begin{aligned}\Delta_h(t)x &= \int_0^{t+h} T_{t+h-s}(T_sx)ds - \int_0^t T_{t-s}(T_sx)ds \\ &= Kf(t+h) - Kf(t)\end{aligned}$$

with $f(s) = T_sx, \|f\|_{L_p([0,\tau],X)} \le \tau^{\frac{1}{p}} C\|x\|$ with $C = \sup\{\|T_s\| : 0 \le s \le \tau\}$. Now (R_p) implies that

$$\int_0^{\tau/2} \|\Delta_h(t)x\|^p dt \to 0 \quad \text{for } h \to 0^+ \tag{2}$$

uniformly for $x \in U_X$.
To relate this condition to the continuity of T_t, we use

$$\begin{aligned}\int_0^t T_{t-s}(\Delta_h(s)x)ds &= \int_0^t [(s+h)T_{t+h}x - sT_tx]ds \\ &= \frac{1}{2}(t+h)^2 T_{t+h}x - \frac{1}{2}t^2T_tx - \frac{1}{2}h^2T_{t+h}x\end{aligned}$$

Hence for $\|x\| \le 1$ and $t \le \tau/2$

$$\begin{aligned}\|(t+h)^2T_{t+h}x - t^2T_tx\| &\le 2\int_0^t \|T_{t-s}(\Delta_h(s)x)\|ds + h^2\|T_{t+h}\| \\ &\le 2C(\tau/2)^{1/p'} \left(\int_0^{\tau/2} \|\Delta_h(s)x\|^p ds\right)^{1/p} + h^2C.\end{aligned}$$

Now (2) implies that t^2T_t is norm continuous on $[0,\tau/2]$ and the claim follows. □

The criterion 2.4. leads to a short proof of the follwing Hilbert space result originally due to P. You.

<u>2.5. Corollary:</u> ([5], [10]) Let X be a Hilbert space. Then a c_0–semigroup T_t with generator A is continuous in the operator norm for $t>0$ if and only if for some $\alpha > s(A)$

$$\|R(\alpha + i\beta, A)\| \to 0 \quad \text{for } |\beta| \to \infty.$$

In the following proof we will use that for a Hilbert space X the Fouriertransform

$$\hat{f}(\beta) = \int e^{-i\beta t} f(t)dt$$

multiplied by $\dfrac{1}{\sqrt{2\pi}}$ defines an isometry on $L_2(\mathbb{R}, X)$. Therefore, for a convolution operator

$$Lf(t) = \int l_{t-s}(f(s))ds \text{ on } L_2(\mathbb{R}, X)$$

where $t \to l_t \in B(X)$ is strongly integrable and $\int \|l_t\|dt < \infty$ we have $t \to \hat{l}_t \in B(X)$ and $\widehat{Lf}(\beta) = (l_t)\hat{}(\beta)(\hat{f}(\beta))$ so that

$$\|L\|_{L_2(\mathbb{R},X)} = \sup_{\beta\in\mathbb{R}} \|(l_t)\hat{}(\beta)\|. \tag{3}$$

<u>Proof:</u> Without loss of generality we can assume that $\omega(T_t) < 0$. We can also choose $\alpha = 0$, since by the resolvent equation

$$R(\alpha + i\beta, A) = [I + (\alpha' - \alpha)R(\alpha + i\beta, A)]R(\alpha' + i\beta, A)$$

for all $\omega(T_t) < \alpha < \alpha'$ and the term [ ] is uniformly bounded in $\beta \in \mathbb{R}$.
It was already checked in [7], Chapter 2, Theorem 3.6. that our assumption on the resolvent is necessary.
On the other hand, since $R(i\beta, A) = (T_t)\hat{}(\beta)$, where $T_t = 0$ for $t < 0$, we may think of $\quad Kf(t) = \int_{-\infty}^{\infty} T_{t-s}(f(s))ds \quad$ as a Fouriermultiplier on $L_2(\mathbb{R}, X)$ with multiplier function $R(i\beta, A)$, i. e.

$$\widehat{Kf}(\beta) = R(i\beta, A)[\hat{f}(\beta)].$$

To see that the restriction $K_\tau = \chi_{[0,\tau]}K\chi_{[0,\tau]}$ of K to $L_2([0,\tau], X)$ satisfies (R_2) we split K for every $n \in \mathbb{N}$ into two convolution operators $K = K_n + L_n$ defined by

$$\begin{aligned}
(K_n f)\hat{}(\beta) &= \varphi(\beta/n)R(i\beta, A)[\hat{f}(\beta)] \\
(L_n f)\hat{}(\beta) &= [1 - \varphi(\beta/n)]R(i\beta, A)[\hat{f}(\beta)]
\end{aligned}$$

where φ is C^∞ with support in [-2,2], $\varphi \equiv 1$ on [-1,1] and $|\varphi| \leq 1$. By (3), our assumptions imply now that $\|L_n\|_{L_2(X)} \to 0$ for $n \to \infty$. The operators K_n are of the form $K_n f(t) = \int k_n(t,s) f(s) ds$ with a kernel

$$k_n(t,s) = [\varphi(\cdot/n) \cdot R(i\cdot, A)]^\vee (t-s)$$

that is Bochner integrable and bounded in $B(X)$.
Therefore, the restriction $K_{\tau,n} = \chi_{[0,\tau]} K_n \chi_{[0,\tau]}$ of K_n to $L_2([0,\tau], X)$ satisfies the assumptions of Example 2.3. with $p=2$ and we can conclude that $K_{\tau,n}$ has property (R_2). Since $\|K_\tau - K_{\tau,n}\| \leq \|L_n\| \to 0$ for $n \to \infty$ it follows from 2.2. b) that K_τ has (R_2). Now the claim follows from the last theorem. $\square$

3 Positive semigroups on $L_p(\Omega, \nu)$

In order to extend the characterization of norm continuous semigroups on Hilbert spaces (Corollary 2.5) to positive semigroups on $L_p(\Omega,\nu)$ we need an extrapolation result for positive operators. (Proposition 3.1 (below)).
It is known (Theorem 2.1 of [9]) that for a positive operator $R : L_p(\Omega,\nu) \to L_p(\Omega,\nu)$ ν σ–finite, there is a density $0 < g \in L_1(\Omega,\nu)$ such that

(1) $$\tilde{R} = J^{-1} R J : L_p(\Omega, \tilde{\nu}) \to L_p(\Omega, \tilde{\nu}),$$

where $d\tilde{\nu} = g d\nu$ and $Jf = g^{1/p} f$ is an isometry of $L_p(\Omega,\tilde{\nu})$ onto $L_p(\Omega,\nu)$, extends to a bounded operator on $L_q(\Omega,\tilde{\nu})$ and

(2) $$\|R\|_{L_q(\tilde{\nu})} \leq 2 \|R\|_{L_p(\nu)} \quad \text{for all } 1 \leq q \leq \infty.$$

It seems to be an open problem, whether it is possible to extrapolate even a positive c_0–semigroup T_t on $L_p(\Omega,\nu)$ in the same way. The next result shows that one can at least extrapolate the convolution operator

$$Kf(t) = \int_0^t T_{t-s}(f(s)) ds$$

on $L_p(\mathbb{R}, L_p(\Omega,\nu))$ to the whole $L_q(L_q)$–scale.

3.1. Theorem: Let T_t be a positive c_0–semigroup on $L_p(\Omega,\nu)$ with $\omega(T_t) < 0$. Then there exists a density $0 < g \in L_1(\Omega,\nu)$ and an isometry $Jf = g^{1/p} f$ of $L_p(\Omega,\tilde{\nu})$ onto $L_p(\Omega,\nu)$ (where $d\tilde{\nu} = g d\nu$) such that

i) The operators $J^{-1}R(\lambda, A)J, Re\lambda \geq 0$, extend to operators on $L_q(\Omega, \tilde{\nu})$ and there is a constant C with $\|J^{-1}R(\lambda, A)J\|_{L_q(\tilde{\nu})} \leq C$ for all $Re\lambda \geq 0$ and $1 \leq q \leq \infty$.

ii) The operator $\tilde{K}f(t) = \int\limits_0^t J^{-1}T_{t-s}J(f(s))ds, \quad t > 0$, extends to a bounded operator on $L_q(\mathbb{R}_+, L_q(\Omega, \tilde{\nu}))$ and there is a $C<\infty$ with $\|\tilde{K}\|_{L_q(L_q)} \leq C$ for all $1 \leq q < \infty$.

iii) For all $f \in L_2(\mathbb{R}_+, L_2(\Omega, \tilde{\nu}))$ we have

$$(\tilde{K}f)\hat{}(\beta) = J^{-1}R(i\beta, A)J(\hat{f}(\beta)).$$

For the proof we need the following result on convolution operators from [8]:

<u>3.2. Lemma:</u> For a fixed $1 \leq q < \infty$, let $t \in \mathbb{R} \to k(t)$ be a function of positive operators on $L_q(\Omega, \nu)$ such that $t \to k(t)f$ is locally Bochner integrable for $f \in L_q(\Omega, \nu)$. Assume that for all $0 \leq h \in L_q(\Omega, \nu)$ and $0 \leq g \in L_{q'}(\Omega, \nu)$ we have

$$\int\limits_{\mathbb{R}} < g, k(t)h > dt \leq C\|g\|_{L_q'}\|h\|_{L_q}.$$

Then the convolution integral

$$Kf(t) = \int\limits_{-\infty}^{\infty} k(t-s)(f(s))ds$$

defined for stepfunctions $f : \mathbb{R} \to L_q(\Omega)$ extends to a bounded operator on $L_q(\Omega, L_r(\mathbb{R}))$ for all $1 \leq r < \infty$ and $\|Kf\|_{L_q(L_r)} \leq C\|f\|_{L_q(L_r)}$.

<u>Proof of 3.1:</u> We apply the extrapolation result (1) to $R = R(-2\varepsilon, A)$ for some ε with $\omega(T_t) < -2\varepsilon < 0$ where A is the generator of T_t. The operators $J^{-1}T_tJ$ may not extend to $L_q(\Omega, \tilde{\nu})$ for all q but we can get around this difficulty by using the Yosida approximation of A and T_t:

$$\begin{aligned} A_\mu &= \mu^2 R(\mu, A) - \mu, \quad \mu > 0 \\ T_{\mu,t} &= \exp(tA_\mu) \end{aligned}$$

$\tilde{A}_\mu = J^{-1}A_\mu J$ and $\tilde{T}_{\mu,t} = J^{-1}T_{\mu,t}J$ extend to bounded operators on $L_q(\Omega, \tilde{\nu})$ for all $1 \leq q \leq \infty$ since $R(\mu, A) \leq R$ for $\mu \geq 0$.

It follows from [7], Chapter 1, Lemma 7.2, that

$$
\begin{aligned}
(3) \qquad R(-\varepsilon,\tilde{A}_\mu) &= (\mu-\varepsilon)^{-1}(\mu I-\tilde{A})R\left(\frac{-\mu\varepsilon}{\mu-\varepsilon},\tilde{A}\right)\\
&= (\mu-\varepsilon)^{-1}I+\left(\frac{\mu}{\mu-\varepsilon}\right)^2 R\left(\frac{-\mu\varepsilon}{\mu-\varepsilon},\tilde{A}\right)\\
&\leq I+4R(-2\varepsilon,\tilde{A})
\end{aligned}
$$

for $\mu \geq \mu_0 := \max(1+\varepsilon, 2\varepsilon)$. Since the semigroups $T_{\mu,t}$ are positive, it follows by (2) that for $\mu \geq \mu_0$, $\lambda \geq -\varepsilon$ and all $1 \leq q \leq \infty$

$$
\begin{aligned}
(4) \qquad \|R(\lambda,\tilde{A}_\mu)\|_{L_q(\tilde{\nu})} &\leq 1+4\|R(-2\varepsilon,\tilde{A})\|_{L_q(\nu)}\\
&\leq 1+8\|R(-2\varepsilon,\tilde{A})\|_{L_p(\nu)} =: C.
\end{aligned}
$$

We will also need an uniform growth estimate for $T_{\mu,t}$ in $L_p(\tilde{\nu})$. Since $\omega(T_t) < -2\varepsilon$ there is a constant D with

$$
(\mu+2\varepsilon)^n\|R(\mu,\tilde{A})^n\|_{L_p(\tilde{\nu})} \leq D \quad \text{for } \mu \geq 0
$$

For $\mu \geq \mu_0$ we obtain then for all $t \geq 0$

$$
\begin{aligned}
(5) \qquad \|T_{\mu,t}\|_{L_p} &\leq e^{-\mu t}\sum_{n=0}^{\infty}\frac{1}{n!}t^n\mu^{2n}\|R(\mu,\tilde{A})^n\|\\
&\leq De^{-\mu t}\sum_{n=0}^{\infty}\frac{1}{n!}t^n\left(\frac{\mu^2}{\mu+2\varepsilon}\right)^n = D\exp\left(\frac{-2\varepsilon\mu}{\mu+2\varepsilon}t\right) \leq De^{-\varepsilon t}
\end{aligned}
$$

Now we consider the convolution integrals

$$
K_\mu f(t) = \int k_\mu(t-s)(f(s))ds
$$

with $k_\mu(t) = \tilde{T}_{\mu,t}$ for $t \geq 0$ and $k_\mu(t) = 0$ for $t < 0$. For all $0 \leq h \in L_q(\Omega,\tilde{\nu}), 0 \leq g \in L_{q'}(\Omega,\tilde{\nu})$ and $\mu \geq \mu_0$ we get by (4)

$$
\begin{aligned}
\int < g, k_\mu(t)h > dt &= \int_0^\infty < g, \tilde{T}_{\mu,t}h > dt\\
&= < g, R(0,\tilde{A}_\mu)h > \leq \|R(0,\tilde{A}_\mu)\|_{L_q(\tilde{\nu})}\|g\|_{L_{q'}}\cdot\|h\|_{L_q}\\
&\leq C\|g\|_{L_{q'}}\cdot\|h\|_{L_q}.
\end{aligned}
$$

Applying Lemma 3.2. to all $L_q(\Omega,\tilde\nu)$, $\quad 1\le q<\infty$, with $r=q$ we get that K_μ is a bounded operator on the spaces $L_q(\mathbb{R},L_q(\Omega,\tilde\nu))$. If we restrict the K_μ's to $L_q(L_q)=L_q(\mathbb{R}_+,L_q(\Omega,\tilde\nu))$ we have for all $1\le q<\infty$ and all $\mu\ge\mu_0$ that

$$\|K_\mu f\|_{L_q(L_q)}\le C\|f\|_{L_q(L_q)} \tag{6}$$

where $t\to f(t)\in L_\infty(\Omega,\tilde\nu)\subset L_q(\Omega,\tilde\nu)$ is a finite step function. Since such step functions are dense in $L_q(L_q)$ it remains to show that

$$\lim_{\mu\to\infty} K_\mu f=\tilde K f \quad \text{ in } L_q(L_q). \tag{7}$$

First we note that (7) holds for $q=p$. Indeed, we have $\lim\limits_{\mu\to\infty}\tilde T_{\mu,s}g=\tilde T_s g$ in $L_p(\Omega,\tilde\nu)$ for all $g\in L_p(\Omega,\tilde\nu)$ and the uniform growth estimate (5). In particular, $K_\mu f$ converges to $\tilde K f$ in measure on $\mathbb{R}\times\Omega$.
Furthermore, for every function $f(t)=\chi_A(t)g$ with a bounded, measurable set $A\subset[0,a]$ and $0\le g\in L_q(\Omega,\tilde\nu)$ the functions $K_\mu f,\mu\ge\mu_0$ are uniformly dominated by a function in $L_q(L_q)$:

$$\begin{aligned}
K_\mu f(t) &= \int_0^t \tilde T_{\mu,s}[\chi_A(t-s)g]ds\\
&= \int_0^\infty e^{-\varepsilon s}\chi_A(t-s)(e^{\varepsilon s}\tilde T_{\mu,s}g)ds\\
&\le e^{-\varepsilon(t-a)}\int_0^\infty e^{\varepsilon s}\tilde T_{\mu,s}g ds = e^{-\varepsilon(t-a)}R(-\varepsilon,\tilde A_\mu)g\\
&\le e^{-\varepsilon(t-a)}[I+4R(-2\varepsilon,\tilde A)]g
\end{aligned}$$

for $\mu\ge\mu_0$ by (3). Lebesgue's convergence theorem and (6) imply now (7) and $\|\tilde K f\|\le C\|f\|$.
iii) For a finite step function $t\to f(t)\in L_p(\Omega,\tilde\nu)\cap L_2(\Omega,\tilde\nu)$ we have that $\beta\to(\tilde K f)\hat{}(\beta)$ and $\beta\to J^{-1}R(i\beta,A)J(\hat f(\beta))$ belong to $L_\infty(L_p)\cap L_2(L_2)$ and are equal a. e. by the definition of $\tilde K$. Since such step functions are dense in $L_2(L_2)$, iii) follows from the boundedness of the Fouriertransform on $L_2(L_2)$ and i). $\square$

Finally we are prepared to prove our main result on normcontinuity in $L_p(\Omega,\nu)$.

<u>3.3. Theorem:</u> Let T_t be a positive semigroup on $L_p(\Omega,\nu)$ for $1<p<\infty$ with generator A. Then T_t is continuous in the operator norm for $t>0$ if and only

if for some $\alpha > s(A)$

(8) $$\|R(\alpha + i\beta, A)\| \to 0 \quad \text{for } |\beta| \to \infty.$$

3.4. Remark: As noted in the introduction, 3.3 implies that for a positive semigroup on $L_p(\Omega, \nu)$ (8) is sufficient for $s(A) = \omega(T_t)$ and for the spectral mapping theorem to hold.

Proof of 3.3.: We may assume that $\omega(T_t) < 0$ and, by Theorem 3.1, that the operators $R(\lambda, A)$ extend to $L_q(\Omega, \nu)$ for all $Re\lambda \geq 0$ and all $1 \leq q \leq \infty$. Furthermore, the operator

$$Kf(t) = \int_0^t T_{t-s}(f(s))ds$$

on $L_p(L_p)$ extends to $L_q(L_q)$ for all $1 \leq q < \infty$, where $L_q(L_q) = L_q(\mathbb{R}_+, L_q(\Omega, \nu))$, and the norm of all these operators is bounded by one constant $C < \infty$.
By applying the Riesz interpolation theorem to the resolvent on $L_p(\Omega, \nu)$ and $L_{p'}(\Omega, \nu)$, $\frac{1}{p} + \frac{1}{p'} = 1$, respectively, we obtain

$$\|R(i\beta, A)\|_{L_2(\Omega,\nu)} \leq \|R(i\beta, A)\|^{1/2}_{L_{p'}(\Omega,\nu)} \|R(i\beta, A)\|^{1/2}_{L_p(\Omega,\nu)}$$

Since $\|R(i\beta, A)\|_{L_{p'}(\Omega,\nu)} \leq C$ the assumption gives

$$\|R(i\beta, A)\|_{L_2(\Omega,\nu)} \to 0 \quad \text{for } |\beta| \to \infty.$$

By 3.1. and the proof of the Hilbert space result 2.5 we conclude that K restricted to $L_2([0,1], L_2(\Omega, \nu))$ satisfies condition (R_2). If $S_h f(s) = f(s+h)$ denotes again the (periodic) shift by h on $L_q(L_q) := L_q([0,1], L_q(\Omega, \nu))$ then (R_q) for $X = L_q(\Omega, \nu)$ is equivalent to

$$\|(I - S_h)K\| \to 0 \quad \text{for } h \to 0.$$

Hence, again by interpolation, K satisfies (R_p) on $L_p(L_p)$. Indeed, if $1 < p < 2$, then by [2], Theorem 5.1.2.,

$$\|(I - S_h)K\|_{L_p(L_p)} \leq \|(I - S_h)K\|^{1-\theta}_{L_1(L_1)} \|(I - S_h)K\|^{\theta}_{L_2(L_2)}$$

with $\theta = \frac{2}{p'}$ and $\|(I - S_h)K\|_{L_1(L_1)} \leq 2C$. For $2 < p < \infty$ we argue similarly. An appeal to Theorem 2.4. completes the proof. □

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