ON THE REGULARITY PROPERTIES OF PERTURBED SEMIGROUPS

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Abstract. We give conditions on a strongly continuous semigroup $\mathcal{T}$ and a bounded operator B, such that the perturbed semigroup inherits the regularity properties of $\mathcal{T}$.

1. Introduction

The aim of this paper is to understand why some regularity properties of strongly continuous semigroups persist under bounded perturbation while others do not.

This is a very old problem. In 1953, R.S. Phillips [8] showed that properties like immediate norm continuity or immediate compactness persist under bounded perturbation. In the same article, he showed that this is not the case for eventual norm continuity. In fact, he found a bounded perturbation of the nilpotent shift semigroup on $L^1([0,1])$, which is no longer eventually norm continuous. This counterexample implies that eventual compactness does not persist either since eventually compact semigroups are eventually norm continuous.

The differentiable case remained unsolved until 1995, when M. Renardy found a counterexample [9].

In this paper, we give conditions guaranteeing the permanence of these regularity properties.

To do so, we consider the space $\mathcal{X}$ of all strongly continuous functions from $[0,\infty)$ to $\mathcal{L}(X)$, where X is a Banach space. For functions F and G in $\mathcal{X}$, we define their convolution $F * G$ by

$$F * G\,(t)x := \int_0^t F(t-s)G(s)x\,ds$$

for $t \geq 0$ and $x \in X$.

It is not difficult to prove that $F * G$ is again an element of $\mathcal{X}$ and that the convolution product "$*$" is associative.

In Sections 2, 3 and 4, we show that if both F and G are eventually norm continuous, or compact, or differentiable, respectively, then their convolution has the same property.

Let now $\mathcal{T} := (T(t))_{t\geq 0}$ be a strongly continuous semigroup on X and let $B \in \mathcal{L}(X)$. On the space $\mathcal{X}$, we define an abstract Volterra operator as

$$(V\,F)(t)x := \int_0^t T(t-s)BF(s)x\,ds \tag{1}$$

We thank Klaus-J. Engel, Jan Poland and Frank Räbiger for many stimulating discussions. The second author gratefully acknowledges support by CNR.

for $F \in \mathcal{X}$, $t \geq 0$ and $x \in X$.
It is well known (see also Section 5) that the semigroup $\mathcal{S} := (S(t))_{t\geq 0}$, obtained from $\mathcal{T}$ by perturbing its generator by the operator B, is given by the Dyson-Phillips series

$$\mathcal{S} = \sum_{k=0}^{\infty} V^k \mathcal{T} = \sum_{k=0}^{\infty} [\mathcal{T}B]^k * \mathcal{T},$$

where $[\mathcal{T}B]^k$ is the k-fold convolution of $\mathcal{T}B$ with itself.
Using the results of Sections 2, 3 and 4, we show in Section 6 that if

$$\begin{aligned} Rg\, V^n \quad \subseteq \quad \{F \in \mathcal{X} : F \text{ is immediately norm continuous (or compact} \\ \text{or differentiable for } t \geq 0)\} \end{aligned}$$

for some $n \in \mathbb{N}$ and $\mathcal{T}$ is eventually norm continuous (or compact or differentiable), then the perturbed semigroup $\mathcal{S}$ is also eventually norm continuous (or compact or differentiable).

2. The Convolution in the Space $\mathcal{C}((\alpha, \infty))$

For each $\alpha \geq 0$, we consider the subspace

$$\mathcal{C}((\alpha,\infty)) := \{F \in \mathcal{X} : \ t \mapsto F(t) \text{ is norm continuous for } t > \alpha\}$$

of $\mathcal{X}$. For these spaces, the following lemma holds.

Lemma 2.1. *Let $F, G \in \mathcal{X}$ and $\alpha, \beta > 0$. Then the following implications hold.*

$$(i)\ F \in \mathcal{C}((0,\infty)) \Rightarrow \left.\begin{array}{c} F * G \\ G * F \end{array}\right\} \in \mathcal{C}((0,\infty));$$

$$(ii)\ \left.\begin{array}{c} F \in \mathcal{C}((\alpha,\infty)) \\ G \in \mathcal{C}((\beta,\infty)) \end{array}\right\} \Rightarrow \left.\begin{array}{c} F * G \\ G * F \end{array}\right\} \in \mathcal{C}((\alpha+\beta,\infty)).$$

Proof.
(i) Let $t > 0$, $h > 0$ and $x \in X$. Then we have

$$\begin{aligned} &\lim_{h\searrow 0} \|F * G\,(t+h)x - F * G\,(t)x\| = \\ &\quad = \lim_{h\searrow 0} \left\| \int_0^t (F(t+h-s) - F(t-s))G(s)x\,ds + \int_t^{t+h} F(t+h-s)G(s)x\,ds \right\| \leq \\ &\quad \leq \lim_{h\searrow 0} \int_0^t \|F(t+h-s) - F(t-s)\| \sup_{s\in[0,t]} \|G(s)\|\,\|x\|\,ds + \\ &\qquad + \lim_{h\searrow 0} \int_t^{t+h} \sup_{s\in[0,t]} \|F(s)\| \sup_{s\in[0,t]} \|G(s)\|\,\|x\|\,ds = 0, \end{aligned}$$

uniformly for $\|x\| \leq 1$. So the map $t \mapsto F(t)$ is right continuous in the uniform operator topology for $t > 0$. In a similar way, we can show that it is also left continuous. The same proof holds for $G * F$ using the identity

$$\int_0^t G(t-s)F(s)x\,ds = \int_0^t G(s)F(t-s)x\,ds. \tag{2}$$

(ii) Let $t > \alpha + \beta$, $0 < h < \min\{t - \alpha - \beta, \alpha\}$ and $x \in X$. Then we have

$$\|F * G(t+h)x - F * G(t)x\| \leq \left\| \int_0^t (F(t+h-s) - F(t-s))G(s)x\, ds \right\| \tag{3}$$
$$+ \int_t^{t+h} \sup_{s\in[0,t]} \|F(s)\| \sup_{s\in[0,t]} \|G(s)\| \, \|x\|\, ds. \tag{4}$$

It is obvious that (4) tends to 0 as $h \searrow 0$ uniformly for $\|x\| \leq 1$. So we only have to estimate (3) and obtain

$$(3) \leq \int_0^{t-\alpha} \|F(t+h-s) - F(t-s)\| \sup_{s\in[0,t]} \|G(s)\| \, \|x\|\, ds + \tag{5}$$
$$+ \left\| \int_{t-\alpha}^t (F(t+h-s) - F(t-s))G(s)x\, ds \right\|. \tag{6}$$

We have that (5) tends to 0 as $h \searrow 0$ uniformly for $\|x\| \leq 1$, because $t+h-s > t-s > \alpha$ for $s \in (0, t-\alpha)$. So it remains to estimate (6) which is

$$(6) = \left\| \int_{t-\alpha-h}^{t-h} F(t-s)\, G(s+h)x\, ds - \int_{t-\alpha}^{t} F(t-s)\, G(s)x\, ds \right\| \leq$$
$$\leq \int_{t-\alpha-h}^{t-\alpha} \|F(t-s)\, G(s+h)x\|\, ds + \tag{7}$$
$$+ \int_{t-\alpha}^{t-h} \|F(t-s)\, (G(s+h) - G(s))x\|\, ds + \tag{8}$$
$$+ \int_{t-h}^{t} \|F(t-s)\, G(s)x\|\, ds. \tag{9}$$

It is obvious that (7) and (9) tend to 0 as $h \searrow 0$ uniformly for $\|x\| \leq 1$. The same holds for (8) because $t - \alpha > \beta$. So $F * G$ is norm right continuous on $(\alpha + \beta, \infty)$.
In the same way, one can show that $F * G$ is norm left continuous.
To prove that $G * F \in \mathcal{C}((\alpha + \beta, \infty))$, we again use the identity (2). □

3. The Convolution in the Space $\mathcal{K}((\alpha, \infty))$

In this section, we denote by $K(X)$ the space of all compact operators on X and define, for each $\alpha \geq 0$, the space

$$\mathcal{K}((\alpha, \infty)) := \{F \in \mathcal{X} : \ F(t) \in K(X) \text{ for all } t > \alpha\}.$$

For the study of the convolution of such functions, we need some results about their strong integrals. It is well known that the subspace of compact operators is a closed ideal of $\mathcal{L}(X)$. This implies that integrals, which exists in the operator norm topology, are compact operators again. This is not true in general for integrals in the strong operator topology. However, J. Voigt [10] showed the following theorem.

Theorem 3.1. *Let X, Y be Banach spaces and let $K(X, Y)$ be the space of all compact operators from X to Y. Let (Ω, μ) be a finite measure space and $F : \Omega \longrightarrow K(X, Y)$ be bounded and strongly measurable. Then the strong integral*

$$X \ni x \mapsto \int_\Omega F(s)x\, d\mu(s) \in Y$$

is a compact operator.

We are now ready to give a result about convolution for compact operator valued functions, which is analogous to Lemma 2.1.

Lemma 3.2. *Let $F, G \in \mathcal{X}$ and let $\alpha, \beta \geq 0$. Then the following implications hold.*

$$(i)\ F \in \mathcal{K}((0,\infty)) \Rightarrow \left.\begin{array}{c} F * G \\ G * F \end{array}\right\} \in \mathcal{K}((0,\infty));$$

$$(ii)\ \left.\begin{array}{c} F \in \mathcal{K}((\alpha,\infty)) \\ G \in \mathcal{K}((\beta,\infty)) \end{array}\right\} \Rightarrow \left.\begin{array}{c} F * G \\ G * F \end{array}\right\} \in \mathcal{K}((\alpha+\beta,\infty)).$$

Proof.
The implication (i) follows directly from Theorem 3.1 and from the fact that the subspace of compact operators $K(X)$ is an ideal in $\mathcal{L}(X)$.
For the implication (ii), we take $t > \alpha+\beta$ and notice that if $0 < s < t-\alpha$, then $t-s > \alpha$, and that if $t-\alpha < s < t$, then $s > \beta$. We can now split the integral of the convolution and obtain

$$\begin{aligned} F * G(t)x &= \int_0^t F(t-s)G(s)x\,ds \\ &= \int_0^{t-\alpha} F(t-s)G(s)x\,ds + \int_{t-\alpha}^t F(t-s)G(s)x\,ds. \end{aligned}$$

If we now apply Theorem 3.1, we see that the claim holds. □

4. The Convolution in the Space $\mathcal{C}^1([\alpha,\infty))$

Finally, we look at eventually differentiable functions. For this regularity property, we consider closed intervals of the form $[\alpha,\infty)$, so that we automatically have boundedness of the derivative in each interval $[\alpha,\beta]$. This will be important in Proposition 4.2 and in Section 6. In the case of open intervals (α,∞), we need additional assumptions on the growth of the derivative near α.

As before, we consider, for $\alpha \geq 0$, the space

$$\mathcal{C}^1([\alpha,\infty)) := \{F \in \mathcal{X} : \text{ the map } t \mapsto F(t) \text{ is strongly continuously differentiable for } t \geq \alpha\}.$$

Remark 4.1. *Let $\alpha \geq 0$ and take $F \in \mathcal{C}^1([\alpha,\infty))$. Then for $t \geq \alpha$ we define $F'(t)x := \frac{d}{dt}F(t)x$ for $x \in X$. By the Principle of Uniform Boundedness, we have that $F'(t)$ is a bounded operator on X.*

We can now formulate the main result of this section.

Proposition 4.2. *Let $F, G \in \mathcal{X}$ and $\alpha, \beta > 0$. Then the following implications hold.*

$$(i)\ F \in \mathcal{C}^1([0,\infty)) \Rightarrow \left.\begin{array}{c} F * G \\ G * F \end{array}\right\} \in \mathcal{C}^1([0,\infty)),$$

$(F*G)'(t)x = F(0)G(t)x + F'*G\,(t)x$ *and*

$(G*F)'(t)x = G(t)F(0)x + G*F'(t)x$ *for every* $x \in X$ *and every* $t \geq 0$*;*

$$(ii)\quad \left.\begin{array}{c} F \in \mathcal{C}^1([\alpha,\infty)) \\ G \in \mathcal{C}^1([\beta,\infty)) \end{array}\right\} \Rightarrow \left.\begin{array}{c} F*G \\ G*F \end{array}\right\} \in \mathcal{C}^1([\alpha+\beta,\infty)),$$

$(F*G)'(t)x = F(\alpha)G(t-\alpha)x + \int_0^{t-\alpha} F'(t-s)G(s)x\,ds + \int_{t-\alpha}^t F(t-s)G'(s)x\,ds$ *and*

$(G*F)'(t)x = G(t-\alpha)F(\alpha)x + \int_0^{t-\alpha} G(s)F'(t-s)x\,ds + \int_{t-\alpha}^t G'(s)F(t-s)x\,ds$

for every $x \in X$ *and every* $t \geq \alpha+\beta$*.*

Proof.
(i) Let $x \in X$ and $t > 0$. Then we have

$$\begin{aligned} &\lim_{h\searrow 0} \frac{F*G\,(t+h)x - F*G\,(t)x}{h} = \\ &= \lim_{h\searrow 0} \frac{1}{h}\int_t^{t+h} F(t+h-s)G(s)x\,ds + \\ &\quad \lim_{h\searrow 0} \int_0^t \frac{F(t+h-s)-F(t-s)}{h} G(s)x\,ds = \\ &= F(0)G(t)x + F'*G\,(t)x. \end{aligned}$$

The same holds for $\lim_{h\nearrow 0}$ and proves that $F*G$ is differentiable for $t>0$. The same proof works for $G*F$, if we use the identity (2).
For $t=0$, we have that

$$\lim_{h\searrow 0} \frac{F*G\,(h)x - F*G\,(0)x}{h} = \lim_{h\searrow 0}\frac{1}{h}\int_0^h F(h-s)G(s)x\,ds = F(0)G(0)x,$$

so $F*G$ is right differentiable in 0. An analogous proof works for $G*F$.

(ii) Let now $t > \alpha+\beta$ and $x \in X$. Then we have

$$\lim_{h\searrow 0} \frac{F*G\,(t+h)x - F*G\,(t)x}{h} = \tag{10}$$

$$= \lim_{h\searrow 0} \frac{1}{h}\int_t^{t+h} F(t+h-s)G(s)x\,ds + \tag{11}$$

$$+ \lim_{h\searrow 0} \int_0^t \frac{F(t+h-s)-F(t-s)}{h} G(s)x\,ds. \tag{12}$$

As in the previous step, we see that (11) is equal to $F(0)G(t)x$, so we have to calculate (12) which yields

$$\begin{aligned} &\lim_{h\searrow 0}\int_0^t \frac{F(t+h-s)-F(t-s)}{h}G(s)x\,ds = \\ &\quad = \lim_{h\searrow 0}\int_0^{t-\alpha} \frac{F(t+h-s)-F(t-s)}{h}G(s)x\,ds + \qquad (13)\\ &\quad + \lim_{h\searrow 0}\int_{t-\alpha}^{t} \frac{F(t+h-s)-F(t-s)}{h}G(s)x\,ds. \qquad (14)\end{aligned}$$

We can immediately see that (13) is equal to $\int_0^{t-\alpha} F'(t-s)G(s)x\,ds$, so we calculate (14) which is

$$\begin{aligned} &\lim_{h\searrow 0}\int_{t-\alpha}^{t} \frac{F(t+h-s)-F(t-s)}{h}G(s)x\,ds = \\ &= \lim_{h\searrow 0}\left(\frac{1}{h}\int_{t-\alpha-h}^{t-h} F(t-s)G(s+h)x\,ds - \frac{1}{h}\int_{t-\alpha}^{t} F(t-s)G(s)x\,ds\right) = \\ &= \lim_{h\searrow 0}\frac{1}{h}\int_{t-\alpha-h}^{t-\alpha} F(t-s)G(s+h)x\,ds + \\ &\quad + \lim_{h\searrow 0}\int_{t-\alpha}^{t-h} F(t-s)\frac{G(s+h)-G(s)}{h}x\,ds + \\ &\quad - \lim_{h\searrow 0}\frac{1}{h}\int_{t-h}^{t} F(t-s)G(s)x\,ds \\ &= F(\alpha)G(t-\alpha)x - F(0)G(t)x + \int_{t-\alpha}^{t} F(t-s)G'(s)x\,ds.\end{aligned}$$

So (10) is equal to

$$F(\alpha)G(t-\alpha)x + \int_0^{t-\alpha} F'(t-s)G(s)x\,ds + \int_{t-\alpha}^{t} F(t-s)G'(s)x\,ds.$$

An analogous proof holds for $\lim_{h\nearrow 0}\frac{F*G\,(t+h)x - F*G\,(t)x}{h}$, and we obtain that $F*G$ is differentiable for $t>\alpha+\beta$.

For $t=\alpha+\beta$ the proof is similar to the previous one, and we obtain

$$\begin{aligned} &\lim_{h\searrow 0}\frac{F*G\,(\alpha+\beta+h)x - F*G\,(\alpha+\beta)x}{h} = \\ &\quad = F(\alpha)G(\beta) + \int_0^{\beta} F'(\alpha+\beta-s)G(s)x\,ds + \int_{\beta}^{\alpha+\beta} F(\alpha+\beta-s)G'(s)x\,ds.\end{aligned}$$

Using the identity (2) in Lemma 2.1, we can prove the statement for $G*F$. ☐

Remark 4.3. *Using the identity (2), we can show that in case* (ii) *of the previous lemma the following holds:*

$$\begin{aligned}(F*G)'(t)x &= F(t-\beta)G(\beta)x + \int_0^{t-\beta} F(s)G'(t-s)x\,ds + \int_{t-\beta}^t F'(s)G(t-s)x\,ds \\ &= F(\alpha)G(t-\alpha)x + \int_0^{t-\alpha} F'(t-s)G(s)x\,ds + \int_{t-\alpha}^t F(t-s)G'(s)x\,ds\end{aligned}$$

for all $x \in X$ *and* $t \geq \alpha + \beta$.

5. The Volterra Operator and Bounded Perturbation

After these preparations, we now turn our attention to the bounded perturbation of strongly continuous semigroups. For more details about this subject, see [3], [4], [7].
Let us consider the Volterra operator V on $\mathcal{X}$ defined in (1) and let $t_0 > 0$. Then the norm estimate

$$\|(V^n F)(t)\| \leq \frac{(M\,\|B\|\,t)^n}{n!}\,\|F\|_\infty \tag{15}$$

holds for all $F \in \mathcal{X}$, $n \in \mathbb{N}$ and $t \in [0, t_0]$ whith $M := \sup\limits_{t\in[0,t_0]} \|T(t)\|$.

Consider now the Banach space $\mathcal{X}_{t_0}$ of all strongly continuous functions from $[0, t_0]$ to $\mathcal{L}(X)$ with the norm $\|\cdot\|_\infty$.
Let $V_{t_0} : \mathcal{X}_{t_0} \longrightarrow \mathcal{X}_{t_0}$, $V_{t_0}(F)(t) := V(F_{t_0})(t)$ for $F \in \mathcal{X}_{t_0}$, where $F_{t_0} \in \mathcal{X}$ with $F_{t_0}(t) = F(t)$ for $t \in [0, t_0]$ and $F(t_0)$ otherwise.
Using the norm estimate (15) and the formula for the spectral radius of a bounded operator, we obtain that the Volterra operator V_{t_0} on $\mathcal{X}_{t_0}$ has spectral radius $r(V_{t_0}) = 0$ for every $t_0 > 0$. So we have that the operator $(1 - V)$ is invertible on each space $\mathcal{X}_{t_0}$, and its inverse is given by the Neumann series

$$(1-V)^{-1} = \sum_{k=0}^{\infty} V^k. \tag{16}$$

Define now $\mathcal{S} := (1-V)^{-1}\mathcal{T} = \sum_{k=0}^{\infty} V^k \mathcal{T}$. This means that $\mathcal{S} = \mathcal{T} + V\mathcal{S}$, i.e. $\mathcal{S}$ satisfies the variation of parameters formula

$$S(t)x = T(t)x + \int_0^t T(t-s)BS(s)x\,ds$$

for every $x \in X$ and every $t \geq 0$. So we have that $\mathcal{S}$ is the strongly continuous semigroup obtained from $\mathcal{T}$ by perturbing its generator A by the operator B.
Using the formula (16), we obtain the well known Dyson-Phillips series

$$S(t)x = \sum_{k=0}^{\infty} (V^k \mathcal{T})(t)x, \qquad x \in X,$$

converging uniformly on compact intervals of $\mathbb{R}_+$.
We note that for $F \in \mathcal{X}$ we have

$$\begin{aligned}(VF)(t)x &= \int_0^t T(t-s)BF(s)x\,ds \\ &= (\mathcal{T}B * F)(t)x\end{aligned}$$

and

$$(V^k F) = [\mathcal{T}B]^k * F,$$

where

$$[\mathcal{T}B]^k := \underbrace{\mathcal{T}B * \mathcal{T}B * \ldots * \mathcal{T}B}_{\text{k-times}}.$$

So the Volterra operator and its powers are just convolution operators.

6. Consequences for the regularity properties of perturbed semigroups

We are now interested in the question: "Which regularity properties of the semigroup $\mathcal{T}$ are inherited from the perturbed semigroup $\mathcal{S}$?".
If $\mathcal{T}$ is analytic, then $\mathcal{S}$ is also analytic (see [7], Corollary 3.2.2). The same permanence property holds if $\mathcal{T} \in \mathcal{C}((0,\infty))$ or $\mathcal{T} \in \mathcal{K}((0,\infty))$. The last claim immediately follows from Lemma 2.1 (i) and Lemma 3.2 (i), since we have $\mathcal{S} = \mathcal{T} + (\mathcal{T}B * \mathcal{S})$.
However, this permanence property does not hold in general for eventually norm continuous, eventually compact and immediately differentiable semigroups (see Phillips [8] and Renardy [9]). So the question arises: "Under which conditions does $\mathcal{S}$ inherit eventual norm continuity or (eventual) differentiability?".

Theorem 6.1. *Let $\mathcal{T} := (T(t))_{t\geq 0}$ be a strongly continuous semigroup such that $\mathcal{T} \in \mathcal{C}((\alpha,\infty))$ for some $\alpha > 0$. Let us suppose that there exists $n \in \mathbb{N}$ with $Rg\, V^n \subseteq \mathcal{C}((0,\infty))$. Then the perturbed semigroup $\mathcal{S}$ belongs to $\mathcal{C}((n\alpha,\infty))$.*

Proof.
Applying Lemma 2.1 (i), we can easily prove that $Rg\, V^k \subseteq \mathcal{C}((0,\infty))$ for every $k \geq n$.
Let $t_0 > n\alpha$. In the space $\mathcal{X}_{t_0}$, we have

$$\mathcal{S} = \sum_{k=0}^{\infty} V_{t_0}^k \mathcal{T} = \sum_{k=0}^{n-1} V_{t_0}^k \mathcal{T} + \sum_{k=n}^{\infty} V_{t_0}^k \mathcal{T}.$$

By Estimate (15) we have that $V_{t_0}^k F$ is norm right continuous in $t = 0$ for every $k \in \mathbb{N}$ and every $F \in \mathcal{X}_{t_0}$.
So we have that $V_{t_0}^k \mathcal{T}$ is norm continuous on $[0, t_0]$ for every $k \geq n$.
Since the series $\sum_{k=n}^{\infty} V_{t_0}^k \mathcal{T}$ converges in the space $\mathcal{X}_{t_0}$, i.e., uniformly in $t \in [0, t_0]$, we have that $\sum_{k=n}^{\infty} V_{t_0}^k \mathcal{T}$ is again norm continuous on $[0, t_0]$.
Using now Lemma 2.1 (ii), we obtain that $V^k \mathcal{T} \in \mathcal{C}(((k+1)\alpha,\infty))$ for $k = 0, 1, \ldots, n-1$.
So we have that $\sum_{k=0}^{n-1} V^k \mathcal{T} \in \mathcal{C}((n\alpha,\infty))$ and
$\mathcal{S} \in \mathcal{C}((n\alpha, t_0]) := \{F \in \mathcal{X}_{t_0} : t \mapsto F(t)$ is norm continuous for $t \in (n\alpha, t_0]\}$.
Since we can choose t_0 as big as we like, we conclude $\mathcal{S} \in \mathcal{C}((n\alpha,\infty))$. ☐

Let now $\mathcal{S}_n := V^n \mathcal{T}$ and $\mathcal{R}_n := \sum_{k=n}^{\infty} V^k \mathcal{T} = \sum_{k=n}^{\infty} \mathcal{S}_k = V^n \mathcal{S} = V\mathcal{R}_{n-1}$ for $n \in \mathbb{N}$.

Lemma 6.2. *Consider the following assertions for each $n \in \mathbb{N}$.*

(a_n) *$\mathcal{S}_n \in \mathcal{C}((0,\infty))$.*

(b_n) *$\mathcal{S}_k \in \mathcal{C}((0,\infty))$ for every $k \geq n$.*

(c_n) *$\mathcal{R}_k \in \mathcal{C}((0,\infty))$ for every $k \geq n$.*

(d_n) $RgV^n \subseteq \mathcal{C}((0,\infty))$.

Then the the following implications hold.

(a_n) $\Rightarrow$ (b_n) $\Rightarrow$ (c_n) $\Rightarrow$ (d_{n+1}) $\Rightarrow$ (a_{n+1}).

Proof.
"(a_n) $\Rightarrow$ (b_n)" Take $n \in \mathbb{N}$ as in (a_n) and let $m \in \mathbb{N}$. Then

$$\mathcal{S}_{n+m} = V^m \mathcal{S}_n = [\mathcal{T}B]^m * \mathcal{S}_n.$$

So by Lemma 2.1 (i), we have $\mathcal{S}_{n+m} \in \mathcal{C}((0,\infty))$ for every $m \in \mathbb{N}$.
"(b_n) $\Rightarrow$ (c_n)" Take $n \in \mathbb{N}$ as in (b_n) and let $k \geq n$. The series $\mathcal{R}_k = \sum_{l=k}^{\infty} \mathcal{S}_l$ converges uniformly on $[0,t_0]$ for all $t_0 \geq 0$, so $\mathcal{R}_k \in \mathcal{C}((0,\infty))$ for every $k \geq n$.
"(c_n) $\Rightarrow$ (d_{n+1})" Take $n \in \mathbb{N}$ as in (c_n) and let $F \in \mathcal{X}$. Then we have

$$\begin{aligned} V^{n+1}F &= [\mathcal{T}B]^{n+1} * F = [\mathcal{T}B]^n * (\mathcal{T} * BF) \\ &= V^n \mathcal{T} * BF = (\mathcal{R}_n - \mathcal{R}_{n+1}) * BF. \end{aligned}$$

As $(\mathcal{R}_n - \mathcal{R}_{n+1}) \in \mathcal{C}((0,\infty))$, we have $V^{n+1}F \in \mathcal{C}((0,\infty))$.
"(d_{n+1}) $\Rightarrow$ (a_{n+1})" Take $n \in \mathbb{N}$ as in (d_{n+1}), then $\mathcal{S}_{n+1} = V^{n+1}\mathcal{T} \in \mathcal{C}((0,\infty))$. $\square$

This shows that the assumption of Theorem 6.1 "$RgV^n \subseteq \mathcal{C}((0,\infty))$ for some $n \in \mathbb{N}$" can be substituted by anyone of the properties in the previous lemma.

The same results hold for eventually compact semigroups. For the proof we just have to substitute $\mathcal{C}$ with $\mathcal{K}$ and apply Lemma 3.2 instead of Lemma 2.1.

Theorem 6.3. *Let $\mathcal{T}:=(T(t))_{t\geq 0}$ be a strongly continuous semigroup such that $\mathcal{T} \in \mathcal{K}((\alpha,\infty))$ for some $\alpha > 0$. Let us suppose that there exists $n \in \mathbb{N}$ with $RgV^n \subseteq \mathcal{K}((0,\infty))$. Then the perturbed semigroup satisfies $\mathcal{S} \in \mathcal{K}((n\alpha,\infty))$.*

Lemma 6.4. *Consider the following assertions for each $n \in \mathbb{N}$.*

(a_n) $\mathcal{S}_n \in \mathcal{K}((0,\infty))$.

(b_n) $\mathcal{S}_k \in \mathcal{K}((0,\infty))$ *for every* $k \geq n$.

(c_n) $\mathcal{R}_k \in \mathcal{K}((0,\infty))$ *for every* $k \geq n$.

(d_n) $RgV^n \subseteq \mathcal{K}((0,\infty))$.

Then the the following implications hold.

(a_n) $\Rightarrow$ (b_n) $\Rightarrow$ (c_n) $\Rightarrow$ (d_{n+1}) $\Rightarrow$ (a_{n+1}).

As already said, differentiability is not preserved under bounded perturbation. In 1968, Pazy showed that a condition on the resolvent of the generator A, which guarantees immediate differentiability of the semigroup $\mathcal{T}$, also implies immediate differentiability of the perturbed semigroup $\mathcal{S}$. One year later, together with Crandall [1], he found a growth condition on the derivative $AT(t)$ of $\mathcal{T}$ for $t \searrow 0$ which gives immediate differentiability of the perturbed semigroup $\mathcal{S}$. Recently, Doytchinov, Hrusa and Watson showed in [2] that the bounded perturbation $\mathcal{S}$ of an immediately differentiable semigroup is differentiable for $t > \alpha$ if

$$0 \leq \alpha := \limsup_{t\searrow 0} \frac{t \log \|AT(t)\|}{\log t} < \infty. \tag{17}$$

We will now give a result for eventually differentiable semigroups in analogy to Theorem 6.1 and 6.3.

Theorem 6.5. *Let $\mathcal{T}:=(T(t))_{t\geq 0}$ be a strongly continuous semigroup such that $\mathcal{T} \in \mathcal{C}^1([\alpha,\infty))$ for some $\alpha > 0$. Let us suppose that there exists $n \in \mathbb{N}$ with $Rg\, V^n \subseteq \mathcal{C}^1([0,\infty))$. Then the perturbed semigroup satisfies $\mathcal{S} \in \mathcal{C}^1([n\alpha,\infty))$.*

Proof. We proceed in 2 steps.
1. step. For $m \in \mathbb{N}$, $x \in X$ and $t \geq 0$ we have

$$(V^{n+m}\mathcal{T})'(t)x = V^m(V^n\mathcal{T})'(t)x. \tag{18}$$

The proof is by induction. For $m = 1$ we have by Proposition 4.2 (i) that

$$\begin{aligned}(V^{n+1}\mathcal{T})'(t)x &= (\mathcal{T}B * V^n\mathcal{T})'(t)x \\ &= T(t)B\,V^n\mathcal{T}(0)x + \mathcal{T}B * (V^n\mathcal{T})'(t)x \\ &= \mathcal{T}B * (V^n\mathcal{T})'(t)x = V(V^n\mathcal{T})'(t)x.\end{aligned}$$

Suppose now that (18) is true for some $m \in \mathbb{N}$. Then we have

$$\begin{aligned}(V^{n+m+1}\mathcal{T})'(t)x &= (\mathcal{T}B * V^{n+m}\mathcal{T})'(t)x \\ &= \mathcal{T}B * (V^{n+m}\mathcal{T})'(t)x \\ &= \mathcal{T}B * [\mathcal{T}B]^m * (V^n\mathcal{T})'(t)x = V^{m+1}(V^n\mathcal{T})'(t)x.\end{aligned}$$

2. step. The rest of the proof is very similar to the proof of Theorem 6.1. By applying Proposition 4.2 (i), we can show that $RgV^k \subseteq \mathcal{C}^1([0,\infty))$ for all $k \geq n$.
Take now $t_0 > n\alpha$. In the space $\mathcal{X}_{t_0}$ we have

$$\mathcal{S} = \sum_{k=0}^{\infty} V_{t_0}^k\mathcal{T} = \sum_{k=0}^{n-1} V_{t_0}^k\mathcal{T} + \sum_{k=n}^{\infty} V_{t_0}^k\mathcal{T}.$$

From Proposition 4.2 (ii) it follows that $\sum_{k=0}^{n-1} V_{t_0}^k\mathcal{T} \in \mathcal{C}^1([n\alpha,\infty))$. It remains to show that $\sum_{m=0}^{\infty} V_{t_0}^{n+m}\mathcal{T}$ converges in the norm of $\mathcal{C}^1([0,t_0])$. As the series already converges uniformly, it is enough to show that $\sum_{m=0}^{\infty}(V_{t_0}^{n+m}\mathcal{T})'$ converges uniformly. By the first step, we have that

$$\sum_{m=0}^{\infty}(V_{t_0}^{n+m}\mathcal{T})' = \sum_{m=0}^{\infty} V_{t_0}^m(V_{t_0}^n\mathcal{T})' = (1 - V_{t_0})^{-1}(V_{t_0}^n\mathcal{T})',$$

which shows that the semigroup $\mathcal{S}$ is differentiable for $t \in [n\alpha, t_0]$. Since we can choose t_0 as big as we like, we conclude that $\mathcal{S}$ is differentiable for all $t \geq n\alpha$. □

Again we have some equivalences.

Lemma 6.6. *Consider the following assertions for each $n \in \mathbb{N}$.*

(a_n) $\mathcal{S}_n \in \mathcal{C}^1([0,\infty))$.

(b_n) $\mathcal{S}_k \in \mathcal{C}^1([0,\infty))$ *for every* $k \geq n$.

(c_n) $\mathcal{R}_k \in \mathcal{C}^1([0,\infty))$ *for every* $k \geq n$.

(d_n) $RgV^n \subseteq \mathcal{C}^1([0,\infty))$.

Then the the following implications hold.

$(a_n) \Rightarrow (b_n) \Rightarrow (c_n) \Rightarrow (d_{n+1}) \Rightarrow (a_{n+1})$.

Proof.
All implications except "$(b_n) \Rightarrow (c_n)$" can be shown as in the proof of Lemma 6.2. So we only show "$(b_n) \Rightarrow (c_n)$". Take $n \in \mathbb{N}$ as in (b_n) and let $k \geq n$. The series $\mathcal{R}_k = \sum_{l=k}^{\infty} \mathcal{S}_l = \sum_{m=0}^{\infty} \mathcal{S}_{n+m}$ converges uniformly on compact intervals.
From step 1 of the proof of Theorem 6.5, we have that

$$\mathcal{S}'_{n+m} = V^m \mathcal{S}'_n$$

for all $m \in \mathbb{N}$. So the series of the derivatives $\sum_{m=0}^{\infty} \mathcal{S}'_{n+m} = \sum_{m=0}^{\infty} V^m \mathcal{S}'_n$ converges uniformly on compact intervals as well, hence $\mathcal{R}_k \in \mathcal{C}^1([0,\infty))$. □

From Theorem 6.5 we immediately obtain the following corollary.

Corollary 6.7. *If the semigroup $\mathcal{T}$ is immediately differentiable and $RgV^n \subseteq \mathcal{C}^1([0,\infty))$ for some $n \in \mathbb{N}$, then the perturbed semigroup $\mathcal{S}$ is immediately differentiable as well.*

Proof.
The semigroup $\mathcal{T}$ belongs to $\mathcal{C}^1([\varepsilon,\infty))$ for every $\varepsilon > 0$. By Theorem 6.5, we have that $\mathcal{S} \in \mathcal{C}^1([n\varepsilon,\infty))$. Since this is true for all $\varepsilon > 0$, $\mathcal{S}$ is immediately differentiable. □

The condition of Lemma 6.6 is an assumption on the Volterra operator derived from the unperturbed semigroup $\mathcal{T}$ and the perturbation B. This means that it is a condition on both $\mathcal{T}$ and B. The condition (17) instead is only on the semigroup $\mathcal{T}$ and gives eventual differentiability for every bounded perturbation.

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