BOUNDED CONVOLUTIONS AND SOLUTIONS OF INHOMOGENEOUS CAUCHY PROBLEMS

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Abstract. Let X be a complex Banach space, $T : \mathbf{R}_+ \to B(X)$ and $f : \mathbf{R}_+ \to X$ be bounded functions, and suppose that the singular points of the Laplace transforms of T and f do not coincide. Under various supplementary assumptions, we show that the convolution $T * f$ is bounded. When $T(t) = I$, this is a classical result of Ingham. Our results are applied to mild solutions of inhomogeneous Cauchy problems on $\mathbf{R}_+$: $u'(t) = Au(t) + f(t)$ $(t \geq 0)$, where A is the generator of a bounded C_0-semigroup on X. For holomorphic semigroups, a result of this type has been obtained by Basit.

1. Introduction

Let $(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$ be bounded sequences, and $a * b$ be their convolution, so that $(a*b)_n = \sum_{r=0}^{n} a_{n-r} b_r$. Consider the holomorphic functions on the open unit disc whose Taylor coefficients are (a_n) and (b_n), respectively, and suppose that, given any point of the unit circle, at least one of these two functions has a holomorphic extension near that point. Then a rather elementary argument shows that $a * b$ is a bounded sequence (see Theorem 8.1).

The corresponding problem for functions on the half-line is more complicated. Let g and f be bounded measurable functions on $\mathbf{R}_+$, so their Laplace transforms $\widehat{g}(z)$ and $\widehat{f}(z)$ are defined and holomorphic on the right half-plane $\mathbf{C}_+ = \{z \in \mathbf{C} : \operatorname{Re} z > 0\}$. Let $\operatorname{Sp}_{\mathbf{R}_+}(f)$ be the set of all real numbers η such that $\widehat{f}$ does not have a holomorpic extension near $i\eta$, and suppose that $\operatorname{Sp}_{\mathbf{R}_+}(g)$ and $\operatorname{Sp}_{\mathbf{R}_+}(f)$ are disjoint. In general, it

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does not follow that the convolution $g * f$ is bounded, but there are some situations where this can be established. For example, if $g(t) = 1$, then $\mathrm{Sp}_{\mathbf{R}_+}(g) = \{0\}$,

$$(g * f)(t) = \int_0^t f(s)\, ds,$$

and a classical theorem of Ingham shows that this function is bounded if f is bounded and $\widehat{f}$ has a holomorphic extension near 0 [11], [12], [4].

In this paper, we shall show that $g * f$ is bounded when $\mathrm{Sp}_{\mathbf{R}_+}(g)$ and $\mathrm{Sp}_{\mathbf{R}_+}(f)$ are disjoint and various supplementary conditions are satisfied. The points of $\mathrm{Sp}_{\mathbf{R}_+}(f)$ represent generalised frequencies of f, so the disjointness assumption indicates an absence of resonance between g and f, which is reflected in $g * f$ being bounded. Our results should be of interest even in the classical (scalar-valued) case, but the most immediate applications are vector-valued. Thus, we shall formulate and prove most of our results for convolutions $T * f$, where f takes values in a complex Banach space X and T takes values in $B(X)$ (and is suitably measurable). Our work has been particularly motivated by current interest in inhomogeneous Cauchy problems:

$$\begin{aligned} u'(t) &= Au(t) + f(t) \quad (t \geq 0), \\ u(0) &= x. \end{aligned} \tag{1.1}$$

Here, A is a closed linear operator on a Banach space X, $f \in L^1_{\mathrm{loc}}(\mathbf{R}_+, X)$, and $x \in X$. We shall assume that A generates a bounded C_0-semigroup $T = \{T(t) : t \geq 0\}$. Then (1.1) has a unique mild solution given by

$$u(t) = T(t)x + \int_0^t T(t-s)f(s)\, ds,$$

which is bounded if and only if the convolution $T * f$ is bounded. In this context, $\widehat{T}(z) = (zI - A)^{-1}$ and $\mathrm{Sp}_{\mathbf{R}_+}(T) = -i\sigma(A) \cap \mathbf{R}$. If T is uniformly exponentially stable, so there exist constants $\alpha > 0$ and M such that $\|T(t)\| \leq Me^{-\alpha t}$ for all $t \geq 0$, then $\|(T * f)(t)\| \leq M\|f\|_\infty/\alpha$ whenever $f \in L^\infty(\mathbf{R}_+, X)$. Conversely, if $T * f$ is bounded whenever $f \in C_0(\mathbf{R}_+, X)$, then the semigroup T is uniformly exponentially stable by an argument of Datko [7] (Datko proved the corresponding fact for L^p-functions but the argument can easily be adapted; a more complicated proof is given in [17, Theorem 3.3.1]). In the terminology of control theory, where f is an input and u the output of a system, this says that the system has BIBO (bounded input, bounded output) stability if and only if T is uniformly exponentially stable. This fact, together with well-known examples in semigroup theory, shows that significant supplementary conditions must be imposed in order to ensure that convolutions on the half-line are bounded (see Section 9), and that our results for semigroups reduce to known results when $\sigma(A) \cap i\mathbf{R}$ is empty.

We shall assume that $\mathrm{Sp}_{\mathbf{R}_+}(T)$ is bounded and disjoint from $\mathrm{Sp}_{\mathbf{R}_+}(f)$. Any supplementary conditions on T concern the decay or growth of derivatives of $\widehat{T}$ along the imaginary axis. For semigroups, these become simple conditions on the resolvent of A. In Section 5, we also assume that f is smooth. Results for semigroups are stated as immediate corollaries of those for general convolutions, but direct proofs of these special cases would not be significantly different from those which we give. In Section 6, we give a result only for semigroups on Hilbert space, which would be excessively complicated to formulate for general convolutions.

Basit [3] has shown that $T * f$ is bounded if T is a bounded holomorphic semigroup and $\widehat{f}$ satisfies a condition which is considerably stronger than ours, and his result is a special case of our Corollary 4.4. Basit's method involves solving an operator equation of the form $AY - YB = C$ for an operator Y from a subspace of $\mathrm{BUC}(\mathbf{R}, X)$ into X. This idea is based on classical results of Lyapunov in the finite-dimensional case, and it has recently been developed and applied by Vũ [23], [22], [24]. We use a very different method, involving Fourier and Laplace transforms and ideas based on distributions.

Applications of our results to evolution equations are not confined to the case when T is a semigroup. Consider a well-posed inhomogeneous linear Volterra equation:

$$u(t) = g(t) + \int_0^t a(t-s) A u(s)\, ds \quad (t \geq 0).$$

When $g(t) = \int_0^t f(s)\, ds$, a mild solution is given by $u = S * f$, where S is the resolvent for the equation, and $\widehat{S}(z) = (I - \widehat{a}(z)A)^{-1}/z$ [20, Section 1]. Thus versions of our results can be formulated for Volterra equations (see [1, Section 4] for a similar application of a Tauberian theorem). In control theory, systems are often governed by equations of the form:

$$u'(t) = Au(t) + Bf(t) \quad (t \geq 0),$$

where A generates a semigroup T, and B is a bounded linear operator. If $u(0) = 0$, the solution is $u = T * (Bf) = (TB) * f$, so one may apply our semigroup results with $f(t)$ replaced by $Bf(t)$, but one can also apply our convolution results with $T(t)$ replaced by $T(t)B$. Since multiplication by B may have a regularizing effect on f, T and their transforms, either method may produce the better result.

The general strategy of our argument is described in the next section, and notation and basic preliminaries are established in Section 3. The main results are given in Sections 4, 5 and 6. In Section 7, we observe that if T and f have certain types of asymptotic behaviour, then so does $T*f$. Suitable types of behaviour include ergodicity, asymptotic almost periodicity or convergence to zero. This complements results for solutions of inhomogeneous Cauchy problems in [1], where it was *assumed* that $T * f$ is bounded and totally ergodic, and that $\sigma(A) \cap i\mathbf{R}$ is countable, but no assumptions were made about $\mathrm{Sp}_{\mathbf{R}_+}(f)$. In Section 8, we give the simple discrete analogue of our results, concerning convolutions of two bounded sequences, mentioned in the first paragraph above. In Section 9, we discuss some open questions and some examples.

2. General strategy

The proofs of our various results have a common structure, but there are technical details which differ substantially. In this section, we give an informal description of the argument without details, and in the next section we give those technical results which are common to the different cases.

If T and f decay exponentially, then $T * f$ has a Fourier transform given by

$$\mathcal{F}(T * f)(\eta) = \widehat{T}(i\eta)\widehat{f}(i\eta).$$

A strategy for proving that $T * f$ is bounded is therefore to try to show that $\widehat{T}(i\eta)\widehat{f}(i\eta)$ has an inverse Laplace transform which is bounded. However, if $\widehat{T}(i\eta)$ or $\widehat{f}(i\eta)$ cannot be defined, this argument runs immediately into difficulty, and one has to find a way to make it precise.

The assumption that $\mathrm{Sp}_{\mathbf{R}_+}(T) \cap \mathrm{Sp}_{\mathbf{R}_+}(f)$ is empty allows us to choose a smooth function ψ on $\mathbf{R}$ such that $\psi = 1$ near $\mathrm{Sp}_{\mathbf{R}_+}(T)$ and $\psi = 0$ near $\mathrm{Sp}_{\mathbf{R}_+}(f)$. Then the functions

$$h(\eta) = \psi(\eta)\widehat{f}(i\eta), \qquad H(\eta) = (1 - \psi(\eta))\widehat{T}(i\eta)$$

make sense and are smooth on $\mathbf{R}$. It suffices to prove that these functions are Fourier transforms of integrable functions g and G on $\mathbf{R}$, with values in X and $B(X)$ respectively, and then to use the decomposition

$$T * f = T * g + G * f$$

to deduce that $T * f$ is bounded and uniformly continuous. Most parts of this argument are straightforward, and they are carried out in the next section.

The function G may only be measurable in the strong operator topology (because T may not be measurable on $\mathbf{R}_+$ in the operator-norm topology, and G has a discontinuity at 0), but this is not a problem. The real difficulty is that H is not the Fourier transform of an integrable function in general. Some additional assumptions are needed for this, and this constitutes the work in Sections 4, 5 and 6. In the discrete case, these assumptions are unnecessary (see Section 8).

3. Preliminaries

Throughout the paper, X will be a complex Banach space (in Section 6, X will be a complex Hilbert space). For $1 \le p \le \infty$, we shall denote by $L^p(\mathbf{R}, X)$ the usual spaces of strongly measurable functions $f : \mathbf{R} \to X$ with the usual norms denoted by $\|f\|_p$. We shall consider $L^p(\mathbf{R}_+, X)$ as a subspace of $L^p(\mathbf{R}, X)$, by regarding a function in $L^p(\mathbf{R}_+, X)$ as having the value zero on $(-\infty, 0)$. When $X = \mathbf{C}$, we shall simply write $L^p(\mathbf{R})$ instead of $L^p(\mathbf{R}, \mathbf{C})$. The space of all infinitely differentiable, complex-valued functions with compact support will be denoted by $C_c^\infty(\mathbf{R})$.

The subspace of $L^\infty(\mathbf{R}, X)$ consisting of all bounded, uniformly continuous functions will be denoted by $\mathrm{BUC}(\mathbf{R}, X)$. Similarly $\mathrm{BUC}(\mathbf{R}_+, X)$ will be the space of all bounded, uniformly continuous functions $f : \mathbf{R}_+ \to X$. By $W^{k,\infty}(\mathbf{R}_+, X)$ we denote the usual Sobolev space of k times differentiable (in the sense of vector-valued distributions) functions f with derivatives $f^{(j)}$ in $L^\infty(\mathbf{R}_+, X)$ for all $0 \le j \le k$.

We shall denote by $\mathcal{F}$ the Fourier transform on $L^1(\mathbf{R}, X)$:

$$(\mathcal{F}g)(\eta) = \int_{-\infty}^{\infty} g(t)e^{-i\eta t}\,dt, \tag{3.1}$$

and by $\mathcal{F}^{-1}$ the inverse Fourier transform on $L^1(\mathbf{R}, X)$:

$$(\mathcal{F}^{-1}g)(\eta) = \frac{1}{2\pi}\int_{-\infty}^{\infty} g(t)e^{i\eta t}\,dt = \frac{1}{2\pi}(\mathcal{F}g)(-\eta). \tag{3.2}$$

We shall also denote by $\mathcal{F}$ and $\mathcal{F}^{-1}$ the Fourier transforms on $L^2(\mathbf{R})$. These are isomorphisms on $L^2(\mathbf{R})$ defined by (3.1) and (3.2) on $L^2(\mathbf{R}) \cap L^1(\mathbf{R})$. We shall use Plancherel's Theorem in the form

$$\int_{-\infty}^{\infty} (\mathcal{F}g_1)(\eta)g_2(\eta)\,d\eta = \int_{-\infty}^{\infty} g_1(t)(\mathcal{F}g_2)(t)\,dt$$

for g_1 and g_2 in $L^1(\mathbf{R})$. This formula is also valid if $g_1 \in L^1(\mathbf{R}, X)$ and $g_2 \in L^1(\mathbf{R})$, as may be seen by composing g_1 with functionals in X^*. We shall make use of the standard formulae for the Fourier transforms of derivatives of f and functions $t^r f(t)$, when applicable.

In the case when X is a Hilbert space, we shall also denote by $\mathcal{F}$ the Fourier transform on $L^2(\mathbf{R}, X)$, so $\frac{1}{\sqrt{2\pi}}\mathcal{F}$ is a unitary operator, a fact which we also refer to as Plancherel's Theorem.

In accordance with our convention of regarding $L^p(\mathbf{R}_+, X)$ as a subspace of $L^p(\mathbf{R}, X)$, we can interpret $\mathcal{F}g$ when $g \in L^1(\mathbf{R}_+, X)$ and convolutions $f * \phi$ when $f \in L^\infty(\mathbf{R}_+, X)$ and $\phi \in L^1(\mathbf{R})$:

$$(\mathcal{F}g)(\eta) = \int_0^{\infty} g(t)e^{-i\eta t}\,dt,$$
$$(f * \phi)(t) = \int_0^{\infty} f(s)\phi(t-s)\,ds = \int_{-\infty}^{t} f(t-s)\phi(s)\,ds.$$

For f in $L^\infty(\mathbf{R}_+, X)$, let $\widehat{f}$ be the Laplace transform of f, defined in the open right half-plane $\mathbf{C}_+$:

$$\widehat{f}(z) = \int_0^{\infty} e^{-tz}f(t)\,dt \quad (\operatorname{Re} z > 0).$$

Then $\widehat{f}$ is a holomorphic function on $\mathbf{C}_+$. Let $\operatorname{Sp}_{\mathbf{R}_+}(f)$ be the set of all real η such that $i\eta$ is a singular point of $\widehat{f}$, that is, $\widehat{f}$ does not have a holomorphic extension to a neighbourhood of $i\eta$ in $\mathbf{C}$. When $\eta \notin \operatorname{Sp}_{\mathbf{R}_+}(f)$, we shall write $\widehat{f}(i\eta)$ for the continuous extension of $\widehat{f}$ to $i\eta$.

Remarks 3.1. (1). We have defined the spectrum $\operatorname{Sp}_{\mathbf{R}_+}(f)$ of f in the same way as [1]. In our proofs, it will suffice that $\widehat{f}$ has a continuous extension to $\mathbf{C}_+ \cup i(\mathbf{R} \setminus \operatorname{Sp}_{\mathbf{R}_+}(f))$ such that $\eta \mapsto \widehat{f}(i\eta)$ has a continuous second derivative at each point of $\mathbf{R} \setminus \operatorname{Sp}_{\mathbf{R}_+}(f)$. Thus if we define the spectrum to be the smallest closed set with this latter property, our results remain true and become slightly more general. Note that the Tauberian theorem of [1] has been extended to allow an even smaller spectrum [6].

(2). For comparison with results in [3] and other papers, we recall that for g in $L^\infty(\mathbf{R}, X)$ the Beurling spectrum $\operatorname{sp}(g)$ is the set of all real η such that $(\mathcal{F}\phi)(\eta) = 0$ for all ϕ in $L^1(\mathbf{R})$ with $g * \phi = 0$. It coincides with the set of all real η such that $i\eta$ is a singular point of the Carleman transform which is defined on $C \setminus i\mathbf{R}$ by:

$$\widetilde{g}(z) = \begin{cases} \int_0^\infty g(t)e^{-tz}\,dt & (\operatorname{Re} z > 0), \\ -\int_0^\infty g(-t)e^{tz}\,dt & (\operatorname{Re} z < 0), \end{cases}$$

[20, Proposition 0.5, p.22]. If $g|_{\mathbf{R}_+} = f$, it is clear that $\operatorname{Sp}_{\mathbf{R}_+}(f) \subseteq \operatorname{sp}(g)$, but this inclusion may be strict. There are many examples where $\operatorname{Sp}_{\mathbf{R}_+}(f)$ is empty, but $\operatorname{sp}(g) = \mathbf{R}$ for all extensions g. Indeed, if $\operatorname{sp}(g)$ is empty, then $g = 0$; if $\operatorname{sp}(g)$ is finite, then g is a trigonometric polynomial; if $\operatorname{sp}(g)$ is compact, then g is smooth.

Lemma 3.2. *Let $f \in \mathrm{BUC}(\mathbf{R}_+, X)$, $\phi \in L^1(\mathbf{R})$. Then $\operatorname{Sp}_{\mathbf{R}_+}((f * \phi)|_{\mathbf{R}_+}) \subseteq \operatorname{supp} \mathcal{F}\phi$ and $\operatorname{Sp}_{\mathbf{R}_+}(f - (f * \phi)|_{\mathbf{R}_+}) \subseteq \operatorname{supp}(1 - \mathcal{F}\phi)$.*

Proof. Let

$$g(t) = \begin{cases} f(t) & (t \geq 0), \\ 0 & (t < 0). \end{cases}$$

Then $g \in L^\infty(\mathbf{R}, X)$. It is well-known, and very easy to see, that $\operatorname{sp}(g * \phi) \subseteq \operatorname{supp} \mathcal{F}\phi$ and $\operatorname{sp}(g - g * \phi) \subseteq \operatorname{supp}(1 - \mathcal{F}\phi)$ [2, Theorem 4.1.4]. Hence

$$\begin{aligned} \operatorname{Sp}_{\mathbf{R}_+}((f * \phi)|_{\mathbf{R}_+}) &= \operatorname{Sp}_{\mathbf{R}_+}((g * \phi)|_{\mathbf{R}_+}) \subseteq \operatorname{sp}(g * \phi) \subseteq \operatorname{supp} \mathcal{F}\phi, \\ \operatorname{Sp}_{\mathbf{R}_+}(f - (f * \phi)|_{\mathbf{R}_+}) &\subseteq \operatorname{sp}(g - g * \phi) \subseteq \operatorname{supp}(1 - \mathcal{F}\phi). \quad \square \end{aligned}$$

Lemma 3.3. *Let $g \in L^\infty(\mathbf{R}_+, X)$, $\theta \in L^1(\mathbf{R})$, $\alpha \in \mathbf{C}$, and suppose that $\mathcal{F}\theta = -\alpha$ on a neighbourhood of $\operatorname{Sp}_{\mathbf{R}_+}(g)$. Let*

$$h(\eta) = \begin{cases} (\alpha + (\mathcal{F}\theta)(\eta))\,\widehat{g}(i\eta) & (\eta \notin \operatorname{Sp}_{\mathbf{R}_+}(g)), \\ 0 & (\eta \in \operatorname{Sp}_{\mathbf{R}_+}(g)). \end{cases}$$

Suppose that $h \in C^k(\mathbf{R}, X)$ *for some* $k \geq 0$ *and that* $h^{(k)} \in L^1(\mathbf{R}, X)$*. Then* $(\mathcal{F}^{-1}h^{(k)})(t) = (-it)^k \cdot (\alpha g + g * \theta)(t)$ *a.e., where*

$$(\alpha g + g * \theta)(t) = \begin{cases} \alpha g(t) + \int_0^\infty g(s)\theta(t-s)\,ds & (t \geq 0), \\ \int_0^\infty g(s)\theta(t-s)\,ds & (t < 0). \end{cases}$$

In particular, $|t|^k \|(\alpha g + g*\theta)(t)\| \leq \|h^{(k)}\|_1$ *for almost all* t*. If* $k \geq 2$*, then* $\alpha g + g * \theta \in L^1(\mathbf{R}, X)$*.*

Proof. Let $\rho \in C_c^\infty(\mathbf{R})$. By two applications of Plancherel's Theorem, the Dominated Convergence Theorem and integration by parts,

$$\begin{aligned} \int_{-\infty}^\infty (\mathcal{F}\rho)(t)(\mathcal{F}^{-1}h^{(k)})(t)\,dt &= \int_{-\infty}^\infty \rho(\eta) h^{(k)}(\eta)\,d\eta \\ &= (-1)^k \int_{-\infty}^\infty \rho^{(k)}(\eta) h(\eta)\,d\eta \\ &= \lim_{\xi \downarrow 0} (-1)^k \int_{-\infty}^\infty \rho^{(k)}(\eta)(\alpha + (\mathcal{F}\theta)(\eta))\widehat{g}(\xi + i\eta)\,d\eta \\ &= \lim_{\xi \downarrow 0} (-1)^k \int_{-\infty}^\infty (\mathcal{F}\rho^{(k)})(t) g_\xi(t)\,dt, \end{aligned}$$

where

$$g_\xi(t) = \alpha e^{-\xi t} g(t) + \int_0^\infty \theta(t-s) e^{-\xi s} g(s)\,ds.$$

By the Dominated Convergence Theorem, $g_\xi(t) \to (\alpha g + g * \theta)(t)$ as $\xi \downarrow 0$. Moreover,

$$\|g_\xi(t)\| \leq (\alpha + \|\theta\|_1)\|g\|_\infty$$

for all $\xi > 0$, $t \in \mathbf{R}$, so, again by the Dominated Convergence Theorem,

$$\int_{-\infty}^\infty (\mathcal{F}\rho)(t)(\mathcal{F}^{-1}h^{(k)})(t)\,dt = \int_{-\infty}^\infty (\mathcal{F}\rho)(t)(-it)^k(\alpha g + g * \theta)(t)\,dt.$$

Since this holds for all ρ in $C_c^\infty(\mathbf{R})$, $(\mathcal{F}^{-1}h^{(k)})(t) = (-it)^k(\alpha g + g * \theta)(t)$ a.e.

The final statements follow simply from this, since g and $g * \theta$ are bounded. □

Let $L_s^\infty(\mathbf{R}_+, B(X))$ be the set of all essentially bounded functions $T : \mathbf{R}_+ \to B(X)$ which are measurable in the strong operator topology. For such a function we put

$$\begin{aligned} \|T\|_\infty &= \operatorname*{ess\,sup}_{t \in \mathbf{R}_+} \|T(t)\|, \\ \widehat{T}(z)x &= \int_0^\infty e^{-tz} T(t)x\,dt \quad (x \in X, \operatorname{Re} z > 0). \end{aligned}$$

Then $\widehat{T} : \mathbf{C}_+ \to B(X)$ is holomorphic. Let $\mathrm{Sp}_{\mathbf{R}_+}(T)$ be the set of all points η such that $i\eta$ is a singular point of $\widehat{T}$.

Suppose that $T = \{T(t) : t \in \mathbf{R}_+\}$ is a bounded C_0-semigroup on X with generator A. Then we shall write $\sigma(A)$ and $\rho(A)$ for the spectrum and resolvent set of A, and $R(z,A) := (zI - A)^{-1}$ for z in $\rho(A)$. Now, $\widehat{T}(z) = R(z,A)$ whenever $\operatorname{Re} z > 0$. Hence, $\mathrm{Sp}_{\mathbf{R}_+}(T) = -i\sigma(A) \cap \mathbf{R}$.

For g in $L^1(\mathbf{R}, X)$ and $T \in L^\infty_s(\mathbf{R}_+, B(X))$, the convolution $T * g$ is defined:

$$(T * g)(t) = \int_0^\infty T(s) g(t-s)\, ds = \int_{-\infty}^t T(t-s) g(s)\, ds.$$

Lemma 3.4. *Let $T \in L^\infty_s(\mathbf{R}_+, B(X))$ and let $g \in L^1(\mathbf{R}, X)$. Then $T*g \in \mathrm{BUC}(\mathbf{R}, X)$.*

Proof. Elementary estimates show that

$$\begin{aligned}
\|(T*g)(t)\| &= \left\| \int_{-\infty}^t T(t-s) g(s)\, ds \right\| \le \|T\|_\infty \int_{-\infty}^\infty \|g(s)\|\, ds, \\
\|(T*g)(t+h) - (T*g)(t)\| &= \left\| \int_0^\infty T(s)(g(t+h-s) - g(t-s))\, ds \right\| \\
&\le \|T\|_\infty \int_{-\infty}^\infty \|g(s+h) - g(s)\|\, ds.
\end{aligned}$$

Thus $T * g \in \mathrm{BUC}(\mathbf{R}, X)$. □

Let $L^1_s(\mathbf{R}, B(X))$ be the set of all functions $G : \mathbf{R} \to B(X)$ such that $t \mapsto G(t)x$ is strongly measurable for each x in X, and $\|G\|_1 := \int_{-\infty}^\infty \|G(t)\|\, dt < \infty$.

Let $G \in L^1_s(\mathbf{R}, B(X))$, and $f \in L^\infty(\mathbf{R}, X)$. We may define $(G * f)(t)$ in X by

$$(G * f)(t) = \int_{-\infty}^\infty G(t-s) f(s)\, ds = \int_{-\infty}^\infty G(s) f(t-s)\, ds.$$

With similar estimates to Lemma 3.4 one proves the following lemma.

Lemma 3.5. *Let $G \in L^1_s(\mathbf{R}, B(X))$, and $f \in L^\infty(\mathbf{R}, X)$. Then $G * f$ is bounded. If $G \in L^1(\mathbf{R}, B(X))$, or if $f \in \mathrm{BUC}(\mathbf{R}_+, X)$, then $G * f \in \mathrm{BUC}(\mathbf{R}, X)$.*

4. L^p-RESOLVENT

Our main objective in this section is to consider the case of a C_0-semigroup whose resolvent satisfies an L^p-condition along vertical lines (Theorem 4.4). First, we establish the corresponding result when T is not assumed to be a semigroup.

Theorem 4.1. *Let $T \in L_s^\infty(\mathbf{R}_+, B(X))$, and suppose that there exist b, $k \geq 2$ such that $\mathrm{Sp}_{\mathbf{R}_+}(T) \subseteq (-b, b)$ and*

$$\int_{|\eta|\geq b} \|\widehat{T}^{(k)}(i\eta)\|\, d\eta < \infty.$$

Let $f \in L^\infty(\mathbf{R}_+, X)$, and suppose that $\mathrm{Sp}_{\mathbf{R}_+}(T) \cap \mathrm{Sp}_{\mathbf{R}_+}(f)$ is empty. Then $T * f$ is bounded. If $T \in L^\infty(\mathbf{R}_+, B(X))$, or if $f \in \mathrm{BUC}(\mathbf{R}_+, X)$, then $T * f \in \mathrm{BUC}(\mathbf{R}_+, X)$.

Proof. Choose ψ in $C_c^\infty(\mathbf{R})$ such that $\psi = 0$ on a neighbourhood of $\mathrm{Sp}_{\mathbf{R}_+}(f)$ and $\psi = 1$ on a neighbourhood of $\mathrm{Sp}_{\mathbf{R}_+}(T)$. Such a ψ exists by the assumptions on $\mathrm{Sp}_{\mathbf{R}_+}(f)$ and $\mathrm{Sp}_{\mathbf{R}_+}(T)$. We define a function $G : \mathbf{R} \to B(X)$ by

$$G(t)x = \begin{cases} T(t)x - (T * \mathcal{F}^{-1}\psi)(t)x & (t \geq 0), \\ -(T * \mathcal{F}^{-1}\psi)(t)x & (t < 0). \end{cases} \tag{4.1}$$

The assumption on the decay of $\widehat{T}^{(k)}$ and Lemma 3.3 with $g = T(\cdot)x$, $\theta = -\mathcal{F}^{-1}\psi$ and $\alpha = 1$ together imply that there exists K, independent of x, such that

$$|t|^k \|G(t)x\| \leq K\|x\| \quad \text{a.e.}$$

Therefore, $G \in L_s^1(\mathbf{R}, B(X))$. By Lemma 3.5, $G*f$ is bounded, and $G*f \in \mathrm{BUC}(\mathbf{R}, X)$ if T is norm-measurable or f is uniformly continuous. By Lemma 3.3, $f * \mathcal{F}^{-1}\psi \in L^1(\mathbf{R}, X)$, and by Lemma 3.4, $T * f * \mathcal{F}^{-1}\psi \in \mathrm{BUC}(\mathbf{R}, X)$. Adding, we obtain the result. $\square$

Remark 4.2. Let $T \in L_s^\infty(\mathbf{R}_+, B(X))$, $f \in L^\infty(\mathbf{R}_+, X)$, and suppose that $\mathrm{Sp}_{\mathbf{R}_+}(T) \cap \mathrm{Sp}_{\mathbf{R}_+}(f)$ is empty and that there exist b, $k \geq 2$ such that $\mathrm{Sp}_{\mathbf{R}_+}(f) \subseteq (-b, b)$ and $\int_{|\eta|\geq b} \|\widehat{f}^{(k)}(i\eta)\|\, d\eta < \infty$. Then $T * f \in \mathrm{BUC}(\mathbf{R}_+, X)$. The proof of this is the same as Theorem 4.1, with the roles of T and f reversed.

Let A be a closed linear operator on X, and let $1 \leq p < \infty$. We say that A *has an L^p-resolvent* if there exist real α and $b \geq 0$ such that $\{\alpha + i\eta : |\eta| \geq b\} \subseteq \rho(A)$ and

$$\int_{|\eta|\geq b} \|R(\alpha + i\eta, A)\|^p\, d\eta < \infty.$$

If A generates a holomorphic semigroup, then $\|R(\alpha+i\eta, A)\| = O(|\eta|^{-1})$ for every α, so A has an L^p-resolvent for every $1 < p < \infty$. It is not difficult to see that the generator of a C_0-semigroup on a non-zero Banach space cannot have an L^1-resolvent.

We record in the following proposition some of the basic properties of a C_0-semigroup whose generator has an L^p-resolvent. Most of these properties will not be needed for the proof of Theorem 4.4, and some of them can be deduced from that result or its proof, but we give more direct arguments here.

Proposition 4.3. *Suppose that A generates a C_0-semigroup T, and that A has an L^p-resolvent for some $1 < p < \infty$. Then*

(1) *For every real ω, there exists b_ω such that $\{\omega + i\eta : |\eta| \geq b_\omega\} \subseteq \rho(A)$, $\|R(\omega + i\eta, A)\| \to 0$ as $|\eta| \to \infty$ and $\int_{|\eta|\geq b_\omega} \|R(\omega + i\eta, A)\|^p \, d\eta < \infty$.*
(2) *T is norm-continuous on $(0, \infty)$.*
(3) *For $t > 0$, $\sigma(T(t)) \setminus \{0\} = \{e^{t\lambda} : \lambda \in \sigma(A)\}$.*
(4) *If $\sigma(A) \subseteq \{\lambda \in \mathbf{C} : \operatorname{Re}\lambda < 0\}$, then T is uniformly exponentially stable.*

Proof. (1). Take α and b such that $R(\alpha + i\eta, A)$ exists whenever $|\eta| > b$ and $\int_{|\eta|>b} \|R(\alpha + i\eta, A)\|^p \, d\eta < \infty$. Take $\varepsilon > 0$, and suppose that $\|R(\alpha + i\mu, A)\| > \varepsilon$ for some real μ with $|\mu| > b$. Suppose $|\eta| > b$ and $|\eta - \mu| < \varepsilon^{-1}$. Then

$$R(\alpha + i\mu, A) = (I + i(\eta - \mu)R(\alpha + i\mu, A))\, R(\alpha + i\eta, A),$$

so

$$\|R(\alpha + i\eta, A)\| \geq \frac{\|R(\alpha + i\mu, A)\|}{1 + |\eta - \mu|\|R(\alpha + i\mu, A)\|} > \frac{\varepsilon}{1 + |\eta - \mu|\varepsilon} > \frac{\varepsilon}{2}.$$

Since $\int_{|\eta|>R} \|R(\alpha + i\eta, A)\|^p \, d\eta < \infty$, it follows that $\{\mu \in \mathbf{R} : |\mu| > b, \|R(\alpha + i\mu, A)\| > \varepsilon\}$ is bounded, so $\|R(\alpha + i\eta, A)\| \to 0$.

Given ω, choose $b_\omega \geq b$ such that $\|R(\alpha + i\eta, A)\| < (2|\omega - \alpha|)^{-1}$ whenever $|\eta| > b_\omega$. Then $R(\omega + i\eta, A)$ exists and $\|R(\omega + i\eta, A)\| < 2\|R(\alpha + i\eta, A)\|$, whenever $|\eta| > b_\omega$. This proves (1).

(2).. This is proved in [10, Theorem 12.6.1], as a consequence of the formula

$$t^{k-1}T(t) = \frac{(k-1)!}{2\pi} \int_{-\infty}^{\infty} e^{t(\omega+i\eta)} R(\omega + i\eta, A)^k \, d\eta,$$

for any $\omega > \omega_0(T)$.

(3) and (4). These are standard consequences of (2) [16, Theorem 6.6, p.87], [17, Theorem 2.3.2]. $\square$

In the following result, when β is not an integer, the fractional power $R(\lambda, A)^\beta$ is defined as in [19, Section 2.6] or [17, Appendix A.1]. Note that $R(\lambda, A)^\beta T * f = T * \big(R(\lambda, A)^\beta f\big)$.

Theorem 4.4. *Let T be a bounded C_0-semigroup on X with generator A, let $f \in L^\infty(\mathbf{R}_+, X)$, and suppose that $\sigma(A) \cap i\,\mathrm{Sp}_{\mathbf{R}_+}(f)$ is empty.*

(1) *Suppose that A has an L^p-resolvent for some $1 < p < \infty$. Then $T * f \in \mathrm{BUC}(\mathbf{R}_+, X)$.*
(2) *Suppose that there exist b and K such that $\{i\eta : |\eta| \geq b\} \subseteq \rho(A)$ and $\|R(i\eta, A)\| \leq K$ whenever $|\eta| \geq b$. Then $R(\lambda, A)^\beta T * f \in \mathrm{BUC}(\mathbf{R}_+, X)$ whenever $\operatorname{Re}\lambda > 0$ and $\beta > 1$.*

Proof. Note first that T is norm-measurable, by Proposition 4.3 (2).

(1). If A has an L^p-resolvent for some $1 < p < \infty$ then, by Proposition 4.3 (1), it also has an L^q-resolvent for $q > p$. This, the formula $(\frac{d}{d\eta})^k R(i\eta, A) = (-i)^k k! R(i\eta, A)^{k+1}$ and again Proposition 4.3 (1) imply that there exist b and $k \geq 2$ such that

$$\int_{|\eta|\geq b} \left\| \left(\frac{d}{d\eta}\right)^k R(i\eta, A) \right\| \, d\eta < \infty.$$

The claim now follows from Theorem 4.1.

(2). Let $T_\beta(t) = R(\lambda, A)^\beta T(t)$. Then $\widehat{T_\beta}(z) = R(\lambda, A)^\beta R(z, A)$ and $\mathrm{Sp}_{\mathbf{R}_+}(T_\beta) = -i\sigma(A) \cap \mathbf{R}$. For $|\eta| \geq \beta$,

$$\left(\widehat{T_\beta}\right)''(i\eta) = -2R(\lambda, A)^\beta R(i\eta, A)^3.$$

For $\beta \geq 2$, the resolvent identity gives

$$\left(\widehat{T_\beta}\right)''(i\eta) = -2(\lambda - i\eta)^{-2} \left(R(\lambda, A) - R(i\eta, A)\right)^2 R(i\eta, A) R(\lambda, A)^{\beta-2},$$

so the assumptions imply that $\left\| \left(\widehat{T_\beta}\right)''(i\eta) \right\| = O(|\eta|^{-2})$. For $1 < \beta < 2$, one obtains as in [17, Lemma 4.3.1] that

$$\left(\widehat{T_\beta}\right)''(i\eta) = \frac{-2\sin\pi(\beta-1)}{\pi(\lambda - i\eta)} (R(i\eta, A) - R(\lambda, A)) R(i\eta, A) \int_0^\infty \frac{R(i\eta, A) - R(\lambda + t, A)}{t^{\beta-1}(\lambda + t - i\eta)} \, dt,$$

so $\left\| \left(\widehat{T_\beta}\right)''(i\eta) \right\| = O(|\eta|^{-\beta})$. In each case, it follows from Theorem 4.1 that $T_\beta * f \in \mathrm{BUC}(\mathbf{R}_+, X)$. $\square$

5. Smooth functions

Let $T \in L^\infty_s(\mathbf{R}_+, B(X))$ and suppose that $\mathrm{Sp}_{\mathbf{R}_+}(T) \cap \mathrm{Sp}_{\mathbf{R}_+}(f)$ is empty for some f in $L^\infty(\mathbf{R}_+, X)$. In this section, we show that $T * f * \phi$ is bounded whenever ϕ is the Fourier transform of a function in $C^2_c(\mathbf{R})$ (Theorem 5.1), and also that $T * f$ is bounded if $\mathrm{Sp}_{\mathbf{R}_+}(T)$ is bounded, $\|\widehat{T}(i\eta)\|$ grows no faster than polynomially as $|\eta| \to \infty$ and f is sufficiently smooth (Theorem 5.3).

Given ϕ in $L^1(\mathbf{R})$, and t in $\mathbf{R}$, we can define $(T * \phi)(t)$ in $B(X)$ by

$$(T * \phi)(t)x = \int_{-\infty}^t \phi(s) T(t-s)x \, ds = \int_0^\infty \phi(t-s) T(s)x \, ds.$$

Since

$$\|(T * \phi)(t+h)x - (T * \phi)(t)x\| \leq M\|x\| \int_{-\infty}^\infty |\phi(s+h) - \phi(s)| \, ds,$$

$T * \phi$ is a uniformly norm-continuous function of $\mathbf{R}$ into $B(X)$.

Theorem 5.1. *Let $T \in L^\infty_s(\mathbf{R}, B(X))$, $f \in L^\infty(\mathbf{R}_+, X)$, and suppose that $\operatorname{Sp}_{\mathbf{R}_+}(T) \cap \operatorname{Sp}_{\mathbf{R}_+}(f)$ is empty. Let ϕ in $L^1(\mathbf{R})$ be such that $\mathcal{F}\phi \in C^2_c(\mathbf{R})$. Then $T * f * \phi \in \mathrm{BUC}(\mathbf{R}, X)$.*

Proof. Choose ψ in $C^\infty_c(\mathbf{R})$ such that $\psi = 0$ on a neighbourhood of $\operatorname{Sp}_{\mathbf{R}_+}(f)$ and $\psi = 1$ on a neighbourhood of $\operatorname{Sp}_{\mathbf{R}_+}(T) \cap \operatorname{supp}(\mathcal{F}\phi)$. In order to carry out the argument outlined in Section 2, we need to show that $T * (\phi - \phi * \mathcal{F}^{-1}\psi)$ is integrable.

Let

$$H(\eta) = \begin{cases} (\mathcal{F}\phi)(\eta)(1 - \psi(\eta))\widehat{T}(i\eta) & (i\eta \notin \operatorname{Sp}_{\mathbf{R}_+}(T)), \\ 0 & (i\eta \in \operatorname{Sp}_{\mathbf{R}_+}(T)). \end{cases}$$

Then H has continuous second derivative and has compact support. Let $x \in X$, and apply Lemma 3.3 with $g(t) = T(t)x$, $\theta = \phi - \phi * \mathcal{F}^{-1}\psi$ and $\alpha = 0$, so that $h(\eta) = H(\eta)x$. This gives

$$|t|^2 \left\| (T * \phi - T * \phi * \mathcal{F}^{-1}\psi)(t)x \right\| \le \|H''\|_1 \cdot \|x\|.$$

Thus, $T * \phi - T * \phi * \mathcal{F}^{-1}\psi \in L^1(\mathbf{R}, B(X))$. By Lemma 3.5, $(T * \phi - T * \phi * \mathcal{F}^{-1}\psi) * f \in \mathrm{BUC}(\mathbf{R}, X)$. By Lemma 3.3 (with $g = f$, $\theta = \phi * \mathcal{F}^{-1}\psi$ and $\alpha = 0$), $f * \phi * \mathcal{F}^{-1}\psi \in L^1(\mathbf{R}, X)$. By Lemma 3.4, $T * f * \phi * \mathcal{F}^{-1}\psi \in \mathrm{BUC}(\mathbf{R}, X)$. Adding, we see that $T * f * \phi \in \mathrm{BUC}(\mathbf{R}, X)$. □

Theorem 5.2. *Let T be a bounded C_0-semigroup on X with generator A, and let $f \in L^\infty(\mathbf{R}_+, X)$. Suppose that $\sigma(A) \cap i \operatorname{Sp}_{\mathbf{R}_+}(f)$ is empty. Let ϕ in $L^1(\mathbf{R})$ be such that $\mathcal{F}\phi \in C^2_c(\mathbf{R})$. Then $T * f * \phi \in \mathrm{BUC}(\mathbf{R}, X)$ and $T * ((f * \phi)|_{\mathbf{R}_+}) \in \mathrm{BUC}(\mathbf{R}_+, X)$.*

Proof. The first result is an immediate consequence of Theorem 5.1. For $t \ge 0$ one obtains

$$\left(T * \left((f * \phi)|_{\mathbf{R}_+}\right)\right)(t) = (T * f * \phi)(t) - T(t)((T * f * \phi)(0))\,.$$

Since every orbit of the bounded semigroup T is uniformly continuous, this gives the second result. □

Theorem 5.3. *Let $T \in L^\infty_s(\mathbf{R}, B(X))$ and suppose that there exist b, K and α such that $\operatorname{Sp}_{\mathbf{R}_+}(T) \subseteq (-b, b)$ and $\|\widehat{T}^{(j)}(i\eta)\| \le K|\eta|^{\alpha+2-j}$ whenever $|\eta| \ge b$ and $j \in \{0, 1, 2\}$. Let $f \in W^{k,\infty}(\mathbf{R}_+, X)$, where $k > \alpha+1$, and suppose that $\operatorname{Sp}_{\mathbf{R}_+}(T) \cap \operatorname{Sp}_{\mathbf{R}_+}(f)$ is empty. Then $T * f \in \mathrm{BUC}(\mathbf{R}_+, X)$.*

Proof. Replacing $T(t)$ by $e^{i\eta t}T(t)$ and $f(t)$ by $e^{i\eta t}f(t)$ for some real η, we can assume that $0 \notin \operatorname{Sp}_{\mathbf{R}_+}(T)$. Choose T_0 in $L^\infty_s(\mathbf{R}, B(X))$ with compact support such that $\widehat{T}_0^{(j)}(0) = \widehat{T}^{(j)}(0)$ for all $0 \le j \le k - 1$. By Lemma 3.5, $T_0 * f \in \mathrm{BUC}(\mathbf{R}_+, X)$, so that, replacing T by $T - T_0$, we can assume that $\widehat{T}^{(j)}(0) = 0$ for all $0 \le j \le k - 1$. Choose $f_0 \in W^{k,\infty}(\mathbf{R}_+, X)$ with compact support such that $f_0^{(j)}(0) = f^{(j)}(0)$ for all $0 \le j \le k-1$. By Lemma 3.4, $T * f_0 \in \mathrm{BUC}(\mathbf{R}_+, X)$, so that, replacing f by $f - f_0$, we can in addition assume that $f^{(j)}(0) = 0$ for all $0 \le j \le k-1$. The spectra $\operatorname{Sp}_{\mathbf{R}_+}(T)$ and $\operatorname{Sp}_{\mathbf{R}_+}(f)$ do not change under these reductions, since $\widehat{T}_0$ and $\widehat{f}_0$ are entire functions.

Let J^jT be the j-fold integral of T, i.e.

$$(J^jT)(t)x = \frac{1}{(j-1)!}\int_0^t (t-s)^{j-1}T(s)x\,ds \quad (x \in X).$$

Since $\widehat{T}(z) = z^j\left(\widehat{J^jT}\right)(z)$ for z in $\mathbf{C}_+$ and since $\widehat{T}^{(j)}(0) = 0$ for all $0 \le j \le k-1$, 0 is a regular point of $\widehat{J^jT}$ for $0 \le j \le k$. Since T is bounded and $0 \notin \mathrm{Sp}_{\mathbf{R}_+}(T)$, Ingham's Tauberian theorem [11, Theorem I] (see [12] or [4, p.27] for a simple proof) shows first that J^1T is bounded, and then, by iteration, that J^kT is bounded. Moreover, $\mathrm{Sp}_{\mathbf{R}_+}(J^kT) = \mathrm{Sp}_{\mathbf{R}_+}(T)$, $\mathrm{Sp}_{\mathbf{R}_+}(f^{(k)}) \subseteq \mathrm{Sp}_{\mathbf{R}_+}(f)$ (the only change might be at 0), and $T * f = (J^kT) * f^{(k)}$ as may be seen by integrating by parts k times.

For $|\eta| \ge b$, the assumptions on the growth of $\widehat{T}(i\cdot)$ and its derivatives imply that

$$\left\|\left(\widehat{J^kT}\right)''(i\eta)\right\| \le \frac{c}{|\eta|^{k-\alpha}}$$

for some constant c. Hence

$$\int_{|\eta|\ge b}\left\|\left(\widehat{J^kT}\right)''(i\eta)\right\|\,d\eta < \infty,$$

and Theorem 4.1 implies that $(J^kT) * f^{(k)} = T * f \in \mathrm{BUC}(\mathbf{R}_+, X)$. □

Remark 5.4. Let $f \in L^\infty(\mathbf{R}_+, X)$ and suppose that there exist b, K and α such that $\mathrm{Sp}_{\mathbf{R}_+}(f) \subseteq (-b, b)$ and $\|\widehat{f}^{(j)}(i\eta)\| \le K|\eta|^{\alpha+2-j}$ whenever $|\eta| \ge b$ and $j \in \{0, 1, 2\}$. Let $T : \mathbf{R}_+ \to B(X)$ be k times differentiable, with bounded continuous derivatives, in the strong operator topology, where $k > \alpha + 1$, and suppose that $\mathrm{Sp}_{\mathbf{R}_+}(T) \cap \mathrm{Sp}_{\mathbf{R}_+}(f)$ is empty. Then $T * f \in \mathrm{BUC}(\mathbf{R}_+, X)$. The proof of this is the same as Theorem 5.3, with the roles of T and f interchanged, and Remark 4.2 replacing Theorem 4.1.

An example of a function T with these properties is: $T(t) = R(\lambda, A)^k S(t)$, where $\{S(t) : t \ge 0\}$ is a bounded C_0-semigroup with generator A and $\lambda \in \rho(A)$.

Theorem 5.5. *Let T be a bounded C_0-semigroup on X with generator A, and suppose that there exist b, K and α such that $\{i\eta : |\eta| \ge b\} \subseteq \rho(A)$ and $\|R(i\eta, A)\| \le K|\eta|^\alpha$ whenever $|\eta| \ge b$. Let $f \in W^{k,\infty}(\mathbf{R}_+, X)$ where $k > 3\alpha + 1$, and suppose that $\sigma(A) \cap i\,\mathrm{Sp}_{\mathbf{R}_+}(f)$ is empty. Then $T * f \in \mathrm{BUC}(\mathbf{R}_+, X)$.*

Proof. The formula $(\frac{d}{d\eta})^j R(i\eta, A) = (-i)^j j! R(i\eta, A)^{j+1}$ implies that (in the terminology of Theorem 5.3) $\|\widehat{T}^{(j)}(i\eta)\| \le K'|\eta|^{(j+1)\alpha}$ for all $|\eta| \ge b$, $j \in \{0, 1, 2\}$ and for a new constant K'. Thus, Theorem 5.3 immediately yields the claim. □

Remark 5.6. There are examples of bounded C_0-semigroups, for which $R(\lambda, A)$ exists and is bounded for $\mathrm{Re}\,\lambda \ge 0$, but which are not uniformly exponentially stable (see Example 9.2). It follows from Datko's result (see the Introduction) that there exist f in $\mathrm{BUC}(\mathbf{R}_+, X)$ for which $T * f \notin \mathrm{BUC}(\mathbf{R}_+, X)$. Thus it is not possible to take $k = 0$ in Theorem 5.5, even if $\alpha = 0$ (see Example 9.1 for an example with $\alpha > 0$). However, we have not attempted to optimise the value of k in Theorems 5.3 and 5.5.

6. Hilbert spaces

Throughout this section, we assume that X is a Hilbert space, T is a bounded C_0-semigroup on X with generator A, and there exist b and K such that $\{i\eta : |\eta| \geq b\} \subseteq \rho(A)$ and $\|R(i\eta, A)\| \leq K$ whenever $|\eta| \geq b$. These conditions will suffice to ensure that $T * f$ is bounded whenever f is bounded and $\sigma(A) \cap i\,\mathrm{Sp}_{\mathbf{R}_+}(f)$ is empty. To establish this, we follow the same general strategy as in the rest of the paper, using the vector-valued Plancherel Theorem and some other Hilbert space arguments from [15], [17, p.40] instead of the L^1 arguments which underlie Sections 4 and 5.

Lemma 6.1. (1). *For each x in X, $\|R(i\eta, A)x\| \to 0$ as $|\eta| \to \infty$.*

(2). *Let S be a closed subset of $\mathbf{R}$ such that $\sigma(A) \cap iS$ is empty. Then there is a constant C_S such that*

$$\left(\int_S \|R(i\eta, A)x\|^2 \, d\eta \right)^{1/2} \leq C_S \|x\|$$

for all x in X.

(3). *Let ψ in $C_c^\infty(\mathbf{R})$ be such that $\psi = 1$ on a neighbourhood of $-i\sigma(A) \cap \mathbf{R}$, and let $G : \mathbf{R} \to B(X)$ be defined by (4.1). For x in X, let*

$$H_x(\eta) = \begin{cases} (1 - \psi(\eta))R(i\eta, A)x & (i\eta \in \rho(A)), \\ 0 & (i\eta \in \sigma(A)). \end{cases} \tag{6.1}$$

Then $H_x \in L^2(\mathbf{R}, X)$ and $(\mathcal{F}^{-1}H_x)(t) = G(t)x$ for almost all t.

Proof. (1) and (2). Take any $\xi > 0$. For any x, the function $t \mapsto e^{-\xi t}T(t)x$ is in $L^1(\mathbf{R}_+, X) \cap L^2(\mathbf{R}_+, X)$, so its Fourier transform $\eta \mapsto R(\xi + i\eta, A)x$ is in $C_0(\mathbf{R}, X) \cap L^2(\mathbf{R}, X)$, with L^2-norm at most $\sqrt{\frac{\pi}{\xi}} M\|x\|$, by the Riemann-Lebesgue Lemma and Plancherel's Theorem.

For $|\eta| > b$,

$$\|R(i\eta, A)x\| = \|(I + \xi R(i\eta, A))R(\xi + i\eta, A)x\| \leq (1 + \xi K)\|R(\xi + i\eta, A)x\|.$$

Now, (1) and (2) follow easily.

(3). The fact that $H_x \in L^2(\mathbf{R}, X)$ follows from (2). For $\xi > 0$, let

$$G_{x,\xi}(t) = \begin{cases} e^{-\xi t}T(t)x - \int_0^\infty (\mathcal{F}^{-1}\psi)(t - s)e^{-\xi s}T(s)x \, ds & (t \geq 0), \\ -\int_0^\infty (\mathcal{F}^{-1}\psi)(t - s)e^{-\xi s}T(s)x \, ds & (t < 0). \end{cases}$$

Then $G_{x,\xi} \in L^1(\mathbf{R}, X) \cap L^2(\mathbf{R}, X)$ and

$$(\mathcal{F}G_{x,\xi})(\eta) = (1 - \psi(\eta))R(\xi + i\eta, A)x := H_{x,\xi}(\eta).$$

By Plancherel's Theorem, $H_{x,\xi} \in L^2(\mathbf{R}, X)$. For $0 < \xi < (2K)^{-1}$ and $|\eta| > b$,

$$\|R(\xi + i\eta, A)x\| = \|(I - \xi R(\xi + i\eta, A))R(i\eta, A)x\| \le 2\|R(i\eta, A)x\|.$$

This estimate enables the Dominated Convergence Theorem to be applied to show that

$$\|H_{x,\xi} - H_x\|_2 \to 0$$

as $\xi \downarrow 0$. By Plancherel's Theorem, $\|G_{x,\xi} - \mathcal{F}^{-1}H_x\|_2 \to 0$. On the other hand, $\|G_{x,\xi}(t) - G(t)x\| \to 0$ as $\xi \downarrow 0$, for all t. Hence $(\mathcal{F}^{-1}H_x)(t) = G(t)x$ a.e. $\square$

Lemma 6.2. *Let ψ in $C_c^\infty(\mathbf{R})$ be such that $\psi = 1$ on a neighbourhood of $-i\sigma(A) \cap \mathbf{R}$, and let $G : \mathbf{R} \to B(X)$ be defined by* (4.1). *There is a constant C such that $\|G(t)\| \le C/t^2$ for all t.*

Proof. Our argument is adapted from [15, Lemma 2.3], [17, Theorem 2.3.5]. The assumptions on ψ imply that there exist

$$-b < a_1 < b_1 < a_2 < b_2 < \cdots < a_n < b_n < b$$

such that $-i\sigma(A) \cap \mathbf{R} \subseteq \bigcup_{r=1}^n (a_r, b_r)$ and $\psi(\eta) = 1$ whenever $\eta \in \bigcup_{r=1}^n [a_r, b_r]$. Let

$$S = (-\infty, -a_1] \cup [b_1, a_2] \cup [b_2, a_3] \cup \cdots \cup [b_{n-1}, a_n] \cup [b_n, \infty).$$

Let $x \in X$, and H_x be defined by (6.1). Take a sufficiently large that $\operatorname{supp} \psi \subseteq [-a, a]$, and let $b_0 = -a$, $a_{n+1} = a$. By integration by parts (twice),

$$\begin{aligned}
&\int_{-a}^{a} e^{i\eta t} H_x(\eta)\, d\eta \\
&= \sum_{r=0}^{n} \int_{b_r}^{a_{r+1}} e^{i\eta t}(1 - \psi(\eta))R(i\eta, A)x\, d\eta \\
&= \frac{e^{iat}}{it} R(ia, A)x - \frac{e^{-iat}}{it} R(-ia, A)x - \frac{ie^{iat}}{t^2} R(ia, A)^2 x + \frac{ie^{-iat}}{t^2} R(-ia, A)^2 x \\
&\qquad - \frac{1}{t^2} \int_{S \cap [-a,a]} e^{i\eta t} \big((1 - \psi(\eta))R(i\eta, A)^3 x + 2i\psi'(\eta)R(i\eta, A)^2 x - \psi''(\eta)R(i\eta, A)x\big)\, d\eta.
\end{aligned}$$

Since $\|R(i\eta, A)\|$ is bounded on S and ψ' and ψ'' have compact support, there is a constant C' (independent of a, x and t) such that

$$\begin{aligned}
\left\| \int_{-a}^{a} e^{i\eta t} H_x(\eta)\, d\eta \right\| \le\ & \frac{1}{t}\|R(ia, A)x\| + \frac{1}{t}\|R(-ia, A)x\| + \frac{C'\|x\|}{t^2} \\
& + \frac{1}{t^2} \left\| \int_{S \cap [-a,a]} e^{i\eta t}(1 - \psi(\eta))R(i\eta, A)^3 x\, d\eta \right\|.
\end{aligned}$$

For any y in X,

$$
\begin{aligned}
\left| \int_{S\cap[-a,a]} e^{i\eta t}(1-\psi(\eta))\langle R(i\eta,A)^3x, y\rangle \, d\eta \right| \\
&\le \int_S \left|\langle R(i\eta,A)^2x, R(i\eta,A)^*y\rangle\right| \, d\eta \\
&\le \int_S \|R(i\eta,A)^2x\| \, \|R(i\eta,A)^*y\| \, d\eta \\
&\le K' \left(\int_S \|R(i\eta,A)x\|^2 \, d\eta\right)^{1/2} \left(\int_S \|R(i\eta,A)^*y\|^2 \, d\eta\right)^{1/2} \\
&\le K' C_S^2 \|x\| \, \|y\|.
\end{aligned}
$$

Here, $K' = \sup\{\|R(i\eta,A)\| : \eta \in S\} < \infty$, and we have used the Cauchy-Schwarz inequality and Lemma 6.1 (2) for T and for the adjoint semigroup T^*. Since this holds for arbitrary y,

$$
\left\| \int_S e^{i\eta t}(1-\psi(\eta))R(i\eta,A)^3x \, d\eta \right\| \le K' C_S^2 \|x\|.
$$

Thus

$$
\left\| \frac{1}{2\pi} \int_{-a}^{a} e^{i\eta t} H_x(\eta) \, d\eta \right\| \le \frac{1}{2\pi t}\|R(ia,A)x\| + \frac{1}{2\pi t}\|R(-ia,A)x\| + \frac{C\|x\|}{t^2},
$$

for all a, $t > 0$ and x, where $C = (C' + K'C_S^2)/2\pi$. Now let $a \to \infty$. By Lemma 6.1, (1) and (3), the left-hand side of this inequality tends to $G(t)x$ in the norm of $L^2(\mathbf{R}, X)$, and the right-hand side tends pointwise to $C\|x\|/t^2$, so

$$
\|G(t)x\| \le \frac{C\|x\|}{t^2}
$$

for all $t > 0$ (since both sides are continuous) and all x. □

Theorem 6.3. *Let T be a bounded C_0-semigroup on a Hilbert space X with generator A, and suppose that there exist b and K such that $\{i\eta : |\eta| \ge b\} \subseteq \rho(A)$ and $\|R(i\eta,A)\| \le K$ whenever $|\eta| \ge b$. Let $f \in L^\infty(\mathbf{R}_+, X)$, and suppose that $\sigma(A) \cap i\,\mathrm{Sp}_{\mathbf{R}_+}(f)$ is empty. Then $T * f$ is bounded. If $f \in \mathrm{BUC}(\mathbf{R}_+, X)$, then $T * f \in \mathrm{BUC}(\mathbf{R}_+, X)$.*

Proof. Choose ψ in $C_c^\infty(\mathbf{R})$ such that $\psi = 1$ on a neighbourhood of $-i\sigma(A) \cap \mathbf{R}$ and $\psi = 0$ on a neighbourhood of $\mathrm{Sp}_{\mathbf{R}_+}(f)$, and let G be defined by (4.1). Then G is bounded and strongly continuous except at 0. By Lemma 6.2, $\int_{-\infty}^{\infty} \|G(t)\| \, dt < \infty$. Now, $T * f - T * f * \mathcal{F}^{-1}\psi = G * f$, which is bounded, and it is uniformly continuous if f is uniformly continuous (Lemma 3.5). By Lemma 3.3, $f * \mathcal{F}^{-1}\psi \in L^1(\mathbf{R}, X)$. By Lemma 3.4, $T * f * \mathcal{F}^{-1}\psi \in \mathrm{BUC}(\mathbf{R}, X)$. □

7. Asymptotic behaviour

In this section, $T : \mathbf{R}_+ \to B(X)$ will be bounded and strongly continuous. We suppose that the function f has some known asymptotic behaviour, and that the orbits $T(\cdot)x$ have similar asymptotic behaviour, and we wish to establish that the convolution $T * f$ has the same type of behaviour. This turns out to be straightforward for many classes of asymptotic behaviour.

As in [1], a closed subspace $\mathcal{E}$ of $\mathrm{BUC}(\mathbf{R}_+, X)$ is said to be *translation-biinvariant* if, for each $t \geq 0$,

$$\mathcal{E} = \{f \in \mathrm{BUC}(\mathbf{R}_+, X) : S(t)f \in \mathcal{E}\}.$$

Here $S(t)f$ is the translate of f: $(S(t)f)(s) = f(s+t)$ $(s, t \geq 0)$. Note that $\mathcal{E}$ is translation-biinvariant if and only if it satisfies condition (P.Λ) of [2, Definition 1.3.1] when $\mathbf{J} = \mathbf{R}_+$. Any translation-biinvariant subspace contains $C_0(\mathbf{R}_+, X)$. Amongst the most interesting classes of translation-biinvariant subspaces are the following, all of which arise in [2] and [1]:

(i) $C_0(\mathbf{R}_+, X)$;

(ii) the space $\mathrm{AAP}(\mathbf{R}_+, X)$ of all asymptotically almost periodic functions;

(iii) the space $W(\mathbf{R}_+, X)$ of all weakly (asymptotically) almost periodic functions (in the sense of Eberlein);

(iv) the space $\mathrm{E}(\mathbf{R}_+, X)$ of all uniformly ergodic functions:

$$\mathrm{E}(\mathbf{R}_+, X) = \left\{ f \in \mathrm{BUC}(\mathbf{R}_+, X) : \lim_{\alpha \downarrow 0} \alpha \int_0^\infty e^{-\alpha t} S(t)f \, dt \text{ exists in } \mathrm{BUC}(\mathbf{R}_+, X) \right\};$$

(v) the space $\mathrm{TE}(\mathbf{R}_+, X)$ of all totally (uniformly) ergodic functions:

$$\mathrm{TE}(\mathbf{R}_+, X) = \{f \in \mathrm{BUC}(\mathbf{R}_+, X) : e^{i\eta t} f(t) \text{ is in } \mathrm{E}(\mathbf{R}_+, X) \text{ for all real } \eta\}.$$

A closed subspace $\mathcal{E}$ of $\mathrm{BUC}(\mathbf{R}_+, X)$ is said to be *operator-invariant* if $U \circ f \in \mathcal{E}$ for all f in $\mathcal{E}$ and U in $B(X)$. Here, $(U \circ f)(t) = U(f(t))$. The examples (i)–(v) above are all operator-invariant.

We shall assume that $\mathcal{E}$ contains all the orbits $T(\cdot)x$ of T. For a bounded C_0-semigroup T, this condition can sometimes be described in terms of properties of the generator A. When $\mathcal{E} = \mathrm{E}(\mathbf{R}_+, X)$, we will be considering the class of bounded, mean-ergodic semigroups [8, Section 5.1], characterised by the property:

(7.1) If $x^* \in \ker A^*$ and $\langle x, x^* \rangle = 0$ for all x in $\ker A$, then $x^* = 0$.

Consequently, the property that all orbits lie in $\mathrm{TE}(\mathbf{R}_+, X)$ is characterised by

(7.2) If $x^* \in \ker(A^* - i\eta I)$ for some real η
and $\langle x, x^* \rangle = 0$ for all x in $\ker(A - i\eta I)$, then $x^* = 0$.

If X is reflexive, then (7.1) and (7.2) hold for any bounded semigroup. If $\sigma(A) \cap i\mathbf{R}$ is countable and (7.2) holds (resp., A^* has no eigenvalues in $i\mathbf{R}$), then every orbit is in $\mathrm{AAP}(\mathbf{R}_+, X)$ (resp., $C_0(\mathbf{R}_+, X)$) [5], [24], [17, Theorems 5.1.5, 5.7.10].

Lemma 7.1. *Let $\mathcal{E}$ be a closed, translation-biinvariant, operator-invariant subspace of* $\mathrm{BUC}(\mathbf{R}_+, X)$, *$G \in L^1_s(\mathbf{R}, B(X))$, and $f \in \mathcal{E}$. Then $(G * f)|_{\mathbf{R}_+} \in \mathcal{E}$.*

Proof. First, assume that G is measurable in the operator-norm topology. For $s \geq 0$, let $f_s = S(s)f \in \mathcal{E}$. For $s < 0$, let

$$f_s(t) = \begin{cases} f(0) & (0 \leq t \leq -s), \\ f(s+t) & (-s \leq t). \end{cases}$$

Then $S(-s)f_s = f \in \mathcal{E}$, so $f_s \in \mathcal{E}$. Now $s \mapsto f_s$ is a continuous function of $\mathbf{R}$ into $\mathcal{E}$, and $s \mapsto G(-s) \circ f_s$ is in $L^1(\mathbf{R}, \mathcal{E})$. For $t \geq 0$, we have the following formula:

$$(G * f)(t) = \left(\int_{-\infty}^{\infty} G(-s) \circ f_s \, ds \right)(t) - \left(\int_t^{\infty} G(s)\, ds \right)(f(0)).$$

The first term (as a function of t) is an $\mathcal{E}$-valued integral, and the second is in $C_0(\mathbf{R}_+, X)$ and hence in $\mathcal{E}$. This shows that $(G * f)|_{\mathbf{R}_+} \in \mathcal{E}$. In particular, putting $G(t) = \phi(t)I$ shows that $(f * \phi)|_{\mathbf{R}_+} \in \mathcal{E}$ for any ϕ in $L^1(\mathbf{R})$, and therefore $\mathcal{E}$ is a Banach $L^1(\mathbf{R}_+)$-module under convolution.

If G is not measurable in the operator-norm topology, we can do the following steps. Choose f_2 in $\mathrm{BUC}(\mathbf{R}_+, X)$ with support in $[0, 1]$ and with $f_2(0) = f(0)$. Then $T * f_2$ is continuous (Lemma 3.4), and $\|(T * f_2)(t)\| \leq \|f_2\|_\infty \int_{t-1}^t \|G(s)\|\, ds \to 0$ as $t \to \infty$. Hence $T * f_2 \in C_0(\mathbf{R}_+, X) \subseteq \mathcal{E}$, so replacing f by $f - f_2$, we may assume that $f(0) = 0$. The Cohen-Hewitt Factorisation Theorem [9, Theorem 32.22] yields the factorisation $f = \phi * g$ for some g in $\mathcal{E}$ and ϕ in $L^1(\mathbf{R}_+)$. Note that $G * \phi$ is measurable in the operator-norm topology: for bounded G in $L^1_s(\mathbf{R}, B(X))$, $G * \phi$ is norm-continuous as in the second paragraph of Section 5, then one argues by density. Now, $(G * f)|_{\mathbf{R}_+} = ((G * \phi) * g)|_{\mathbf{R}_+} \in \mathcal{E}$, by the previous paragraph. □

Lemma 7.2. *Let $T : \mathbf{R}_+ \to B(X)$ be bounded and strongly continuous, $\mathcal{E}$ be a closed, translation-biinvariant subspace of* $\mathrm{BUC}(\mathbf{R}_+, X)$ *containing all the orbits $T(\cdot)x$, and $g \in L^1(\mathbf{R}, X)$. Then $(T * g)|_{\mathbf{R}_+} \in \mathcal{E}$.*

Proof. The map $g \mapsto (T * g)|_{\mathbf{R}_+}$ is bounded from $L^1(\mathbf{R}, X)$ into $\mathrm{BUC}(\mathbf{R}_+, X)$. By linearity and density, it suffices to prove the result when $g = \mathbf{1}_{(a,b)} \otimes x$. Then $T * g = T(\cdot)x * \mathbf{1}_{(a,b)}$ and the claim follows from Lemma 7.1 or the first part of its proof. □

Theorem 7.3. *Let $\mathcal{E}$ be a closed, translation-biinvariant, operator-invariant subspace of* $\mathrm{BUC}(\mathbf{R}_+, X)$. *Let $T : \mathbf{R}_+ \to B(X)$ be bounded and strongly continuous, and suppose that $T(\cdot)x \in \mathcal{E}$ for every $x \in X$, and let $f \in \mathcal{E}$. Assume that $\mathrm{Sp}_{\mathbf{R}_+}(T) \cap \mathrm{Sp}_{\mathbf{R}_+}(f)$ is empty. Then*

1. *$(T * f * \phi)|_{\mathbf{R}_+} \in \mathcal{E}$ whenever $\phi \in L^1(\mathbf{R})$ is such that $\mathcal{F}\phi \in C^2_c(\mathbf{R})$.*
2. *$T * f \in \mathcal{E}$ whenever $T * f \in \mathrm{BUC}(\mathbf{R}_+, X)$.*

Proof. (1) . The proof of Theorem 5.2 is modified simply by replacing Lemmas 3.4 and 3.5 by Lemmas 7.2 and 7.1.

(2) . Choose an approximate identity (ϕ_n) in $L^1(\mathbf{R}_+)$ such that $\mathcal{F}\phi_n \in C_c^2(\mathbf{R})$ for every n. Since $(T * f)(0) = 0$, we can consider $T * f$ as an element of $\mathrm{BUC}(\mathbf{R}, X)$ and we obtain that $T * f * \phi_n \to T * f$ uniformly as $n \to \infty$. The result follows from (1). □

In the case when $T(t) = I$ and $\mathcal{E}$ contains the constant functions, Theorem 7.3 (2) is somewhat reminiscent of the Bohr-Kadets Theorem, which states that if X does not contain c_0 and the primitive F of an almost periodic function $f : \mathbf{R} \to X$ is bounded, then F is almost periodic [13, Theorem 2, p.86]. Whereas the Bohr-Kadets Theorem involves no explicit spectral assumption on f, the assumption that $\mathrm{Sp}_{\mathbf{R}_+}(T) \cap \mathrm{Sp}_{\mathbf{R}_+}(f)$ is empty cannot be omitted from Theorem 7.3 (2), as there exist functions f in $\mathrm{AAP}(\mathbf{R}_+, \mathbf{C})$ whose primitives are bounded but not asymptotically almost periodic (see [21, Section 4]).

8. Power series

There is also a discrete analogue of our results, in which there is no need of supplementary conditions. We doubt that this result is new (at least in the scalar-valued case), but we have been unable to find it in the literature.

Theorem 8.1. *Let $(T_n)_{n \geq 0}$ be a bounded sequence in $B(X)$, $(x_n)_{n \geq 0}$ be a bounded sequence in X, and $y_n = \sum_{j=0}^{n} T_{n-j} x_j$. Let $f_1 : D \to B(X)$ and $f_2 : D \to X$ be the holomorphic functions on the open unit disc defined by*

$$f_1(z) = \sum_{n=0}^{\infty} z^n T_n, \qquad f_2(z) = \sum_{n=0}^{\infty} z^n x_n \qquad (|z| < 1),$$

and let E_1 and E_2 be the singular sets of f_1 and f_2, respectively, in the unit circle Γ. Suppose that E_1 and E_2 are disjoint. Then (y_n) is bounded.

Proof. Let ψ in $C^\infty(\Gamma)$ be such that $\psi = 0$ on a neighbourhood of E_1 and $\psi = 1$ on a neighbourhood of E_2. Let

$$h_1(z) = \begin{cases} \psi(z) f_1(z) & (z \in \Gamma \setminus E_1), \\ 0 & (z \in E_1), \end{cases} \qquad h_2(z) = \begin{cases} (1 - \psi(z)) f_2(z) & (z \in \Gamma \setminus E_2), \\ 0 & (z \in E_2), \end{cases}$$

Let $(\alpha_n)_{n \in \mathbf{Z}}$, $(T'_n)_{n \in \mathbf{Z}}$ and $(x'_n)_{n \in \mathbf{Z}}$ be the Fourier coefficients of ψ, h_1 and h_2, respectively. Since all three functions are infinitely differentiable, their Fourier coefficients are absolutely summable. Moreover,

$$\begin{aligned} T'_n = \frac{1}{2\pi} \int_0^{2\pi} h_1(e^{it}) e^{-int}\, dt &= \lim_{r \uparrow 1} \frac{1}{2\pi} \int_0^{2\pi} \psi(e^{it}) f_1(re^{it}) e^{-int}\, dt \\ &= \lim_{r \uparrow 1} \sum_{k=0}^{\infty} \alpha_{n-k} r^k T_k = \sum_{k=0}^{\infty} \alpha_{n-k} T_k. \end{aligned}$$

Similarly,

$$x'_n = \begin{cases} x_n - \sum_{j=0}^{\infty} \alpha_{n-j} x_j & (n \geq 0), \\ -\sum_{j=0}^{\infty} \alpha_{n-j} x_j & (n < 0). \end{cases}$$

Hence

$$\sum_{j=0}^{\infty} T'_{n-j} x_j + \sum_{k=0}^{\infty} T_k x'_{n-k} = y_n \quad (n \geq 0),$$

so

$$\|y_n\| \leq \left(\sum_{k=-\infty}^{\infty} \|T'_k\| \right) \sup_{j \geq 0} \|x_j\| + \sup_{k \geq 0} \|T_k\| \left(\sum_{j=-\infty}^{\infty} \|x'_j\| \right). \quad \square$$

9. Examples and questions

An example.

As remarked in the Introduction, any bounded C_0-semigroup which is not uniformly exponentially stable, but for which $\sigma(A) \cap i\mathbf{R}$ is empty, provides examples of functions f in $\mathrm{BUC}(\mathbf{R}_+, X)$ (with $\sigma(A) \cap i\,\mathrm{Sp}_{\mathbf{R}_+}(f)$ empty) for which $T * f \notin \mathrm{BUC}(\mathbf{R}_+, X)$. Indeed, it is possible to find such functions with $\mathrm{Sp}_{\mathbf{R}_+}(f)$ empty, as the following concrete example shows.

Example 9.1. Let $X = c_0$, $0 < \beta < \alpha < 1$,

$$T(t)x = (e^{\lambda_n t} x_n),$$

where $\lambda_n = -n^{-\alpha} + in$. Then $\sigma(A) = \{\lambda_n : n \geq 1\}$, so $\sigma(A) \cap i\mathbf{R}$ is empty. Moreover, $\|R(i\eta, A)\| = \sup_n \left(|\eta - \lambda_n|^{-1} \right) = O(|\eta|^{\alpha})$ as $|\eta| \to \infty$. Let

$$f(t) = (n^{-\beta} e^{\lambda_n t}).$$

Then $f \in C_0(\mathbf{R}_+, c_0)$, but $f \notin \mathrm{BUC}^1(\mathbf{R}_+, c_0)$. Moreover, $\widehat{f}(z) = (n^{-\beta}(z - \lambda_n)^{-1})$, which has a holomorphic extension to $\mathbf{C} \setminus \{\lambda_n\}$, so $\mathrm{Sp}_{\mathbf{R}_+}(f)$ is empty. However

$$(T * f)(t) = (n^{-\beta} t e^{\lambda_n t}),$$
$$\|(T * f)(n^{\alpha})\| \geq n^{\alpha - \beta} e^{-1},$$

so $T * f$ is unbounded.

Some general questions.

One can easily modify Example 9.1 to obtain an example where $\sigma(A) \cap i\mathbf{R}$ is a singleton, $\mathrm{Sp}_{\mathbf{R}_+}(f)$ is empty, and $T * f$ is unbounded (just redefine the first coordinate of $T(t)x$ to be x_1 for all t). Nevertheless, if $\mathcal{C}$ is a natural class of semigroups for which it is known that

$$\sigma(A) \cap \{\lambda \in \mathbf{C} : \operatorname{Re}\lambda = \omega_0(T)\} \text{ is non-empty} \tag{9.1}$$

(where $\omega_0(T)$ is the growth bound of T), then one might hope to prove that, for any bounded semigroup in $\mathcal{C}$,

$$T * f \in \mathrm{BUC}(\mathbf{R}_+, X) \text{ whenever } f \in \mathrm{BUC}(\mathbf{R}_+, X) \text{ and } \sigma(A) \cap i\,\mathrm{Sp}_{\mathbf{R}_+}(f) \text{ is empty.} \tag{9.2}$$

Theorem 4.4 is an example of a result of this type (see Proposition 4.3). A variant of this is to take $\mathcal{C}$ to be a class of C_0-semigroups for which it is known that

$$\text{If } R(\lambda, A) \text{ exists whenever } \operatorname{Re}\lambda > \alpha \text{ and } \sup_{\operatorname{Re}\lambda>\alpha} \|R(\lambda, A)\| < \infty, \text{ then } \omega_0(T) < \alpha. \tag{9.3}$$

Then one might hope to prove that if T is a bounded semigroup in $\mathcal{C}$, $\sigma(A) \cap i\mathbf{R}$ is bounded and $\|R(i\eta, A)\|$ is bounded at infinity, then $T * f$ is bounded whenever f is bounded and $\sigma(A) \cap i\,\mathrm{Sp}_{\mathbf{R}_+}(f)$ is empty. In Section 6, we proved this for the class of all semigroups on Hilbert spaces, for which (9.3) is well-known [16, Corollary 7.11, p.96], [17, Corollary 2.2.5].

It would also be interesting to know whether Theorem 4.4 (2) holds for $\beta = 1$. This is true under the additional assumption that $\sigma(A) \cap i\mathbf{R}$ is empty, because there then exist positive constants c and α such that $\|R(\lambda, A)T(t)\| \le ce^{-\alpha t}$ for all t [28], [17, Corollary 4.2.7], [18], [27].

We now give some specific classes of semigroups for which (9.1) is known, but for which we do not know whether (9.2) holds.

Eventually norm-continuous semigroups.

Let $\mathcal{C}_0$ be the class of semigroups which are norm-continuous on $(0, \infty)$. For this class, the Spectral Mapping Theorem holds [16, Theorem 6.6, p.87], [17, Theorem 2.3.2], so (9.1) is true. Moreover, $\sigma(A) \cap i\mathbf{R}$ is bounded, and $\|R(i\eta, A)\| \to 0$ as $|\eta| \to \infty$ [19, Theorem 3.6, p.50]. However, we do not know whether (9.2) holds for all bounded semigroups in the class $\mathcal{C}_0$.

There are several variants of this class, including the following:

- $\mathcal{C}_e$: semigroups which are norm-continuous on (t_0, ∞) for some $t_0 \ge 0$;
- $\mathcal{C}_\infty$: semigroups which are norm-continuous at infinity [15], [17, p.37];
- $\mathcal{C}_0'$: semigroups for which $\|R(\alpha+i\eta, A)\| \to 0$ as $|\eta| \to \infty$ (this condition is independent of α—see Proposition 4.3).

Then $\mathcal{C}_0 \subset \mathcal{C}_e \subset \mathcal{C}_\infty$, and (9.1) holds for the class $\mathcal{C}_\infty$ [15], [17, Theorem 2.3.3]. Moreover, $\mathcal{C}_0 \subseteq \mathcal{C}_0'$, but we do not know whether (9.1) holds for $\mathcal{C}_0'$. Nor do we know whether the classes $\mathcal{C}_0$ and $\mathcal{C}_0'$ coincide, except in special cases such as semigroups on Hilbert space [29], [17, Corollary 2.3.6] and multiplier semigroups on L^p-spaces and on $C_0(\Omega)$ [16, Example 1.28, p.42].

The classes considered here all include the semigroups with L^p-resolvents considered in Section 5. It is easy to modify Example 9.1 to obtain examples of multiplier semigroups on c_0 or ℓ^q such that the resolvent tends to 0 along the imaginary axis, but is not in L^p. A bounded semigroup of multipliers is norm-continuous at infinity if and only if $\sigma(A) \cap i\mathbf{R}$ is bounded and $\|R(i\eta, A)\|$ is bounded at infinity (see [15, Example 1.11]). For these multiplier semigroups, it is easy to see that (9.2) holds.

Positive semigroups on L^p-spaces.

Another class for which (9.1) is known is the class of positive semigroups on L^p-spaces for $1 \leq p < \infty$ [26], [17, Theorem 3.5.3]. If the semigroup is bounded, then $\sigma(A) \cap i\mathbf{R}$ is either unbounded or $\{0\}$ [16, Theorem 2.10, p.302]. It would be interesting to know whether (9.2) holds for the following classes:

$\mathcal{P}$: bounded, positive semigroups on L^p-spaces such that $\sigma(A) \cap i\mathbf{R} = \{0\}$;

$\mathcal{P}'$: bounded, positive semigroups on L^p-spaces such that $\sigma(A) \cap i\mathbf{R} = \{0\}$ and $\sup_{|\eta| \geq 1} \|R(i\eta, A)\| < \infty$.

The following modification of a well-known example provides counterexamples to (9.2) for (non-positive) semigroups on L^p-spaces and for positive semigroups on Banach lattices.

Example 9.2. Let $X_0 = L^p(1,\infty) \cap L^2(1,\infty)$, where $2 < p < \infty$, $\|f\| = \max(\|f\|_p, \|f\|_2)$, and

$$(T_0(t)f)(s) = e^{t/2} f(se^t), \quad (s > 1, t \geq 0).$$

Then T_0 is a bounded, positive semigroup on the Banach lattice X_0, with $\sigma(A_0) = \{\lambda \in \mathbf{C} : \operatorname{Re} \lambda < \frac{1}{2} - \frac{1}{p}\}$ and $\|T_0(t)\| = 1$ [17, Example 1.4.4]. So $\sigma(A) \cap i\mathbf{R}$ is empty, and there exists $f_0 \in \mathrm{BUC}(\mathbf{R}_+, X_0)$ with $T_0 * f_0$ unbounded. Moreover, X_0 is isomorphic as a Banach space to an L^p-space [14, Corollary 2.e.8], [26].

Now let $X = X_0 \oplus \mathbf{C}$ and $T(t) = T_0(t) \oplus I$. Then T is a bounded, positive semigroup on the Banach lattice X (which is also isomorphic as a Banach space to an L^p-space), $\sigma(A) \cap i\mathbf{R} = \{0\}$, and $\sup_{|\eta| \geq 1} \|R(i\eta, A)\| < \infty$. Regarding X_0 as a subspace of X in the natural way, we consider f_0 in $\mathrm{BUC}(\mathbf{R}_+, X)$. Let $\phi \in L^1(\mathbf{R})$ such that $\mathcal{F}\phi \in C_c^\infty(\mathbf{R})$ and $\mathcal{F}\phi = 1$ on a neighbourhood of 0. Let $f = f_0 - (f_0 * \phi)|_{\mathbf{R}_+} \in \mathrm{BUC}(\mathbf{R}_+, X)$. By Lemma 3.1, $0 \notin \mathrm{Sp}_{\mathbf{R}_+}(f)$. By Theorem 5.2, $T * ((f_0 * \phi)|_{\mathbf{R}_+}) \in \mathrm{BUC}(\mathbf{R}_+, X)$. Hence, $T * f$ is unbounded.

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