ON THE ASYMPTOTIC BEHAVIOUR OF PERTURBED SEMIGROUPS

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Abstract. We give conditions on a strongly continuous semigroup $\mathcal{T}$ and a bounded perturbation B such that the perturbed semigroup $\mathcal{S}$ preserves asymptotic properties as boundedness, asymptotic almost periodicity, uniform ergodicity and total uniform ergodicity.

1. Introduction

In the first systematic treatise on perturbation of semigroups, R.S.Phillips [10] proved in 1953 that, if A is the infinitesimal generator of a strongly continuous semigroup $\mathcal{T}$ of linear bounded operators on a Banach space X and B is a linear continuous operator on X, then $A+B$ again generates a strongly continuous semigroup $\mathcal{S}$. This result has been generalized into various directions. On one hand, the hypotheses have been weakened, admitting B to be an unbounded operator. On the other hand, the following question has been studied. Which properties of $\mathcal{T}$ are preserved under perturbation?

R.S.Phillips proved that immediate norm continuity and immediate compactness of $\mathcal{T}$ are inherited by the perturbed semigroup $\mathcal{S}$. This is not the case for eventual norm continuity, eventual compactness and, as shown only in 1995 by M.Renardy [11], for immediate differentiability. Nonetheless, M.G.Crandall and A.Pazy [4,9] found conditions under which the perturbed semigroup is immediately differentiable. More recently, R.Nagel and the second author of this paper found in [7] other conditions guaranteeing the permanence of these regularity properties. Their method, based on the representation of $\mathcal{S}$ through the Dyson-Phillips series, can also be applied to the study of the asymptotic behaviour of $\mathcal{S}$. This is the main purpose of this paper.

In particular, a semigroup $\mathcal{T}$ is considered, such that every orbit $t \mapsto T(t)x$, $x \in X$, belongs to a closed, translation-invariant subspace $\mathcal{E}$ of the set $\mathcal{C}_{ub}(\mathbb{R}_+, X)$ of all uniformly continuous bounded functions from $\mathbb{R}_+$ to X (see section 3 for definitions). For example, $\mathcal{T}$ can be asymptotically almost periodic (also in the sense of Eberlein) or uniformly ergodic or such that $t \mapsto T(t)x$ vanishes at infinity for all $x \in X$. In all these cases, the semigroup $\mathcal{T}$ is, as a consequence of the uniform boundedness principle, uniformly bounded. Thus, if $\mathcal{S}$ inherits the same property, $\mathcal{S}$ is uniformly bounded as well. This means that, first of

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all, one has to find conditions, under which a bounded perturbation preserves the uniform boundedness of $\mathcal{T}$. We obtain such a result in Corollary 2.2.
The second step consists in proving the permanence of certain asymptotic properties under a bounded perturbation B. In particular, in Theorem 3.5 we prove that

$$\mathcal{S} = \sum_{n=0}^{\infty} \mathcal{T} * [B\mathcal{T}]^n ,$$

where the symbol $[B\mathcal{T}]^n$ denotes the k-fold convolution product $B\mathcal{T} * B\mathcal{T} * ... * B\mathcal{T}$ and the series converges uniformly on $\mathbb{R}_+$.
Then the permanence problem can be rewritten as: If $\mathcal{T}$ is such that $t \mapsto T(t)x$ belongs to the space $\mathcal{E}$ for all $x \in X$, under which conditions does the convolution product $t \mapsto \mathcal{T} * [B\,\mathcal{T}]^n(t)x$ belong to $\mathcal{E}$?
Some recent results of C.J.K.Batty and R.Chill [3] yield an answer to this question. Bounded convolutions were also considered by B.Basit [2]. In particular, he found conditions on the Laplace transform $\hat{f}$ of a measurable function f on $\mathbb{R}_+$, such that $T * f$ is bounded, when $\mathcal{T}$ is a uniformly bounded holomorphic semigroup. However, using the result of [3], we do not need any spectral assumption on $\hat{f}$ or $\sigma(A)$.
We then discuss some examples using operator matrices. Moreover, we expect to extend these results to the case of unbounded perturbations.

2. The convergence of the Dyson-Phillips series

Let X denote a complex Banach space. The symbol $\mathcal{L}(X)$ will denote the Banach algebra of all linear bounded operators on X. The symbol $\mathbb{R}_+$ will denote the set $[0, +\infty)$.
Let $\mathcal{T} = (T(t))_{t\geq 0}$ be a strongly continuous semigroup of linear bounded operators on X, with generator $(A,\, D(A))$ and let $B \in \mathcal{L}(X)$. For the strongly continuous semigroup $\mathcal{S} = (S(t))_{t\geq 0}$, generated by $(A + B, D(A))$, the representation formula

$$S(t) = \sum_{n=0}^{\infty} \big(V^n\mathcal{T}\big)(t)\ ,\ t \geq 0 \tag{2.1}$$

holds, where $(V^0\mathcal{T})(t)x := T(t)x$ and

$$\big(V^{n+1}\mathcal{T}\big)(t)x := \int_0^t \big(V^n\mathcal{T}\big)(s)BT(t-s)x\, ds \text{ for every } x \in X\ .$$

This series is called the Dyson-Phillips series and it is well-known that it converges uniformly on compact intervals of $\mathbb{R}_+$ ([5,Ch.III]). In order to deduce asymptotic properties of $\mathcal{S}$ from those of $V^n\mathcal{T}$ the uniform convergence on $\mathbb{R}_+$ of the Dyson-Phillips series is necessary. This convergence does not hold in general , unless suitable hypotheses are made. For example, in [6] this has been proved in the framework of Banach lattices and for a not necessarily bounded one-dimensional perturbation B. We present the following result.

Theorem 2.1. *Let $\mathcal{T} = (T(t))_{t\geq 0}$ be a strongly continuous semigroup of linear bounded operators on X, generated by $A : D(A) \subseteq X \to X$, satisfying*

$$||T(t)|| \leq M \text{ for all } t \geq 0 \text{ and some } M \geq 1.$$

Let $B \in \mathcal{L}(X)$. Assume that there exists a constant $0 < C < 1$, such that

$$\int_0^t ||BT(s)x||ds \leq C||x||, \tag{2.2}$$

for all $t \geq 0$ and $x \in X$.
Then the series (2.1) converges uniformly on $\mathbb{R}_+$.

Proof. Let $\mathcal{L}_s(X)$ denote the space $\mathcal{L}(X)$ endowed with the strong operator topology.
Let $\mathcal{X}$ denote the operator valued function space $\mathcal{C}_{ub}(\mathbb{R}_+, \mathcal{L}_s(X))$ of all uniformly continuous, bounded functions from $\mathbb{R}_+$ to $\mathcal{L}_s(X)$, that is $F \in \mathcal{X}$ if and only if $F(t) \in \mathcal{L}(X)$ for $t \geq 0$, $t \mapsto F(t)x$ is uniformly continuous for every $x \in X$ and $\sup_{t\geq 0} ||F(t)|| < +\infty$.
This space is a Banach space for the norm

$$||F||_\infty := \sup_{t\geq 0} ||F(t)||, \quad F \in \mathcal{X}.$$

Define the abstract Volterra operator V by

$$VF(t)x := \int_0^t F(s)BT(t-s)x\, ds, \;\; F \in \mathcal{X}, t \geq 0, x \in X\ .$$

We first prove that V is a bounded operator from $\mathcal{X}$ to $\mathcal{X}$.
Let $t \geq 0$ and $x \in X$. Then

$$\begin{aligned} ||VF(t)x|| &\leq ||F||_\infty \int_0^t ||BT(t-s)x||ds \\ &= ||F||_\infty \int_0^t ||BT(s)x||ds \leq ||F||_\infty C||x||, \end{aligned}$$

so that $||VF(t)|| \leq C||F||_\infty$, whence also

$$||VF||_\infty \leq C||F||_\infty \text{ and } ||V||_{\mathcal{L}(\mathcal{X})} \leq C.$$

To show that $t \mapsto VF(t)x$ is uniformly continuous, take $t \in \mathbb{R}_+$, $h > 0$ and $x \in X$. Then

$$\begin{aligned} ||VF(t+h)x - VF(t)x|| &= || \int_0^{t+h} F(s)BT(t+h-s)x\,ds - \int_0^t F(s)BT(t-s)x\,ds|| \\ &\le || \int_0^{t+h} F(s)BT(t+h-s)x\,ds - \int_0^t F(s)BT(t+h-s)x\,ds|| \\ &\quad + || \int_0^t F(s)BT(t+h-s)x\;ds - \int_0^t F(s)BT(t-s)x\,ds|| \\ &= || \int_t^{t+h} F(s)BT(t+h-s)x\,ds|| + || \int_0^t F(s)B\big(T(t+h-s) - T(t-s)\big)x\,ds|| \\ &\le h \cdot M \cdot ||F||_\infty \cdot ||B|| \cdot ||x|| + ||F||_\infty \cdot \int_0^t ||BT(t-s)\big(T(h)x - x\big)||ds \\ &\le h \cdot M||F||_\infty \cdot ||B|| \cdot ||x|| + ||F||_\infty \cdot C \cdot ||T(h)x - x||\,. \end{aligned}$$

This last expression converges to zero, as $h \to 0^+$, uniformly with respect to t. In an analogous way one can show that

$$\lim_{h \to 0^-} ||VF(t+h)x - VF(t)x|| = 0$$

uniformly in $t > 0$ and for every $x \in X$.
Thus $V \in \mathcal{L}(\mathcal{X})$ and $||V||_{\mathcal{L}(\mathcal{X})} \le C$.
By induction one can verify that

$$V^n F(t)x = \int_0^t dt_n \int_0^{t_n} dt_{n-1} \dots \int_0^{t_2} dt_1 F(t_1)BT(t_2 - t_1) \dots BT(t_n - t_{n-1})BT(t - t_n)x$$

for every $t \ge 0$ and $x \in X$, whence it follows that $||V^n||_{\mathcal{L}(\mathcal{X})} \le C^n$ for all $n \in \mathbb{N}$. Thus the Hadamard formula for the spectral radius yields $r(V) \le C$. Since $C < 1$, we conclude that $r(V) < 1$ and $1 \in \rho(V)$.
Observe now that $\mathcal{T}$ belongs to $\mathcal{X}$, since $\mathcal{T}$ is uniformly bounded and, therefore, $t \mapsto T(t)x$ is uniformly continuous for every $x \in X$. Since $I - V$ is bijective on $\mathcal{X}$, there exists a unique $\mathcal{S} \in \mathcal{X}$, such that

$$(I - V)\mathcal{S} = \mathcal{T}.$$

Moreover, the resolvent $R(\lambda, V)$ in $\lambda = 1$ exists and is given by the Neumann series, that is

$$R(1, V) = \sum_{n=0}^{+\infty} V^n.$$

Thus $\mathcal{S} = R(1, V)\mathcal{T} = \sum_{n=0}^{\infty} V^n \mathcal{T}$, where the series converges in the Banach space $\mathcal{X}$, *i.e.* uniformly on $\mathbb{R}_+$, and

$$\sup_{t \ge 0} ||S(t)|| \le \sum_{n=0}^{+\infty} \sup_{t \ge 0} ||\big(V^n \mathcal{T}\big)(t)|| \le \sum_{n=0}^{+\infty} C^n \sup_{t \ge 0} ||T(t)|| \le \frac{M}{1 - C}. \quad \square$$

Since one knows that $\mathcal{S}$ is the semigroup generated by $A+B$, we obtain the following result as well.

Corollary 2.2. *Under the same assumptions of Theorem* 2.1*, the strongly continuous semigroup* $\mathcal{S} = (S(t))_{t\geq 0}$ *generated by* $(A+B, D(A))$ *is uniformly bounded.*

Remarks 2.3
1. If the operator $B \in \mathcal{L}(X)$ satisfies condition (2.2) and commutes with every $T(t)$, $t \geq 0$, then $||BT(t)|| \to 0$ for $t \to +\infty$. Indeed, under these assumptions one has

$$\begin{aligned} t||BT(t)x|| &= \int_0^t ||BT(t)x||ds = \int_0^t ||T(t)Bx||ds \\ &= \int_0^t ||T(t-s)T(s)Bx||ds \leq M \int_0^t ||T(s)Bx||ds \leq MC||x||, \end{aligned}$$

so that $||BT(t)x|| \leq \dfrac{MC}{t}||x||$ for all $x \in X$ and $t > 0$, whence $||BT(t)|| \to 0$ as $t \to +\infty$. In particular, if the range of B is dense in X, then $||T(t)|| \to 0$ for $t \to +\infty$, and therefore, in view of Proposition V.1.7 in [5], $\mathcal{T}$ is uniformly exponentially stable. In this particular case, Theorem 3.1.8 in [8] provides an estimate for the growth bound ω_0 of $\mathcal{T}$, and we have that $\omega_0 \leq -\dfrac{1}{C}$.

2. Suppose now $(A, D(A))$ to be the infinitesimal generator of a strongly continuous, uniformly bounded group $(T(t))_{t\in\mathbb{R}}$ on X. If $B \in \mathcal{L}(X)$ satisfies

$$\int_{-t}^t ||BT(s)x||ds \leq C||x||$$

for all $t \in \mathbb{R}_+$, $x \in X$, and some constant $0 < C < 1$, then it is easy to verify that $(A+B, D(A))$ is the infinitesimal generator of a strongly continuous, uniformly bounded group $(S(t))_{t\in\mathbb{R}}$.

Examples 2.4
Three examples will now be presented. The first shows that every uniformly exponentially stable semigroup $\mathcal{T}$ fulfills condition (2.2), provided B has sufficiently small norm. The second one shows that, when the semigroup $\mathcal{T}$ is expressed by a matrix, the behaviour and the growth of only one entry can be essential, in order to verify condition (2.2). In the last example, a particular perturbation of a matrix-semigroup $\mathcal{T}$ is illustrated, preserving the uniform boundedness of $\mathcal{T}$.

1. Condition (2.2) is satisfied if the semigroup $\mathcal{T}$ is uniformly exponentially stable, *i.e.* $||T(t)|| \leq M\, e^{-\omega t}$ for all $t \geq 0$, some $M \geq 1$ and $\omega > 0$, and, moreover, the linear bounded operator B satisfies $\dfrac{||B||M}{\omega} < 1$.

2. Let X_1 and X_2 be complex Banach spaces. Let $\mathcal{T}_1 = (T_1(t))_{t\geq 0}$ and $\mathcal{T}_2 = (T_2(t))_{t\geq 0}$ be strongly continuous semigroups on X_1 and X_2 respectively, with generator $(A_1, D(A_1))$ and $(A_2, D(A_2))$. Suppose that $||T_1(t)|| = 1$ and $||T_2(t)|| \leq Me^{-\omega t}$ for every $t \geq 0$ and some $M \geq 1$ and $\omega > 0$.
Let $(U(t))_{t\geq 0}$ be the strongly continuous semigroup on $X_1 \times X_2$ given by

$$U(t) := \begin{pmatrix} T_1(t) & 0 \\ 0 & T_2(t) \end{pmatrix} \text{ with generator } \mathcal{A} := \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} .$$

Take a linear bounded operator $B : X_2 \to X_1$ and define the operator matrix

$$\mathcal{B} := \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix} \in \mathcal{L}(X_1 \times X_2) .$$

Then

$$\mathcal{A} + \mathcal{B} = \begin{pmatrix} A_1 & B \\ 0 & A_2 \end{pmatrix} \text{ with domain } D(A_1) \times D(A_2)$$

generates a strongly continuous semigroup $\mathcal{S} = (S(t))_{t\geq 0}$ on $X_1 \times X_2$. Since

$$\mathcal{B}\, U(t) = \begin{pmatrix} 0 & BT_2(t) \\ 0 & 0 \end{pmatrix}$$

and $\mathcal{T}_2$ is uniformly exponentially stable, condition (2.2) in Theorem 2.1 is satisfied, independently on $(T_1(t))_{t\geq 0}$, provided that $\dfrac{M||B||}{\omega} < 1$. By choosing B with sufficiently small norm, the perturbed semigroup $(S(t))_{t\geq 0}$ is then bounded.

3. On the Banach space

$$E := X \times L^1(\mathbb{R}_+, X)$$

define the operators

$$U(t) := \begin{pmatrix} T(t) & R(t) \\ 0 & S(t) \end{pmatrix} , \; t \geq 0 ,$$

where $\mathcal{T} = (T(t))_{t\geq 0}$ is a semigroup on X, generated by $(A, D(A))$, $(S(t))_{t\geq 0}$ is the left translation semigroup on $L^1(\mathbb{R}_+, X)$ and $R(t)$ is defined by

$$R(t)f := \int_0^t T(t-s)f(s)ds \quad \text{for every } f \in L^1(\mathbb{R}_+, X).$$

If $\mathcal{T}$ is bounded, then $\mathcal{U} := (U(t))_{t\geq 0}$ is bounded, since in this case

$$||R(t)f|| \leq M \int_0^t |f(s)|ds \leq M\, ||f||_{L^1(\mathbb{R}_+,X)}$$

for every $f \in L^1(\mathbb{R}_+, X)$.
It is well known [5, Ch.VI.6] that $\mathcal{U}$ is a strongly continuous semigroup and that its infinitesimal generator is given by

$$\mathcal{A} := \begin{pmatrix} A & \delta_0 \\ 0 & \dfrac{d}{ds} \end{pmatrix} \text{ with domain } D(\mathcal{A}) := D(A) \times W^{1,1}(\mathbb{R}_+, X).$$

Take a linear bounded operator $B : L^1(\mathbb{R}_+, X) \to X$ such that $||B|| < 1$. Define the operator matrix

$$\mathcal{B} = \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix} \in \mathcal{L}(E).$$

Since

$$\int_0^t ||BS(s)f||_X ds \leq ||B|| \cdot ||f||_{L^1(\mathbb{R}_+, X)},$$

for every $t \geq 0$ and $f \in L^1(\mathbb{R}_+, X)$, condition (2.2) is satisfied. From Theorem 2.1 we obtain that the operator matrix

$$\mathcal{A} + \mathcal{B} = \begin{pmatrix} A & \delta_0 + B \\ 0 & \frac{d}{ds} \end{pmatrix} \text{ with domain } D(\mathcal{A} + \mathcal{B}) = D(\mathcal{A})$$

is the infinitesimal generator of a uniformly bounded semigroup on $\mathcal{X}$.

3. The permanence of asymptotic properties

Suppose now that certain asymptotic properties of the strongly continuous semigroup $\mathcal{T}$ are known. Is it possible to deduce informations about the asymptotic behaviour of the perturbed semigroup $\mathcal{S}$?

We first consider asymptotic almost periodicity. Recall that a function $f : \mathbb{R}_+ \to X$ is called *asymptotically almost periodic* (a.a.p.) if the set $H(f) = \{f(\cdot + t) : t \in \mathbb{R}_+ \}$ is relatively compact in the space $\mathcal{C}_b(\mathbb{R}_+, X)$ of all continuous, bounded maps from $\mathbb{R}_+$ to X.
The following decomposition theorem holds:

Theorem 3.1 [12]. *A function $f \in \mathcal{C}_b(\mathbb{R}_+, X)$ is asymptotically almost periodic if and only if there exist a unique almost periodic function $g \in \mathcal{C}_b(\mathbb{R}, X)$ and a unique function $h \in \mathcal{C}_b(\mathbb{R}_+, X)$, vanishing at infinity, such that $f = h + g_{|\mathbb{R}_+}$.*

A strongly continuous semigroup is called *strongly asymptotically almost periodic* if the function $t \mapsto T(t)x$, from $\mathbb{R}_+$ to X, is a.a.p. for every $x \in X$.
A first, simple result about asymptotic almost periodicity follows by combining Theorem 2.1 and the following, well-known proposition (see, for example, [1]).

Proposition 3.2. *Let $(A, D(A))$ be the infinitesimal generator of a strongly continuous, uniformly bounded semigroup $\mathcal{T}$. If A has compact resolvent, then $\mathcal{T}$ is strongly asymptotically almost periodic.*

Since $R(\lambda, A+B)$ remains compact, if $R(\lambda, A)$ is compact, then the following result can be stated.

Proposition 3.3. *Let $(A, D(A))$ be the infinitesimal generator of a strongly continuous semigroup $\mathcal{T}$ and let $B \in \mathcal{L}(X)$. Suppose $R(\lambda, A)$ to be compact for some $\lambda \in \rho(A)$.*
If condition (2.2) of Theorem 2.1 holds, then the semigroup $(S(t))_{t\geq 0}$, generated by $(A+B, D(A))$, is strongly asymptotically almost periodic.

Indeed, recent results of C.J.K.Batty and R.Chill [3] yield much more general information on the asymptotic behaviour of the perturbed semigroup. In particular, this method yields information on the asymptotic behaviour of $(S(t))_{t\geq 0}$ independently on the special class of functions considered.

Recall that a closed subspace $\mathcal{E}$ of $C_{ub}(\mathbb{R}_+, X)$ is said to be *translation invariant* if

$$\mathcal{E} = \{f \in C_{ub}(\mathbb{R}_+, X) : \; f(\cdot + t) \in \mathcal{E} \; \} \text{ for all } t \geq 0.$$

A closed subspace $\mathcal{E}$ of $C_{ub}(\mathbb{R}_+, X)$ is said to be *operator invariant* if $M \circ f \in \mathcal{E}$ for every $f \in \mathcal{E}$ and $M \in \mathcal{L}(X)$, where $M \circ f$ is defined by $\big(M \circ f\big)(t) = M(f(t))$, $t \geq 0$.
As remarked by Batty and Chill [3], the following classes of X-valued functions are closed, translation invariant and operator invariant subspaces of $C_{ub}(\mathbb{R}_+, X)$:
- $C_0(\mathbb{R}_+)$
- the class of all asymptotically almost periodic functions from $\mathbb{R}_+$ to X;
- the class of all weakly asymptotically almost periodic functions in the sense of Eberlein;
- the class of totally ergodic functions from $\mathbb{R}_+$ to X;
- the class of totally uniformly ergodic functions from $\mathbb{R}_+$ to X.

The following lemma will be used.

Lemma 3.4 [3]. *Let $L : \mathbb{R}_+ \to \mathcal{L}(X)$ be a bounded strongly continuous function. Let $\mathcal{E}$ be a closed, translation invariant, operator invariant subspace of $C_{ub}(\mathbb{R}_+, X)$, such that the functions $\mathbb{R}_+ \ni t \mapsto L(t)x \in X$ belong to $\mathcal{E}$ for every $x \in X$.*
If $g \in L^1(\mathbb{R}, X)$, then $(L * g)_{|\mathbb{R}_+} \in \mathcal{E}$.

Here, the convolution between L and g is defined as

$$(L * g)(t) = \int_0^{+\infty} L(s)g(t-s)ds = \int_{-\infty}^{t} L(t-s)g(s)ds \, , \; t \geq 0 \, .$$

We can now prove the main result of this paper.

Theorem 3.5. *Let $\mathcal{T} = (T(t))_{t\geq 0}$ be a strongly continuous semigroup of linear bounded operators on X, generated by $(A, D(A))$ and let $B \in \mathcal{L}(X)$, such that $A+B$ generates $(S(t))_{t\geq 0}$. Assume that condition (2.2) in Theorem 2.1 is satisfied. Let $\mathcal{E}$ be a translation invariant and operator invariant closed subspace of $C_{ub}(\mathbb{R}_+, X)$.*
If $t \mapsto T(t)x$ belongs to $\mathcal{E}$ for every $x \in \mathcal{E}$, then $t \mapsto S(t)x$ belongs to $\mathcal{E}$ for every $x \in \mathcal{E}$.

Proof. Let V be the "Volterra-type" operator introduced in Theorem 2.1. In virtue of that theorem, the semigroup $\mathcal{S} = (S(t))_{t\geq 0}$ is uniformly bounded and is given by

$$\mathcal{S} = \sum_{n=0}^{+\infty} V^n \mathcal{T}, \tag{3.1}$$

the convergence being in the norm of $\mathcal{X} = \mathcal{C}_{ub}(\mathbb{R}_+, \mathcal{L}_s(X))$.
First of all, it will be proved by induction that

$$V^n \mathcal{T} = \mathcal{T} * [B\mathcal{T}]^n, \tag{3.2}$$

where the symbol $[B\mathcal{T}]^n$ denotes the convolution $B\mathcal{T} * B\mathcal{T} * ... * B\mathcal{T}$ k-times.
For $n = 1$, one has

$$(V\mathcal{T})(t)x = \int_0^t T(s)BT(t-s)x\,ds = \int_0^t T(t-s)BT(s)x\,ds,$$

for all $t \geq 0$ and $x \in X$, so that $V\mathcal{T} = \mathcal{T} * B\mathcal{T}$.
Suppose now that (3.2) holds for $n-1$. Then

$$(V^n\mathcal{T})(t)x = V(V^{n-1}\mathcal{T})(t)x = \int_0^t \big(V^{n-1}\mathcal{T}\big)(s)BT(t-s)x\,ds, \quad \text{for all } t \geq 0 \text{ and } x \in X\ ,$$

and therefore, by using the associative law for the convolution product $*$,

$$\begin{aligned} V^n\mathcal{T} &= V^{n-1}\mathcal{T} * B\mathcal{T} = \big(\mathcal{T} * [B\mathcal{T}]^{n-1}\big) * B\mathcal{T} \\ &= \mathcal{T} * \big([B\mathcal{T}]^{n-1} * B\mathcal{T}\big) = \mathcal{T} * [B\mathcal{T}]^n, \end{aligned}$$

proving (3.2).

Take now $x \in X$ and define

$$g(t) := \begin{cases} BT(t)x & \text{if } t \geq 0 \\ 0 & \text{if } t < 0\ . \end{cases}$$

Since

$$\int_{-\infty}^{+\infty} ||g(t)||dt = \int_0^{+\infty} ||BT(t)x||dt \leq C||x||,$$

we obtain that g belongs to $L^1(\mathbb{R}, X)$. Since $\mathcal{T}$ is bounded, strongly continuous and, by hypothesis, $t \mapsto T(t)x$ belongs to $\mathcal{E}$ for every $x \in \mathcal{E}$, Lemma 3.4 entails that $(\mathcal{T} * g)_{|\mathbb{R}_+} \in \mathcal{E}$. Observe now that

$$(\mathcal{T} * g)_{|\mathbb{R}_+}(t) = \int_{-\infty}^{t} T(t-s)g(s)ds = \int_0^t T(t-s)BT(s)xds,$$

i.e. $(\mathcal{T} * g)_{|\mathbb{R}_+} = \mathcal{T} * B\mathcal{T} \in \mathcal{E}$.
It will now be shown by induction that $\mathcal{T} * [B\mathcal{T}]^n$ is bounded, strongly continuous (from $\mathbb{R}_+$ to $\mathcal{L}(X)$) and such that the maps $t \mapsto \big(\mathcal{T} * [B\mathcal{T}]^n\big)(t)x$ belong to $\mathcal{E}$ for all $x \in X$.
For $n = 1$, the result has been proved above.
Suppose now $V^{n-1}\mathcal{T}$ to be bounded, strongly continuous and such that every map $t \mapsto \big(V^{n-1}\mathcal{T}\big)(t)x$ belongs to $\mathcal{E}$ (since $\mathcal{E}$ is a subspace of $\mathcal{C}_{ub}(\mathbb{R}_+, X)$, this last condition implies, in particular, that $t \mapsto \big(V^{n-1}\mathcal{T}\big)(t)x$ is uniformly continuous for every $x \in X$).
It has been proved in Theorem 2.1 that the operator V maps $\mathcal{X}$ into $\mathcal{X}$, so that $V\big(V^{n-1}\mathcal{T}\big)$ belongs to $\mathcal{X}$; in particular, $V^n\mathcal{T}$ is strongly continuous and bounded. Moreover, we have

$$V^n\mathcal{T} = \mathcal{T} * [B\mathcal{T}]^n = \big(\mathcal{T} * [B\mathcal{T}]^{n-1}\big) * B\mathcal{T} = V^{n-1}\mathcal{T} * B\mathcal{T}\ .$$

Since by hypothesis $t \mapsto \big(\mathcal{T} * [B\mathcal{T}]^{n-1}\big)(t)x = \big(V^{n-1}\mathcal{T}\big)(t)x$ belongs to $\mathcal{E}$ for all $x \in X$, by defining g as above and by applying again Lemma 3.4, one finally obtains that $t \mapsto \big(\mathcal{T} * [B\mathcal{T}]^n\big)(t)x$ belongs to $\mathcal{E}$ for all $x \in X$.
Thus every term of the series

$$S(t)x = \sum_{n=0}^{+\infty} \big(V^n\mathcal{T}\big)(t)x\ ,\ x \in X\ ,\ t \geq 0\ ,$$

belongs to $\mathcal{E}$. Since $\mathcal{E}$ is closed in $\mathcal{C}_{ub}(\mathbb{R}_+, X)$ and (2.1) converges in the norm of $\mathcal{X}$, every map $t \mapsto S(t)x$ belongs to $\mathcal{E}$. $\square$

Example 3.6
Define, as in Example 2.4.2, the semigroup

$$U(t) := \begin{pmatrix} T_1(t) & 0 \\ 0 & T_2(t) \end{pmatrix},$$

where $(T_1(t))_{t\geq 0}$ is such that the map $t \mapsto T_1(t)x$, is, for every $x \in X_1$, the restriction of an almost periodic function and $(T_2(t))_{t\geq 0}$ is such that $||T_2(t)|| \leq Me^{-\omega t}$ for every $t \geq 0$, for some $M \geq 1$ and $\omega > 0$.
Then $\mathcal{U}$ is strongly asymptotically almost periodic.
Take a linear bounded operator $B : X_2 \to X_1$ with $\dfrac{M||B||}{\omega} < 1$, and define the operator matrix $\mathcal{B}$ as in Example 2.4.2. Then $\mathcal{A} + \mathcal{B}$ is again the infinitesimal generator of a strongly asymptotically almost periodic semigroup, which is given by

$$\left(\begin{pmatrix} T_1(t) & \int_0^t T_1(t-s)BT_2(s)\,ds \\ 0 & T_2(t) \end{pmatrix}\right)_{t\geq 0}.$$

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