New Spectral Criteria for Almost Periodic Solutions of Evolution Equations *

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Abstract

We present a general spectral decomposition technique for bounded solutions to inhomogeneous linear periodic evolution equations of the form $\dot{x} = A(t)x + f(t)$ $(*)$, with f having precompact range, which will be then applied to find new spectral criteria for the existence of almost periodic solutions with specific spectral properties in the resonnant case where $\overline{e^{isp(f)}}$ may intersect the spectrum of the monodromy operator P of $(*)$ (here $sp(f)$ denotes the Carleman spectrum of f). We show that if $(*)$ has a bounded uniformly continuous mild solution u and $\sigma_\Gamma(P)\backslash\overline{e^{isp(f)}}$ is closed, where $\sigma_\Gamma(P)$ denotes the part of $\sigma(P)$ on the unit circle, then $(*)$ has a bounded uniformly continuous mild solution w such that $\overline{e^{isp(w)}} = \overline{e^{isp(f)}}$. Moreover, w is a "spectral component" of u. This allows to solve the general Massera-typed problem for almost periodic solutions. Various spectral criteria for the existence of almost periodic, quasi-periodic mild solutions to $(*)$ are given.

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Key words and **phrases**: *periodic evolution equation, spectrum of functions, spectral decomosition, almost periodic mild solution, quasi-periodic mild solution, Fourier coefficient.*

Abbreviated title: *New Spectral Criteria*

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1 Introduction

We consider in this paper the following linear inhomogeneous integral equation

$$x(t) = U(t,s)x(s) + \int_s^t U(t,\xi)g(\xi)d\xi, \forall t \geq s; t,s \in \mathbf{R}, \tag{1}$$

where f is continuous, $x(t) \in \mathbf{X}$, $\mathbf{X}$ is a Banach space, $(U(t,s))_{t\geq s}$ is assumed to be a 1-periodic evolutionary process on $\mathbf{X}$ (see Definition 2.2 below). In turn, this notion of evolutionary processes arises naturally from the well-posed evolution equations

$$\frac{dx}{dt} = A(t)x + f(t), t \in \mathbf{R}, x \in \mathbf{X}, \tag{2}$$

where $A(t)$ is a (in general, unbounded) linear operator for every fixed t and is 1-periodic in t (see e.g. [Paz]).

Recently, there is an increasing interest in investigating the asymptotic behavior of solutions to the well-posed equation (2) by means of Eq.(1) (see e.g. [Na], [Ne]). An interesting problem to be studied in the qualitative theory of solutions to Eq.(1) is to find conditions for the existence of (almost) periodic solutions to Eq.(1). In this direction, it is known (see e.g. [Pr], [V-S] for the autonomous case and [N-M], [M-N-M] and also [B-H-R] for the periodic case) that if the following *nonresonnant condition* holds

$$(\sigma(P) \cap S^1) \cap \overline{e^{isp(f)}} = \oslash, \tag{3}$$

where $P := U(1,0)$, S^1 denotes the unit circle of the complex plane, and f is almost periodic, then there exists an almost periodic solution x_f to Eq.(1) which is unique if one requires

$$\overline{e^{isp(x_f)}} \subset \overline{e^{isp(f)}}.$$

We may ask a question as what happens in *the resonnant case* where condition (3) fails. In fact, in the particular case where *the forcing term f is 1-periodic and the monodromy operator P is compact* this question has been answered with an additional assumption that there exists a bounded uniformly continuous solution to Eq.(1). Historically, this question goes back to a classical result by Massera [Ma] saying that for Eq.(2) in the finite dimensional case to have a 1-periodic solution it is necessary and sufficient that it has a bounded solution (see [C-H], [D-M, Thrm 11.20], [S-N], [N-M-M-S] for extensions to the infinite dimensional case, and also [F, Example 12.5], [J], [S-Y], [Ya] for related results and counterexamples for almost periodic equations).

It is the purpose of this paper to give an answer to the general problem as mentioned above (*Massera-typed problem*): *Let Eq.(1) have a bounded (uniformly continuous) solution x_f with given almost periodic forcing term f. Then, when does Eq.(1) have an almost periodic solution w (which may be different from x_f) such that*

$$\overline{e^{isp(w)}} \subset \overline{e^{isp(f)}} \quad ?$$

In connection with this problem we note that various conditions are found on the bounded solution, itself, and the countability of the part of spectrum $\sigma(P) \cap S^1$ so that the bounded solution itself is almost periodic, or more generally, together with f belongs to a given function space $\mathcal{F}$ (see e.g. [L-Z], [A-B], [R-V], [Ba]...). Here we note that this philosophy in general does not apply to the Massera-typed problem for almost periodic solutions. In fact, it is not difficult to give a simple example in which f is 1-periodic and a bounded (uniformly continuous) solution to Eq.(1) exists, but this bounded solution itself is not 1-periodic.

Our method is to employ the evolution semigroup (see Definition 2.3 below) associated with $(U(t,s))_{t\geq s}$ to study the harmonic analysis of bounded solutions to Eq.(1). As a result we will prove a spectral decomposition theorem for bounded solutions (Theorem 3.3 and Theorem 3.5) which seems to be useful in dealing with the above Massera-typed problem. In fact, we will apply the spectral decomposition theorem to find new spectral criteria for the existence of almost periodic solutions and will consider particular cases to show the usefulness of this spectral decomposition technique. More concretely, even in the case where condition (3) fails we can still prove the existence of a bounded uniformly continuous solution w to Eq.(1) such that $\overline{e^{isp(w)}} = \overline{e^{isp(f)}}$ provided that $(\sigma(P)\cap S^1)\backslash\overline{e^{isp(f)}}$ is closed, and that Eq.(1) has a bounded uniformly continuous solution u (Corollary 4.2). Since w is a "spectral component" of u in case u is almost periodic the Fourier series of w is part of that of u (Corollary 4.3). Our Corollary 4.4 will deal with a particular autonomous case in which Corollary 4.2 fails to give a spectral criterion for the existence of quasi-periodic mild solutions.

2 Preliminaries

Throughout the paper we will use the following notations: $\mathbf{N},\mathbf{Z},\mathbf{R},\mathbf{C}$ stand for the set of natural, integer, real, complex numbers, respectively. S^1 denotes the unit circle in the complex plane $\mathbf{C}$. $\mathbf{X}$ will denote a given complex Banach space. If T is a linear operator on $\mathbf{X}$, then $D(T)$ stands for its domain. Given two Banach spaces $\mathbf{X},\mathbf{Y}$ by $L(\mathbf{X},\mathbf{Y})$ we will denote the space of all bounded linear operators from $\mathbf{X}$ to $\mathbf{Y}$. As usual, $\sigma(T),\rho(T),R(\lambda,T)$ are the notations of the spectrum, resolvent set and resolvent of the operator T . The notations $BC(\mathbf{R},\mathbf{X}),BUC(\mathbf{R},\mathbf{X}),AP(\mathbf{X})$ will stand for the space of all $\mathbf{X}$-valued bounded, bounded uniformly continuous functions on $\mathbf{R}$ and its subspace of almost periodic (in Bohr's sense) functions, respectively. Recall that a function $f\in BUC(\mathbf{R},\mathbf{X})$ is called *almost periodic (in Bohr's sense)* if $\{S(\tau)f,\tau\in\mathbf{R}\}$ is relatively compact in $BUC(\mathbf{R},\mathbf{X})$, where $(S(t))_{t\in\mathbf{R}}$ is the group of translations on $BUC(\mathbf{R},\mathbf{X})$. It is known (see e.g. [L-Z]) that the set of Fourier-Bohr exponents of an almost periodic function f is defined to be the set of all nonzero real numbers of the form

$$\alpha=\lim_{N\to\infty}\frac{1}{2N}\int_{-N}^{N}e^{-i\rho t}f(t)dt,\rho\in\mathbf{R}.$$

The set of Fourier-Bohr exponents of an almost periodic function f is at most countable and will be denoted by $exp(f)$.

In the paper we will use the notion of *Carleman spectrum* of a bounded continuous function u on the whole line, denoted by $sp(u)$, consisting of $\xi\in\mathbf{R}$ such that the Fourrier-

Carleman transform of u

$$\hat{u}(\lambda) = \begin{cases} \int_0^\infty e^{-\lambda t} u(t)dt, & (Re\lambda > 0); \\ \\ -\int_0^\infty e^{\lambda t} u(-t)dt, & (Re\lambda < 0) \end{cases} \tag{4}$$

has no holomorphic extension to a neighborhood of $i\xi$ (see e.g. [Pr, pp. 19-27]). We refer the reader to [D], [Ka], [Pr], [A-B] for more information on this notion as well as its relations with other notions of spectrum. Basic properties of the Carleman spectrum of a function and its relation with the behavior of the function are listed below for the reader's convenience.

Proposition 2.1 *Let* $f, g_n \in BUC(\mathbf{R}, \mathbf{X})$ *such that* $\lim_{n\to\infty} \|g_n - f\| = 0$. *Then*

i) $sp(f)$ *is closed,*

ii) $sp(f(\cdot + h)) = sp(f)$,

iii) If $\alpha \in \mathbf{C}\backslash\{0\}$, *then* $sp(\alpha f) = sp(f)$,

iv) If $sp(g_n) \subset \Lambda$ *for all* $n \in \mathbf{N}$, *then* $sp(f) \subset \overline{\Lambda}$,

v) $sp(\psi + f) \subset sp(f) \cup sp(\psi), \forall \psi \in BC(\mathbf{R}, \mathbf{X})$,

vi) If u *is uniformly continuous,* $sp(u)$ *is countable and* $\mathbf{X}$ *does not contains any subspace which is isomorphic to the space of sequences* c_0 , *then* u *is almost periodic,*

vii) If u *is uniformly continuous,* $sp(u)$ *is discrete, then* u *is almost periodic.*

Proof For the proof we refer the reader to [Pr, Proposition 0.4, p. 20 , Theorem 0.8 , p. 21] and [A-S], [L-Z, Chap.6].□

For the sake of simplicity of notations we will use throughout the paper the following notation: $\sigma(g) := \overline{e^{isp(g)}}$ for every bounded uniformly continuous function g. Throughout the paper we will denote by $\sigma_\Gamma(P) = \sigma(P) \cap S^1$. If f is almost periodic, then $sp(f) = \overline{exp(f)}$.

The following notion will be used throughout the paper:

Definition 2.2 A family of bounded linear operators $(U(t,s))_{t\geq s}, (t, s \in \mathbf{R})$ from a Banach space $\mathbf{X}$ to itself is called *1-periodic strongly continuous evolutionary process* if the following conditions are satisfied

i) $U(t,t) = I$ for all $t \in \mathbf{R}$,

ii) $U(t,s)U(s,r) = U(t,r)$ for all $t \geq s \geq r$,

iii) The map $(t,s) \mapsto U(t,s)x$ is continuous for every fixed $x \in \mathbf{X}$,

iv) $U(t+1, s+1) = U(t,s)$ for all $t \geq s$,

v) $\|U(t,s)\| < Ne^{\omega(t-s)}$ for some positive N, ω independent of $t \geq s$.

The operator $U(1,0)$ will be called the *monodromy operator* of the evolutionary process $(U(t,s))_{t\geq s}$ and will be denoted by P throughout this paper. Note that in this paper the period of the evolutionary processes is assumed to be 1 merely for the sake of simplicity.

Definition 2.3 Let $(U(t,s))_{t\geq s}, (t,s \in \mathbf{R})$ be a 1-periodic strongly continuous evolutionary process and $\mathbf{F}$ be a closed subspace of $BUC(\mathbf{R},\mathbf{X})$ such that for every fixed $h > 0, g \in \mathbf{F}$ the map $t \mapsto U(t,t-h)g(t-h)$ belongs to $\mathbf{F}$. Then the semigroup of operators $(T^h)_{h\geq 0}$ on $\mathbf{F}$, defined by the formula

$$T^h g(t) = U(t,t-h)g(t-h), \forall t \in \mathbf{R}, h \geq 0, g \in \mathbf{F},$$

is called *evolution semigroup* associated with the process $(U(t,s))_{t\geq s}$.

We refer the reader to the papers [A-M], [N-M], [M-N-M] for more information on the applications of this notion to study the existence of bounded and almost periodic solutions as well as the references therein for information on other applications of it to study exponential dichotomy and stability of solutions to Eq.(1).

In the case where the evolution semigroup $(T^h)_{h\geq 0}$ is strongly continuous on $\mathbf{F}$ [A-M, Thrm 2] yields the explicit formula for the generator $\mathcal{A}$ of $(T^h)_{h\geq 0}$. In fact, similarly to [N-M, Lemma 2] we have the following:

Lemma 2.4 *Let $(T^h)_{h\geq 0}$ be strongly continuous on $\mathbf{F}$, a closed subspace of $BUC(\mathbf{R},\mathbf{X})$. Then its generator $\mathcal{A}$ is the operator on $\mathbf{F}$ with $D(\mathcal{A})$ consisting of all $g \in \mathbf{F}$ such that g is a solution to Eq.(1) with some $f \in \mathbf{F}$ (in this case such a function f is unique), by definition, $\mathcal{A}g = -f$.*

3 Spectral Decomposition

Let us consider the subspace $\mathcal{M} \subset BUC(\mathbf{R},\mathbf{X})$ consisting of all functions $v \in BUC(\mathbf{R},\mathbf{X})$ such that

$$\overline{e^{isp(v)}} := \sigma(v) \subset S_1 \cup S_2 \ , \tag{5}$$

where $S_1, S_2 \subset \mathbf{S}^1$ are disjoint closed subsets of the unit circle. We denote by $\mathcal{M}_v = \overline{spann\{S(t)v, t \in \mathbf{R}\}}$, where $(S(t))_{t\in\mathbf{R}}$ is the translation group on $BUC(\mathbf{R},\mathbf{X})$, i.e. $S(t)v(s) = v(t+s), \forall t,s \in \mathbf{R}$.

Theorem 3.1 *Under the above notations and assumptions the function space $\mathcal{M}$ can be split into a direct sum $\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2$ such that $v \in \mathcal{M}_i$ if and only if $\sigma(v) \subset S_i$ for $i = 1,2$.*

Proof Let $v \in \mathcal{M}$. Then, as is known (see e.g.[A-B])

$$isp(v) = \sigma(\mathcal{D}_{\mathcal{M}_v}). \tag{6}$$

Thus, by the Weak Spectral Mapping Theorem (see e.g. [Na])

$$\sigma(S(1)|_{\mathcal{M}_v}) = \overline{e^{\sigma(\mathcal{D}_{\mathcal{M}_v})}} = \sigma(v) \subset S_1 \cup S_2. \tag{7}$$

Hence there is a spectral projection in $\mathcal{M}_v$ (note that in general we do not claim that this projection is defined on the whole space $\mathcal{M}$)

$$P_v^1 := \frac{1}{2i\pi}\int_\gamma R(\lambda, S(1)|_{\mathcal{M}_v})d\lambda \ ,$$

where γ is a contour enclosing S_1 and disjoint from S_2, (or in general a union of fintely many such countours) by which we have

$$\sigma(P_v^1 S(1) P_v^1) \subset S_1. \tag{8}$$

On the other hand, denote $\Lambda_i \subset BUC(\mathbf{R}, \mathbf{X})$ consisting of all functions u such that $\sigma(u) \subset S_i$ for $i = 1, 2$. Then obviously, $\Lambda_i \subset \mathcal{M}$. Moreover, they are closed subspaces of $\mathcal{M}$, $\Lambda_1 \cap \Lambda_2 = \{0\}$. Now we show that if $v \in \mathcal{M}$, then $P_v^1 v \in \Lambda_1$ and $v - v_1 := v_2 \in \Lambda_2$. If this is true, then it yields that

$$\mathcal{M} = \Lambda_1 \oplus \Lambda_2.$$

To this end, we will prove

$$\sigma(v_j) \subset S_j, \forall j = 1, 2. \tag{9}$$

In fact we show that $\mathcal{M}_{v_1} = ImP_v^1$. Obviously, in view of the invariance of ImP_v^1 under translations we have $\mathcal{M}_{v_1} \subset ImP_v^1$. We now show the inverse. To this end, let $y \in ImP_v^1 \subset \mathcal{M}_v$. Then, by definition, we have

$$y = \lim_{n\to\infty} x_n,$$

where x_n can be represented in the form

$$x_n = \sum_{k=1}^{N(n)} \alpha_{k,n} S(t_{k,n}) v, \alpha_{k,n} \in \mathbf{C}, t_{k,n} \in \mathbf{R}\ \forall n.$$

Hence, since $y, x_n \in \mathcal{M}_v$

$$\begin{aligned} y = P_v^1 y &= \lim_{n\to\infty} \sum_{k=1}^{N(n)} \alpha_{k,n} S(t_{k,n}) P_v^1 v \\ &= \lim_{n\to\infty} \sum_{k=1}^{N(n)} \alpha_{k,n} S(t_{k,n}) v_1 \ . \end{aligned} \tag{10}$$

This shows that $y \in \mathcal{M}_{v_1}$. Thus, by the Weak Spectral Mapping Theorem and (8),

$$\overline{e^{isp(v_1)}} = \sigma(S(1)|_{\mathcal{M}_{v_1}}) = \sigma(S(1)|_{ImP_v^1}) \subset S_1.$$

By definition, $v_1 \in \Lambda_1$ and similarly, $v_2 \in \Lambda_2$. Thus the theorem is proved.□

Remarks Below for every $v \in \mathcal{M}$ we will call $v_j, j = 1, 2$, as defined in the proof of Theorem 3.1, *spectral components* of the functions v. It is easily seen that if in the proof of Theorem 3.1, v is assumed to be almost periodic, then both spectral components v_j are almost periodic.

Lemma 3.2 *Let u be a bounded uniformly continuous solution to Eq.(1). Then the following assertions hold true:*

i)

$$\sigma(u) \subset \sigma_\Gamma(P) \cup \sigma(f) \ , \tag{11}$$

ii) If furthermore we assume that f is of precompact range, then

$$\sigma(u) \supset \sigma(f) \ . \tag{12}$$

Proof (i) For the proof see [B-H-R, Proposition 3.2].
(ii) Under the assumptions it may be seen that the evolution semigroup $(T^h)_{h\geq 0}$ associated with $(U(t,s))_{t\geq s}$ is strongly continuous at $u \in BUC(\mathbf{R}, \mathbf{X})$ (this can be checked directly using Eq.(1)), and f (see [N-M, Lemma 2]). Hence, by Lemma 2.4

$$\lim_{h\to 0^+} \frac{T^h u - u}{h} = \mathcal{A}u = -f \ . \tag{13}$$

Hence, to prove (12) it suffices to show that $\sigma(T^h u) \subset \sigma(u)$. In turn, this inclusion can be shown using the definition of the spectrum of a function as in [V, Lemma 4.3], [B-H-R, Lemma 3.6]. □

Remarks We refer the reader to [L-Z], [Ba], [A-B] ... for results related to the estimate (11) . The estimate (12) seems to be new which yields a necessary condition for the existence of bounded solutions to Eq.(1).

We are now in a position to state the main result of this paper.

Theorem 3.3 *(Spectral Decomposition Theorem) Let u be a bounded, uniformly continuous solution to Eq.(1). Moreover, let f have precompact range and the sets $\sigma(f)$ and $\sigma(P)\cap S^1$ be contained in a disjoint union of the closed subsets $S_1, \cdots, S_k$ of the unit circle. Then the solution u can be decomposed into a sum of k spectral components $u_j, j = 1, \cdots, k$ such that each $u_j, j = 1, \cdots, k$ is a solution to Eq.(1) with $f = f_j, j = 1, \cdots, k$, respectively, where $f = \sum_{j=1}^k f_j$ is the decomposition of f into the sum of spectral components as described in Theorem 3.1, i.e. $u = \sum_{j=1}^k u_j, \sigma(u_j), \sigma(f_j) \subset S_j, j = 1, \cdots, k$ and $u_j \in BUC(\mathbf{R}, \mathbf{X})$ is a solution to Eq.(1) with $f := f_j$ for $j = 1, \cdots, k$.*

Proof Let us denote by $\mathcal{N}$ the subspace of $BUC(\mathbf{R}, \mathbf{X})$ consisting of all functions u such that $\sigma(u) \subset \cup_{j=1}^k S_j$. Then, by assumptions and Theorem 3.1 there are corresponding spectral projections $P_1, \cdots, P_k$ on $\mathcal{N}$ with properties that

i) $P_j P_n = 0$ if $j \neq n$,

ii) $\Sigma_{j=1}^k P_j = I$,

iii) If $u \in ImP_j$, then $\sigma(P_j u) \subset S_j$ for all $j = 1, \cdots, k$.

Note that by Lemma 3.2 for every positive h and $j = 1, \cdots, k$ the operator T^h leaves ImP_j invariant. Hence, $\mathcal{N}$ and $ImP_1, \cdots, ImP_k$ are invariant under the semigroup $(T^h)_{h\geq 0}$. Consequently, since u is a solution to Eq.(1) and f has precompact range the evolution semigroup $(T^h)_{h\geq 0}$ is strongly continuous at u and f. Using the explicit formula for the generator of $(T^h)_{h\geq 0}$ as described in Lemma 2.4 we have

$$\begin{aligned} P_j f = P_j \lim_{h\to 0^+} \frac{T^h u - u}{h} &= P_j \lim_{h\to 0^+} \Sigma_{n=1}^k P_n \frac{T^h u - u}{h} \\ &= \lim_{h\to 0^+} \frac{T^h P_j u - P_j u}{h}. \end{aligned} \tag{14}$$

This yields that $P_j u$ is a solution to Eq.(1) with corresponding $f_j = P_j f$.$\Box$

Remarks If in Theorem 3.3 we assume furthermore that f and u are both almost periodic, then the spectral components $u_j, j = 1, \cdots, k$ are all almost periodic.

Now we are going to focus our special attention on the autonomous equations of the form

$$dx/dt = Ax + f(t), \tag{15}$$

where A is the generator of a C_0-semigroup $(T(t))_{t\geq 0}$, and $f \in BUC(\mathbf{R}, \mathbf{X})$ has compact range. Below we will use the following notation: $\sigma_i(A) = \{\lambda \in \mathbf{R} : i\lambda \in (\sigma(A) \cap i\mathbf{R})\}$. *By mild solutions* of Eq.(15) we will understand in a standard way (see [Paz]) that they are solutions to Eq.(1) with $U(t, s) := T(t - s), \forall t \geq s$. As shown below, in this case we can refine the spectral decomposition technique to get stronger assertions which usefulness will be shown in the next section when we deal with quasi-periodic solutions. To this purpose, we now prove the following lemma.

Lemma 3.4 *Let Eq.(15) satisfy the above conditions, i.e. A generates a C_0-semigroup and $f \in BUC(\mathbf{R}, \mathbf{X})$ has compact range. Moreover, let u be a bounded uniformly continuous mild solution to Eq.(15). Then the following assertions hold:*

i)
$$sp(u) \subset \sigma_i(A) \cup sp(f) \ , \tag{16}$$

ii)
$$sp(u) \supset sp(f). \tag{17}$$

Proof (i) For (16) we can use the standard way by computing the Carleman transform of u as done in [A-B, p. 373]. Then there is no difficulty to get (16) if one uses the definition of the spectrum of a function.
(ii) Note that since for every $h > 0$ the operator T^h is a multiplication by a bounded

operator $T(h)$ we have $sp(T^h u) \subset sp(u)$. Thus, using the argument of the proof of Lemma 3.2 (ii) we have

$$\begin{aligned} sp(f) = sp(-f) &= sp(\lim_{h\to 0^+} \frac{T^h u - u}{h}) \\ &\subset sp(u) \ . \end{aligned}$$

This completes the proof of (ii).$\Box$

The main result for the autonomous case is the following theorem.

Theorem 3.5 *Let A generate a C_0-semigroup and $f \in BUC(\mathbf{R}, \mathbf{X})$ have compact range. Moreover, let u be a bounded uniformly continuous mild solution to Eq.(15). Then the following assertions hold true:*

i) If

$$\overline{e^{i\sigma_i(A)}} \backslash \sigma(f)$$

is closed, Eq.(15) has a bounded uniformly continuous mild solution w such that $\sigma(w) = \sigma(f)$,

ii) If $\sigma_i(A)$ is bounded and

$$\sigma_i(A) \backslash sp(f) \tag{18}$$

is closed, then Eq.(15) has a bounded uniformly continuous mild solution w such that $sp(w) = sp(f)$.

Proof (i) Note that in this case together with (16) the proof of Theorem 3.3 applies.
(ii) Under the assumptions there exists a continuous function ψ which belongs to the Schwartz space of all C^∞-functions on $\mathbf{R}$ with each of its derivatives decaying faster than any polynomial such that its Fourier transform $\tilde{\psi}$ has $\sigma_i(A)\backslash sp(f)$ as its support (which is compact in view of the assumptions). Hence, every bounded uniformly continuous function g such that $sp(g) \subset \sigma_i(A) \cup sp(f)$ can be decomposed into the sum of two spectral components as follows:

$$g = g_1 + g_2 = \psi * g + (g - \psi * g),$$

where $g_1 = \psi * g, g_2 = (g - \psi * g)$. Moreover, this decomposition is continuous in the following sense: If $g^{(n)}, n = 1, 2, \cdots$ is a sequence in $BUC(\mathbf{R}, \mathbf{X})$ with $sp(g^{(n)}) \subset \sigma_i(A) \cup sp(f)$ such that $\lim_n g^{(n)} = g$ in $BUC(\mathbf{R}, \mathbf{X})$, then $\lim_n g_1^{(n)} = g_1, \lim_n g_2^{(n)} = g_2$. Hence we have in fact proved a version of Theorem 3.1 which allows us to employ the proof of Theorem 3.3 for this assertion (ii). $\Box$

Remarks

i) In view of the failure of the Spectral Mapping Theorem for general C_0-semigroups the condition in the assertion (i) is a little more general than that formulated in terms of $\sigma(T(1))$.

ii) If we know beforehand that u is almost periodic, then in the statement of Theorem 3.5 we can claim that the spectral component w is almost periodic.

4 Spectral Criteria For Almost Periodic Solutions

This section will be devoted to some applications of the spectral decomposition theorem to prove the existence of almost periodic solutions with specific spectral properties. In particular, we will revisit the classical result by Massera on the existence of periodic solutions as well as its extensions. To this end, the following notion will play the key role.

Definition 4.1 Let $\sigma(f)$ and $\sigma_\Gamma(P)$ be defined as above. We say that the set $\sigma(f)$ and $\sigma_\Gamma(P)$ satisfy *the spectral separation condition* if the set $\sigma_\Gamma(P)\backslash\sigma(f)$ is closed.

Corollary 4.2 *Let f be almost periodic, $\sigma(f)$ and $\sigma_\Gamma(P)$ satisfy the spectral separation condition. Moreover, let $\sigma(f)$ be countable and $\mathbf{X}$ not contain any subspace which is isomorphic to c_0. Then if there exists a bounded uniformly continuous solution u to Eq.(1), there exists an almost periodic solution w to Eq.(1) such that $\sigma(w) = \sigma(f)$.*

Proof We define in this case $S_1 := \sigma(f)$, $S_2 := \sigma_\Gamma(P)\backslash\sigma(f)$. Then, by Theorem 3.3 there exists a solution w to Eq.(1) such that $\sigma(w) \subset \sigma(f)$. Using the estimate (12) we have $\sigma(w) = \sigma(f)$. In particular, by Proposition 1.1 (vi), w is almost periodic. □

Remarks

i) If $\sigma(f)$ is finite, then by Proposition 1.1 (vii) the condition that $\mathbf{X}$ does not contain any subspace isomorphic to c_0 can be dropped.

ii) In the case where $\sigma_\Gamma(P)$ is countable it is known that with additional ergodic conditions on u the solution u has "similar spectral properties" as f (see e.g. [L-Z], [Ba], [A-B], [B-H-R],...). However, in many cases it is not expected that the solution u itself has similar spectral properties as f as in the Massera-typed problem (see [Ma], [C-H], [S-N], [N-M-M-S], ...).

iii) In the case where P is compact (or merely $\sigma_\Gamma(P)$ is finite) the spectral separation condition is always satisfied. Hence, we have a natural extension of Massera's result for almost periodic solutions. In this case see also Corollary 4.3 below.

iv) We emphasize that the solution w in the statement of Corollary 4.2 is a "$\sigma(f)$-spectral component" of the bounded solution u. This will be helpful to find the Fourier coefficients of w as part of those of u.

v) In view of estimate (12) w may be seen as a "minimal" solution in some sense.

Corollary 4.3 *Let all assumptions of Corollary 4.2 be satisfied. Moreover, let $\sigma_\Gamma(P)$ be countable. Then if there exists a bounded uniformly continuous solution u to Eq.(1), it is almost periodic. Moreover, the following part of the Fourier series of u*

$$\Sigma b_\lambda e^{i\lambda t} \quad , b_\lambda = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} e^{-i\lambda\xi}u(\xi)d\xi, \tag{19}$$

where $e^{i\lambda} \in \sigma(f)$, is again the Fourier series of another almost periodic solution to Eq.(1).

Proof The assertion that u is almost periodic is standard in view of (11) (see e.g. [L-Z], [A-B], [Ba]). It may be noted that in the case u is almost periodic, the spectral decomposition can be carried out in the function space $AP(\mathbf{X})$ instead of the larger space $BUC(\mathbf{R}, \mathbf{X})$. Hence, we can decompose the solution u into the sum of two almost periodic solutions with spectral properties described in Theorem 3.3. Using the definition of Fourier series of almost periodic functions we arrive at the next assertion of the corollary. $\square$

The following corollary will show the advantage of Theorem 3.5 which allows us to take into account the structure of $sp(f)$ rather than that of $\sigma(f)$. To this end, we introduce the following terminology. A set of reals S is said to have an *integer and finite basis* if there is a finite subset $T \subset S$ such that any element $s \in S$ can be represented in the form $s = n_1 b_1 + \cdots + n_m b_m$, where $n_j \in \mathbf{Z}, j = 1, \cdots, m,\ b_j \in T, j = 1, \cdots, m$. If f is quasi-periodic, the set of its Fourier-Bohr exponents is discrete (which coincides with $sp(f)$ in this case), then the spectrum $sp(f)$ has an integer and finite basis (see [L-Z, p.48]). Conversely, if f is almost periodic and $sp(f)$ has an integer and finite basis, then f is quasi-periodic. We refer the reader to [L-Z, pp. 42-48] more information on the relation between quasi-periodicity and spectrum, Fourier-Bohr exponents of almost periodic functions.

Corollary 4.4 *Let all assumptions of the second assertion of Theorem 3.5 be satisfied. Moreover, assume that* $\mathbf{X}$ *does not contain* c_0. *Then if* $sp(f)$ *has an integer and finite basis, Eq.(15) has a quasi-periodic mild solution* w *with* $sp(w) = sp(f)$.

Proof Under the corollary's assumptions the spectrum $sp(w)$ of the solution w, as described in Theorem 3.5, is in particular countable. Hence w is almost periodic. Since $sp(w) = sp(f)$, $sp(w)$ has an integer and finite basis. Thus w is quasi-periodic.$\square$

Below we will consider some particular cases

Example 1. **Periodic solutions**. If $\sigma(f) = \{1\}$ we are actually concerned with the existence of periodic solutions. Hence, Corollary 4.2 extends the classical result by Massera to a large class of evolution equations which has 1 as an isolated point of $\sigma_\Gamma(P)$. Moreover, Corollary 4.3 provides a way to approximate the periodic solution. (See [N-M-M-S] for the case the monodromy operator P is compact.)

Example 2. **Anti-periodic solutions**. An anti-periodic (continuous) function f is defined to be a continuous one which satisfies $f(t+\omega) = -f(t), \forall t \in \mathbf{R}$ and here $\omega > 0$ is given. Thus, f is $2-\omega$-periodic. It is known that, the space of anti-periodic functions f with antiperiod ω, which is denoted by $\mathcal{AP}(\omega)$, is a subspace of $BUC(\mathbf{R}, \mathbf{X})$ with spectrum

$$sp(f) \subset \{\frac{2k+1}{\omega}, k \in \mathbf{Z}\}.$$

Without loss of generality we can assume that $\omega = 1$. Obviously, $\sigma(f) = \{-1\}, \forall f \in \mathcal{AP}(\omega)$ (for more information on anti-periodic solutions problem see [A-P], [Nak], [O], [V-S]). In this case the spectral separation condition as mentioned in Definition 3 is nothing but the condition that $\{-1\}$ is an isolated point of $\sigma_\Gamma(P)$. Hence, we have extended Massera's result to anti-periodic solutions.

Example 3. Let u be a bounded uniformly continuous solution to Eq.(1) with f 2-periodic. Let us define

$$F(t) = \frac{f(t) - f(t+1)}{2}, G(t) = \frac{f(t) + f(t+1)}{2}, \forall t \in \mathbf{R}.$$

Then, it is seen that F is 1-anti-periodic and G is 1-periodic. Applying Theorem 3.3 we see that there exist two solutions to Eq.(1) as two components of u which are 1-antiperiodic and 1-periodic with forcing terms F, G, respectively. In particular, the sum of these solutions is a 2-periodic solution of Eq.(1) with forcing term f.

Example 4. Let A be a sectorial operator in a Banach space $\mathbf{X}$ and the map $t \mapsto B(t) \in L(\mathbf{X}^\alpha, \mathbf{X})$ be Hölder continuous and 1-periodic. Then, as shown in [He, Thrm 7.1.3] the equation

$$\frac{dx}{dt} = (-A + B(t))x, \tag{20}$$

generates an 1-periodic strongly continuous evolutionary process $(U(t,s))_{t\geq s}$. If, furthermore, A has compact resolvent, then the monodromy operator P of the process is compact. Hence, for every almost periodic function f the sets $\sigma(f)$ and $\sigma_\Gamma(P)$ always satisfy spectral separation condition. In [N-M] we have shown that if $\sigma_\Gamma(P) \cap \sigma(f) = \oslash$, then there is a unique almost periodic solution x_f to the inhomogeneous equation

$$\frac{dx}{dt} = (-A + B(t))x + f(t) \tag{21}$$

with property that $\sigma(x_f) \subset \sigma(f)$. Now suppose that $\sigma_\Gamma(P) \cap \sigma(f) \neq \oslash$. By Corollary 4.2, if u is any bounded solution (the uniform continuity follows from the boundedness of such a solution to Eq.(21)), then there exists an almost periodic solution w such that $\sigma(w) = \sigma(f)$. We refer the reader to [He] and [Paz] for examples from parabolic differential equations which can be included into the abstract equation (21).

Example 5. Consider the heat equation in materials

$$\begin{cases} v_t(t,x) = \Delta v(t,x) + f(t,x), t \in \mathbf{R}, x \in \Omega \\ \\ v(t,x) = 0, t \in \mathbf{R}, x \in \partial\Omega, \end{cases} \tag{22}$$

where $\Omega \subset \mathbf{R}^n$ denotes a bounded domain with smooth boundary $\partial\Omega$. Let $\mathbf{X} = L^2(\Omega), A = \Delta$ with $D(A) = W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$. Then A is selfadjoint and negative definite (see e.g. [Paz]). Hence $\sigma(A) \subset (-\infty, 0)$. In particular $\sigma_i(A) = \oslash$. Eq.(22) now becomes

$$\frac{dv}{dt} = Av + f. \tag{23}$$

We assume further that $f(t,x) = a(t)g(x)$ where a is a bounded uniformly continuous real function with $sp(a) = \mathbf{Z} \cup \pi\mathbf{Z}$, $g \in L^2(\Omega), g \neq 0$. It may be seen that $\sigma(f) = S^1$ and $sp(f)$ has an integer and finite basis. Hence, Theorem 3.3 does not give any information on the existence of a solution w with specific spectral properties. However, in this case Theorem 3.5 applies.

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