Elastic and exchange interaction of kinks in some bubbly mediums

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Abstract

The properties of nonlinear waves in one of nonlinear partial differential equations were investigated. The kinks with solitonic, but unusual from the classic viewpoint, behaviour were found. The equation itself does not possess the Painlevé property.

1 Introduction and preliminary

Since the discovery of soliton interaction by Zabusky and Kruskal [1] and the invention of the Inverse Scattering Transform by Gardner, Greene, Kruskal and Miura [2] at the end of the sixties nonlinear partial differential equations have extensively been studied in mathematics and physics. Much efforts have been made to understand their properties, and much progress has been achieved up to now. However, only so-called integrable systems have so far been investigated in reality. A number of methods and structures are applicable and associate with them: the above IST, Bäcklund transformations, master-symmetries, infinite hierarchies of conservation laws, etc. Furthermore, though such systems usually comprise some simplification, they do play an important role in many applications from traditional hydrodynamics or optics to biology and neurophysiology. For these reasons, the overwhelming majority of research works deals with namely 'integrable' equations despite the fact that their behaviour is very similar, and main features (e.g., phase shift or, say, a Hirota's ansatz) are well known and were widely described in the literature.

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In contrast to them, ‘nonintegrable’ models are a significantly less studied area of nonlinear science. Any theory is still missing, and a remarkable fact that they may differ radically. One major surprise is the existence of systems with solutions and behaviour analogous to classical solitonic cases [3, 4], although this kind of waves (they were called ‘soliton structures’ in [5]) is very seldom realized in practice.

In the present work, one more nonlinear partial differential equation is proposed and investigated both via numerical and by analytical techniques. The equation may arise as a model for wave processes in specific bubbly mediums [6]. It can be classified as a ‘nonintegrable’ one and admits solutions in the form of kinks with special properties. These structures play an important role in the system behaviour and can be generated from appropriate initial disturbances. They are of solitonic features. To be precise, they elastically interact with other localized waves. While, as a result of a collision between two such kinks, new kinks with other wave numbers and velocities are shaped. In other words, after an interaction we have not only a phase shift, but change-over from one state to another as well. This switch always keeps the structure itself, however. To our knowledge, this is the only equation with such properties, which could be considered as 'exchange interaction'. In this respect the results obtained here are important, because this is in agreement with the view to the above-mentioned ‘soliton-structures’ proposing in [5, 7, 8].

The paper is further organized as follows: First, the ground for the nonlinear equation is first adduced in short. In doing so, the derivation is mainly following the same procedure as [9, 10], but taking into account the additional assumptions relevant in our case. This is done in section 2. In section 3, the derived equation is investigated by means of the invariant singular manifold approach due to [11]. We will start just with a recapitulation and restrict the analysis to a brief sketch. (An excellent survey is given in [12]). Section 4 is devoted to computer simulation for various initial data and properties of the kinks derived previously. The paper is ended with some discussion and remarks in Conclusion.

2 The physical model

In [6] the existence of peculiar submicrobubbles in aqueous electrolytes was theoretically predicted, and a number of experimental data were presented, which could be treated as the evidence. Such submicrobubbles constitute gaps in the fluid structure charged because of ion hydration of the solution. Their diameter was estimated as 100–1000 Å. Here, we propose a nonlinear equation for describing of pressure waves in such hypothetical mediums. The derivation mainly follows the same line as the preceding ones for conventional cases [9, 10], but some assumptions valid for the above scales and situation are making.

For simplicity the plane flow and an incompressible viscous liquid are considered. In the one-velocity approximation for the mixture, the hydrodynamic system takes the form

$$\rho_t + (\rho u)_x = 0 \tag{1}$$

$$\rho(u_t + uu_x) = -p_x + \frac{4}{3}\nu u_{xx}, \qquad \nu > 0 \tag{2}$$

(ρ, u, p, ν are the density and velocity of the medium and the liquid pressure and viscosity, respectively.) It is closed by the Rayleigh-Plesset equation for a bubble and by the expression for the mean density

$$p + \frac{2\sigma}{R} + \rho_l\left(R\ddot{R} + \frac{3}{2}\dot{R}^2\right) + 2\nu\frac{\dot{R}}{R} - \frac{q^2}{4\pi R^4} = 0, \qquad \sigma > 0 \tag{3}$$

$$\rho = \rho_l\left(1 - \frac{4}{3}\pi R^3 n\right) \tag{4}$$

Where the surface tension, bubble and liquid densities are denoted by σ, n and ρ_l, as well as a bubble radius and charge by R and q. In our approach the gas pressure in a gap becomes too small and is negligible.

Next, let us introduce the new independent variables instead of R and ρ in (3) and (4) as follows

$$\begin{aligned} &R = R_0 + \delta, \quad |\delta| \ll R_0 \ll 1 \\ &\rho = \rho_0 - a\delta + \mathrm{O}(\delta^2), \quad a = 4\pi R_0^2 n \rho_l > 0 \end{aligned} \tag{5}$$

Obviously, R_0 and ρ_0 are the undisturbed bubble radius and mixture density. From equation (3) one concludes

$$p = p_0 - b\delta - c\dot{\delta} + \mathrm{O}(\delta^2) \tag{6}$$

(The dynamic terms are small enough in this case.) Here p_0 (the undisturbed pressure), b and c are used as the abbreviations

$$p_0 = \frac{q^2}{4\pi R_0^4} - \frac{2\sigma}{R_0}$$

$$b = \frac{q^2}{\pi R_0^5} - \frac{2\sigma}{R_0^2}$$

$$c = \frac{2\nu}{R_0}$$

It can be checked easily that $b > 0$ for physically relevant values of p_0 and R_0, because the expression for it is written down as

$$b = \frac{3q^2}{4\pi R_0^5} + \frac{p_0}{R_0}$$

Substituting (5) into (1), in the lowest order one arrives at the equation

$$a\rho_t - \rho_0 u_x = 0 \tag{7}$$

This is clear to be valid for u of the order of one only for long enough waves. The latter confirms with the natural assumption that the characteristic length of the perturbation is much greater that the distance between bubbles in order to particularize their distribution. After that, taking into account (5), (6), and retaining the leading terms, equation (2) is replaced by

$$\rho_0(u_t + uu_x) = b\delta_x + c\delta_{tx} + \frac{4}{3}\nu u_{xx} \tag{8}$$

for $u \sim \mathrm{O}(1)$, sufficiently long waves and considerable viscosity.

One can eventually rearrange (7) and (8) into the system for δ

$$\begin{aligned} \delta_t &= \frac{\rho_0}{a} u_x \\ \delta_x &= \frac{\rho_0}{b}(u_t + uu_x) - \left(\frac{c\rho_0}{ba} + \frac{4\nu}{3b}\right) u_{xx} \end{aligned} \tag{9}$$

Moreover, its compatibility condition, after scaling as

$$\begin{aligned} u &\longrightarrow \pm 2 \left(\sqrt{\frac{b}{a}}\right) u \\ t &\longrightarrow \frac{a}{b}\left(\frac{c}{a} + \frac{4\nu}{3\rho_0}\right) t \\ x &\longrightarrow \pm\sqrt{\frac{a}{b}}\left(\frac{c}{a} + \frac{4\nu}{3\rho_0}\right) x \end{aligned}$$

to normalize the coefficients to one, results in

$$u_{tt} - u_{xx} + (u^2)_{tx} - u_{txx} = 0 \tag{10}$$

The latter equation is subject to study thereafter.

3 The invariant Painlevé analysis and the singular manifold equations

Although, as is now becoming clear, the Painlevé property (having poles as the only movable singular point, i.e. whose location depends on the initial conditions) is neither sufficient nor necessary for integrability, and there are reasons for doubting its general validity (see [12, 13] and references therein), all the classical soliton equations possess it. And, thus, a negative result may indicate another nonlinear dynamics for the system under consideration.

Acting in correspondence with the main scheme of the singular manifold method [12] and using the representation of the invariant Painlevé analysis due to [11], one expands u in (10) as the power series

$$u(x,t) = \sum_{i=-n}^{+\infty} w_i(x,t) \cdot \chi^i(x,t), \qquad n \in N \tag{11}$$

Where w_i are functions of independent variables being than determined in the recursive manner, and χ obeys to the Riccati equations

$$\chi_x = 1 + \frac{S}{2}\chi^2$$
$$\chi_t = -C + C_x\chi - \frac{1}{2}(CS + C_{xx})\chi^2$$

with the compatibility condition to $S(x,t)$ and $C(x,t)$

$$S_t + C_{xxx} + 2SC_x + CS_x = 0$$

Let us substitute the above into the equation of interest and equate the same powers of χ. The leading term is seen to be $-\chi^{-1}$. There is no difficulty in evaluation other ones except at the resonances $i = 1, 2$. In those case, the following constrains arise

$$C_t + CC_x = 0$$
$$\left(2C_x - C\frac{\partial}{\partial x}\right)(C_t + CC_x) = 0$$

that indicates just the weak Painlevé property.

The next step that we now have to consider is the truncation of (11) at the constant level

$$u = -\chi^{-1} + \frac{C^2 - 1}{2C}$$

It yields another set of equations, which reduce to the simple relations to S and C via Differential Gröbner Bases [14]

$$
\begin{aligned}
S_{xx} &= 0 \\
S_t + CS_x &= 0 \\
C_x &= 0 \\
C_t &= 0
\end{aligned}
$$

And we obtain the simplest bounded solutions of the kink type to the nonlinear equation (10)

$$
u_{\text{kink}}(x,t) = \frac{k}{2}\left(\frac{1-\exp(k(x-vt))}{1+\exp(k(x-vt))}\right) + \frac{v^2-1}{2v}; \quad k, v = Const \tag{12}
$$

Their interactions with other waves and with each other are numerically investigated in the next section.

4 Computer simulation

In order to investigate properties of the kinks obtained previously and evolution of other nonlinear waves, equation (10) was numerically studied by the finite-difference methods. In our computations, the following implicit scheme

$$
\begin{aligned}
&\frac{u_i^{n+1}-2u_i^n+u_i^{n-1}}{\tau^2} - \frac{u_{i+1}^n-2u_i^n+u_{i-1}^n}{h^2} + \frac{u_i^n}{\tau}\left(\frac{u_{i+1}^{n+1}-u_{i-1}^{n+1}}{2h} - \frac{u_{i+1}^{n-1}-u_{i-1}^{n-1}}{2h}\right) \\
&\quad + 2\left(\frac{u_i^{n+1}-u_i^{n-1}}{2\tau}\right)\left(\frac{u_{i+1}^n-u_{i-1}^n}{2h}\right) \\
&\quad - \frac{1}{2\tau}\left(\frac{u_{i+1}^{n+1}-2u_i^{n+1}+u_{i-1}^{n+1}}{h^2} - \frac{u_{i+1}^{n-1}-2u_i^{n-1}+u_{i-1}^{n-1}}{h^2}\right) = 0, \\
&\qquad\qquad\qquad\qquad n = \overline{0,N};\, i = \overline{0,I}
\end{aligned}
\tag{13}
$$

was employed with an appropriate choice of an initial disturbance and the boundary conditions $u_0^n, u_I^n = Const$. Here, obviously, u_i^n is a mesh function at points ($x = hi$, $t = \tau n$); h and τ are the grid spacings.

The difference equation (13) reduces to the set of algebraic ones with a band matrix simple for solving

$$
a_i^n u_{i-1}^n + b_i^n u_i^n + c_i^n u_{i+1}^n = 0
$$

The above scheme is found to give the best result among other, both implicit and explicit ones with various approximation for the nonlinear addent in

(10). (The schemes for (9) were considered as well, by the way.) It is of second-order accuracy $\mathrm{O}(\tau^2) + \mathrm{O}(h^2)$ and is while very economic, efficient and simple for realization. And, though it is not absolutely stable ($\tau < 1.1h$ is needed), this seen to do not constitute any difficulties in practice.

In order to verify real accuracy of calculations, a number of experiments were performed both with the test solution (12) and with various time and space intervals. The results were compared with analytical ones and each other, and the errors are in good agreement with the theoretical prediction. The ration $\tau = 2h$ would be the best choice from many points of view. Moreover, there is no indication of any numerical instability in those and other computations because of roundoff errors. The errors do not increase with the laps of time and remind within the scheme accuracy after sufficiently long time, although we used up to 10 000 steps in time and points on x-axis. All were done in long double precision (18 figures).

In the sequence, some concrete results of computer modeling are adduced. The most of them were carried our at $h = 0.01$ and $t = 0.005$, which gives accurate enough results.

First of all, investigate interaction of the kink solution (12) with another localized wave or kink. In the former case, the various situations were considered to examine solitonic properties for the solution (12). In doing so, we varied both the direction of kink moving along the x-axis, its wave number, and the form and type of a perturbation. One of such collisions is illustrated in figure 1, where the waveforms are presented versus time. The only visible effect for the kink is the phase shift. In opposition to this, in the latter cases, wave numbers and velocities of the kinks are also changed after a collision, although in both cases the kinks always keep their structures, and none of them loses its identity after overlapping. This is demonstrated in figures 2, 3, for example. Figure 2 depicts one interesting instance of such a kink-kink collision. The matter is that, in a general case, the link between kink parameters before and after cannot be determined from analytical or numerical considerations at present. However, the problem is simplified in that rather restrictive case. Here from the boundary conditions and the symmetry we have the relation

$$\frac{v_{\text{before}}^2 - 1}{v_{\text{before}}} = \frac{v_{\text{after}}^2 - 1}{v_{\text{after}}}$$

This enables one to interpret the process as switching from one stationary state to another, that may be impotent for some technological applications.

One more circumstance should also be pointed out: the smaller the wave number of a kink generated after an interaction, the larger time for its final formation; and this time exponentially grows for small values.

The next series of the experiments were devoted to evolution of shock waves in the medium, and the initial data were given by the expression

$$u(x,0) = c_1 \frac{\exp kx}{1+\exp kx} + c_0 \tag{14}$$

with some initial velocity $v = -\frac{\partial v}{\partial t} / \frac{\partial v}{\partial x}\Big|_{t=0}$. The key observation is here that for both shock- ($c_1 k < 0$) and for 'antishock-' ($c_1 k > 0$) waves there exist three distinct regions of v separated by the critical points v_- and v_+. This takes place irrespectively to the parameters in (14), although v_- and v_+ depend on c_1 and c_0. Within each of the regions a scenario is qualitatively the same: the wave splits to two new ones. Their type depends on the initial perturbation type and region. On the other hand, there are just two types: A — the above kink solution (12) and B — a quasi-steady structure. At the critical values v_- and v_+, the only wave (and, maybe, some noise quickly dissipating) shapes as a result. All this could be illustrated by the diagrams

$$\begin{array}{ll}
\text{shock-wave, } v > v_+ \longrightarrow B + A & (a) \\
\text{shock-wave, } v = v_+ \longrightarrow A & (b) \\
\text{shock-wave, } v_- < v < v_+ \longrightarrow A + A & (c) \\
\text{shock-wave, } v = v_- \longrightarrow A & (d) \\
\text{shock-wave, } v < v_- \longrightarrow A + B & (e) \\
\text{antishock-wave, } v > v_+ \longrightarrow A + B & (f) \\
\text{antishock-wave, } v = v_+ \longrightarrow B & (g) \\
\text{antishock-wave, } v_- < v < v_+ \longrightarrow B + B & (h) \\
\text{antishock-wave, } v = v_- \longrightarrow B & (i) \\
\text{antishock-wave, } v < v_- \longrightarrow B + A & (j)
\end{array}$$

Some of them are shown in figures 4–7. It is clear that kink-kink formation considered before (figures 2, 3) and the kinks themselves can be fell to the categories (c) and (b),(d), respectively.

Finally, it is necessary to note that the wave number in (14) has no effect on evolution or, more precisely, plays a minimal role and just during a short initial period. The Heaviside step function could be used instead without loss of generality.

Collapse of a localized disturbance was also considered in our investigation. Such structures dissipate with time, perhaps after preliminary splitting into several impulses. Figure 8 exhibits such a situation.

Unfortunately, we have no possibility to verify solitonic properties of the above quasi-steady structure. This appears to be a rather difficult problem for computer modeling due to its specificity and the computer facility limit.

5 Conclusion

Before we have studied the nonlinear partial differential equation, which could arise in the theory of bubbly mediums, analytically and numerically and discovered its interesting properties. It admits the solitonic solutions of the kink type, and they elastically interact with localized perturbations in the manner standard for solitons. But their mutual interaction is unusual from the classical standpoint. The shock-wave behaviour is remarkable as well.

The equation does not possess the Painlevé properties and could be classified as 'nonintegrable' one, though this has not proved rigorously. Also, it seems to have no conservation laws besides itself and the following

$$\frac{\partial}{\partial t}\left(2xu^2 + 2(t+1)u - (t^2+x^2)u_t\right) + \\ \frac{\partial}{\partial x}\left((t^2+x^2)(u_x - 2uu_t + u_{tx}) - 2x(u+u_t)\right) = 0$$

$$\frac{\partial}{\partial t}(u - tu_t) + \frac{\partial}{\partial x}(-2tuu_t + tu_x + tu_{tx}) = 0$$
$$\frac{\partial}{\partial t}(u^2 - xu_t) + \frac{\partial}{\partial t}(xu_x - 2xuu_t - u - u_t + xu_{tx}) = 0$$

And its Lie-point symmetries are trivial and correspond just to x- and t-translations.

The research was initiated in [15] and was partially presented at the 6th International Conference on Evolution Equations and their Applications in Physics and Life Sciences, 14-19 September 1998, Bad Herrenalb, FRG [16].

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Figure captions

Figure 1. Interaction of a kink ($k = 7$, $v = 6.5$) with a localized perturbation.
Figure 2. Kink-kink interaction ($k_1 = 2.1$, $v_1 = 2.5$, $k_2 = 2.1$, $v_2 = -2.5$).
Figure 3. Kink-kink interaction ($k_1 = 1.7$, $v_1 = 2.5$, $k_2 = 2.5$, $v_2 = -2.5$).
Figure 4. Collapse of a shock-wave ($k = 10$, $v = 2 > v_+$).
Figure 5. Collapse of an antishock-wave ($k = 10$, $v_- < v = 0 < v_+$).
Figure 6. Collapse of an antishock-wave ($k = 10$, $v = -0.68 = v_-$).
Figure 7. Collapse of an antishock-wave ($k = 10$, $v = -2.5 < v_-$).
Figure 8. Dissipation of a localized perturbation.